Discontinuous Initial Value Problems for Functional Differential-Algebraic Equations of Mixed Type

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Abstract

We study the well-posedness of initial value problems for nonlinear functional differential-algebraic equations of mixed type. We are interested in solutions to such problems that admit a single jump discontinuity at time zero. We focus specially on the question whether unstable equilibria can be stabilized by appropriately choosing the size of the jump discontinuity. We illustrate our techniques by analytically studying an economic model for the interplay between inflation and interest rates. In particular, we investigate under which circumstances the central bank can prevent runaway inflation by appropriately hiking the interest rate.

Key words: functional differential equations, advanced and retarded arguments, interest rates, inflation rates, initial value problems, indeterminacy, impulsive equations.

1 Introduction

In this paper we consider a class of initial value problems that includes the prototypes

$$\begin{array}{rclr} \mathcal{I}x'(\xi) & = & x(\xi) + \int_{-1}^{1} x(\xi+\sigma)d\sigma + f\big(x(\xi)\big) & \text{for all } \xi \geq 0, \\ x(\tau) & = & \phi(\tau) & \text{for all } -1 \leq \tau \leq 0, \end{array} \tag{1.1}$$

in which $\mathcal{I}$ is a $n \times n$ diagonal matrix that is allowed to be either singular or invertible, while the nonlinearity f is required to have $f(0) = Df(0) = 0$. We require the solution x to be continuous for all $\xi > 0$, but we will allow it to have a single discontinuity at $\xi = 0$. More precisely, we require that the limit $x(0+) := \lim_{\xi \downarrow 0} x(\xi)$ exists but allow $x(0+) \neq \phi(0)$. Our main goal is to characterize the set of initial conditions $\big(\phi, x(0+)\big) \in C([-1,0], \mathbb{R}^n) \times \mathbb{R}^n$ that lead to a bounded solution to (1.1) that decays to zero as $\xi \to \infty$ and to determine whether such solutions are unique.

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If the matrix $\mathcal{I}$ is non-singular, the first line of (1.1) is called a functional differential equation of mixed type (MFDE). On the other hand, if $\mathcal{I}$ is singular, we refer to this equation as a functional differential-algebraic equation of mixed type (MFDAE). The word 'mixed' here reflects the fact that the nonlocal term in (1.1) involves shifts in the argument of x that are both positive and negative.

The current paper should be seen as a continuation of the program initiated in [16, 17]. In particular, in [16] solutions to the initial value problem (1.1) were not allowed to have jumps. In addition, we restricted ourselves to purely differential systems where $\mathcal{I}$ is invertible and purely algebraic systems where $\mathcal{I} = 0$. Furthermore, we only considered linear equations. By contrast, in [17] we considered smooth nonlinear algebraic equations and constructed local center manifolds around equilibrium solutions. Special care needed to be taken to address the intricate compatibility conditions that the nonlinear terms must satisfy. We continue this analysis here for (1.1) and construct local stable manifolds for the zero equilibrium, under less restrictive conditions on the nonlinear terms.

Due to the possibility for solutions to (1.1) to admit jumps at $\xi = 0$, our work here is closely related to the theory of impulsive differential equations. The main difference is that models involving impulsive equations typically prescribe the size of the jump as part of the dynamics, whereas we treat this jump as a parameter that can be freely chosen to control the further evolution of the system. Impulsive ODEs have been widely studied in the literature; see for example [1, 2, 20–22] and the references therein. Technical complications related to the smoothness of solutions prevented these ODE results from being easily generalized to impulsive delay differential equations. Nevertheless, the basic theory concerning existence and uniqueness of solutions to such equations has been successfully developed in [3, 4, 24].

One of the more intriguing phenomena that has been uncovered by these studies, is that jumps can stabilize systems that would be unstable under the usual smooth dynamics. Considerable effort has been spent to establish criteria that can guarantee equilibria for impulsive delay differential equations to be asymptotically stable [23, 26]. The typical tools used for this purpose are Lyapunov functions and Razumikhin techniques.

We take a completely different route in this paper and focus directly on the linear part of (1.1), using Green's functions to account for the jumps. This allows us to quantify the stabilizing effect that these jumps have. In particular, depending on the choice of $\phi \in C([-1, 0], \mathbb{R}^n)$, there may be zero, one, or an entire family of values for $x(0+)$ that lead to a bounded solution. The tools developed in this paper should help to distinguish between these situations.

As discussed extensively in [16], the fact that we consider MFDEs and MFDAEs poses additional complications besides those that are encountered for impulsive delay differential equations. Even in the purely differential case where $\mathcal{I}$ is invertible, the problem (1.1) differs from traditional initial value problems in the sense that the initial condition ϕ does not provide sufficient information to calculate $x'(0)$. In essence, the natural mathematical state space for (1.1) is given by $C([-1, 1], \mathbb{R}^n)$, which of course differs from the space $C([-1, 0], \mathbb{R}^n)$ that our initial condition ϕ belongs to. One of the interesting consequences of this discrepancy is that even if ϕ and $x(0+)$ are both fixed, the problem (1.1) can still have multiple bounded solutions.

Motivation

Differential equations involving both advanced and retarded arguments arise in a number of different settings. They have played a very visible role in the study of travelling wave solutions to lattice differential equations [27, 28]. In addition, a result due to Hughes [14] shows that these equations arise as the Euler-Lagrange optimality conditions when studying optimal control problems that involve time delays [15, 32]. In this paper however, our primary motivation for the study of (1.1) comes from the area of macro-economic modelling.

Inspired by Benhabib and his coworkers [5, 7], let us consider a toy model that describes the interaction between inflation and interest rates in a simple closed economy under the supervision of a central bank. Writing $R(t)$ for the interest rate and $\pi(t)$ for the inflation rate at time t, the dynamical behaviour of R for $t \geq 0$ is given by

$$\Lambda'(R(t))R'(t) \quad = \quad \Lambda(R(t))\big[r + \pi(t) - R(t)\big]. \tag{1.2}$$

Here the constant r denotes the temporal discount factor and the function Λ is related to the utility function, which roughly describes the welfare that is associated to a marginal increase in consumption. For our purposes here, we note that $\Lambda > 0$ and either $\Lambda' \equiv 0$ or $\Lambda' \neq 0$.

In order to close the system, we need to describe the relation between π and R. In order to do this, we introduce the two functions

$$\begin{aligned} \pi^b(t) &= \frac{\int_{-\Omega^b}^0 e^{\beta^b \sigma} \pi(t+\sigma) d\sigma}{\int_{-\Omega^b}^0 e^{\beta^b \sigma} d\sigma}, \\ \pi^f(t) &= \frac{\int_0^{\Omega^f} e^{-\beta^f \sigma} \pi(t+\sigma) d\sigma}{\int_0^{\Omega^f} e^{-\beta^f \sigma} d\sigma}. \end{aligned} \tag{1.3}$$

Here $\pi^b(t)$ describes the recent inflation averaged over the past $\Omega^b > 0$ time units and $\pi^f(t)$ describes the expectation for the inflation over the upcoming $\Omega^f > 0$ time units, weighted by the exponential rates $\beta^b > 0$ and $\beta^f > 0$. Our key assumption is that π and R are constrained on the manifold in function space that is described by

$$R(t) = \rho\Big(\chi_1 \pi^b(t) + \chi_2 \pi^f(t) + \chi_3 \pi(t)\Big). \tag{1.4}$$

Here ρ is a smooth function that has $\rho' > 0$, while the additional parameters satisfy $\chi_1 \geq 0$, $\chi_2 \geq 0$, $\chi_3 > 0$ and $\chi_1 + \chi_2 + \chi_3 = 1$. Roughly speaking, the requirement (1.4) states that the interest rate is determined directly by weighing the present, past and future expectations for the inflation.

Note that the assumptions above imply that (1.4) can be inverted locally to yield $\pi(t)$ as a function of $R(t)$, $\pi^b(t)$ and $\pi^f(t)$. In particular, the variable $\pi(t)$ can be eliminated from (1.2) and (1.3) to yield a three dimensional system of the form (1.1), with one differential and two algebraic components. The appropriate initial condition for this system can be written as

$$\big(R(\tau), \pi^b(\tau), \pi^f(\tau)\big) = \big(\phi_R(\tau), \phi_b(\tau), \phi_f(\tau)\big), \qquad -\Omega^b \leq \tau \leq 0, \tag{1.5}$$

where $(\phi_R, \phi_b, \phi_f) \in C([-\Omega^b, 0], \mathbb{R})^3$.

Let us consider the situation at time $t = 0$ and suppose that the initial functions (ϕ_R, ϕ_b, ϕ_f) have been fixed. Our toy economy has a central bank, which has the power[1] to make an instantaneous adjustment to the interest rate at $t = 0$. In the terminology of (1.1), it has the ability to choose $R(0+) = \lim_{t \downarrow 0} R(t)$. In keeping with its mandate to stabilize the inflation rate, the central bank wishes to make this decision in such a way that $R(t)$ and $\pi(t)$ converge as $t \to \infty$ to an equilibrium state (R_*, π_*) of our model.

The question we are interested in is whether such a step is possible for arbitrary initial conditions (ϕ_R, ϕ_b, ϕ_f) in the vicinity of such an equilibrium. The secondary concern is to determine if such a choice for $R(0+)$ is unique or if multiple trajectories exist for a single initial condition that all converge to the same equilibrium.

Indeterminacy

A well-known problem in the area of macro-economic research is that societies that have seemingly similar economic structures and initial conditions often experience markedly disparate growth paths. The term indeterminacy is widely used to refer to economic models that reproduce this uncertainty in some fashion. This topic has attracted significant interest and we refer the reader to [6] for an informative survey paper.

[1]The description of the model given here is a simplified version of the discussion in [5, 7] and ignores many economic subtleties in the derivation of (1.2)-(1.4).

Roughly speaking, two main mechanisms have been used to introduce indeterminacy into models. On the one hand, economically relevant variables have been coupled with additional external variables in order to incorporate extrinsic uncertainty. The dynamics of these external variables can be either random or deterministic. This approach was pioneered by Cass and Shell [9], who referred to such external variables as 'sunspots' in honor of a nineteenth century attempt by Jevons to attribute business crises to solar anomalies [19].

On the other hand, it has long been believed that expectations of market participants play a major role in the evolution of markets. Venditti and his coworkers have studied the concept of indeterminacy along these lines by analyzing systems where several different sequences of self-fulfilling expectations exist simultaneously [25]. The model (1.2) discussed above should be seen in a similar spirit.

Many authors have considered the issue of indeterminacy in temporally discrete models with finite degrees of freedom. For example, in [5] a discrete version of the model (1.2) described above is considered. In particular, the past inflation rate is sampled at a small number of equidistant time intervals and subsequently used to calculate the interest and inflation rates at the next time step.

Such models are typically written as discrete dynamical systems on $\mathbb{R}^n$, for which $m \le n$ initial conditions can be freely chosen. The remaining variables are considered to be external and hence are excluded from the 'economic' initial condition. Restricting oneself to trajectories that converge to an equilibrium, the degree of indeterminacy can be readily computed by subtracting the number of initial conditions m from the dimension of the stable manifold around the equilibrium under consideration, assuming that suitable non-degeneracy conditions are satisfied. Detailed discussions of these concepts can be found in [8, 12, 30].

Such dimension counting arguments no longer suffice to study the indeterminacy of continuous time models such as (1.2), since the dimension of the space of initial conditions $C([-\Omega^b, 0], \mathbb{R}^3)$ and the dimension of the natural state space $C([-\Omega^b, \Omega^f], \mathbb{R}^3)$ are both infinite. Nevertheless, using the techniques developed in this paper, the notion of indeterminacy can be quantified and analytically calculated under a number of different assumptions on the relevant parameters.

Characteristic Equations

Our analysis of the nonlinear problem (1.1) will rely heavily upon our understanding of linear initial value problems of the form

$$\begin{array}{lcll} x'(\xi) & = & L\,\mathrm{ev}_\xi\, x & \text{for all } \xi \ge 0, \\ x(\tau) & = & \phi(\tau) & \text{for all } -1 \le \tau \le 0. \end{array} \tag{1.6}$$

Assume for the moment that x is a continuous $\mathbb{R}^n$-valued function on the interval $[-1, \infty)$ and that the operator L is a bounded linear map from $C\big([-1, 1], \mathbb{R}^n\big)$ into $\mathbb{R}^n$. We will use the notation $\mathrm{ev}_\xi\, x \in C([-1, 1], \mathbb{R}^n)$ to denote the state of x evaluated at ξ, defined by $[\mathrm{ev}_\xi\, x](\theta) = x(\xi + \theta)$ for all $-1 \le \theta \le 1$. The initial condition ϕ is taken from the set $C([-1, 0], \mathbb{R}^n)$.

In the special case that $L\psi$ only depends on the part of ψ in the interval $[-1, 0]$, the problem (1.6) reduces to an initial value problem for a retarded functional differential equation (RFDE). Such systems have been studied extensively during the last three decades, resulting in a rich and diverse literature on the subject. Using the theory described in [13], the well-posedness of (1.6) can be read off directly from the characteristic function $\Delta_L : \mathbb{C} \to \mathbb{C}^{n \times n}$, that can be written as

$$\Delta_L(z) = z - L\, e^{z\cdot}. \tag{1.7}$$

Indeed, if the characteristic equation $\det \Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z \ge 0$, then any $\phi \in C([-1, 0], \mathbb{R}^n)$ can be extended to a bounded continuous solution. If this property fails, one can characterize the set of initial conditions that can be extended by studying the number and multiplicity of the roots of $\det \Delta_L(z) = 0$ that have $\mathrm{Re}\, z \ge 0$ and subsequently employing spectral projections.

As we shall see in §5.1, the characteristic function Δ_L also plays a major role when studying discontinuous solutions to (1.6). Indeed, one can define a Green's function $\widehat{G}_L$ for this equation by writing

$$\widehat{G}_L(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\nu\xi} \Delta_L(i\nu)^{-1} \, d\nu. \tag{1.8}$$

This Green's function satisfies (1.6) and has a jump precisely at $\xi = 0$, which makes it ideally suited for our analysis.

Let us now return to the general case where $L\psi$ depends on the values of ψ in the entire interval $[-1, 1]$. In this case, our investigation is complicated by the fact that the characteristic equation $\det \Delta_L(z) = 0$ will in general have an infinite number of roots on both sides of the imaginary axis. In addition, we reiterate that in this situation the natural mathematical state space $C([-1, 1], \mathbb{R}^n)$ differs from the space $C([-1, 0], \mathbb{R}^n)$ containing our initial condition ϕ.

The key result that we exploit heavily in this paper was obtained by Mallet-Paret and Verduyn Lunel in [29]. In particular, restricting ourselves to the scalar situation where $n = 1$, the authors show that for every $\nu \in \mathbb{R}$ there exists a Wiener-Hopf factorization

$$(z - \nu)\Delta_L(z) = \Delta_{L_-}(z)\Delta_{L_+}(z), \tag{1.9}$$

in which Δ_{L_-} and Δ_{L_+} are the characteristic functions associated to a retarded respectively advanced functional differential equation, i.e. $\Delta_{L_\pm}(z) = z - L_\pm \exp(z\cdot)$ for some pair of operators $L_- \in \mathcal{L}(C([-1, 0], \mathbb{C}), \mathbb{C})$ and $L_+ \in \mathcal{L}(C([0, 1], \mathbb{C}), \mathbb{C})$. In §2.2 we describe how the well-posedness of the initial value problem (1.6) can be understood once such a factorization has been obtained, by separately studying initial value problems for the operators L_- and L_+. We also recall our results from [16] that show how one can proceed in the typical situation that a Wiener-Hopf factorization is not easily computable. In §5.2 we describe how the Green's function $\widehat{G}_L$ can again be used to obtain results in the discontinuous setting.

Returning to our prototype equation (1.1), let us now assume that $\mathcal{I}$ is singular so that we are dealing with a MFDAE. A single differentiation yields

$$\mathcal{I}x''(\xi) = x'(\xi) + x(\xi + 1) - x(\xi - 1) + Df(x(\xi))x'(\xi). \tag{1.10}$$

The linear part of this equation can easily be seen to be a pure differential equation. This smoothening property is a key requirement that we will impose on the differential-algebraic systems that we consider in this paper.

By construction, any solution to (1.1) will automatically satisfy (1.10). However, the converse is not true and care has to be taken to isolate the superfluous solutions to (1.10). This is a relative minor point when dealing with algebraic delay equations, since one can simply incorporate the algebraic condition into the initial condition space $C([-1, 0], \mathbb{R}^n)$. This is no longer true for (1.1), since the algebraic conditions need to be verified on intervals of the form $[\xi - 1, \xi + 1]$. In particular, the initial condition $\phi \in C([-1, 0], \mathbb{R}^n)$ itself no longer provides sufficient information to verify these conditions. In §5.3 we address this issue for the linear part of (1.1) by using spectral projections and Laplace transform techniques, generalizing prior results that were obtained in [16]. Finally, in §5.4 we show how the nonlinear terms can be incorporated.

We conclude this introduction by giving a brief overview of the structure of this paper. In §2 we recall the existing theory for initial value problems involving RFDEs and MFDEs in the situation that only continuous solutions are allowed. This discussion sets the stage for our main results, which we state in §3. By explicitly describing the consequences of our results in a number of different scenarios, we hope that they will be accessible to a wide audience. With this in mind, we further illustrate our results by applying them in §4 to the economic model (1.2) described above. Finally, in §5 we provide the technical proofs for our results.

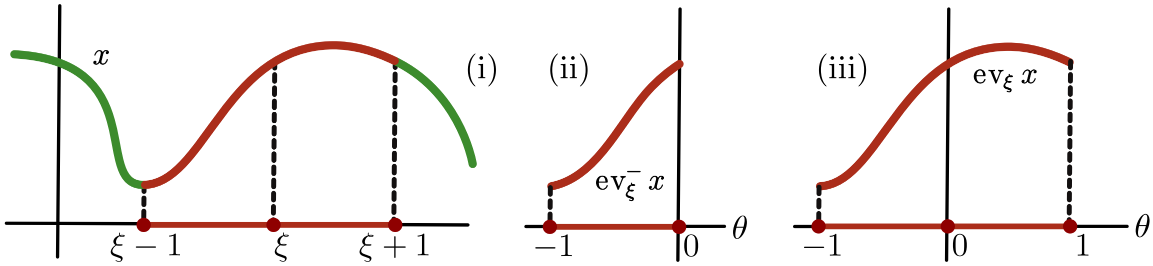

Fig. 1: *Panels (ii) and (iii) illustrate how the operators* ev_ξ^- *and* ev_ξ *act on the continuous function depicted in panel (i).*

2 Continuous Solutions to Initial Value Problems

In this section we review the existing theory concerning initial value problems for linear delay differential equations and functional differential equations of mixed type. At present, we are concerned only with solutions of such equations that do not admit discontinuities. The results for linear delay differential equations are well-known, but we include them here to set the stage for our main results and develop the notation we will use throughout this paper.

2.1 Delay Differential Equations

Let us start by considering the autonomous linear homogeneous delay differential equation

$$x'(\xi) = L_- \, \mathrm{ev}_\xi^- \, x. \tag{2.1}$$

Here L_- is a bounded linear operator from $C([-1,0],\mathbb{C}^n)$ into $\mathbb{C}^n$ and the notation $\mathrm{ev}_\xi^- x$ stands for the function in $C([-1,0],\mathbb{C}^n)$ that has

$$[\mathrm{ev}_\xi^- x](\theta) = x(\xi+\theta), \qquad -1 \le \theta \le 0, \tag{2.2}$$

as illustrated in Figure 1(ii). To ease our notation, we will write

$$X^- = C([-1,0],\mathbb{C}^n). \tag{2.3}$$

We are primarily interested in determining which functions $\phi \in X^-$ can be extended to solutions to (2.1) that can be bounded by prescribed exponentials. To this end, we introduce the following two families of function spaces, parametrized by $\eta \in \mathbb{R}$,

$$\begin{array}{lcl} BC_\eta^\oplus & = & \{x \in C([-1,\infty),\mathbb{C}^n), \|x\|_\eta := \sup_{\xi \ge -1} e^{-\eta\xi} |x(\xi)| < \infty\}, \\ BC_\eta^- & = & \{x \in C((-\infty,0],\mathbb{C}^n), \|x\|_\eta := \sup_{\xi \le 0} e^{-\eta\xi} |x(\xi)| < \infty\}. \end{array} \tag{2.4}$$

Using these function spaces, we introduce two families of solution sets for (2.1),

$$\begin{array}{lcl} \mathfrak{P}_{L_-}(\eta) & = & \big\{x \in BC_\eta^- \mid x'(\xi) = L_- \, \mathrm{ev}_\xi^- \, x \text{ for all } \xi \le 0\big\}, \\ \mathfrak{Q}_{L_-}(\eta) & = & \big\{x \in BC_\eta^\oplus \mid x'(\xi) = L_- \, \mathrm{ev}_\xi^- \, x \text{ for all } \xi \ge 0\big\}. \end{array} \tag{2.5}$$

As in [29], we also need the spaces

$$\begin{array}{lcl} P_{L_-}(\eta) & = & \big\{\phi \in X^- \mid \phi = \mathrm{ev}_0^- \, x \text{ for some } x \in \mathfrak{P}_{L_-}(\eta)\big\}, \\ Q_{L_-}(\eta) & = & \big\{\phi \in X^- \mid \phi = \mathrm{ev}_0^- \, x \text{ for some } x \in \mathfrak{Q}_{L_-}(\eta)\big\}, \end{array} \tag{2.6}$$

which contain the initial segments of the solution sets $\mathfrak{P}_{L_-}$ and $\mathfrak{Q}_{L_-}$ in the natural state space $X^- = C([-1,0],\mathbb{C}^n)$. For any $\phi \in Q_{L_-}(\eta)$, we write $x = E_{L_-}\phi$ for the unique $x \in \mathfrak{Q}_{L_-}(\eta)$ that has $\mathrm{ev}_0^- x = \phi$. We also use this extension operator E_{L_-} to map $P_{L_-}(\eta)$ back into $\mathfrak{P}_{L_-}(\eta)$ in the analogous fashion.

It is well known that crucial information concerning $P_{L_-}(\eta)$ and $Q_{L_-}(\eta)$ is encoded in the characteristic function $\Delta_{L_-}(z) : \mathbb{C} \to \mathbb{C}^{n\times n}$ associated to (2.1), which is given by

$$\Delta_{L_-}(z) = zI - L_- e^{z\cdot} I, \tag{2.7}$$

where I is the $n \times n$ identity matrix. Indeed, upon introducing the notation

$$n^+_{L_-}(\eta) = \#\{z \in \mathbb{C} \mid \det \Delta_{L_-}(z) = 0 \text{ and } \operatorname{Re} z > \eta\}, \tag{2.8}$$

where all roots are counted according to their multiplicity, we have the following result.

Proposition 2.1 (See [13, Thms. 7.2.1 and 7.6.1]). *Consider the linear system (2.1) and choose $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\det \Delta_{L_-}(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. Then the spaces $P_{L_-}(\eta)$ and $Q_{L_-}(\eta)$ are closed and satisfy*

$$C([-1,0],\mathbb{C}^n) = P_{L_-}(\eta) \oplus Q_{L_-}(\eta). \tag{2.9}$$

In addition, the space $P_{L_-}(\eta)$ is spanned by the $n^+_{L_-}(\eta)$ generalized eigenfunctions associated to the roots of $\det \Delta_{L_-}(z) = 0$ that have $\operatorname{Re} z > \eta$.

2.2 Mixed Type Equations

We now turn our attention to the linear homogeneous functional differential equation of mixed type (MFDE)

$$x'(\xi) = L\,\mathrm{ev}_\xi\, x. \tag{2.10}$$

In this case, L is a bounded linear operator from $C([-1,1],\mathbb{C}^n)$ into $\mathbb{C}^n$ and the notation $\mathrm{ev}_\xi\, x$ now stands for the function in $C([-1,1],\mathbb{C}^n)$ that has

$$[\mathrm{ev}_\xi x](\theta) = x(\xi+\theta), \qquad -1 \le \theta \le 1, \tag{2.11}$$

as illustrated in Figure 1(iii). To ease our notation, we will write

$$X = C([-1,1],\mathbb{C}^n) \tag{2.12}$$

and add the following family of function spaces to the list (2.4),

$$BC^\ominus_\eta \;=\; \{x \in C((-\infty,1],\mathbb{C}^n), \|x\|_\eta := \sup_{\xi\le 1} e^{-\eta\xi}\,|x(\xi)| < \infty\}, \tag{2.13}$$

where again $\eta \in \mathbb{R}$. As before, we are interested in the following solution sets for (2.10),

$$\begin{aligned} \mathfrak{P}_L(\eta) &= \big\{x \in BC^\ominus_\eta \mid x'(\xi) = L\,\mathrm{ev}_\xi\, x \text{ for all } \xi \le 0\big\}, \\ \mathfrak{Q}_L(\eta) &= \big\{x \in BC^\oplus_\eta \mid x'(\xi) = L\,\mathrm{ev}_\xi\, x \text{ for all } \xi \ge 0\big\}. \end{aligned} \tag{2.14}$$

The initial segments of these solutions in the natural state space $X = C([-1,1],\mathbb{C}^n)$ are contained in the spaces

$$\begin{aligned} P_L(\eta) &= \big\{\psi \in X \mid \psi = \mathrm{ev}_0\, x \text{ for some } x \in \mathfrak{P}_L(\eta)\big\}, \\ Q_L(\eta) &= \big\{\psi \in X \mid \psi = \mathrm{ev}_0\, x \text{ for some } x \in \mathfrak{Q}_L(\eta)\big\}. \end{aligned} \tag{2.15}$$

We will write E_L for the extension operator that maps the initial condition $\psi \in Q_L(\eta)$ back to the unique $x = E_L\psi \in \mathfrak{Q}_L(\eta)$ that has $\mathrm{ev}_0\, x = \psi$, while acting similarly on $P_L(\eta)$. The characteristic function associated to (2.10) is given by

$$\Delta_L(z) = zI - L\, e^{z\cdot} I. \tag{2.16}$$

Although X is the natural state space for the MFDE (2.10), our applications require us to consider initial values in the 'shorter' spaces $C([-1,0],\mathbb{C}^n)$ and $C([0,1],\mathbb{C}^n)$. An important role is therefore reserved for the restriction operators

$$\begin{aligned} \pi^+_{P_L(\eta)} &: P_L(\eta) \to C\big([0,1],\mathbb{C}^n\big), \qquad && \psi \mapsto \psi_{|[0,1]}, \\ \pi^-_{Q_L(\eta)} &: Q_L(\eta) \to C\big([-1,0],\mathbb{C}^n\big), \qquad && \psi \mapsto \psi_{|[-1,0]}. \end{aligned} \tag{2.17}$$

The following result shows that for appropriate values of η, the state space X is decomposed by $P_L(\eta)$ and $Q_L(\eta)$. In addition, the restriction operators (2.17) are Fredholm, which means that their kernels are finite dimensional, while their ranges are closed and of finite codimension. We recall that the index of a Fredholm operator F is determined by the formula

$$\mathrm{ind}(F) = \dim \mathrm{Ker}(F) - \mathrm{codim}\,\mathrm{Range}(F). \tag{2.18}$$

Proposition 2.2 (see [29, 31]). *Consider the linear system (2.10) and choose $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\det \Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z = \eta$. Then the spaces $P_L(\eta)$ and $Q_L(\eta)$ are closed and satisfy*

$$C([-1,1],\mathbb{C}^n) = P_L(\eta) \oplus Q_L(\eta). \tag{2.19}$$

In addition, the operators $\pi^+_{P_L(\eta)}$ and $\pi^-_{Q_L(\eta)}$ defined in (2.17) are Fredholm, with

$$\mathrm{ind}(\pi^+_{P_L(\eta)}) + \mathrm{ind}(\pi^-_{Q_L(\eta)}) = -n. \tag{2.20}$$

To obtain more detailed information on the restriction operators $\pi^+_{P_L(\eta)}$ and $\pi^-_{Q_L(\eta)}$, we need to impose the following additional restriction on the linear operator L. We remark that this condition is significantly weaker than the atomicity condition used in [29, Eq. (2.3)], which requires $s_\pm = 0$ to hold in (HL).

(HL) There exist quantities $s_\pm \geq 0$ and non-singular $n \times n$ matrices $J_\pm$ such that the following asymptotic expansions hold,

$$\begin{aligned} \Delta_L(z) &= z^{-s_+} e^{z}(J_+ + o(1)) \qquad && \text{as } z \to \infty, \\ \Delta_L(z) &= z^{-s_-} e^{-z}(J_- + o(1)) \qquad && \text{as } z \to -\infty. \end{aligned} \tag{2.21}$$

Proposition 2.3 (see [29, Thm. 5.2] and [16, Prop. 2.2]). *Consider the linear system (2.10) and suppose that (HL) is satisfied. Then for any monic[2] polynomial p of degree n, there exist linear operators*

$$L_- \in \mathcal{L}\big(C([-1,0],\mathbb{C}^n),\mathbb{C}^n\big), \qquad L_+ \in \mathcal{L}\big(C([0,1],\mathbb{C}^n),\mathbb{C}^n\big), \tag{2.22}$$

with associated characteristic matrices

$$\Delta_{L_\pm}(z) = zI - L_\pm e^{z\cdot}\, I \tag{2.23}$$

for which the splitting

$$p(z) \det \Delta_L(z) = \det \Delta_{L_-}(z) \det \Delta_{L_+}(z) \tag{2.24}$$

holds.

[2]A polynomial $p(z)$ is called monic if the coefficient associated to the highest power of z is one, e.g. $p(z) = z + 2$.

The splitting (2.24) is referred to as a Wiener-Hopf factorization for the symbol Δ_L and we will call any such triplet (p, L_-, L_+) a Wiener-Hopf triplet for L. In general, such triplets need not be unique. Indeed, in [29] a mechanism is given by which pairs of roots of the characteristic equations $\det \Delta_{L_\pm}(z) = 0$ may be interchanged. Nevertheless, it turns out to be possible to extract a quantity that does not depend on the chosen splitting (2.24). To this end, let us consider any Wiener-Hopf triplet (p, L_-, L_+) for L and pick an $\eta \in \mathbb{R}$ for which the equation $p(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. We now introduce the integer

$$n_L^\sharp(\eta) = n_{L_+}^-(\eta) - n_{L-}^+(\eta) + n_p^+(\eta) \tag{2.25}$$

that is defined by

$$\begin{array}{lcl} n_{L_-}^+(\eta) & = & \#\{z \in \mathbb{C} \mid \det \Delta_{L_-}(z) = 0 \text{ and } \operatorname{Re} z > \eta\}, \\ n_{L_+}^-(\eta) & = & \#\{z \in \mathbb{C} \mid \det \Delta_{L_+}(z) = 0 \text{ and } \operatorname{Re} z < \eta\}, \\ n_p^+(\eta) & = & \#\{z \in \mathbb{C} \mid p(z) = 0 \text{ and } \operatorname{Re} z > \eta\}. \end{array} \tag{2.26}$$

This quantity $n_L^\sharp(\eta)$ is invariant in the following sense.

Proposition 2.4 (see [29, Thm. 5.2] and [16, Prop. 2.3]). *Consider the linear system (2.10) and suppose that (HL) is satisfied. Fix any $\eta \in \mathbb{R}$ for which the characteristic equation $\det \Delta_L(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. Then the quantity $n_L^\sharp(\eta)$ is invariant across all Wiener-Hopf triplets (p, L_-, L_+) for L that have $p(\eta + i\nu) \neq 0$ for all $\nu \in \mathbb{R}$.*

In the special case that (2.10) is scalar (i.e., for which $n = 1$), the quantities $n_L^\sharp(\eta)$ can be used to characterize the kernels and ranges of the Fredholm operators $\pi^-_{Q_L(\eta)}$ and $\pi^+_{P_L(\eta)}$. This dimension restriction is related to the fact that the splitting (2.24) only features the determinant of Δ_L.

Proposition 2.5 (see [29, Thms. 6.1-6.2] and [16, Prop. 2.4]). *Consider a scalar version of the linear system (2.10) and suppose that (HL) is satisfied. Fix any $\eta \in \mathbb{R}$ for which the characteristic equation $\Delta_L(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. Then the following identities hold,*

$$\begin{array}{lcllcl} \dim \operatorname{Ker} \pi^+_{P_L(\eta)} & = & \max\{-n_L^\sharp(\eta), 0\}, & \operatorname{codim} \operatorname{Range} \pi^+_{P_L(\eta)} & = & \max\{n_L^\sharp(\eta), 0\}, \\ \dim \operatorname{Ker} \pi^-_{Q_L(\eta)} & = & \max\{n_L^\sharp(\eta) - 1, 0\}, & \operatorname{codim} \operatorname{Range} \pi^-_{Q_L(\eta)} & = & \max\{1 - n_L^\sharp(\eta), 0\}. \end{array} \tag{2.27}$$

To aid the reader, the next three results explore the implications of the identities (2.27) for initial value problems involving the MFDE (2.10).

Corollary 2.6. *Consider a scalar version of (2.10) that satisfies (HL) and suppose that $n_L^\#(\eta) \leq 0$. Then there is a subspace $\mathcal{V} \subset C([-1, 0], \mathbb{C})$ of codimension $1 - n_L^\#(\eta)$ such that for every $\phi \in \mathcal{V}$ there is a unique $y \in BC_\eta^\oplus$ that solves (2.10) and has $\mathrm{ev}_0^- y = \phi$.*

Corollary 2.7. *Consider a scalar version of (2.10) that satisfies (HL) and suppose that $n_L^\#(\eta) = 1$. Then for every $\phi \in C([-1, 0], \mathbb{C})$ there is a unique $y \in BC_\eta^\oplus$ that solves (2.10) and has $\mathrm{ev}_0^- y = \phi$.*

Corollary 2.8. *Consider a scalar version of (2.10) that satisfies (HL) and suppose that $n_L^\#(\eta) \geq 2$. Then for any $\phi \in C([-1, 0], \mathbb{C})$ there is a $y \in BC_\eta^\oplus$ that solves (2.10) and has $\mathrm{ev}_0^- y = \phi$. In addition, $z \in BC_\eta^\oplus$ also solves (2.10) with $\mathrm{ev}_0^- y = \phi$ if and only if $z - y = E_L\psi$, for some $\psi \in C([-1, 1], \mathbb{C})$ that is contained in the $(n_L^\#(\eta) - 1)$-dimensional space $\operatorname{Ker} \pi^-_{Q_L(\eta)}$.*

Unfortunately, it is often intractable to find Wiener-Hopf triplets for a prescribed operator L. This often prevents us from computing $n^{\sharp}_{L}(\eta)$ directly from (2.25). The following result addresses this difficulty and allows $n^{\sharp}_{L}(\eta)$ to be calculated in settings where a Wiener-Hopf triplet is not readily available for the system (2.10) under consideration. The only requirement is that a Wiener-Hopf triplet is available for some reference system that can be continuously transformed into the original system without violating (HL). Please note however that the exponents $s_{\pm}$ appearing in this condition (HL) need not remain constant during this transformation.

Proposition 2.9 (see [16, Thm. 2.5]). *Consider a continuous path*

$$\Gamma : [0,1] \to \mathcal{L}\big(C([-1,1],\mathbb{C}^n),\mathbb{C}^n\big) \tag{2.28}$$

and suppose that the operators $\Gamma(\mu)$ satisfy (HL) for all $0 \le \mu \le 1$. Fix any $\eta \in \mathbb{R}$ and suppose that the characteristic equation $\det \Delta_{\Gamma(\mu)}(z) = 0$ admits roots with $\operatorname{Re} z = \eta$ for only finitely many values of $\mu \in [0,1]$ and that $\mu \in (0,1)$ for all such μ. Then we have the identity

$$n^{\sharp}_{\Gamma(1)}(\eta) - n^{\sharp}_{\Gamma(0)}(\eta) = -\mathrm{cross}(\Gamma,\eta), \tag{2.29}$$

in which the crossing number $\mathrm{cross}(\Gamma,\eta)$ denotes the net number of roots of the characteristic equation $\det \Delta_{\Gamma(\mu)}(z) = 0$, counted with multiplicity, that cross the line $\operatorname{Re} z = \eta$ from left to right as μ increases from 0 to 1.

The situation that we will encounter in §4.1.3 during the analysis of our model is covered by the following special case.

Corollary 2.10. *Consider a continuous path*

$$\Gamma : [0,1] \to \mathcal{L}\big(C([-1,1],\mathbb{C}^n),\mathbb{C}^n\big) \tag{2.30}$$

and suppose that the operators $\Gamma(\mu)$ satisfy (HL) for all $0 \le \mu \le 1$. Fix any $\eta \in \mathbb{R}$ and suppose that for all $0 \le \mu \le 1$ the characteristic equation $\det \Delta_{\Gamma(\mu)}(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. Then we have

$$n^{\sharp}_{\Gamma(1)}(\eta) = n^{\sharp}_{\Gamma(0)}(\eta). \tag{2.31}$$

3 Main Results

In this section we present our main results, which we will prove in §5. We start by focussing on initial value problems for linear delay differential equations and MFDEs in §3.1 and §3.2, where we now allow solutions to admit a single discontinuity at $\xi = 0$. The presence of these jumps forces us to take care that the linear operators L_- and L featuring in our systems are well-defined even when acting on discontinuous solution segments. In §3.3 we shift our attention to smooth differential-algebraic systems. In particular, we establish a connection between initial value problems for such equations and their non-algebraic counterparts studied in §3.1 and §3.2. Finally, in §3.4 we turn to our chief interest and study initial value problems for nonlinear differential-algebraic systems.

3.1 Differential Delay Equations

We return to the delay equation (2.1), but now wish to allow solutions that have a discontinuity at $\xi = 0$. To make this more precise, let us introduce the two families of function spaces

$$\begin{array}{lcl} BC^{+}_{\eta} & = & \{x \in C\big([0,\infty),\mathbb{C}^n\big), \|x\|_{\eta} := \sup_{\xi \ge 0} e^{-\eta\xi}\,|x(\xi)| < \infty\}, \\ \widehat{BC}^{\oplus}_{\eta} & = & C([-1,0],\mathbb{C}^n) \times BC^{+}_{\eta}, \end{array} \tag{3.1}$$

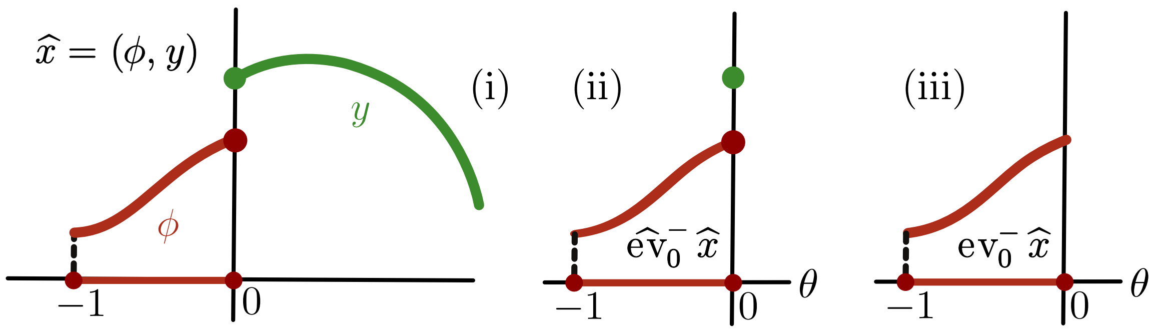

Fig. 2: *Panel (i) depicts a function* $\widehat{x}$ *that is multi-valued at* $\xi = 0$*. Panels (ii) and (iii) illustrate how the operators* $\widehat{\mathrm{ev}}_0^-$ *and* ev_0^- *act on* $\widehat{x}$*.*

parametrized by $\eta \in \mathbb{R}$. We will seek our solutions to (2.1) from the latter family. Any $\widehat{x} = (\phi, y) \in \widehat{BC}_\eta^\oplus$ can be interpreted as a regular function on $[-1, \infty)$ that is multi-valued at $\xi = 0$. In particular, we will use the notation $\widehat{x}(\xi) = y(\xi)$ whenever $\xi > 0$, $\widehat{x}(\theta) = \phi(\theta)$ for $-1 \leq \theta < 0$ and write $\widehat{x}(0+) = y(0)$ and $\widehat{x}(0-) = \phi(0)$ to distinguish between the two values at zero.

The evaluation operators (2.2) applied to a function $\widehat{x} \in BC_\eta^\oplus$ will no longer all map into $C([-1, 0], \mathbb{C}^n)$. Let us therefore introduce the family of function spaces

$$\widehat{X}_\alpha^- = C([-1, \alpha], \mathbb{C}^n) \times C([\alpha, 0], \mathbb{C}^n), \tag{3.2}$$

for any $-1 < \alpha < 0$, together with the special case

$$\widehat{X}_0^- = C([-1, 0], \mathbb{C}^n) \times \mathbb{C}^n. \tag{3.3}$$

Using these new spaces, we can introduce an appropriate set of evaluation operators $\widehat{\mathrm{ev}}_\xi^-$ for $\xi \geq 0$. For $0 \leq \xi < 1$, we define

$$\widehat{\mathrm{ev}}_\xi^- : \widehat{BC}_\eta^\oplus \to \widehat{X}_{-\xi}, \qquad \widehat{\mathrm{ev}}_\xi^-(\phi, y) = \Big(\phi(\xi + \cdot), y(\xi + \cdot)\Big), \tag{3.4}$$

which is depicted in Figure 3(ii). In particular, if $\widehat{x} = (\phi, y) \in \widehat{BC}_\eta^\oplus$, then in the special case $\xi = 0$ we have

$$\widehat{\mathrm{ev}}_0^-\, \widehat{x} = \big(\phi, y(0)\big), \tag{3.5}$$

as illustrated in Figure 2(ii). For $\xi \geq 1$ we will write

$$\widehat{\mathrm{ev}}_\xi^- : \widehat{BC}_\eta^\oplus \to X^- = C([-1, 0], \mathbb{C}^n), \qquad [\widehat{\mathrm{ev}}_\xi^-(\phi, y)](\theta) = y(\xi + \theta), \tag{3.6}$$

since there is no need to distinguish between $\widehat{\mathrm{ev}}_\xi^-$ and ev_ξ^-. Let us emphasize here that we will continue to use the operator ev_0^-, which maps into the space $X^- = C([-1, 0], \mathbb{C}^n)$ and acts on a function $\widehat{x} = (\phi, y) \in \widehat{BC}_\eta^\oplus$ as

$$\mathrm{ev}_0^-\, \widehat{x} = \phi, \tag{3.7}$$

as illustrated in Figure 2(iii). The reader should contrast this definition to (3.5).

Let us now consider a bounded linear operator $L_- : C([-1, 0], \mathbb{C}^n) \to \mathbb{C}^n$. In order to formulate a well-posed delay differential equation, we need to specify how L_- should be extended to the spaces

$\widehat{X}_\alpha^-$ introduced above. To accomplish this task, we note that the Riesz representation theorem implies that there exists a unique

$$\mu \in \mathrm{NBV}([-1,0], \mathbb{C}^{n\times n}) \tag{3.8}$$

such that we have the representation

$$L_-\phi = \int_{-1}^{0} d\mu(\sigma)\phi(\sigma), \tag{3.9}$$

for all $\phi \in C([-1,0], \mathbb{C}^n)$. We will refer to μ as the measure associated to L_-.

We recall here that the set $\mathrm{NBV}([-1,0], \mathbb{C}^{n\times n})$ contains all $\mathbb{C}^{n\times n}$-valued functions μ that are right-continuous on $(-1,0)$, are normalized to have $\mu(-1) = 0$ and have bounded variation on $[-1,0]$; see [11, App. I]. Let us summarize some important properties that these measures have.

Lemma 3.1. *If $\mu \in \mathrm{NBV}([-1,0], \mathbb{C}^{n\times n})$ then for any $-1 \le \vartheta < 0$ the right-hand limits $\mu(\vartheta+)$ are well-defined, while for any $-1 \le \vartheta \le 0$ the left-hand limits $\mu(\vartheta-)$ are well-defined. In addition, μ is continuous on $[-1,0]$ except possibly at a countable number of points.*

Notice that for any $\mu \in \mathrm{NBV}([-1,0], \mathbb{C}^{n\times n})$ and any $-1 < \alpha \le 0$, we can define two new measures

$$\mu_\alpha^l \in \mathrm{NBV}([-1,\alpha], \mathbb{C}^{n\times n}), \qquad \mu_\alpha^r \in \mathrm{NBV}([\alpha,0]) \tag{3.10}$$

by writing

$$\begin{array}{rcll} \mu_\alpha^l(\vartheta) & = & \mu(\vartheta), & -1 \le \vartheta < \alpha, \\ \mu_\alpha^l(\alpha) & = & \mu(\alpha-), & \\ \mu_\alpha^r(\alpha) & = & 0 & \\ \mu_\alpha^r(\vartheta) & = & \mu(\vartheta) - \mu(\alpha-), & \alpha < \vartheta \le 0. \end{array} \tag{3.11}$$

Using these new measures, we extend L_- to $\widehat{X}_0^-$ by writing

$$L_-(\phi, v) = \int_{-1}^{0} d\mu_0^l(\sigma)\phi(\sigma) + \big(\mu(0) - \mu(0-)\big)v, \tag{3.12}$$

for any $(\phi, v) \in \widehat{X}_0^- = C([-1,0], \mathbb{C}^n) \times \mathbb{C}^n$. In addition, for any $-1 < \alpha < 0$ we extend L_- to $\widehat{X}_\alpha^-$ by writing

$$L_-(\phi^l, \phi^r) = \int_{-1}^{\alpha} d\mu_\alpha^l(\sigma)\phi^l(\sigma) + \int_{\alpha}^{0} d\mu_\alpha^r(\sigma)\phi^r(\sigma) \tag{3.13}$$

for any $(\phi^l, \phi^r) \in \widehat{X}_\alpha^- = C([-1,\alpha], \mathbb{C}^n) \times C([\alpha,0], \mathbb{C}^n)$.

The best way to grasp the implications of this extension is to consider an example. Suppose therefore that for $\phi \in C([-1,0], \mathbb{C})$ we have

$$L_-\phi = \phi(0) + \phi(-1) + \int_{-1}^{0} \sigma\phi(\sigma)d\sigma \tag{3.14}$$

In this case, the extension of L_- to $\widehat{X}_0^-$ would be given by

$$L_-(\phi, v) = v + \phi(-1) + \int_{-1}^{0} \sigma\phi(\sigma)d\sigma. \tag{3.15}$$

Stated more practically, if $\widehat{x} \in \widehat{BC}_\eta^\oplus$, then any reference to $\widehat{x}(0)$ in the right hand side of (2.1) should be interpreted as $\widehat{x}(0+)$. We emphasize that this bias for right-hand limits is a deliberate choice on our part. This decision is motivated by our interest in initial value problems that are posed in forward time. When studying differential equations this choice is not material, but in §3.3 it will turn out to be crucial.

Our preparations complete, we are now ready to study the delay differential equation

$$\widehat{x}'(\xi) = L_- \, \widehat{\mathrm{ev}}_\xi^- \, \widehat{x} \tag{3.16}$$

and look for solutions $\widehat{x} \in \widehat{BC}_\eta^\oplus$. Before we proceed, we need to comment on the notion of a solution to (3.16), aided by the following observation.

Lemma 3.2 (see §5.1). *Consider any $\widehat{x} \in \widehat{BC}_\eta^\oplus$. Then the map $\xi \mapsto L_- \, \widehat{\mathrm{ev}}_\xi^- \, \widehat{x}$ is continuous for all $\xi \geq 0$ except possibly at a countable number of points.*

In particular, we will need to look for functions $\widehat{x} \in \widehat{BC}_\eta^\oplus$ that satisfy (3.16) for almost all $\xi \geq 0$. Note that any such solution $\widehat{x} = (\phi, y)$ automatically satisfies $y \in W_\eta^{1,\infty}\big([0,\infty), \mathbb{C}^n\big)$, where we have introduced the function space

$$W_\eta^{1,\infty}\big([0,\infty), \mathbb{C}^n\big) = \{x : e^{-\eta \cdot} x(\cdot) \in W^{1,\infty}\big([0,\infty), \mathbb{C}^n\big)\}. \tag{3.17}$$

This motivates the introduction of the solution spaces

$$\begin{array}{lcl} \widehat{\mathfrak{Q}}_{L_-}(\eta) & = & \big\{\widehat{x} \in \widehat{BC}_\eta^\oplus \mid \widehat{x}'(\xi) = L_- \, \widehat{\mathrm{ev}}_\xi^- \, \widehat{x} \text{ for almost all } \xi \geq 0\big\}, \\ \widehat{Q}_{L_-}(\eta) & = & \big\{\widehat{\phi} \in \widehat{X}_0^- \mid \widehat{\phi} = \widehat{\mathrm{ev}}_0^- \, \widehat{x} \text{ for some } \widehat{x} \in \widehat{\mathfrak{Q}}_{L_-}(\eta)\big\}. \end{array} \tag{3.18}$$

As before, we will write $\widehat{E}_{L_-}$ to map $\widehat{\phi} \in \widehat{Q}_{L_-}(\eta)$ back to the function $\widehat{x} \in \widehat{\mathfrak{Q}}_{L_-}(\eta)$ that has $\widehat{\mathrm{ev}}_0^- \widehat{x} = \widehat{\phi}$. Our main result describes the space $\widehat{Q}_{L_-}(\eta)$ in considerable detail.

Theorem 3.3 (see §5.1). *Consider the delay equation (3.16), pick $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\det \Delta_{L_-}(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$ and recall the integer $n_{L_-}^+(\eta)$ defined in (2.8).*

If $n_{L_-}^+(\eta) = 0$, then we have

$$\widehat{Q}_{L_-}(\eta) = \widehat{X}_0^- = C([-1,0], \mathbb{C}^n) \times \mathbb{C}^n. \tag{3.19}$$

On the other hand, if $n_{L_-}^+(\eta) > 0$, then there exist bounded linear operators

$$M_\ell : X^- = C([-1,0], \mathbb{C}^n) \to \mathbb{C}, \qquad 1 \leq \ell \leq n_-(\eta), \tag{3.20}$$

together with an integer

$$1 \leq s \leq \min\{n_{L_-}^+(\eta), n\} \tag{3.21}$$

and a set of s linearly independent vectors

$$\alpha_\ell \in \mathbb{C}^n, \qquad 1 \leq \ell \leq s, \tag{3.22}$$

such that any pair $(\phi, v) \in \widehat{X}_0^-$ satisfies $(\phi, v) \in \widehat{Q}_{L_-}(\eta)$ if and only if

$$\begin{array}{lcll} \alpha_\ell^\dagger v & = & M_\ell \phi & \textit{for } 1 \leq \ell \leq s, \\ 0 & = & M_\ell \phi & \textit{for } s < \ell \leq n_{L_-}^+(\eta). \end{array} \tag{3.23}$$

Note that this result does not specify how the integer s, the vectors α_ℓ and the operators M_ℓ can be computed. This situation is remedied in §5.1, where we develop the machinery to attack this problem. For the moment however, the following result gives a more explicit characterization of $\widehat{Q}_{L_-}(\eta)$ in the special case that $\det \Delta_{L_-}(z) = 0$ only has simple roots to the right of the line $\operatorname{Re} z = \eta$.

Corollary 3.4 (see §5.1). *Pick any $\eta \in \mathbb{R}$. Consider the delay equation (3.16) and suppose that the collection of roots of the characteristic equation $\det \Delta_{L_-}(z) = 0$ that have $\operatorname{Re} z \geq \eta$ consists of a set of $n^+_{L_-}(\eta)$ distinct simple roots $\{z_\ell\}$ that all have $\operatorname{Re} z_\ell > \eta$. For all $1 \leq \ell \leq n^+_{L_-}(\eta)$, choose a non-zero vector $\alpha_\ell \in \mathbb{C}^n$ that satisfies*

$$\Delta^\dagger_{L_-}(z_\ell)\alpha_\ell = 0. \tag{3.24}$$

Then for any $\phi \in C([-1,0],\mathbb{C}^n)$ and any $v \in \mathbb{C}^n$ that satisfies

$$\alpha^\dagger_\ell v = -\alpha^\dagger_\ell \int_{-1}^0 d\mu(\sigma) e^{z_\ell \sigma} \int_\sigma^0 e^{-z_\ell \tau}\phi(\tau)d\tau \tag{3.25}$$

for all $1 \leq \ell \leq n^+_{L_-}(\eta)$, there exists a unique $\widehat{x} \in \widehat{BC}^\oplus_\eta$ that solves (5.4) and has $\widehat{\mathrm{ev}}^-_0\, \widehat{x} = (\phi, v)$.

In particular, if the set of vectors $\{\alpha_\ell\}$ is linearly independent, the quantity s appearing in Theorem 3.3 satisfies $s = n^+_{L_-}(\eta)$.

To illustrate the consequences of the results above for initial value problems involving (3.16), we explicitly describe a number of scenarios.

Corollary 3.5. *Consider the delay equation (3.16) and pick $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\det \Delta_{L_-}(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. If $n^+_{L_-}(\eta) = 0$, then for every $v \in \mathbb{C}^n$ and every $\phi \in C([-1,0],\mathbb{C}^n)$ there is a unique $\widehat{x} \in \widehat{BC}^\oplus_\eta$ that solves (3.16) and has $\widehat{\mathrm{ev}}^-_0\, \widehat{x} = (\phi, v)$.*

Corollary 3.6. *Consider the delay equation (3.16) and pick $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\det \Delta_{L_-}(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. Suppose that $n^+_{L_-}(\eta) \geq 1$.*

Then there exists a subspace $\mathcal{V} \subset C([-1,0],\mathbb{C}^n)$ of codimension $n^+_{L_-}(\eta) - s$ and a subspace $\mathcal{W} \subset \mathbb{C}^n$ of codimension s such that the following holds true.

For every $\phi \in \mathcal{V}$ there exists a $\widehat{x} \in \widehat{BC}^\oplus_\eta$ that solves (3.16) and has $\mathrm{ev}^-_0\, \widehat{y} = \phi$. In addition, $\widehat{y} \in \widehat{BC}^\oplus_\eta$ also solves (3.16) with $\mathrm{ev}^-_0\, \widehat{y} = \phi$, if and only if $\widehat{y} - \widehat{x} = \widehat{E}_{L_-}(0,w)$ for some $w \in \mathcal{W}$.

When we are dealing with scalar versions of (3.16), i.e., when $n = 1$, the ambiguity with respect to the integer s and the vectors α_ℓ disappears. As a consequence, the results simplify considerably.

Corollary 3.7. *Consider a scalar version of the delay equation (3.16) and pick $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\Delta_{L_-}(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. Suppose that $n^+_{L_-}(\eta) \geq 2$. Then there is a subspace $\mathcal{V} \subset C([-1,0],\mathbb{C})$ of codimension $n^+_{L_-}(\eta) - 1$ such that for every $\phi \in \mathcal{V}$ there is a unique $\widehat{x} \in \widehat{BC}^\oplus_\eta$ that solves (3.16) and has $\mathrm{ev}^-_0\, \widehat{x} = \phi$.*

Corollary 3.8. *Consider a scalar version of the delay equation (3.16) and pick $\eta \in \mathbb{R}$ in such a way that the characteristic equation $\Delta_{L_-}(z) = 0$ admits no roots with $\operatorname{Re} z = \eta$. If $n^+_{L_-}(\eta) = 1$, then for every $\phi \in C([-1,0],\mathbb{C}^n)$ there is a unique $\widehat{x} \in \widehat{BC}^\oplus_\eta$ that solves (3.16) and has $\mathrm{ev}^-_0\, \widehat{x} = \phi$.*

Before we conclude this section, we introduce some terminology that will help us to compare our results here to similar results in the sequel. In particular, we introduce the operators

$$\begin{aligned} \pi^-_{\widehat{Q}_{L_-}(\eta)} &: \widehat{Q}_{L_-}(\eta) \to C\big([-1,0],\mathbb{C}^n\big), && (\phi, v) \mapsto \phi, \\ \widehat{\pi}^-_{\widehat{Q}_{L_-}(\eta)} &: \widehat{Q}_{L_-}(\eta) \to C\big([-1,0],\mathbb{C}^n\big) \times \mathbb{C}^n, && (\phi, v) \mapsto (\phi, v). \end{aligned} \tag{3.26}$$

Corollary 3.9. *Consider the setting of Theorem 3.3 and recall the integer s introduced there, with the identification $s = 0$ in the special case $n^{+}_{L_-}(\eta) = 0$. Then we have the identities*

$$\operatorname{codim} \operatorname{Range} \widehat{\pi}^{-}_{\widehat{Q}_L(\eta)} = n^{+}_{L_-}(\eta), \qquad \dim \operatorname{Ker} \widehat{\pi}^{-}_{\widehat{Q}_L(\eta)} = 0, \tag{3.27}$$

together with

$$\operatorname{codim} \operatorname{Range} \pi^{-}_{\widehat{Q}_L(\eta)} = n^{+}_{L_-}(\eta) - s, \qquad \dim \operatorname{Ker} \pi^{-}_{\widehat{Q}_L(\eta)} = n - s. \tag{3.28}$$

3.2 Mixed Type Equations

We now turn our attention back to the MFDE (2.10) and study solutions that admit a discontinuity at $\xi = 0$. Much as in §3.1, we need to introduce the family of function spaces

$$\widehat{X}_\alpha = C([-1, \alpha], \mathbb{C}^n) \times C([\alpha, 1], \mathbb{C}^n), \tag{3.29}$$

again parametrized by $-1 < \alpha \leq 0$. Any bounded linear operator $L : C([-1, 1], \mathbb{C}^n) \to \mathbb{C}^n$ can be represented by a measure $\mu \in NBV([-1, 1], \mathbb{C}^{n\times n})$ by means of

$$L\psi = \int_{-1}^{1} d\mu(\sigma)\psi(\sigma). \tag{3.30}$$

For any $-1 < \alpha \leq 0$ we can define a pair of measures

$$\mu^l_\alpha \in \mathrm{NBV}([-1, \alpha], \mathbb{C}^{n\times n}), \qquad \mu^r_\alpha \in \mathrm{NBV}([\alpha, 1]) \tag{3.31}$$

by writing

$$\begin{array}{rcll} \mu^l_\alpha(\vartheta) & = & \mu(\vartheta), & -1 \leq \vartheta < \alpha, \\ \mu^l_\alpha(\alpha) & = & \mu(\alpha-), & \\ \mu^r_\alpha(\alpha) & = & 0 & \\ \mu^r_\alpha(\vartheta) & = & \mu(\vartheta) - \mu(\alpha-), & \alpha < \vartheta \leq 1. \end{array} \tag{3.32}$$

Using these new measures, we extend L to $\widehat{X}_\alpha$ by writing

$$L(\psi^l, \psi^r) = \int_{-1}^{\alpha} d\mu^l_\alpha(\sigma)\psi^l(\sigma) + \int_{\alpha}^{0} d\mu^r_\alpha(\sigma)\psi^r(\sigma), \tag{3.33}$$

for any $(\psi^l, \psi^r) \in \widehat{X}_\alpha = C([-1, \alpha], \mathbb{C}^n) \times C([\alpha, 1], \mathbb{C}^n)$.

The MFDE we are interested in can now be written as

$$\widehat{x}'(\xi) = L\, \widehat{\mathrm{ev}}_\xi\, \widehat{x}. \tag{3.34}$$

Here the operators $\widehat{\mathrm{ev}}_\xi$ map $\widehat{BC}^{\oplus}_{\eta}$ into $\widehat{X}_{-\xi}$ for $0 \leq \xi < 1$ in a fashion similar to (3.4); see Figure 3(iii). As before, we have $\widehat{\mathrm{ev}}_\xi = \mathrm{ev}_\xi$ for $\xi \geq 1$. The solution sets we are interested in are defined by

$$\begin{array}{rcl} \widehat{\mathfrak{Q}}_L(\eta) & = & \big\{\widehat{x} \in \widehat{BC}^{\oplus}_{\eta} \mid \widehat{x}'(\xi) = L\, \widehat{\mathrm{ev}}_\xi\, \widehat{x} \text{ for almost all } \xi \geq 0\big\}, \\ \widehat{Q}_L(\eta) & = & \big\{\widehat{\psi} \in \widehat{X}_0 \mid \widehat{\psi} = \widehat{\mathrm{ev}}_0\, \widehat{x} \text{ for some } \widehat{x} \in \widehat{\mathfrak{Q}}_L(\eta)\big\}, \end{array} \tag{3.35}$$

where as before, we write $\widehat{E}_L$ to map $\widehat{\psi} \in \widehat{Q}_L(\eta)$ back to the function $\widehat{x} \in \widehat{\mathfrak{Q}}_L(\eta)$ that has $\widehat{\mathrm{ev}}_0\, \widehat{x} = \widehat{\psi}$.

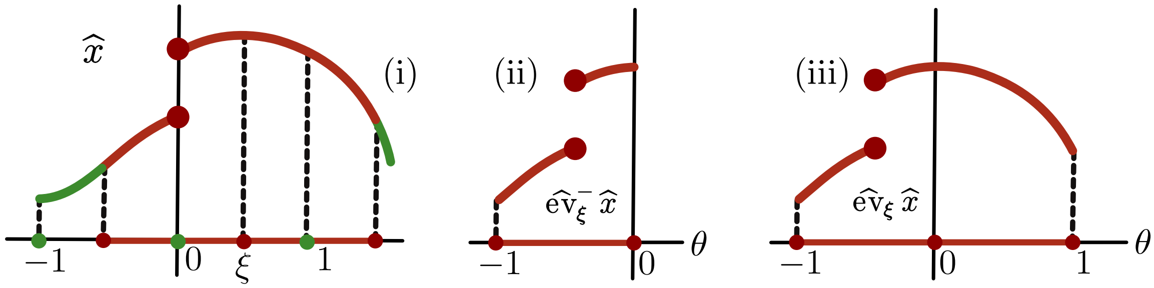

Fig. 3: *Panel (i) depicts a function $\widehat{x}$ that is multi-valued at $\xi = 0$. Panels (ii) and (iii) illustrate how the operators $\widehat{\mathrm{ev}}_{\xi}^{-}$ and $\widehat{\mathrm{ev}}_{\xi}$ act on $\widehat{x}$, for $0 \le \xi < 1$.*

Our main result in this section describes the restriction operators

$$\begin{array}{lll} \pi^{-}_{\widehat{Q}_L(\eta)} : \widehat{Q}_L(\eta) \to C\big([-1,0],\mathbb{C}\big) & & (\psi^l,\psi^r) \mapsto \psi^l, \\ \widehat{\pi}^{-}_{\widehat{Q}_L(\eta)} : \widehat{Q}_L(\eta) \to C\big([-1,0],\mathbb{C}\big) \times \mathbb{C} & & (\psi^l,\psi^r) \mapsto \big(\psi^l,\psi^r(0)\big), \end{array} \tag{3.36}$$

which generalize those introduced in (2.17). We emphasize that our results at present only work for scalar versions of (3.34), since we need to build on the theory described in §2.2.

Theorem 3.10 (see §5.2). *Consider a scalar version of the MFE (3.34) and suppose that (HL) is satisfied. Suppose furthermore that the characteristic equation $\Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\, z = \eta$. Then the operator $\widehat{\pi}^{-}_{\widehat{Q}_L(\eta)}$ is Fredholm and the following identities hold,*

$$\mathrm{codim}\,\mathrm{Range}\,\widehat{\pi}^{-}_{\widehat{Q}_L(\eta)} = \max\{1 - n_L^{\#}(\eta), 0\}, \qquad \dim \mathrm{Ker}\,\widehat{\pi}^{-}_{\widehat{Q}_L(\eta)} = \max\{n_L^{\#}(\eta) - 1, 0\}. \tag{3.37}$$

In addition, the operator $\pi^{-}_{\widehat{Q}_L(\eta)}$ is Fredholm and we have

$$\mathrm{codim}\,\mathrm{Range}\,\pi^{-}_{\widehat{Q}_L(\eta)} = \max\{-n_L^{\#}(\eta), 0\}, \qquad \dim \mathrm{Ker}\,\pi^{-}_{\widehat{Q}_L(\eta)} = \max\{n_L^{\#}(\eta), 0\}. \tag{3.38}$$

To illustrate the implications of this theorem for initial value problems involving (3.34), we analyze the different scenarios in the following results.

Corollary 3.11. *Consider the setting of Theorem 3.10. If $n_L^{\#}(\eta) < 0$, then there is a subspace $\mathcal{V} \subset C([-1,0],\mathbb{C})$ of codimension $-n_L^{\#}(\eta)$ such that for every $\phi \in \mathcal{V}$ there is a unique $\widehat{x} \in \widehat{BC}_{\eta}^{\oplus}$ that solves (3.34) and has $\mathrm{ev}_0^{-}\,\widehat{x} = \phi$.*

Corollary 3.12. *Consider the setting of Theorem 3.10. If $n_L^{\#}(\eta) = 0$, then for every $\phi \in C([-1,0],\mathbb{C})$ there is a unique $\widehat{x} \in \widehat{BC}_{\eta}^{\oplus}$ that solves (3.34) and has $\mathrm{ev}_0^{-}\,\widehat{x} = \phi$.*

Corollary 3.13. *Consider the setting of Theorem 3.10. If $n_L^{\#}(\eta) = 1$, then for any $\phi \in C([-1,0],\mathbb{C})$ and any $v \in \mathbb{C}$ there is a unique $\widehat{x} \in \widehat{BC}_{\eta}^{\oplus}$ that solves (3.34) and has $\widehat{\mathrm{ev}}_0^{-}\,\widehat{x} = (\phi, v)$.*

Corollary 3.14. *Consider the setting of Theorem 3.10. If $n_L^{\#}(\eta) > 1$, then for any $\phi \in C([-1,0],\mathbb{C})$ and any $v \in \mathbb{C}$ there is a $\widehat{x} \in \widehat{BC}_{\eta}^{\oplus}$ that solves (3.34) and has $\widehat{\mathrm{ev}}_0^{-}\,\widehat{x} = (\phi, v)$. In addition, $\widehat{y} \in \widehat{BC}_{\eta}^{\oplus}$ also solves (3.34) with $\widehat{\mathrm{ev}}_0^{-}\,\widehat{y} = (\phi, v)$ if and only if $\widehat{y} - \widehat{x} = E_L\psi$, for some $\psi \in C([-1,1],\mathbb{C})$ that is contained in the $(n_L^{\#}(\eta) - 1)$-dimensional space $\mathrm{Ker}\,\pi^{-}_{Q_L(\eta)}$ mentioned in Proposition 2.5.*

3.3 Differential-Algebraic Equations

We are now ready to turn our attention to linear differential-algebraic equations of the form

$$\mathcal{I}x'(\xi) = M\,\widehat{\mathrm{ev}}_\xi\, x. \tag{3.39}$$

Here $\mathcal{I}$ is a diagonal $(n \times n)$-matrix that has $\mathcal{I}^2 = \mathcal{I}$ and M is a bounded linear operator from $C([-1,1],\mathbb{C}^n)$ into $\mathbb{C}^n$ that we extend to the spaces $\widehat{X}_\alpha$ with $-1 < \alpha \leq 0$ in the same fashion as in §3.2. We remark that our results in this section do not require the condition (HL) to hold. In particular, the theory presented here is also applicable to delay equations. However, as explained in §1, the true power of these results come to light only when studying equations that have both delayed and advanced terms.

We are interested in systems that can be closely related to a differential system of the form (3.34). In order to clarify this relationship, we introduce the characteristic function

$$\delta_{\mathcal{I},M}(z) = \mathcal{I}z - Me^{z\cdot}I \tag{3.40}$$

that is associated to (3.39). The restriction on M that we need in this paper can now be captured by the following condition on the characteristic function.

(HM) There exist a bounded linear operator $L : C([-1,1],\mathbb{C}^n) \to \mathbb{C}^n$, a constant $\alpha \in \mathbb{C}$ and a set of non-negative integers $\ell_1, \ldots, \ell_n \in \mathbb{Z}_{\geq 0}$ such that

$$\mathcal{J}_\alpha(z)\delta_{\mathcal{I},M}(z) = \Delta_L(z), \tag{3.41}$$

where $\mathcal{J}_\alpha : \mathbb{C} \to \mathbb{C}^{n\times n}$ denotes the diagonal matrix function

$$\mathcal{J}_\alpha(z) = \mathrm{diag}\big((z-\alpha)^{\ell_1}, \ldots, (z-\alpha)^{\ell_n}\big). \tag{3.42}$$

Please note that the purely algebraic components of the system (3.39) are unaffected if the corresponding components of M are multiplied by a non-zero factor. In particular, the corresponding rows of $\delta_{\mathcal{I},M}$ can be rescaled without affecting the dynamics of (3.39). Adjusting M in such a manner will typically be necessary in order to show that all the terms $(z-\alpha)^\ell$ appearing in (3.42) have coefficient one. Furthermore, we remark that a simple matching of asymptotics along the imaginary axis shows that

$$\mathcal{J}_\alpha(\alpha) = \mathcal{I}, \tag{3.43}$$

or alternatively, that $\ell_i = 0$ if and only if $\mathcal{I}_{ii} = 1$.

The condition (HM) is related to the fact that we require any solution to the differential-algebraic equation (3.39) to also satisfy the MFDE (3.34) with the operator L featuring in (HM).

We will be interested in the solution spaces

$$\begin{array}{lcl} \mathfrak{p}_{\mathcal{I},M}(\eta) & = & \big\{x \in BC^\ominus_\eta \mid \mathcal{I}x'(\xi) = M\,\mathrm{ev}_\xi x \text{ for all } \xi \leq 0\big\}, \\ \mathfrak{q}_{\mathcal{I},M}(\eta) & = & \big\{x \in BC^\oplus_\eta \mid \mathcal{I}x'(\xi) = M\,\mathrm{ev}_\xi x \text{ for all } \xi \geq 0\big\}, \end{array} \tag{3.44}$$

which both contain solutions to (3.39) that do not admit a discontinuity at $\xi = 0$. However, when considering functions that are allowed to be multi-valued at $\xi = 0$, care must be taken to ensure that (3.39) is well-posed. The following result is important in this respect, as it shows that the part of the right-hand side of (3.39) that corresponds to the purely algebraic equations is continuous.

Lemma 3.15 (see §5.3). *Suppose that (HM) is satisfied. Then for any $\widehat{x} \in \widehat{BC}^\oplus_\eta$, the function*

$$\xi \mapsto (I - \mathcal{I})M\widehat{\mathrm{ev}}_\xi\,\widehat{x} \tag{3.45}$$

is continuous.

We recall that Lemma 3.2 implies that the map $\xi \mapsto \mathcal{I} M \,\widehat{\mathrm{ev}}_\xi \,\widehat{x}$ is continuous except possibly at a countable number of points. This motivates the definition

$$\begin{array}{lcl}\widehat{\mathfrak{q}}_{\mathcal{I},M}(\eta) & = & \big\{ \widehat{x} \in \widehat{BC}^{\oplus}_{\eta} \mid \mathcal{I}\widehat{x}'(\xi) = \mathcal{I} M \,\widehat{\mathrm{ev}}_\xi \,\widehat{x} \text{ for almost all } \xi \geq 0 \\ & & \qquad \text{and } 0 = (I - \mathcal{I}) M \,\widehat{\mathrm{ev}}_\xi \,\widehat{x} \text{ for all } \xi \geq 0 \big\}. \end{array} \tag{3.46}$$

Our main result relates these spaces $\mathfrak{p}_{\mathcal{I},M}(\eta)$, $\mathfrak{q}_{\mathcal{I},M}(\eta)$ and $\widehat{\mathfrak{q}}_{\mathcal{I},M}(\eta)$ to their counterparts $\mathfrak{P}_L(\eta)$, $\mathfrak{Q}_L(\eta)$ and $\widehat{\mathfrak{Q}}_L(\eta)$ that were defined for the differential equation (3.34). In particular, initial value problems for (3.39) can be studied using the techniques outlined in §3.1 and §3.2.

Theorem 3.16 (see §5.3). *Consider the differential-algebraic equation (3.39) and suppose that (HM) is satisfied. Choose any $\eta_* \in \mathbb{R}$ for which the characteristic equation $\det \delta_{\mathcal{I},M}(z) = 0$ admits no roots with $\mathrm{Re}\, z = \eta_*$. Then there exists a bounded linear operator $L' : C([-1,1], \mathbb{C}^n) \to \mathbb{C}^n$ such that*

$$\mathcal{J}_{\eta_*}(z)\delta_M(z) = \Delta_{L'}(z). \tag{3.47}$$

In addition, for every $\eta < \eta_*$ we have

$$\mathfrak{q}_M(\eta) = \mathfrak{Q}_{L'}(\eta), \tag{3.48}$$

while for every $\eta > \eta_*$ we have

$$\mathfrak{p}_M(\eta) = \mathfrak{P}_{L'}(\eta). \tag{3.49}$$

Finally, for any $\eta < \eta_*$ we have

$$\widehat{\mathfrak{q}}_M(\eta) = \widehat{\mathfrak{Q}}_{L'}(\eta). \tag{3.50}$$

3.4 Nonlinear Equations

We are now ready to turn to the main goal of this paper and discuss initial value problems for nonlinear differential-algebraic equations. In particular, we will set out to study the nonlinear equation

$$\mathcal{I}\widehat{x}'(\xi) = M\widehat{\mathrm{ev}}_\xi \widehat{x} + \mathcal{M}\big(\widehat{\mathrm{ev}}_\xi \widehat{x}\big) \tag{3.51}$$

and look for solutions in the spaces $\widehat{BC}^{\oplus}_{-\eta}$ for $\eta \geq 0$. In particular, we will assume that $x = 0$ is an equilibrium for (3.51) and investigate how the stable manifold of this equilibrium intersects the space of initial conditions $C([-1,0], \mathbb{C}^n)$.

As in §3.3, the matrix $\mathcal{I}$ needs to be diagonal with $\mathcal{I}^2 = \mathcal{I}$, while the bounded linear operator M that maps $C([-1,1], \mathbb{C}^n)$ into $\mathbb{C}^n$ should satisfy (HM). The conditions on the nonlinearity $\mathcal{M}$ that we need are described below.

(H$\mathcal{M}$) Recall the constants $\ell_1, \ldots, \ell_n$ appearing in (HM). There exists an integer $k \geq 1$ and a set of operators

$$\mathcal{M}^{(s)}_i : \bigcup_{-1 < \alpha \leq 0} \widehat{X}_{-\alpha} \times \mathbb{C}^n \to \mathbb{C}, \qquad 0 \leq s \leq \ell_i, \qquad 1 \leq i \leq n \tag{3.52}$$

that satisfies the following properties for each $1 \leq i \leq n$ and each $0 \leq s \leq \ell_i$.

(i) For each $-1 < \alpha \leq 0$, the map $\mathcal{M}^{(s)}_i$ is C^k-smooth as a map from $\widehat{X}_{-\alpha} \times \mathbb{C}^n$ into $\mathbb{C}$.

(ii) For every $K > 0$, there exists a constant C such that for any $-1 < \alpha \leq 0$ and any $(\widehat{\psi}, v) \in \widehat{X}_\alpha \times \mathbb{C}^n$ that has $\left\| (\widehat{\psi}, v) \right\| < K$, we have

$$\left\| D^j \mathcal{M}^{(s)}_i (\widehat{\psi}, v) \right\| < C, \qquad 0 \leq j \leq k. \tag{3.53}$$

(iii) We have $\mathcal{M}_i^{(s)}(0,0) = D\mathcal{M}_i^{(s)}(0,0) = 0$.

(iv) Pick a function $\widehat{x} \in \widehat{BC}_0^{\oplus} \cap W^{1,1}_{\mathrm{loc}}([0,\infty),\mathbb{C}^n)$. Then the function f_i defined by

$$f_i : \xi \mapsto \Big(\mathcal{M}\big(\widehat{\mathrm{ev}}_\xi\, \widehat{x}\big)\Big)_i \tag{3.54}$$

satisfies $f_i \in W^{\ell_i,1}_{\mathrm{loc}}([0,\infty),\mathbb{C})$. Furthermore, we have

$$D^s f_i(\xi) = \mathcal{M}_i^{(s)}\big(\widehat{\mathrm{ev}}_\xi\, \widehat{x}, \widehat{x}'(\xi)\big) \tag{3.55}$$

for almost all $\xi \geq 0$.

We remark that this condition is a little more involved than one usually encounters for invariant manifold results. Conditions (i) and (ii) basically state that the maps $\mathcal{M}_i^{(s)}$ are C^k-smooth, but care must be taken to account for the fact that these operators must be defined simultaneously on the different spaces $\widehat{X}_\alpha$. On the other hand, condition (iv) is necessary to ensure that the nonlinearity respects the differentiability structure that the linear part of the differential-algebraic equation imposes. A simpler version of this condition that does not involve the derivative $\widehat{x}'(\xi)$ can be found in [17, (HR1)-(HR2)], but the version stated here is necessary to cover the applications we consider in this paper.

Before we state our results, we have to comment on what we consider to be a solution to (3.51). In particular, we say that $\widehat{x} \in \widehat{BC}_\eta^{\oplus}$ solves (3.51) if it satisfies

$$\mathcal{I}\widehat{x}'(\xi) = \mathcal{I}M\, \widehat{\mathrm{ev}}_\xi\, \widehat{x} + \mathcal{I}\mathcal{M}\big(\widehat{\mathrm{ev}}_\xi\, \widehat{x}\big) \tag{3.56}$$

for almost all $\xi \geq 0$, together with

$$0 = (I - \mathcal{I})M\, \widehat{\mathrm{ev}}_\xi\, \widehat{x} + (I - \mathcal{I})\mathcal{M}\big(\widehat{\mathrm{ev}}_\xi\, \widehat{x}\big) \tag{3.57}$$

for all $\xi \geq 0$.

Recalling the solution spaces $\widehat{\mathfrak{q}}_{\mathcal{I},M}(\eta)$ defined by (3.46), we introduce the sets

$$\widehat{q}_{\mathcal{I},M}(\eta) = \big\{\widehat{\psi} \in \widehat{X}_0 \mid \widehat{\psi} = \widehat{\mathrm{ev}}_0\, \widehat{x} \text{ for some } \widehat{x} \in \widehat{\mathfrak{q}}_{\mathcal{I},M}(\eta)\big\} \tag{3.58}$$

that contain the initial segments of these solutions, together with the open balls

$$\widehat{\mathcal{V}}_\eta^\delta = \{\widehat{\psi} \in \widehat{q}_{\mathcal{I},M}(\eta) \text{ for which } \big\|\widehat{\psi}\big\| < \delta\} \subset \widehat{X}_0 \tag{3.59}$$

in these spaces, defined for any $\delta > 0$.

Our main result describes the local stable manifold of the zero equilibrium of (3.51). In particular, it shows that this stable manifold is a graph over $\widehat{\mathcal{V}}_0^\delta$.

Theorem 3.17 (see §5.4). *Consider the nonlinear system (3.51) and suppose that (HM) and (H$\mathcal{M}$) are satisfied. Suppose furthermore that the characteristic equation* $\det \delta_{\mathcal{I},M}(z) = 0$ *admits no solutions with* $\mathrm{Re}\, z = 0$*. Then there exist constants $\delta > 0$, $\delta_* > 0$ and $\eta_* > 0$, a bounded linear projection operator*

$$\widehat{\Pi}_{\widehat{q}_{\mathcal{I},M}(0)} : \widehat{X}_0 \to \widehat{q}_{\mathcal{I},M}(0) \tag{3.60}$$

and a C^k-smooth map

$$\widehat{u}^* : \widehat{\mathcal{V}}_0^\delta \to \widehat{BC}_{-\eta_*}^{\oplus} \tag{3.61}$$

that satisfy the following properties.

(i) For any $\widehat{\psi} \in \widehat{\mathcal{V}}_0^{\delta}$, the function $\widehat{u}^*(\widehat{\psi})$ satisfies the nonlinear equation (3.51).

(ii) For any $\widehat{\psi} \in \widehat{\mathcal{V}}_0^{\delta}$, we have

$$\widehat{\Pi}_{\widehat{q}_{\mathcal{I},M}(0)}\, \widehat{\mathrm{ev}}_0\, \widehat{u}^*(\widehat{\psi}) = \widehat{\psi}. \tag{3.62}$$

(iii) Suppose that $\widehat{x} \in \widehat{BC}_0^{\oplus}$ satisfies the nonlinear equation (3.51) and that $|\widehat{x}(\xi)| \leq \delta_*$ for all $\xi \geq -1$. Then we have

$$\widehat{x} = \widehat{u}^*\Big(\widehat{\Pi}_{\widehat{q}_{\mathcal{I},M}(0)}\, \widehat{\mathrm{ev}}_0\, \widehat{x}\Big). \tag{3.63}$$

(iv) We have the derivative

$$D[\widehat{\mathrm{ev}}_0\, \widehat{u}^*](0) = I. \tag{3.64}$$

In order to analyze the consequences of this theorem, we introduce the familiar restriction operators

$$\begin{aligned} \pi^-_{\widehat{q}_{\mathcal{I},M}(\eta)} &: \widehat{q}_{\mathcal{I},M}(\eta) \to C\big([-1,0],\mathbb{C}^n\big), && (\psi^l,\psi^r) \mapsto \psi^l, \\ \widehat{\pi}^-_{\widehat{q}_{\mathcal{I},M}(\eta)} &: \widehat{q}_{\mathcal{I},M}(\eta) \to C\big([-1,0],\mathbb{C}^n\big) \times \mathbb{C}^n, && (\psi^l,\psi^r) \mapsto \big(\psi^l,\psi^r(0)\big). \end{aligned} \tag{3.65}$$

The results below outline six scenarios that may arise for initial value problems associated to (3.51). We start by picking a small initial condition $\phi \in C([-1,0],\mathbb{C}^n)$ and discussing whether ϕ can be extended to a solution to (3.51) that decays to zero at a specified exponential rate. We note that this solution is allowed to have a discontinuity at $\xi = 0$.

In our first scenario, every small $\phi \in C([-1,0],\mathbb{C}^n)$ can be uniquely extended. In particular, the discontinuity depends directly on ϕ. We note that there may exist other extensions of ϕ, but these will all leave a fixed small neighbourhood of the zero equilibrium at some point.

Definition 3.18 (Stabilizable). *The equilibrium $x = 0$ of (3.51) is of type $(S)_\eta$ if there exist constants $\delta > 0$ and $\epsilon > 0$ such that the following holds true.*

For every $\phi \in C([-1,0],\mathbb{C}^n)$ that has $\|\phi\| < \delta$, there exists a unique $\widehat{x} = \widehat{x}(\phi) \in \widehat{BC}_\eta^{\oplus}$ that has $\|\widehat{x}\|_\eta < \epsilon$, solves the nonlinear system (3.51) and satisfies $\mathrm{ev}_0^-\, \widehat{x} = \phi$.

Corollary 3.19 (see §5.4). *Pick any $\eta \geq 0$. Consider the nonlinear system (3.51), suppose that (HM) and (H$\mathcal{M}$) are satisfied and that $\det \delta_{\mathcal{I},M}(z) = 0$ admits no solutions with $\mathrm{Re}\, z = -\eta$.*

Suppose furthermore that

$$\mathrm{Range}\, \pi^-_{\widehat{q}_{\mathcal{I},M}(-\eta)} = C([-1,0],\mathbb{C}^n), \qquad \mathrm{Ker}\, \pi^-_{\widehat{q}_{\mathcal{I},M}(-\eta)} = \{0\}. \tag{3.66}$$

Then the equilibrium $x = 0$ of (3.51) is of type $(S)_{-\eta}$.

In our second scenario, any $\phi \in C([-1,0],\mathbb{C}^n)$ can still be extended, but no longer in a unique fashion. In fact, there exists a whole continuous family of possible extensions that all decay to zero.

Definition 3.20 (Multi-Stabilizable). *The equilibrium $x = 0$ of (3.51) is of type $(MS)_{\eta,n_{\mathrm{ker}}}$ if there exist constants $\delta > 0$ and $\epsilon > 0$ such that the following holds true.*

For every pair $\phi \in C([-1,0],\mathbb{C}^n)$ and $\rho \in \mathbb{C}^{n_{\mathrm{ker}}}$ that have $\|\phi\| < \delta$ and $|\rho| < \delta$, there exists a function $\widehat{x} = \widehat{x}(\phi,\rho) \in \widehat{BC}_\eta^{\oplus}$ that has $\|\widehat{x}\|_\eta < \epsilon$, solves the nonlinear system (3.51) and satisfies $\mathrm{ev}_0^-\, \widehat{x} = \phi$. These functions are all distinct continuations of ϕ, in the sense that $\widehat{x}(\phi,\rho_1) \neq \widehat{x}(\phi,\rho_2)$ whenever $\rho_1 \neq \rho_2$.

Corollary 3.21 (see §5.4). *Pick any $\eta \geq 0$. Consider the nonlinear system (3.51), suppose that (HM) and (H$\mathcal{M}$) are satisfied and that* $\det \delta_{\mathcal{I},M}(z) = 0$ *admits no solutions with* $\operatorname{Re} z = -\eta$.

Suppose furthermore that

$$\operatorname{Range} \pi^{-}_{\widehat{q}_{\mathcal{I},M}(-\eta)} = C([-1,0],\mathbb{C}^n), \qquad n_{\mathrm{ker}} := \dim \operatorname{Ker} \pi^{-}_{\widehat{q}_{\mathcal{I},M}(-\eta)} > 0. \tag{3.67}$$

Then the equilibrium $x = 0$ of (3.51) is of type $(MS)_{-\eta,n_{\mathrm{ker}}}$.

Our third scenario covers the situation in which there exists a branch of $\phi \in C([-1,0],\mathbb{C}^n)$ for which no extension exists that decays to zero.

Definition 3.22 (Unstable). *The equilibrium $x = 0$ of (3.51) is of type $(U)_\eta$ if there exists a constant $\epsilon > 0$ such that the following holds true.*

For every sufficiently small $\delta > 0$, there exists a function $\phi \in C([-1,0],\mathbb{C}^n)$ that has $\|\phi\| < \delta$ such that any function $\widehat{x} \in \widehat{BC}^{\oplus}_{-\eta}$ that has $\|\widehat{x}\|_{-\eta} < \epsilon$ and solves the nonlinear system (3.51), must have

$$\mathrm{ev}_0^- \, \widehat{x} \neq \phi. \tag{3.68}$$

Corollary 3.23 (see §5.4). *Pick any $\eta \geq 0$. Consider the nonlinear system (3.51), suppose that (HM) and (H$\mathcal{M}$) are satisfied and that* $\det \delta_{\mathcal{I},M}(z) = 0$ *admits no solutions with* $\operatorname{Re} z = -\eta$.

Suppose furthermore that

$$\dim \operatorname{Ker} \pi^{-}_{\widehat{q}_{\mathcal{I},M}(-\eta)} < \operatorname{codim} \operatorname{Range} \pi^{-}_{\widehat{q}_{\mathcal{I},M}(-\eta)}. \tag{3.69}$$

Then the equilibrium $x = 0$ of (3.51) is of type $(U)_{-\eta}$.

Notice that we have given no results in the case that

$$0 < \operatorname{codim} \operatorname{Range} \pi^{-}_{\widehat{q}_{\mathcal{I},M}(-\eta)} \leq \dim \operatorname{Ker} \pi^{-}_{\widehat{q}_{\mathcal{I},M}(-\eta)}. \tag{3.70}$$

This can be understood by observing that nonlinear effects play a role here. For example, if one has equality in (3.70), one would like to conclude that the zero equilibrium is of type $(S)_{-\eta}$. In order to verify this however, one would need to compute the second order derivative $[D^2 \mathrm{ev}_0^- \widehat{u}^*](0)$ and check that no degeneracies occur. For instance, if $\mathcal{M} = 0$, then obviously the zero equilibrium is of type $(U)_{-\eta}$.

Naturally, this entire analysis can be repeated for the restriction operators $\widehat{\pi}^{-}_{\widehat{q}_{\mathcal{I},M}(-\eta)}$. This time, we consider a small initial condition $\phi \in C([-1,0],\mathbb{C}^n)$ together with a small $v \in \mathbb{C}^n$. The question at hand is then whether the function ϕ can be extended to a solution to (3.51) that decays to zero at a specified exponential rate and has a jump at $\xi = 0$ of size $v - \phi(0)$.

Definition 3.24 (Gap-Stabilizable). *The equilibrium $x = 0$ of (3.51) is of type $(GS)_\eta$ if there exist constants $\delta > 0$ and $\epsilon > 0$ such that the following holds true.*

For every pair $(\phi, v) \in C([-1,0],\mathbb{C}^n) \times \mathbb{C}^n$ that has $\|(\phi, v)\| < \delta$, there exists a unique $\widehat{x} = \widehat{x}(\phi, v) \in \widehat{BC}^{\oplus}_{\eta}$ that has $\|\widehat{x}\|_\eta < \epsilon$, solves the nonlinear system (3.51) and satisfies $\widehat{\mathrm{ev}}_0^- \, \widehat{x} = (\phi, v)$.

Corollary 3.25 (see §5.4). *Pick any $\eta \geq 0$. Consider the nonlinear system (3.51), suppose that (HM) and (H$\mathcal{M}$) are satisfied and that* $\det \delta_{\mathcal{I},M}(z) = 0$ *admits no solutions with* $\operatorname{Re} z = -\eta$.

Suppose furthermore that

$$\operatorname{Range} \widehat{\pi}^{-}_{\widehat{q}_{\mathcal{I},M}(-\eta)} = C([-1,0],\mathbb{C}^n) \times \mathbb{C}^n, \qquad \operatorname{Ker} \widehat{\pi}^{-}_{\widehat{q}_{\mathcal{I},M}(-\eta)} = \{0\}. \tag{3.71}$$

Then the equilibrium $x = 0$ of (3.51) is of type $(GS)_{-\eta}$.

Definition 3.26 (Gap-Multi-Stabilizable). *The equilibrium $x = 0$ of (3.51) is of type $(GMS)_{\eta,n_{\mathrm{ker}}}$ if there exist constants $\delta > 0$ and $\epsilon > 0$ such that the following holds true.*

For every pair $(\phi, v) \in C([-1,0],\mathbb{C}^n) \times \mathbb{C}^n$ and $\rho \in \mathbb{C}^{n_{\mathrm{ker}}}$ that have $\|(\phi, v)\| < \delta$ and $|\rho| < \delta$, there exists a function $\widehat{x} = \widehat{x}(\phi, v, \rho) \in \widehat{BC}_{\eta}^{\oplus}$ that has $\|\widehat{x}\|_{\eta} < \epsilon$, solves the nonlinear system (3.51) and satisfies $\widehat{\mathrm{ev}}_0^- \,\widehat{x} = (\phi, v)$. These functions are all distinct continuations of (ϕ, v), in the sense that $\widehat{x}(\phi, v, \rho_1) \neq x(\phi, v, \rho_2)$ whenever $\rho_1 \neq \rho_2$.

Corollary 3.27 (see §5.4). *Pick any $\eta \geq 0$. Consider the nonlinear system (3.51), suppose that (HM) and (H$\mathcal{M}$) are satisfied and that $\det \delta_{\mathcal{I},M}(z) = 0$ admits no solutions with $\mathrm{Re}\, z = -\eta$.*

Suppose furthermore that

$$\mathrm{Range}\, \widehat{\pi}^-_{\widehat{q}_{\mathcal{I},M}(-\eta)} = C([-1,0],\mathbb{C}^n) \times \mathbb{C}^n, \qquad n_{\mathrm{ker}} := \dim \mathrm{Ker}\, \widehat{\pi}^-_{\widehat{q}_{\mathcal{I},M}(-\eta)} > 0. \tag{3.72}$$

Then the equilibrium $x = 0$ of (3.51) is of type $(GMS)_{-\eta,n_{\mathrm{ker}}}$.

Definition 3.28 (Gap-Unstable). *The equilibrium $x = 0$ of (3.51) is of type $(GU)_{\eta}$ if there exists a constant $\epsilon > 0$ such that the following holds true.*

For every sufficiently small $\delta > 0$, there exists a pair $(\phi, v) \in C([-1,0],\mathbb{C}^n) \times \mathbb{C}^n$ that has $\|(\phi, v)\| < \delta$ such that any function $\widehat{x} \in \widehat{BC}_{\eta}^{\oplus}$ that has $\|\widehat{x}\|_{\eta} < \epsilon$ and solves the nonlinear system (3.51), must have

$$\widehat{\mathrm{ev}}_0^- \,\widehat{x} \neq (\phi, v). \tag{3.73}$$

Corollary 3.29 (see §5.4). *Pick any $\eta \geq 0$. Consider the nonlinear system (3.51), suppose that (HM) and (H$\mathcal{M}$) are satisfied and that $\det \delta_{\mathcal{I},M}(z) = 0$ admits no solutions with $\mathrm{Re}\, z = -\eta$.*

Suppose furthermore that

$$\dim \mathrm{Ker}\, \widehat{\pi}^-_{\widehat{q}_{\mathcal{I},M}(-\eta)} < \mathrm{codim}\, \mathrm{Range}\, \widehat{\pi}^-_{\widehat{q}_{\mathcal{I},M}(-\eta)}. \tag{3.74}$$

Then the equilibrium $x = 0$ of (3.51) is of type $(GU)_{-\eta}$.

4 Examples

In this section we return to the model from economic theory that we discussed in the introduction. Our goal is to illustrate the application range of the results described in §3. In particular, we discuss how the technical issues that arise in practice can be handled and hope that our analysis here will prove insightful to readers that encounter similar problems.

The full nonlinear model that we set out to analyze is given by

$$\begin{array}{rcl} \Lambda'(R(t))\,\frac{dR(t)}{dt} & = & \Lambda(R(t))\,[r + \pi(t) - R(t)] \\ \pi^b(t) & = & \frac{\int_{-\Omega^b}^{0} e^{\beta^b \sigma}\pi(t+\sigma)d\sigma}{\int_{-\Omega^b}^{0} e^{\beta^b \sigma} d\sigma}, \\ \pi^f(t) & = & \frac{\int_{0}^{\Omega^f} e^{-\beta^f \sigma}\pi(t+\sigma)d\sigma}{\int_{0}^{\Omega^f} e^{-\beta^f \sigma} d\sigma}, \end{array} \tag{4.1}$$

where the additional identity

$$\pi(t) = (1 - \chi_1 - \chi_2)^{-1}[\rho^{-1}(R(t)) - \chi_1 \pi^b(t) - \chi_2 \pi^f(t)] \tag{4.2}$$

serves to close the system. This latter equation was obtained by isolating $\pi(t)$ from the expression

$$R(t) = \rho\left((1 - \chi_1 - \chi_2)\,\pi(t) + \chi_1 \pi^b(t) + \chi_2 \pi^f(t)\right). \tag{4.3}$$

Since we will be studying solutions that are allowed to have a discontinuity at $t = 0$, we emphasize that any references to $R(0)$, $\pi^b(0)$ and $\pi^f(0)$ in the system (4.1) should be interpreted as $R(0+)$, $\pi^b(0+)$ and $\pi^f(0+)$.

Looking for equilibria $\big(R(t), \pi^b(t), \pi^f(t)\big) = (R_*, \pi^b_*, \pi^f_*)$, we see that we must have

$$R_* - r = \rho^{-1}(R_*) = \pi^b_* = \pi^f_*. \tag{4.4}$$

Assuming for the moment that (4.4) has a solution, we introduce the quantities

$$\begin{array}{lcl} \pi_* & = & \pi^b_* = \pi^f_*, \\ \rho'_* & = & \rho'(\rho^{-1}(R_*)) = \rho'(\pi_*), \\ \kappa_\chi & = & 1 - \chi_1 - \chi_2, \\ \mathcal{A}_* & = & [(\rho'_*)^{-1} - \kappa_\chi]^{-1}, \\ \kappa_b & = & \big[\int_{-\Omega^b}^0 e^{\beta^b \sigma} d\sigma\big]^{-1}, \\ \kappa_f & = & \big[\int_0^{\Omega^f} e^{-\beta^f \sigma} d\sigma\big]^{-1}, \\ \Lambda_* & = & -\frac{\Lambda(R_*)}{\Lambda'(R_*)}. \end{array} \tag{4.5}$$

We will make the following assumption concerning these parameters.

(hp) We have $\chi_1 \geq 0$, $\chi_2 \geq 0$ and $\kappa_\chi > 0$. In addition, we have $\Omega^b > 0$, $\Omega^f > 0$, $\beta^b > 0$ and $\beta^f > 0$. Finally, the function ρ is C^1-smooth, with $\rho'_* > 0$ but $\rho'_* \neq (1 - \chi_1 - \chi_2)^{-1}$.

Let us make a couple of observations concerning (hp) before we proceed. First of all, notice that sufficient conditions that guarantee that (4.4) has a unique solution are not hard to find. For example, it suffices to require that $\rho'(\pi) > 1$ for all $\pi \in \mathbb{R}$ or alternatively that $\rho'(\pi) < 1$ for all $\pi \in \mathbb{R}$.

Second, we note that the assumption on ρ'_* ensures that $\rho^{-1}(R)$ is well-defined for all R sufficiently close to the equilibrium R_*. Furthermore, the constant $\mathcal{A}_*$ can have both signs, depending on the size of ρ'_*.

Finally, we remark that we have not made any assumptions concerning Λ_* yet. In fact, we will split our analysis into two separate parts that are covered by the following mutually exclusive assumptions.

(HPS) The equilibrium equation (4.4) has a solution that satisfies (hp). In addition, the function Λ is constant, i.e., $\Lambda'(R) = 0$ for all $R \in \mathbb{R}$.

(HPF) The equilibrium equation (4.4) has a solution that satisfies (hp). In addition, we have $\Lambda_* > 0$.

Note that (HPS) describes the singular situation where the first line of (4.1) reduces to an algebraic equation, while (HPF) leaves the full mixed differential-algebraic system intact.

4.1 The Singular System

In this section, we concentrate on the dynamics of (4.1) under the singular condition (HPS). In this case, the first line of (4.1) reduces to

$$R(t) = r + \pi(t), \tag{4.6}$$

from which we derive the nonlinear algebraic equation

$$R(t) = r + \kappa_\chi^{-1}[\rho^{-1}\big(R(t)\big) - \chi_1 \pi^b(t) - \chi_2 \pi^f(t)]. \tag{4.7}$$

Using the two remaining lines of (4.1), we can eliminate the variables π^b and π^f to arrive at

$$\begin{array}{lcl} \kappa_\chi (R(t) - r) & = & \rho^{-1}\big(R(t)\big) - \chi_1 \kappa_b \int_{-\Omega^b}^0 e^{\beta^b \sigma} (R(t+\sigma) - r)\, d\sigma \\ & & \qquad - \chi_2 \kappa_f \int_0^{\Omega^f} e^{-\beta^f \sigma} (R(t+\sigma) - r)\, d\sigma, \end{array} \tag{4.8}$$

which in turn can be rewritten as

$$\begin{array}{lcl} \kappa_\chi R(t) - \rho^{-1}\big(R(t)\big) & = & r - \chi_1 \kappa_b \int_{-\Omega^b}^0 e^{\beta^b \sigma} R(t+\sigma)\, d\sigma \\ & & \qquad - \chi_2 \kappa_f \int_0^{\Omega^f} e^{-\beta^f \sigma} R(t+\sigma)\, d\sigma. \end{array} \tag{4.9}$$

Our main goal is to determine which of the scenario's outlined in §3.4 is applicable to the equilibrium $R(t) = R_*$. In particular, given an initial condition $\phi_R \in C([-\Omega^b, 0], \mathbb{R})$ that is close to R_*, we wish to determine if (4.9) admits a solution $\widehat{R}(t)$ that decays to R_* as $t \to \infty$ and has $\widehat{R}(\sigma) = \phi_R(\sigma)$ for $-\Omega^b \le \sigma < 0$. This solution $\widehat{R}(t)$ is allowed to have a single discontinuity at $t = 0$. In addition, we wish to know whether this solution is unique. If not, does it become unique if $\widehat{R}(0+)$ is also specified besides ϕ_R?

In order to answer these questions, we need to separate the system (4.9) into its linear and nonlinear parts. To do this, we linearize around the equilibrium $R(t) = R_*$ by making the replacement $R(t) \mapsto R_* + R(t)$. This leads to the system

$$\begin{array}{lcl} \kappa_\chi R(t) & = & \frac{1}{\rho'_*} R(t) + \mathcal{N}\big(R(t)\big) - \chi_1 \kappa_b \int_{-\Omega^b}^0 e^{\beta^b \sigma} R(t+\sigma)\, d\sigma \\ & & \qquad - \chi_2 \kappa_f \int_0^{\Omega^f} e^{-\beta^f \sigma} R(t+\sigma)\, d\sigma, \end{array} \tag{4.10}$$

where $\mathcal{N} : \mathbb{R} \to \mathbb{R}$ stands for the nonlinear function

$$\mathcal{N}(R) = \rho^{-1}(R_* + R) - \rho^{-1}(R_*) - \frac{1}{\rho'_*} R. \tag{4.11}$$

Note first that the nonlinearity depends only on the value $R(t)$, which together with the smoothness of ρ^{-1} shows that $(H\mathcal{M})$ is satisfied.

Let us rewrite the system as

$$0 = M \widehat{\mathrm{ev}}_t R - \mathcal{A}_* \mathcal{N}\big(R(t)\big), \tag{4.12}$$

where M acts on a function $\widehat{\psi}$ as

$$M \widehat{\psi} = -\widehat{\psi}(0+) + \chi_1 \kappa_b \mathcal{A}_* \int_{-\Omega^b}^0 e^{\beta^b \sigma} \widehat{\psi}(\sigma)\, d\sigma + \chi_2 \kappa_f \mathcal{A}_* \int_0^{\Omega^f} e^{-\beta^f \sigma} \widehat{\psi}(\sigma)\, d\sigma. \tag{4.13}$$

The characteristic function associated to the linear part of (4.12) is given by

$$\delta_{0,M}(z) \quad = \quad 1 - \chi_1 \kappa_b \mathcal{A}_* \int_{-\Omega^b}^0 e^{(z+\beta^b)\sigma}\, d\sigma - \chi_2 \kappa_f \mathcal{A}_* \int_0^{\Omega^f} e^{(z-\beta^f)\sigma}\, d\sigma. \tag{4.14}$$

In order to show that condition (HM) is satisfied, we pick any $\gamma \in \mathbb{R}$ and introduce the operator $L(\gamma) : C([-\Omega^b, \Omega^f], \mathbb{R}) \to \mathbb{R}$ that acts as

$$\begin{array}{lcl} L(\gamma)\psi & = & \gamma \psi(0) + \chi_1 \kappa_b \mathcal{A}_* \big[\psi(0) - e^{-\beta^b \Omega^b} \psi(-\Omega^b) - (\gamma + \beta^b) \int_{-\Omega^b}^0 e^{\beta^b \sigma} \psi(\sigma)\, d\sigma\big] \\ & & \quad + \chi_2 \kappa_f \mathcal{A}_* \big[e^{-\beta^f \Omega^f} \psi(\Omega^f) - \psi(0) - (\gamma - \beta^f) \int_0^{\Omega^f} e^{-\beta^f \sigma} \psi(\sigma)\, d\sigma\big]. \end{array} \tag{4.15}$$

A short calculation now yields the identity

$$(z - \gamma) \delta_{0,M}(z) = \Delta_{L(\gamma)}(z), \tag{4.16}$$

which shows that (HM) is indeed satisfied. We now proceed to analyze the characteristic function $\delta_{0,M}(z)$ in a number of different scenarios.

4.1.1 Backward-Looking Variables Only

In this part we will study the singular problem (4.9) under the additional assumption that $\chi_2 = 0$. Notice that this turns (4.9) into a delayed algebraic equation. The characteristic function now reads

$$\delta_{0,M}(z) \quad = \quad 1 - \chi_1 \kappa_b \mathcal{A}_* \int_{-\Omega^b}^{0} e^{(z+\beta^b)\sigma} d\sigma. \tag{4.17}$$

We start by characterizing the roots of $\delta_{0,M}(z) = 0$.

Lemma 4.1. *Suppose that (HPS) holds and that $\chi_2 = 0$. If $\rho_*' > (1-\chi_1)^{-1}$, then any root of the characteristic equation $\delta_{0,M}(z) = 0$ satisfies $\operatorname{Im} z \neq 0$ and $\operatorname{Re} z < -\beta^b$.*

On the other hand, if $0 < \rho_*' < (1-\chi_1)^{-1}$, then $\delta_{0,M}(z) = 0$ has a simple root at $z = z_*$ for some $z_* \in \mathbb{R}$. We have $z_* < 0$ if $0 < \rho_*' < 1$, $z_* = 0$ if $\rho_*' = 1$ and $z_* > 0$ if $1 < \rho_*' < (1-\chi_1)^{-1}$. If $z \neq z_*$ also solves $\delta_{0,M}(z) = 0$, then we have $\operatorname{Im} z \neq 0$ and $\operatorname{Re} z < \min\{-\beta^b, z_*\}$.

Proof. We will start by looking for solutions to $\delta_{0,M}(p) = 0$ that have $p \in \mathbb{R}$. If $\rho_*' > (1-\chi_1)^{-1}$, we have $\mathcal{A}_* < 0$ and hence $\delta_{0,M}(p) > 0$ for all $p \in \mathbb{R}$. If on the other hand $\rho_*' < (1-\chi_1)^{-1}$, we have $\mathcal{A}_* > 0$ which implies $\delta_{0,M}'(p) > 0$ for all $p \in \mathbb{R}$. In view of the limits

$$\lim_{p \to -\infty} \delta_{0,M}(p) = -\infty, \qquad \lim_{p \to \infty} \delta_{0,M}(p) = 1, \tag{4.18}$$

there now exists a unique $z_* \in \mathbb{R}$ with $\delta_{0,M}(z_*) = 0$. Using the identity

$$\delta_{0,M}(0) = 1 - \chi_1 \kappa_b \mathcal{A}_* \int_{-\Omega^b}^{0} e^{\beta^b \sigma} d\sigma = \frac{1-\rho_*'}{1-(1-\chi_1)\rho_*'}, \tag{4.19}$$

we conclude that $z_* < 0$ for $0 < \rho_*' < 1$, $z_* = 0$ for $\rho_*' = 1$ and $z_* > 0$ for $1 < \rho_*' < (1-\chi_1)^{-1}$.

It remains to consider complex roots of $\delta_{0,M}(z) = 0$. For any pair $p, q \in \mathbb{R}$, we may write

$$\begin{aligned} \operatorname{Re} \delta_{0,M}(p+iq) \quad &= \quad 1 - \chi_1 \kappa_b \mathcal{A}_* \int_{-\Omega^b}^{0} e^{(\beta^b+p)\sigma} \cos(q\sigma) d\sigma, \\ \operatorname{Im} \delta_{0,M}(p+iq) \quad &= \quad -\chi_1 \kappa_b \mathcal{A}_* \int_{-\Omega^b}^{0} e^{(\beta^b+p)\sigma} \sin(q\sigma) d\sigma. \end{aligned} \tag{4.20}$$

For any $p > -\beta^b$, the map $\sigma \mapsto e^{(\beta^b+p)\sigma}$ is a strictly increasing function. In particular, if we also have $q > 0$ then

$$\int_{-\Omega^b}^{0} e^{(\beta^b+p)\sigma} \sin(q\sigma) d\sigma < 0. \tag{4.21}$$

A similar identity for $q < 0$ shows that $\operatorname{Im} \delta_{0,M}(p+iq) \neq 0$ for all $p > -\beta^b$ and $q \neq 0$. Directly computing

$$q\delta_{0,M}(-\beta^b + iq) = q + i\chi_1 \kappa_b \mathcal{A}_* (1 - e^{-iq\Omega^b}) \tag{4.22}$$

shows that in fact $\delta_{0,M}(p+iq) \neq 0$ for all $p \geq -\beta^b$ and $q \neq 0$.

Finally, notice that for any $q \neq 0$, we have

$$\int_{-\Omega^b}^{0} e^{(\beta^b+p)\sigma} \cos(q\sigma) d\sigma < \int_{-\Omega^b}^{0} e^{(\beta^b+p)\sigma} d\sigma. \tag{4.23}$$

In particular, if $\mathcal{A}_* > 0$, $p \geq z_*$ and $q \neq 0$, we may estimate

$$\operatorname{Re} \delta_{0,M}(p+iq) > \delta_{0,M}(p) \geq 0, \tag{4.24}$$

which concludes the proof. □

Corollary 4.2. *Suppose that (HPS) is satisfied and that $\chi_2 = 0$. If $\rho'_* > (1-\chi_1)^{-1}$ or $0 < \rho'_* < 1$, then the equilibrium R_* of the nonlinear system (4.9) is of type $(S)_0$.*

Proof. Fix $\gamma > 0$. Recalling the operator $L(\gamma)$ from (4.15), Theorem 3.16 implies that

$$\widehat{q}_{\mathcal{I},M}(0) = \widehat{Q}_{L(\gamma)}(0). \tag{4.25}$$

Note that $n^+_{L(\gamma)}(0) = 1$, which due to Corollary 3.9 implies that

$$\operatorname{codim} \operatorname{Range} \pi^-_{\widehat{q}_{\mathcal{I},M}(0)} = 0, \qquad \dim \operatorname{Ker} \pi^-_{\widehat{q}_{\mathcal{I},M}(0)} = 0. \tag{4.26}$$

The result now follows from Corollary 3.19. □

Corollary 4.3. *Suppose that (HPS) is satisfied and that $\chi_2 = 0$. If $1 < \rho'_* < (1-\chi_1)^{-1}$, then the equilibrium R_* of the nonlinear system (4.9) is of type $(U)_0$.*

Proof. For any $\gamma > 0$, we now have $n^+_{L(\gamma)}(0) = 2$. Corollary 3.9 hence implies that

$$\operatorname{codim} \operatorname{Range} \pi^-_{\widehat{q}_{\mathcal{I},M}(0)} = 1, \qquad \dim \operatorname{Ker} \pi^-_{\widehat{q}_{\mathcal{I},M}(0)} = 0. \tag{4.27}$$

The result now follows from Corollary 3.23. □

In the critical case $\rho'_* = 1$, the characteristic equation satisfies $\delta_{0,M}(0) = 0$. One would have to construct a center manifold to properly analyze the dynamics near the equilibrium R_*.

4.1.2 Forward-Looking Variables Only

We now study the singular problem (4.9) under the additional assumption that $\chi_1 = 0$. This turns (4.9) into an advanced algebraic equation. The charactistic function now reads

$$\delta_{0,M}(z) \quad = \quad 1 - \chi_2 \kappa_f \mathcal{A}_* \int_0^{\Omega^f} e^{(z-\beta^f)\sigma} d\sigma \tag{4.28}$$

and as before we proceed by studying the roots of $\delta_{0,M}(z) = 0$.

Lemma 4.4. *Suppose that (HPS) holds and that $\chi_1 = 0$. If $\rho'_* > (1-\chi_2)^{-1}$, then any root of the characteristic equation $\delta_{0,M}(z) = 0$ must have $\operatorname{Im} z \neq 0$ and $\operatorname{Re} z > \beta^f$.*

On the other hand, if $0 < \rho'_* < (1-\chi_2)^{-1}$, then $\delta_{0,M}(z)$ has a simple root at $z = z_*$ for some $z_* \in \mathbb{R}$. We have $z_* > 0$ if $0 < \rho'_* < 1$, $z_* = 0$ if $\rho'_* = 1$ and $z_* < 0$ if $1 < \rho'_* < (1-\chi_2)^{-1}$. If $z \neq z_*$ also solves $\delta_{0,M}(z) = 0$, then we have $\operatorname{Im} z \neq 0$ and $\operatorname{Re} z > \max\{\beta^f, z_*\}$.

Proof. The statements follow from Lemma 4.1 after making the substitutions $z \mapsto -z$ and $\sigma \mapsto -\sigma$. □

Corollary 4.5. *Suppose that (HPS) holds and that $\chi_1 = 0$. If $0 < \rho'_* < 1$ or $\rho'_* > (1-\chi_2)^{-1}$, then the equilibrium R_* of (4.9) is of type $(GU)_0$. In particular, there exists $\epsilon > 0$ such that the only solution R to (4.9) that has $|R(t) - R_*| < \epsilon$ for all $t \geq 0$ is $R(t) = R_*$.*

Proof. Fix $\gamma > 0$, which ensures that

$$\widehat{q}_{0,M}(0) = \widehat{Q}_{L(\gamma)}(0). \tag{4.29}$$

Notice that the part of any initial condition on $[-\Omega^b, 0]$ plays no role in the dynamics of (4.9). In particular, since $n^-_{L(\gamma)} = 0$, we have

$$\widehat{q}_{0,M}(0) = \widehat{Q}_{L(\gamma)}(0) = C([-\Omega^b, 0], \mathbb{C}) \times \{0\}. \tag{4.30}$$

The conclusion now follows from Corollary 3.29. □

Corollary 4.6. *Suppose that (HPS) holds and that $\chi_1 = 0$. If $1 < \rho'_* < (1-\chi_2)^{-1}$, then the equilibrium $R = R_*$ of (4.9) is of type $(GS)_0$. In particular, there exists $\delta > 0$ such that for any R_0 that has $|R_0 - R_*| < \delta$, (4.9) admits a solution R that has $R(0) = R_0$ and $\lim_{t\to\infty} R(t) = R_*$.*

Proof. Fix $\gamma > 0$. In this, case we have $n^-_{L(\gamma)}(0) = 1$, which implies that

$$\widehat{q}_{0,M}(0) = C([-\Omega^b, 0], \mathbb{C}) \times \mathrm{span}\{\phi_*\}, \tag{4.31}$$

where $\phi_*(\sigma) = e^{z_*\sigma}$ is the eigenfunction corresponding to the eigenvalue z_* of $L(\gamma)$ that has $\mathrm{Re}\, z_* < 0$. Since $\phi_*(0) = 1$, the conditions of Corollary 3.25 are all satisfied. □

4.1.3 Mixed Variables

We now assume that both $\chi_1 > 0$ and $\chi_2 > 0$. In this case, it no longer suffices to analyze the characteristic equation $\delta_{0,M}(z) = 0$, since it will have an infinite number of roots on both sides of the imaginary axis. Instead, we will need to apply the results from §3.2, which require us to compute the quantity $n^{\#}_{L(\gamma)}(0)$. We have found it to be intractable to compute a Wiener-Hopf factorization directly for the operators $L(\gamma)$, so we use the path following technique outlined in Proposition 2.9.

Recalling the operators $L(\gamma)$ from (4.15), a short calculation yields

$$\begin{array}{lcl} \Delta_{L(\gamma)}(z) & = & z - \gamma - \chi_1\kappa_b\mathcal{A}_*\frac{z-\gamma}{z+\beta^b}\big(1 - e^{-(z+\beta^b)\Omega^b}\big) \\ & & +\chi_2\kappa_f\mathcal{A}_*\frac{z-\gamma}{z-\beta^f}\big(1 - e^{(z-\beta^f)\Omega^f}\big). \end{array} \tag{4.32}$$

We introduce the operators

$$L_-(\gamma) : C([-\Omega^b, 0], \mathbb{R}) \to \mathbb{R}, \qquad L_+(\gamma) : C([0, \Omega^f], \mathbb{R}) \to \mathbb{R} \tag{4.33}$$

that are defined by

$$\begin{array}{lcl} L_-(\gamma)\phi_R & = & \chi_1\kappa_b\mathcal{A}_*\Big(\phi_R(0) - e^{-\beta^b\Omega^b}\phi_R(-\Omega^b) - (\gamma+\beta^b)\int_{-\Omega^b}^0 e^{\beta^b\sigma}\phi_R(\sigma)\,d\sigma\Big) \\ & & +\gamma\phi_R(0), \\ L_+(\gamma)\phi_R & = & -\chi_2\kappa_f\mathcal{A}_*\Big(\phi_R(0) - e^{-\beta^f\Omega^f}\phi_R(\Omega^f) - (\gamma-\beta^f)\int_{\Omega^f}^0 e^{-\beta^f\sigma}\phi_R(\sigma)\,d\sigma\Big) \\ & & +\gamma\phi_R(0). \end{array} \tag{4.34}$$

A short calculation shows that

$$\begin{array}{lcl} \Delta_{L_-(\gamma)}(z) & = & z - \gamma - \chi_1\kappa_b\mathcal{A}_*\dfrac{z-\gamma}{z+\beta^b}\left(1 - e^{-(z+\beta^b)\Omega^b}\right), \\ \Delta_{L_+(\gamma)}(z) & = & z - \gamma + \chi_2\kappa_f\mathcal{A}_*\dfrac{z-\gamma}{z-\beta^f}\left(1 - e^{(z-\beta^f)\Omega^f}\right). \end{array}$$

In particular, we have

$$\Delta_{L_-(\gamma)}(z)\Delta_{L_+(\gamma)}(z) = (z-\gamma)\big(\Delta_{L(\gamma)}(z) - \chi_1\chi_2\kappa_b\kappa_f\mathcal{A}_*^2\,\Xi_\gamma(z)\big). \tag{4.35}$$

Here we have introduced the function $\Xi_\gamma(z)$ that is given by

$$\begin{array}{lcl} \Xi_\gamma(z) & = & \frac{z-\gamma}{(z+\beta^b)(z-\beta^f)}\left(1 - e^{-(z+\beta^b)\Omega^b}\right)\left(1 - e^{(z-\beta^f)\Omega^f}\right) \\ & = & -\left(\beta^f+\beta^b\right)^{-1}\left(\frac{\gamma-\beta^f}{z-\beta^f} - \frac{\gamma+\beta^b}{z+\beta^b}\right)\left(1 - e^{-(z+\beta^b)\Omega^b}\right)\left(1 - e^{(z-\beta^f)\Omega^f}\right) \\ & = & -\left(\beta^f+\beta^b\right)^{-1}\left((\gamma+\beta^b)\Xi^b(z) + (\gamma-\beta^f)\Xi^f(z)\right), \end{array} \tag{4.36}$$

in which

$$
\begin{array}{rcl}
\Xi^b(z) & = & -\frac{1}{z+\beta^b}\Big(1 - e^{-\left(z+\beta^b\right)\Omega^b} - e^{\left(z-\beta^f\right)\Omega^f} + e^{-\beta^b\Omega^b-\beta^f\Omega^f+z\left(\Omega^f-\Omega^b\right)}\Big), \\
\Xi^f(z) & = & \frac{1}{z-\beta^f}\Big(1 - e^{-\left(z+\beta^b\right)\Omega^b} - e^{\left(z-\beta^f\right)\Omega^f} + e^{-\beta^b\Omega^b-\beta^f\Omega^f+z\left(\Omega^f-\Omega^b\right)}\Big).
\end{array}
\tag{4.37}
$$

Upon introducing two operators $L_f, L_b \in \mathcal{L}\big(C([-\Omega^b, \Omega^f], \mathbb{R}), \mathbb{R}\big)$ that act as

$$
\begin{array}{rcl}
L_b R_t & = & \int_{-\Omega^b}^{\Omega^f} e^{\beta_b\sigma} R(t+\sigma)\, d\sigma + \Big(1 + e^{-\left(\beta^f+\beta^b\right)\Omega_f}\Big) \int_{\Omega^f}^{0} e^{\beta^b\sigma} R(t+\sigma)\, d\sigma \\
 & & + e^{-\left(\beta^f+\beta^b\right)\Omega_f} \int_0^{\Omega_f-\Omega_b} e^{\beta^b\sigma} R(t+\sigma)\, d\sigma, \\
L_f R_t & = & \int_{-\Omega^b}^{\Omega^f} e^{-\beta^f\sigma} R(t+\sigma) d\sigma + \big(1 + e^{-(\beta^f+\beta^b)\Omega^b}\big) \int_0^{-\Omega^b} e^{-\beta^f\sigma} R(t+\sigma)\, d\sigma \\
 & & + e^{-(\beta^f+\beta^b)\Omega^b} \int_{\Omega^f-\Omega^b}^{0} e^{-\beta^f\sigma} R(t+\sigma)\, d\sigma,
\end{array}
\tag{4.38}
$$

we can compute

$$
\begin{array}{rcl}
L_b e^{z\cdot} & = & \frac{1}{z+\beta^b}\big[e^{(z+\beta^b)\Omega^f} - e^{-(z+\beta^b)\Omega^b} + (1 + e^{-(\beta^f+\beta^b)\Omega^f})(1 - e^{(z+\beta^b)\Omega^f}) \\
 & & \quad + e^{-(\beta^f+\beta^b)\Omega^f}\big(e^{(z+\beta^b)(\Omega^f-\Omega^b)} - 1)\big] \\
 & = & -\Xi^b(z) \\
L_f e^{z\cdot} & = & \frac{1}{z-\beta^f}\big[e^{(z-\beta^f)\Omega^f} - e^{-(z-\beta^f)\Omega^b} + (1 + e^{-(\beta^f+\beta^b)\Omega^b})(e^{-(z-\beta^f)\Omega^b} - 1) \\
 & & \quad + e^{-(\beta^f+\beta^b)\Omega^b}(1 - e^{(z-\beta^f)(\Omega^f-\Omega^b)})\big] \\
 & = & -\Xi^f(z).
\end{array}
\tag{4.39}
$$

In particular, upon introducing the path of operators

$$
\Gamma_\gamma(\mu) = L + (1-\mu)\chi_1\chi_2\kappa_b\kappa_f\mathcal{A}_*^2(\beta^f + \beta^b)^{-1}\Big((\gamma - \beta^f)L_f + (\gamma + \beta^b)L_b\Big), \tag{4.40}
$$

we find that

$$
\Delta_{\Gamma_\gamma(\mu)}(z) \quad = \quad \Delta_{L(\gamma)}(z) - (1-\mu)\chi_1\chi_2\kappa_b\kappa_f\mathcal{A}_*^2\Xi_\gamma(z). \tag{4.41}
$$

This in turn implies that

$$
\begin{array}{rcl}
\Delta_{\Gamma_\gamma(0)} & = & (z-\gamma)^{-1}\Delta_{L_+(\gamma)}(z)\Delta_{L_-(\gamma)}(z), \\
\Delta_{\Gamma_\gamma(1)} & = & \Delta_{L(\gamma)}(z).
\end{array}
\tag{4.42}
$$

In particular, the path $\Gamma_\gamma(\mu)$ interpolates between an operator $\Gamma_\gamma(0)$ for which a Wiener-Hopf factorization is available and the operator $\Gamma_\gamma(1)$ that we want to analyze.

We proceed by studying the roots of the two characteristic equations $\Delta_{L_\pm(\gamma)}(z) = 0$. To this end, we introduce the analytic functions

$$
\begin{array}{rcccl}
\Psi_{L_-}(z) & = & (z-\gamma)^{-1}\Delta_{L_-(\gamma)}(z) & = & 1 - \chi_1\kappa_b\mathcal{A}_*\int_{-\Omega^b}^{0} e^{(z+\beta^b)\sigma} d\sigma, \\
\Psi_{L_+}(z) & = & (z-\gamma)^{-1}\Delta_{L_+(\gamma)}(z) & = & 1 + \chi_2\kappa_f\mathcal{A}_*\int_{\Omega^f}^{0} e^{(z-\beta^f)\sigma} d\sigma.
\end{array}
\tag{4.43}
$$

Observing the similarity with (4.17) and (4.28), we can repeat the proof of Lemma 4.1 to obtain the following two results.

Lemma 4.7. *Suppose that (HPS) holds and that both $\chi_1 > 0$ and $\chi_2 > 0$. If $\rho_*' > (1 - \chi_1 - \chi_2)^{-1}$, then any root of $\Psi_{L_-}(z) = 0$ must have $\mathrm{Im}\, z \neq 0$ and $\mathrm{Re}\, z < -\beta^b$.*

On the other hand, if $0 < \rho_*' < (1 - \chi_1 - \chi_2)^{-1}$, then $\Psi_{L_-}(z) = 0$ has a simple root at $z = z_*$ for some $z_* \in \mathbb{R}$. We have $z_* < 0$ if $0 < \rho_*' < (1 - \chi_2)^{-1}$, $z_* = 0$ if $\rho_*' = (1 - \chi_2)^{-1}$ and $z_* > 0$ if $(1 - \chi_2)^{-1} < \rho_*' < (1 - \chi_1 - \chi_2)^{-1}$. If $z \neq z_*$ also solves $\Psi_{L_-}(z) = 0$, then we have $\mathrm{Im}\, z \neq 0$ and $\mathrm{Re}\, z < \min\{-\beta^b, z_*\}$.

Lemma 4.8. *Suppose that (HPS) holds and that both $\chi_1 > 0$ and $\chi_2 > 0$. If $\rho_*' > (1 - \chi_1 - \chi_2)^{-1}$, then any root of $\Psi_{L_+}(z) = 0$ must have $\mathrm{Im}\, z \neq 0$ and $\mathrm{Re}\, z > \beta^f$.*

On the other hand, if $0 < \rho_*' < (1 - \chi_1 - \chi_2)^{-1}$, then $\Psi_{L_+}(z) = 0$ has a simple root at $z = z_*$ for some $z_* \in \mathbb{R}$. We have $z_* > 0$ if $0 < \rho_*' < (1 - \chi_1)^{-1}$, $z_* = 0$ if $\rho_*' = (1 - \chi_1)^{-1}$ and $z_* < 0$ if $(1 - \chi_1)^{-1} < \rho_*' < (1 - \chi_1 - \chi_2)^{-1}$. If $z \neq z_*$ also solves $\Psi_{L_+}(z) = 0$, then we have $\mathrm{Im}\, z \neq 0$ and $\mathrm{Re}\, z > \max\{\beta^f, z_*\}$.

The next step is to analyze the equation $\Delta_{\Gamma_\gamma(\mu)}(z) = 0$ and search for roots that cross the imaginary axis as μ is increased from zero to one. To this end, we introduce the function

$$\Psi_\mu(z) \quad = \quad (z - \gamma)^{-1} \Delta_{\Gamma_\gamma(\mu)}(z) \tag{4.44}$$

and compute

$$\begin{array}{rcl} \Psi_\mu(z) & = & 1 - \chi_1 \kappa_b \mathcal{A}_* \int_{-\Omega^b}^0 e^{(\beta^b + z)\sigma} d\sigma + \chi_2 \kappa_f \mathcal{A}_* \int_{\Omega^f}^0 e^{(z - \beta^f)\sigma} d\sigma \\ & & \quad -(1 - \mu)\chi_1 \chi_2 \kappa_b \kappa_f \mathcal{A}_*^2 \int_{-\Omega^b}^0 e^{(\beta^b + z)\sigma} d\sigma \int_{\Omega^f}^0 e^{(z - \beta^f)\sigma} d\sigma. \end{array} \tag{4.45}$$

Lemma 4.9. *Suppose that (HPS) holds and that both $\chi_1 > 0$ and $\chi_2 > 0$. Suppose furthermore that $0 < \rho_*' < 1$ or that $\rho_*' > (1 - \chi_1 - \chi_2)^{-1}$. Then the equation $\Psi_\mu(z) = 0$ has no roots with $\mathrm{Re}\, z = 0$ for any $0 \leq \mu \leq 1$.*

Proof. Using (4.45) we compute

$$\begin{array}{rcl} \mathrm{Re}\, \Psi_\mu(iq) & = & 1 - \chi_1 \kappa_b \mathcal{A}_* \int_{-\Omega^b}^0 e^{\beta^b \sigma} \cos(q\sigma) d\sigma - \chi_2 \kappa_f \mathcal{A}_* \int_0^{\Omega^f} e^{-\beta^f \sigma} \cos(q\sigma) d\sigma \\ & & \quad +(1 - \mu)\chi_1 \chi_2 \kappa_b \kappa_f \mathcal{A}_*^2 \Big(\int_{-\Omega^b}^0 e^{\beta^b \sigma} \cos(q\sigma) d\sigma \int_0^{\Omega^f} e^{-\beta^f \sigma} \cos(q\sigma) d\sigma \\ & & \qquad + \int_0^{-\Omega^b} e^{\beta^b \sigma} \sin(q\sigma) d\sigma \int_0^{\Omega^f} e^{-\beta^f \sigma} \sin(q\sigma) d\sigma \Big), \\ \mathrm{Im}\, \Psi_\mu(iq) & = & \chi_1 \kappa_b \mathcal{A}_* \int_0^{-\Omega^b} e^{\beta^b \sigma} \sin(q\sigma) d\sigma - \chi_2 \kappa_f \mathcal{A}_* \int_0^{\Omega^f} e^{-\beta^f \sigma} \sin(q\sigma) d\sigma \Big) \\ & & \quad +(1 - \mu)\chi_1 \chi_2 \kappa_b \kappa_f \mathcal{A}_*^2 \Big(\int_{-\Omega^b}^0 e^{\beta^b \sigma} \cos(q\sigma) d\sigma \int_0^{\Omega^f} e^{-\beta^f \sigma} \sin(q\sigma) d\sigma \\ & & \qquad + \int_0^{-\Omega^b} e^{\beta^b \sigma} \sin(q\sigma) d\sigma \int_0^{\Omega^f} e^{-\beta^f \sigma} \cos(q\sigma) d\sigma \Big). \end{array} \tag{4.46}$$

For any $q > 0$, we have the inequalities

$$\begin{array}{rclcl} 0 & < & \int_{-\Omega^b}^0 e^{\beta^b \sigma} \cos(q\sigma) d\sigma & < & \int_{-\Omega^b}^0 e^{\beta^b \sigma} d\sigma, \\ 0 & < & \int_0^{\Omega^f} e^{-\beta^f \sigma} \cos(q\sigma) d\sigma & < & \int_0^{\Omega^f} e^{-\beta^f \sigma} d\sigma, \\ 0 & < & \int_0^{-\Omega^b} e^{\beta^b \sigma} \sin(q\sigma) d\sigma, & & \\ 0 & < & \int_0^{\Omega^f} e^{-\beta^f \sigma} \sin(q\sigma) d\sigma. & & \end{array} \tag{4.47}$$

Note that $\mathcal{A}_* > 0$ whenever $\rho'_* < (1-\chi_1-\chi_2)^{-1}$. For any $0 \le \mu \le 1$, this allows us to compute

$$\begin{array}{lcl} \operatorname{Re}\Psi_\mu(iq) & \geq & 1 - \chi_1\kappa_b\mathcal{A}_*\int_{-\Omega^b}^0 e^{\beta^b\sigma}\cos(q\sigma)d\sigma - \chi_2\kappa_f\mathcal{A}_*\int_0^{\Omega^f} e^{-\beta^f\sigma}\cos(q\sigma)d\sigma \\ & \geq & 1 - \chi_1\kappa_b\mathcal{A}_*\int_{-\Omega^b}^0 e^{\beta^b\sigma}d\sigma - \chi_2\kappa_f\mathcal{A}_*\int_0^{\Omega^f} e^{-\beta^f\sigma}d\sigma \\ & = & \frac{1-\rho'_*}{1-\rho'_*(1-\chi_1-\chi_2)}, \end{array} \tag{4.48}$$

with equality if and only if $q = 0$ and $\mu = 1$. In particular, if $0 < \rho'_* < 1$, the characteristic equation $\Psi_{\Gamma(\mu)}(z) = 0$ has no roots with $\operatorname{Re} z = 0$.

On the other hand, we have $\mathcal{A}_* < 0$ if $\rho'_* > (1-\chi_1-\chi_2)^{-1}$. This immediately implies that $\operatorname{Re}\Psi_\mu(iq) \ge 1$ for all $q \in \mathbb{R}$, which completes the proof. □

In the special cases that $0 < \rho'_* < 1$ or that $\rho'_* > (1-\chi_1-\chi_2)^{-1}$, we now have sufficient information to compute the quantity $n^{\#}_{L(\gamma)}(0)$. This in turn allows us to draw conclusions concerning the equilibrium solution $R(t) = R_*$ to (4.9).

Corollary 4.10. *Suppose that (HP) holds and that both $\chi_1 > 0$ and $\chi_2 > 0$. Suppose furthermore that $0 < \rho'_* < 1$ or that $\rho'_* > (1-\chi_1-\chi_2)^{-1}$. If $\gamma > 0$, then we have $n^{\#}_{L(\gamma)}(0) = 0$, while if $\gamma < 0$, then we have $n^{\#}_{L(\gamma)}(0) = 1$.*

Proof. Since no roots cross the imaginary axis as μ is increased from zero to one, we may apply Proposition 2.9 to find

$$n^{\#}_{L(\gamma)}(0) = n^{\#}_{\Gamma_\gamma(1)}(0) = n^{\#}_{\Gamma_\gamma(0)}(0) = n^{-}_{L_+(\gamma)}(0) - n^{+}_{L_-(\gamma)}(0) + n^{+}_{p_\gamma}(0), \tag{4.49}$$

where $p_\gamma(z) = (z-\gamma)$.

If $\gamma > 0$, then we have $n^{-}_{L_+(\gamma)}(0) = 0$, $n^{+}_{L_-(\gamma)}(0) = 1$ and $n^{+}_{p_\gamma}(0) = 1$, which gives $n^{\#}_{L(\gamma)}(0) = 0$. On the other hand, if $\gamma < 0$, then we have $n^{-}_{L_+(\gamma)}(0) = 1$, $n^{+}_{L_-(\gamma)}(0) = 0$ and $n^{+}_{p_\gamma}(0) = 0$, which gives $n^{\#}_{L(\gamma)}(0) = 1$. □

Corollary 4.11. *Suppose that (HPS) holds and that both $\chi_1 > 0$ and $\chi_2 > 0$. Suppose furthermore that $0 < \rho'_* < 1$ or that $\rho'_* > (1-\chi_1-\chi_2)^{-1}$. Then the equilibrium $R = R_*$ of (4.9) is of type $(S)_0$.*

Proof. We pick $\gamma > 0$, which gives $n^{\#}_{L(\gamma)}(0) = 0$. Since $\widehat{q}_{\mathcal{I},M}(0) = \widehat{Q}_{L(\gamma)}(0)$, we can use Theorem 3.10 to conclude that

$$\operatorname{codim}\operatorname{Range}\pi^{-}_{\widehat{q}_{\mathcal{I},M}(0)} = 0, \qquad \dim\operatorname{Ker}\pi^{-}_{\widehat{q}_{\mathcal{I},M}(0)} = 0. \tag{4.50}$$

This allows us to apply Corollary 3.19. □

In the remaining case $1 < \rho'_* < (1-\chi_1-\chi_2)^{-1}$, we can only draw definitive conclusions concerning real roots to $\Psi_\mu(z) = 0$ that cross the imaginary axis.

Lemma 4.12. *Suppose that (HPS) holds and that both $\chi_1 > 0$ and $\chi_2 > 0$. If $1 < \rho'_* < \min\{(1-\chi_1)^{-1}, (1-\chi_2)^{-1}\}$, there is precisely one $\mu_* \in [0,1]$ such that $\Psi_{\mu_*}(0) = 0$ and in fact $0 < \mu_* < 1$. The same conclusion holds if $\max\{(1-\chi_1)^{-1}, (1-\chi_2)^{-1}\} < \rho'_* < (1-\chi_1-\chi_2)^{-1}$.*

On the other hand, if $\min\{(1-\chi_1)^{-1}, (1-\chi_2)^{-1}\} < \rho'_* < \max\{(1-\chi_1)^{-1}, (1-\chi_2)^{-1}\}$, then $\Psi_\mu(0) < 0$ for all $0 \le \mu \le 1$.

Proof. The conclusion follows from the identity

$$\begin{array}{lcl} \Psi_0(0) & = & (1 - \chi_1\kappa_b\mathcal{A}_*\int_{-\Omega^b}^0 e^{\beta^b\sigma}d\sigma)(1 + \chi_2\kappa_f\mathcal{A}_*\int_{\Omega^f}^0 e^{-\beta^f\sigma}d\sigma) \\ & = & \frac{\big(1-\rho'_*(1-\chi_2)\big)\big(1-\rho'_*(1-\chi_1)\big)}{\big(1-\rho'_*(1-\chi_1-\chi_2)\big)^2}, \\ \Psi_1(0) & = & 1 - \chi_1\kappa_b\mathcal{A}_*\int_{-\Omega^b}^0 e^{\beta^b\sigma}d\sigma + \chi_2\kappa_f\mathcal{A}_*\int_{\Omega^f}^0 e^{-\beta^f\sigma}d\sigma \\ & = & \frac{1-\rho'_*}{1-\rho'_*(1-\chi_1-\chi_2)}. \end{array} \tag{4.51}$$

together with $\frac{d}{d\mu}\Psi_\mu(0) < 0$. $\square$

Besides the root crossings at zero mentioned above, we typically also expect complex roots to cross the imaginary axis if $1 < \rho'_* < (1 - \chi_1 - \chi_2)^{-1}$. Consider for example the symmetric case $\chi_1 = \chi_2$, $\Omega^b = \Omega^f$ and $\beta^f = \beta^b$. Fixing $\mu = 1$, we have $\operatorname{Im}\Psi_1(iq) = 0$ for all $q \in \mathbb{R}$. Together with the limits $\lim_{q\to\pm\infty}\operatorname{Re}\Psi_1(iq) = 1$ and $\operatorname{Re}\Psi_1(0) < 0$, we find that $\Delta_{L(\gamma)}(z) = 0$ has at least one pair of purely imaginary roots. In particular, in order to compute the quantities $n^{\#}_{L(\gamma)}(0)$, the easiest approach appears to be to resort to numerics.

4.2 The Full System

In this part we return to the full problem (4.1) under the assumption (HPF). Our goal is to consider an initial condition

$$\Phi = (\phi_R, \phi_b, \phi_f) \in C([-\Omega^b, 0], \mathbb{R})^3 \tag{4.52}$$

and determine in which sense this initial condition can be extended to a solution to (4.1) that decays to the equilibrium (R_*, π_*, π_*).

We linearize around this equilibrium by making the replacements $R(t) \mapsto R_* + R(t)$, $\pi^b(t) \mapsto \pi_* + \pi^b(t)$ and $\pi^f(t) \mapsto \pi_* + \pi^f(t)$, which transforms the system (4.1) into

$$\begin{array}{rcl} R'(t) & = & -\Lambda_*\big[-R(t) + \kappa_\chi^{-1}\big(\frac{1}{\rho'_*}R(t) - \chi_1\pi^b(t) - \chi_2\pi^f(t)\big)\big] \\ & & +\mathcal{M}_R\big(R(t), \pi^b(t), \pi^f(t)\big) \\ \pi^b(t) & = & \kappa_b\kappa_\chi^{-1}\int_{-\Omega^b}^0 e^{\beta^b\sigma}\big(\frac{1}{\rho'_*}R(t+\sigma) - \chi_1\pi^b(t+\sigma) - \chi_2\pi^f(t+\sigma)\big)\,d\sigma \\ & & +\mathcal{M}_b\big(\mathrm{ev}_t\,R\big) \\ \pi^f(t) & = & \kappa_f\kappa_\chi^{-1}\int_0^{\Omega^f} e^{-\beta^f\sigma}\big(\frac{1}{\rho'_*}R(t+\sigma) - \chi_1\pi^b(t+\sigma) - \chi_2\pi^f(t+\sigma)\big)\,d\sigma \\ & & +\mathcal{M}_f\big(\mathrm{ev}_t\,R\big). \end{array} \tag{4.53}$$

Here we have used the function $\mathcal{N}$ defined in (4.11) to introduce the nonlinearities

$$\begin{array}{rcl} \mathcal{M}_R\big(R, \pi^b, \pi^f\big) & = & -\Lambda_*\kappa_\chi^{-1}\mathcal{N}(R) + \big[\frac{\Lambda(R_*+R)}{\Lambda'(R_*+R)} - \frac{\Lambda(R_*)}{\Lambda'(R_*)}\big] \\ & & \big[-R + \kappa_\chi^{-1}\big(\mathcal{N}(R) + (\rho'_*)^{-1}R - \chi_1\pi^b - \chi_2\pi^f\big)\big], \\ \mathcal{M}_b(\psi_R) & = & \kappa_b\kappa_\chi^{-1}\int_{-\Omega^b}^0 e^{\beta^b\sigma}\mathcal{N}\big(\psi_R(\sigma)\big)\,d\sigma, \\ \mathcal{M}_f(\psi_R) & = & \kappa_f\kappa_\chi^{-1}\int_0^{\Omega^f} e^{-\beta^f\sigma}\mathcal{N}\big(\psi_R(\sigma)\big)\,d\sigma. \end{array} \tag{4.54}$$

Introducing the notation $\Psi(t) = \big(R(t), \pi^b(t), \pi^f(t)\big)$, we can write the system as

$$\mathcal{I}\Psi'(t) = M\mathrm{ev}_t\Psi + \mathcal{M}(\mathrm{ev}_t\Psi). \tag{4.55}$$

Here $\mathcal{I}$ is the diagonal matrix $\mathcal{I} = \operatorname{diag}(1, 0, 0)$, while $\mathcal{M} : C([-\Omega^b, \Omega^f], \mathbb{R})^3 \to \mathbb{R}^3$ denotes the nonlinear function

$$\mathcal{M}\big(\psi_R, \psi_b, \psi_f\big) = \Big(\mathcal{M}_R\big(\psi_R(0), \psi_b(0), \psi_f(0)\big), \mathcal{M}_b(\psi_R), \mathcal{M}_f(\psi_R)\Big) \tag{4.56}$$

and the linear operator $M : C([-\Omega^b, \Omega^f], \mathbb{R})^3 \to \mathbb{R}^3$ captures the remaining terms in (4.53). The characteristic function for the linear part of this system is given by

$$\delta_{\mathcal{I},M}(z) = \begin{pmatrix} z - \Lambda_* \frac{\kappa_\chi \rho'_* - 1}{\kappa_\chi \rho'_*} & -\Lambda_* \frac{\chi_1}{\kappa_\chi} & -\Lambda_* \frac{\chi_2}{\kappa_\chi} \\ -\frac{\kappa_b}{\rho'_* \kappa_\chi} \int_{-\Omega^b}^0 e^{(z+\beta^b)\sigma} d\sigma & 1 + \frac{\chi_1 \kappa_b}{\kappa_\chi} \int_{-\Omega^b}^0 e^{(z+\beta^b)\sigma} d\sigma & \frac{\chi_2 \kappa_b}{\kappa_\chi} \int_{-\Omega^b}^0 e^{(z+\beta^b)\sigma} d\sigma \\ -\frac{\kappa_f}{\rho'_* \kappa_\chi} \int_0^{\Omega^f} e^{(z-\beta^f)\sigma} d\sigma & \frac{\chi_1 \kappa_f}{\kappa_\chi} \int_0^{\Omega^f} e^{(z-\beta^f)\sigma} d\sigma & 1 + \frac{\chi_2 \kappa_f}{\kappa_\chi} \int_0^{\Omega^f} e^{(z-\beta^f)\sigma} d\sigma \end{pmatrix}. \tag{4.57}$$

In order to show that (HM) is satisfied, we pick any $\gamma \in \mathbb{R}$ and define the linear operator $L(\gamma) : C([-\Omega^b, \Omega^f], \mathbb{R})^3 \to \mathbb{R}^3$ that acts on $\Psi = (\psi_R, \psi_b, \psi_f)$ as

$$\begin{array}{rcl} (L(\gamma)\Psi)_1 & = & -\Lambda_* \big[-\psi_R(0) + \kappa_\chi^{-1} \big(\frac{1}{\rho'_*} \psi_R(0) - \chi_1 \psi_b(0) - \chi_2 \psi_f(0) \big) \big], \\ (L(\gamma)\Psi)_2 & = & \gamma \psi_b(0) + \frac{\kappa_b}{\rho' \kappa_\chi} \big[\psi_R(0) - e^{-\beta^b \Omega^b} \psi_R(-\Omega^b) - (\gamma + \beta^b) \int_{-\Omega^b}^0 e^{\beta^b \sigma} \psi_R(\sigma)\, d\sigma \big] \\ & & - \frac{\chi_1 \kappa_b}{\kappa_\chi} \big[\psi_b(0) - e^{-\beta^b \Omega^b} \psi_b(-\Omega^b) - (\gamma + \beta^b) \int_{-\Omega^b}^0 e^{\beta^b \sigma} \psi_b(\sigma)\, d\sigma \big] \\ & & - \frac{\chi_2 \kappa_b}{\kappa_\chi} \big[\psi_f(0) - e^{-\beta^b \Omega^b} \psi_f(-\Omega^b) - (\gamma + \beta^b) \int_{-\Omega^b}^0 e^{\beta^b \sigma} \psi_f(\sigma)\, d\sigma \big], \\ (L(\gamma)\Psi)_3 & = & \gamma \psi_f(0) + \frac{\kappa_f}{\rho' \kappa_\chi} \big[e^{-\beta^f \Omega^f} \psi_R(\Omega^f) - \psi_R(0) - (\gamma - \beta^f) \int_0^{\Omega^f} e^{-\beta^f \sigma} \psi_R(\sigma)\, d\sigma \big] \\ & & - \frac{\chi_1 \kappa_f}{\kappa_\chi} \big[e^{-\beta^f \Omega^f} \psi_b(\Omega^f) - \psi_b(0) - (\gamma - \beta^f) \int_0^{\Omega^f} e^{-\beta^f \sigma} \psi_b(\sigma)\, d\sigma \big] \\ & & - \frac{\chi_2 \kappa_f}{\kappa_\chi} \big[e^{-\beta^f \Omega^f} \psi_f(\Omega^f) - \psi_f(0) - (\gamma - \beta^f) \int_0^{\Omega^f} e^{-\beta^f \sigma} \psi_f(\sigma)\, d\sigma \big]. \end{array} \tag{4.58}$$

Upon writing

$$\mathcal{J}_\gamma(z) = \mathrm{diag}(1, z - \gamma, z - \gamma), \tag{4.59}$$

it is not hard to see that

$$\mathcal{J}_\gamma(z) \delta_{\mathcal{I},M}(z) = \Delta_{L(\gamma)}(z), \tag{4.60}$$

which verifies (HM). As expected, any solution to the linear equation $\mathcal{I}\Psi'(t) = M\,\mathrm{ev}_t\,\Psi(t)$ will automatically also satisfy $\Psi'(t) = L(\gamma)\,\mathrm{ev}_t\,\Psi(t)$ for any $\gamma \in \mathbb{R}$.

In order to verify condition (H$\mathcal{M}$), we need to exploit the explicit structure of the nonlinearity $\mathcal{M}$. In particular, the integrals that are present in the definitions of $\mathcal{M}_b$ and $\mathcal{M}_f$ ensure that the time derivative of the map $t \to \mathcal{M}(\mathrm{ev}_t\Psi)$ depends only on $\mathrm{ev}_t\Psi$ and $\Psi'(t)$. The remaining required properties follow immediately from the construction of the nonlinearities and the smoothness of the function ρ^{-1}.

In the companion paper [10], we analyze the characteristic function $\delta_{\mathcal{I},M}(z)$ in the two special cases that either $\chi_1 = 0$ or $\chi_2 = 0$. In combination with the results in §3.4, this analysis yields the following characterizations for the equilibrium (R_*, π_*, π_*).

Corollary 4.13 (see [10]). *Suppose that (HPF) is satisfied and that $\chi_2 = 0$. If $\rho'_* > 1$, then the equilibrium $(R(t), \pi_b(t)) = (R_*, \pi_*)$ of (4.1) is of type $(S)_0$.*

Corollary 4.14 (see [10]). *Suppose that (HPF) is satisfied and that $\chi_2 = 0$. If $0 < \rho'_* < 1$, then the equilibrium $(R(t), \pi_b(t)) = (R_*, \pi_*)$ of (4.1) is of type $(MS)_0$.*

Corollary 4.15 (see [10]). *Suppose that (HPF) is satisfied and that $\chi_1 = 0$. If $\rho'_* > (1 - \chi_2)^{-1}$, then the equilibrium (R_*, π_*) of (4.1) is of type $(GU)_0$. In fact, there exists $\epsilon > 0$ such that the only solution (R, π_f) to (4.1) that has $|R(t) - R_*| + |\pi_f(t) - \pi_*| < \epsilon$ for all $t \geq 0$ is $(R(t), \pi_f(t)) = (R_*, \pi_*)$.*

The complexity of the characteristic equation $\det \delta_{\mathcal{I},M}(z) = 0$ forces us to resort to numerical studies to analyze the situation where $\chi_1 = 0$ and $0 < \rho'_* < (1-\chi_2)^{-1}$. In particular, let us suppose that $\det \delta_{\mathcal{I},M}(z) = 0$ has $n_* \geq 0$ roots to the left of the imaginary axis and write $\psi_1, \ldots, \psi_{n_*}$ for the associated eigenfunctions. Upon introducing the space

$$\mathcal{V} := \operatorname{span}\{\psi_1(0), \ldots, \psi_{n_*}(0)\} \subset \mathbb{R}^2, \tag{4.61}$$

we note that the equilibrium (R_*, π_*) is of type $(GU)_0$ if $n_* \leq 1$, of type $(GS)_0$ if $\mathcal{V} = \mathbb{R}^2$ and $n_* = 2$ and of type $(GMS)_0$ if $\mathcal{V} = \mathbb{R}^2$ but $n_* > 2$. If $\mathcal{V} \neq \mathbb{R}^2$ but $n_* \geq 2$, then we can draw no conclusions without a more detailed analysis that includes the nonlinear terms.

We conclude by remarking that the mixed case where both $\chi_1 > 0$ and $\chi_2 > 0$ is out of our reach for the moment, because the results in §3.2 are only available for scalar systems. We are hopeful that this situation can be remedied in the near future.

5 Proof of Main Results

In this section we set out to prove the main results outlined in §3. The main tool that we use to analyze linear initial value problems is the Fourier transform, which transforms time shifts into multiplication operators. We recall here that the Fourier transform $\mathcal{F}^+(f)$ of a function $f \in L^2(\mathbb{R}, \mathbb{C}^n)$ and the inverse Fourier transform $\mathcal{F}^-(g)$ of any $g \in L^2(\mathbb{R}, \mathbb{C}^n)$ are given by

$$\mathcal{F}^+(f)(\nu) = \int_{-\infty}^{\infty} e^{-i\nu\xi} f(\xi) d\xi, \qquad \mathcal{F}^-(g)(\xi) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\nu\xi} g(\nu) d\nu. \tag{5.1}$$

Our analysis will focus on the Green's function $\widehat{G}_L$ associated to a linear operator $L : C([-1,1], \mathbb{C}^n) \to \mathbb{C}^n$, which is given by

$$\mathcal{F}^+(\widehat{G}_L)(\nu) = \Delta_L^{-1}(i\nu) \tag{5.2}$$

and solves

$$\widehat{G}'_L(\xi) = L \operatorname{ev}_\xi \widehat{G}_L + \delta(\xi) \tag{5.3}$$

in the sense of distributions. In particular, $\widehat{G}_L$ has a discontinuity at $\xi = 0$. We exploit this fact heavily in order to build a bridge between the solution spaces $\widehat{Q}_L(\eta)$ that allow for such discontinuities and their traditional counterparts $Q_L(\eta)$.

5.1 Delay Differential Equations

In this section we set out to prove Theorem 3.3, which concerns the differential delay equation

$$\widehat{x}'(\xi) = L_- \, \widehat{\operatorname{ev}}_\xi^- \, \widehat{x}, \tag{5.4}$$

where L_- is a bounded linear operator from $X^- = C([-1,0], \mathbb{C}^n)$ to $\mathbb{C}^n$ that acts on the larger spaces $\widehat{X}_\alpha^-$ as explained in §3.1. We start by providing a proof for Lemma 3.2, which states that for any $\widehat{x} \in \widehat{BC}_\eta^\oplus$, the map $\xi \mapsto L_- \widehat{\operatorname{ev}}_\xi^- x$ is continuous for almost all $\xi \geq 0$.

Proof of Lemma 3.2. Note that we only have to establish continuity for almost all $\xi \in (0,1)$. Lemma 3.1 implies that the set of points in $(-1,0)$ at which μ is discontinuous is at most countable. We will show that if μ is continuous at some $\theta \in (-1,0)$, then the function w defined by $w(\xi) = L_- \, \widehat{\operatorname{ev}}_\xi^- \widehat{x}$ is continuous at $\xi = -\theta$. Indeed, writing $\widehat{x} = (\phi, y)$, note that for any sufficiently small h we have

$$\begin{array}{lcl} w(\xi) & = & \int_{-1}^{\theta} d\mu(\sigma)\phi(\xi+\sigma) + \int_{\theta}^{0} d\mu(\sigma) y(\xi+\sigma) \\ w(\xi+h) & = & \int_{-1}^{\theta-h} d\mu(\sigma)\phi(\xi+h+\sigma) + \int_{\theta-h}^{0} d\mu(\sigma) y(\xi+h+\sigma), \end{array} \tag{5.5}$$

where μ is the measure associated to L_- via (3.9). In particular, we may compute

$$\begin{array}{rcl} w(\xi+h) - w(\xi) & = & \int_{-1}^{\theta-h} d\mu(\sigma)[\phi(\xi+h+\sigma) - \phi(\xi+\sigma)] \\ & & \quad + \int_{\theta}^{0} d\mu(\sigma)[y(\xi+h+\sigma) - y(\xi+\sigma)] \\ & & \quad + \int_{\theta-h}^{\theta} d\mu(\sigma)[y(\xi+h+\sigma) - \phi(\xi+\sigma)]. \end{array} \tag{5.6}$$

The first two lines are of order $o(1)$ as $h \to 0$ on account of the continuity of ϕ and y. The third line is also of order $o(1)$ as $h \to 0$, because the total variation of μ over the interval $[\theta - h, \theta]$ vanishes as $h \to 0$. This latter fact is a consequence of our assumption that μ is continuous at θ. □

Returning to (5.4), we are ready to study the Green's function for this equation. To aid us, we introduce the two special functions

$$\begin{array}{rcl} H_z^+(\xi) & = & \left\{ \begin{array}{ll} e^{-z\xi} & \text{for } \xi \geq 0, \\ 0 & \text{for } \xi < 0, \end{array} \right. \\ H_z^-(\xi) & = & \left\{ \begin{array}{ll} e^{z\xi} & \text{for } \xi \leq 0, \\ 0 & \text{for } \xi > 0, \end{array} \right. \end{array} \tag{5.7}$$

in which we will always choose $z \in \mathbb{C}$ in such a way that $\operatorname{Re} z > 0$. The Fourier transforms of these functions are hence given by

$$\mathcal{F}^+(H_z^+)(\nu) = \frac{1}{z + i\nu}, \qquad \mathcal{F}^+(H_z^-)(\nu) = \frac{1}{z - i\nu}. \tag{5.8}$$

In addition, for any pair $z_1, z_2 \in \mathbb{C}$ with $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, we introduce the combined function

$$H_{z_1,z_2}(\xi) = \left\{ \begin{array}{ll} e^{z_1\xi} & \text{for } \xi \leq 0, \\ e^{-z_2\xi} & \text{for } \xi \geq 0, \end{array} \right. \tag{5.9}$$

which is continuous on $\mathbb{R}$. The Fourier transform is given by

$$\mathcal{F}^+(H_{z_1,z_2})(\nu) = \frac{1}{z_1 - i\nu} + \frac{1}{z_2 + i\nu}, \tag{5.10}$$

which is a function of class $L^1(\mathbb{R}, \mathbb{C})$.

Proposition 5.1. *Consider the delay equation (5.4) and suppose that the characteristic equation* $\det \Delta_{L_-}(z) = 0$ *admits no roots with* $\operatorname{Re} z = 0$*. Then there exists a function*

$$\widehat{G}_{L_-} = (G_{L_-}^l, G_{L_-}^r) \in C((-\infty, 0], \mathbb{C}^{n\times n}) \times C([0, \infty), \mathbb{C}^{n\times n}) \tag{5.11}$$

that satisfies the following properties.

(i) For almost every $\xi \geq 0$, $\widehat{G}_{L_-}$ *satisfies the differential equation*

$$\widehat{G}'_{L_-}(\xi) = L_- \,\widehat{\mathrm{ev}}_\xi^- \widehat{G}_{L_-}, \tag{5.12}$$

while $G_{L_-}^l$ *satisfies*

$$(G_{L_-}^l)'(\xi) = L_- \,\mathrm{ev}_\xi^- G_{L_-}^l, \tag{5.13}$$

for every $\xi \leq 0$.

(ii) There exist constants $K > 0$ and $\kappa > 0$ such that

$$\left|\widehat{G}_{L_-}(\xi)\right| \le K e^{-\kappa|\xi|}, \qquad \xi \in \mathbb{R}. \tag{5.14}$$

(iii) Writing I for the $n \times n$-identity matrix, we have

$$G^r_{L_-}(0) - G^l_{L_-}(0) = I. \tag{5.15}$$

(iv) The Fourier transform of $\widehat{G}_{L_-}$ is given by

$$\mathcal{F}^+(\widehat{G}_{L_-})(\nu) = \Delta_{L-}(i\nu)^{-1}. \tag{5.16}$$

Proof. The estimates [18, Lem. 3.1] imply that the map $\nu \mapsto \Delta_{L_-}(i\nu)^{-1}$ is of class $L^2(\mathbb{R}, \mathbb{C}^{n\times n})$, which allows us to apply the inverse Fourier transform to (5.16) in order to define $\widehat{G}_{L_-}(\xi)$ as a function in $L^2(\mathbb{R}, \mathbb{C}^{n\times n})$.

To show that $\widehat{G}_{L_-}$ is continuous on the half-lines $\mathbb{R}_-$ and $\mathbb{R}_+$, we pick an arbitrary $\alpha > 0$ and introduce the function

$$\Psi(\nu) = \mathcal{F}^+(\widehat{G}_{L_-})(\nu) - \mathcal{F}^+(H^+_\alpha I)(\nu) = \Delta(i\nu)^{-1} - \frac{1}{\alpha + i\nu} I. \tag{5.17}$$

Again using the estimates [18, Lem. 3.1] one finds the asymptotic expansion

$$\Psi(\nu) = O(1/\nu^2), \qquad \nu \to \pm\infty, \tag{5.18}$$

which implies that $\Psi \in L^1(\mathbb{R}, \mathbb{C}^{n\times n})$. In particular, standard properties of the Fourier transform now imply that

$$\mathcal{F}^-(\Psi) = \widehat{G}_{L_-} - H^+_\alpha I \in C(\mathbb{R}, \mathbb{C}^{n\times n}) \cap L^\infty(\mathbb{R}, \mathbb{C}^{n\times n}). \tag{5.19}$$

This establishes (iii) and (iv). The remaining properties (i) and (ii) are standard properties of the Green's function that can be established as in the proof of [27, Thm. 4.1]. □

The Green's function $\widehat{G}_{L_-}$ is the key ingredient that allows us to compare solutions to the delay equation (2.1) and its counterpart (5.4) that allows for jumps at $\xi = 0$. Indeed, if $x \in BC_0^\oplus$ satisfies (2.1), then for any $v \in \mathbb{C}^n$ the function $x + \widehat{G}_{L_-} v$ is contained in $\widehat{BC}_0^\oplus$ and solves (5.4). Conversely, if $\widehat{y} \in \widehat{BC}_0^\oplus$ solves (5.4), then the function

$$x = \widehat{y} - \widehat{G}_{L_-}\big(\widehat{y}(0+) - \widehat{y}(0-)\big) \tag{5.20}$$

solves (2.1) and satisfies $x \in BC_0^\oplus$, since it is continuous. In particular, this shows that

$$\widehat{\mathfrak{Q}}_{L_-}(0) = \mathfrak{Q}_{L_-}(0) \oplus \mathrm{span}_{\mathbb{C}^n}\{\widehat{G}_{L_-}\}. \tag{5.21}$$

In order to exploit this and establish a relationship between $Q_{L_-}(0)$ and $\widehat{Q}_{L_-}(0)$, we need to closely study the function

$$\mathrm{ev}_0^- \, \widehat{G}_{L_-} \in C([-1, 0], \mathbb{C}^n). \tag{5.22}$$

We proceed by picking any root z_* of the characteristic equation $\det \Delta_{L_-}(z) = 0$ and recalling the associated spectral projection

$$\Pi^{\mathrm{sp}}_{L_-}(z_*) : C([-1, 0], \mathbb{C}^n) \to C([-1, 0], \mathbb{C}^n) \tag{5.23}$$

that is given by

$$[\Pi^{\mathrm{sp}}_{L_-}(z_*)\phi](\theta) = \mathrm{Res}_{z=z_*}\, e^{z\theta}\Delta_{L_-}(z)^{-1}\big[\phi(0) + \int_{-1}^{0} d\mu(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\phi(\tau)d\tau\big]. \tag{5.24}$$

This operator projects onto the generalized eigenspace associated to the root z_* and plays an important role in characterizing $Q_{L_-}(0) \subset C([-1,0],\mathbb{C}^n)$. Indeed, we have the following representation.

Lemma 5.2 (see [11, Chp. IV]). *Consider the delay differential equation (2.1) and suppose that the characteristic equation* $\det \Delta_{L_-}(z) = 0$ *admits no roots with* $\mathrm{Re}\, z = 0$*. Then any function* $\phi \in C([-1,0],\mathbb{C}^n)$ *satisfies* $\phi \in Q_{L_-}(0)$ *if and only if*

$$\Pi^{\mathrm{sp}}_{L_-}(z)\phi = 0 \tag{5.25}$$

for all $z \in \mathbb{C}$ *that have* $\mathrm{Re}\, z > 0$ *and* $\det \Delta_{L_-}(z) = 0$.

To simplify our notation in the sequel, we now introduce the spectral set

$$\Sigma^+_{L_-} = \{z \in \mathbb{C} \mid \det \Delta_{L_-}(z) = 0 \text{ and } \mathrm{Re}\, z > 0\} \tag{5.26}$$

together with its associated spectral projection

$$\Pi^{\mathrm{sp}}_{L_-}(\Sigma^+_{L_-}) = \sum_{z_* \in \Sigma^+_{L_-}} \Pi^{\mathrm{sp}}_{L_-}(z_*). \tag{5.27}$$

Lemma 5.2 implies that

$$Q_{L_-}(0) = \{\phi \in C([-1,0],\mathbb{C}^n) \text{ for which } \Pi^{\mathrm{sp}}_{L_-}(\Sigma^+_{L_-})\phi = 0\}. \tag{5.28}$$

In order to describe $\Pi^{\mathrm{sp}}_{L_-}(\Sigma^+_{L_-})$ in a bit more detail, let us introduce a set of functions

$$\psi_\ell \in C(\mathbb{R},\mathbb{C}^n), \qquad 1 \le \ell \le n^+_{L_-}(0), \tag{5.29}$$

that span the space of generalized eigenfunctions corresponding to the spectral set $\Sigma^+_{L_-}$. There exists a set of linearly independent linear operators

$$\widetilde{M}_\ell \in \mathcal{L}\big(C([-1,0],\mathbb{C}^n),\mathbb{C}\big), \qquad 1 \le \ell \le n^+_{L_-}(0), \tag{5.30}$$

that allow us to write

$$\Pi^{\mathrm{sp}}_{L_-}(\Sigma^+_{L_-})\phi = \sum_{\ell=1}^{n^+_{L_-}(0)} \psi_\ell \widetilde{M}_\ell \phi. \tag{5.31}$$

In order to characterize $\widehat{Q}_{L_-}(0)$, we need to determine how $\mathrm{ev}_0^-\, \widehat{G}_{L_-}$ projects onto the generalized eigenspace associated to the eigenvalues in $\Sigma^+_{L_-}$. The following technical result shows that these projections typically do not vanish.

Lemma 5.3. *Consider the delay differential equation (2.1) and suppose that the characteristic equation* $\det \Delta_{L_-}(z) = 0$ *admits no roots with* $\mathrm{Re}\, z = 0$*. Then the following identity holds for all* z_* *that have* $\mathrm{Re}\, z_* > 0$ *and* $\det \Delta_{L_-}(z_*) = 0$*,*

$$[\Pi^{\mathrm{sp}}_{L_-}(z_*)\,\mathrm{ev}_0^-\, \widehat{G}_{L_-}](\theta) = -\mathrm{Res}_{z=z_*} e^{z\theta}\Delta_{L_-}(z)^{-1}. \tag{5.32}$$

Proof. First of all, notice that $\mathrm{ev}_0^- \widehat{G}_{L_-} = \mathrm{ev}_0^- G^l_{L_-}$. In addition, for any $\alpha > 0$ we recall from (5.17) the $L^1(\mathbb{R}, \mathbb{C}^{n\times n})$ function

$$\Psi(\nu) = \Delta_{L_-}(i\nu)^{-1} - \frac{1}{\alpha + i\nu} I. \tag{5.33}$$

We note that $\mathcal{F}^-(\Psi_{L_-})$ is a continuous function that agrees with $G^l_{L_-}$ on $\mathbb{R}_-$. In particular, for any $\xi \le 0$ we may write

$$G^l_{L_-}(\xi) = [\mathcal{F}^-(\Psi)](\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\nu\xi} \Psi(\nu)\, d\nu. \tag{5.34}$$

We now proceed by computing the integral

$$\mathcal{T}_1(z) = \int_{-1}^{0} d\mu(\sigma) e^{z\sigma} \int_{\sigma}^{0} e^{-z\tau} G^l_{L_-}(\tau)\, d\tau, \tag{5.35}$$

which using the representation (5.34) can be written as

$$\begin{array}{lcl} \mathcal{T}_1(z) & = & \frac{1}{2\pi} \int_{-1}^{0} d\mu(\sigma) e^{z\sigma} \int_{\sigma}^{0} e^{-z\tau} \int_{-\infty}^{\infty} e^{i\nu\tau} [\Delta_{L_-}^{-1}(i\nu) - \frac{1}{\alpha+i\nu} I]\, d\nu\, d\tau \\ & = & \frac{1}{2\pi} \int_{-\infty}^{\infty} \big[\int_{-1}^{0} d\mu(\sigma) e^{z\sigma} \int_{\sigma}^{0} e^{(i\nu - z)\tau}\, d\tau \big] \big[\Delta_{L_-}^{-1}(i\nu) - \frac{1}{\alpha+i\nu} I\big]\, d\nu, \end{array} \tag{5.36}$$

where we used Fubini's theorem to change the order of integration. Proceeding with the τ-integration, we obtain

$$\begin{array}{lcl} \mathcal{T}_1(z) & = & \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\nu - z} \big[\int_{-1}^{0} d\mu(\sigma)(e^{z\sigma} - e^{i\nu\sigma}) \big] \big[\Delta_{L_-}^{-1}(i\nu) - \frac{1}{\alpha+i\nu} I\big]\, d\nu \\ & = & \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\nu - z} \big[(z - i\nu)I + \Delta_{L_-}(i\nu) - \Delta_{L_-}(z)\big] \big[\Delta_{L_-}^{-1}(i\nu) - \frac{1}{\alpha+i\nu} I\big]\, d\nu \\ & = & -\frac{1}{2\pi} \int_{-\infty}^{\infty} \big[\Delta_{L_-}^{-1}(i\nu) - \frac{1}{\alpha+i\nu} I\big]\, d\nu \\ & & \quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\nu - z} \big[\Delta_{L_-}(i\nu) - \Delta_{L_-}(z)\big] \big[\Delta_{L_-}^{-1}(i\nu) - \frac{1}{\alpha+i\nu} I\big]\, d\nu \\ & = & -G^l_{L_-}(0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(i\nu - z)(\alpha + i\nu)} [(\alpha + i\nu)I - \Delta_{L_-}(i\nu)]\, d\nu \\ & & \quad -\frac{1}{2\pi} \Delta_{L_-}(z) \int_{-\infty}^{\infty} \frac{1}{i\nu - z} \big[\Delta_{L_-}^{-1}(i\nu) - \frac{1}{\alpha+i\nu} I\big]\, d\nu. \end{array} \tag{5.37}$$

Notice that when computing the residual in (5.24), any terms that are analytic in z can be ignored. This allows us to neglect the final term in the last line above and compute

$$[\Pi^{\mathrm{sp}}_{L_-}(z_*)\, \mathrm{ev}_0^- \widehat{G}_{L_-}](\theta) \quad = \quad \mathrm{Res}_{z=z_*}\, e^{z\theta} \Delta_{L_-}(z)^{-1} \mathcal{T}_2(z), \tag{5.38}$$

in which we have

$$\begin{array}{lcl} \mathcal{T}_2(z) & = & \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{z+\alpha} [\frac{1}{i\nu - z} - \frac{1}{\alpha + i\nu}] [\alpha + \int_{-1}^{0} d\mu(\sigma) e^{i\nu\sigma}] I\, d\nu \\ & = & \frac{\alpha}{2\pi(z+\alpha)} \int_{-\infty}^{\infty} [\frac{1}{i\nu - z} - \frac{1}{\alpha + i\nu}] I\, d\nu \\ & & \quad + \frac{1}{2\pi(z+\alpha)} \int_{-1}^{0} d\mu(\sigma) \int_{-\infty}^{\infty} e^{i\nu\sigma} [\frac{1}{i\nu - z} - \frac{1}{\alpha + i\nu}] I\, d\nu. \end{array} \tag{5.39}$$

Restricting ourselves to situations where $\mathrm{Re}\, z > 0$ and remembering that $\alpha > 0$, we can use the special function (5.9) to write

$$\begin{array}{lcl} \mathcal{T}_2(z) & = & -\frac{\alpha}{z+\alpha} H_{z,\alpha}(0) I - \frac{1}{z+\alpha} \int_{-1}^{0} d\mu(\sigma) H_{z,\alpha}(\sigma) I \\ & = & -\frac{\alpha}{z+\alpha} I - \frac{1}{z+\alpha} \int_{-1}^{0} d\mu(\sigma) e^{z\sigma} I \\ & = & -\frac{\alpha}{z+\alpha} I - \frac{1}{z+\alpha} \big(zI - \Delta_{L_-}(z)\big) \\ & = & -I + \frac{1}{z+\alpha} \Delta_{L_-}(z). \end{array} \tag{5.40}$$

Recalling the identity (5.38), we see that we can again neglect the term involving $\Delta_{L_-}(z)$, which establishes (5.32) and completes the proof. $\square$

Notice that in the proof of Lemma 5.3 the condition $\operatorname{Re} z_* > 0$ is explicitly needed. Indeed, if we assume $\operatorname{Re} z_* < 0$, a similar calculation shows that $[\Pi^{\mathrm{sp}}_{L_-}(z_*)\, \mathrm{ev}_0^-\, \widehat{G}_{L_-}] = 0$. This can be understood by noting that $G^l_{L_-}$ satisfies (2.1) for all $\xi \leq 0$, which shows that $\mathrm{ev}_0^-\, G^l_{L_-} \in P_{L_-}(0)$.

Proof of Theorem 3.3. We restrict ourselves to the case where $\eta = 0$. Our starting point is the identity

$$\widehat{\mathfrak{Q}}_{L_-}(0) = \mathfrak{Q}_{L_-}(0) \oplus \mathrm{span}_{\mathbb{C}^n}\{\widehat{G}_{L_-}\} \tag{5.41}$$

discussed above. If $n^+_{L_-}(0) = 0$, then Lemma 5.2 implies that $Q_{L_-}(0) = C([-1,0],\mathbb{C}^n)$, which together with (5.41) and the jump condition (5.15) immediately implies

$$\widehat{Q}_{L_-}(0) = C([-1,0],\mathbb{C}^n) \times \mathbb{C}^n \tag{5.42}$$

as desired.

Moving on to the case where $n^+_{L_-}(0) > 0$, let us introduce a set of vectors

$$\widetilde{\alpha}_\ell \in \mathbb{C}^n, \qquad 1 \leq \ell \leq n^+_{L_-}(0) \tag{5.43}$$

such that

$$\Pi^{\mathrm{sp}}_{L_-}(\Sigma^+_{L_-})\, \mathrm{ev}_0^-\, G^l_{L_-} w = \sum_{\ell=1}^{n^+_{L_-}(0)} \psi_\ell \widetilde{\alpha}^\dagger_\ell w. \tag{5.44}$$

Lemma 5.3 guarantees that at least one of these vectors $\widetilde{\alpha}_\ell$ is non-zero.

Observe that a pair

$$(\phi, v) \in \widehat{X}^-_0 = C([-1,0],\mathbb{C}^n) \times \mathbb{C}^n \tag{5.45}$$

satisfies $(\phi, v) \in \widehat{Q}_{L_-}(0)$ if and only if

$$\phi - \mathrm{ev}_0^-\, G^l_{L_-}\big(v - \phi(0)\big) \in Q_{L_-}(0), \tag{5.46}$$

which is equivalent to the requirement

$$\Pi^{\mathrm{sp}}_{L_-}(\Sigma^+_{L_-})\phi = \Pi^{\mathrm{sp}}_{L_-}(\Sigma^+_{L_-})\, \mathrm{ev}_0^-\, G^l_{L_-}\big(v - \phi(0)\big). \tag{5.47}$$

Using the operators (5.30) this can be rewritten as

$$\widetilde{M}_\ell \phi = \widetilde{\alpha}^\dagger_\ell\big(v - \phi(0)\big), \qquad 1 \leq \ell \leq n^+_{L_-}(0). \tag{5.48}$$

Let $1 \leq s \leq n$ be the dimension of the space spanned by the vectors $\{\widetilde{\alpha}_\ell\}$ and reorder these vectors in such a way that the set $\{\widetilde{\alpha}_\ell\}_{\ell=1}^s$ is linearly independent. Since the remaining vectors $\widetilde{\alpha}_\ell$ for $s < \ell \leq n^+_{L_-}$ can all be expressed as a linear combination of the s vectors in this set, the system (5.48) can be rewritten in the required form (3.23). $\square$

Proof of Corollary 3.4. In the neighbourhood of a simple root $z = z_*$ of the characteristic equation $\det \Delta_{L_-}(z) = 0$, the inverse of the characteristic function Δ_{L_-} can be expanded as [11, Chp. IV]

$$\Delta^{-1}_{L_-}(z) = (z - z_*)^{-1}\beta_*\alpha^\dagger_* + O(1), \qquad z \to z_*, \tag{5.49}$$

where β_* and α_* are two vectors that satisfy

$$\Delta_{L_-}(z_*)\beta_* = 0, \qquad \Delta^\dagger_{L_-}(z_*)\alpha_* = 0, \qquad \alpha^\dagger_*\Delta'_{L_-}(z_*)\beta_* = 1. \tag{5.50}$$

The spectral projection $\Pi^{\mathrm{sp}}_{L_-}(z_*)$ is now given by

$$[\Pi^{\mathrm{sp}}_{L_-}(z_*)\phi](\theta) = e^{z_*\theta}\beta_*\big[\alpha_*^\dagger\phi(0) + \alpha_*^\dagger\int_{-1}^{0} d\mu(\sigma)e^{z_*\sigma}\int_{\sigma}^{0} e^{-z_*\tau}\phi(\tau)d\tau\big]. \tag{5.51}$$

In particular, for any $w \in \mathbb{C}^n$ we have

$$[\Pi^{\mathrm{sp}}_{L_-}(z_*)\mathrm{ev}_0^-\widehat{G}_{L_-}w](\theta) = -e^{z_*\theta}\beta_*\alpha_*^\dagger w. \tag{5.52}$$

The identity (3.25) can now be read off from (5.47). □

5.2 Mixed Type Equations

We now set out to prove Theorem 3.10, which features the MFDE

$$x'(\xi) = L\,\widehat{\mathrm{ev}}_\xi x. \tag{5.53}$$

Here L a bounded linear operator that maps $X = C([-1,1],\mathbb{C})$ into $\mathbb{C}$ and, for any $-1 < \alpha \le 0$, acts on the spaces $\widehat{X}_\alpha$ as explained in §3.2. We start by remarking that a Green's function

$$\widehat{G}_L = (G_L^l, G_L^r) \in C((-\infty,0],\mathbb{C}) \times C([0,\infty),\mathbb{C}) \tag{5.54}$$

can be defined for (5.53) by writing

$$\mathcal{F}^+(\widehat{G}_L)(\nu) = \Delta_L^{-1}(i\nu). \tag{5.55}$$

This function satisfies properties that are similar to those mentioned in Proposition 5.1. In particular, $\widehat{G}_L$ satisfies (5.53) for almost all $\xi \ge 0$ and admits the jump

$$G_L^r(0) - G_L^l(0) = 1. \tag{5.56}$$

Following the same arguments as in §5.1, one can conclude that

$$\widehat{\mathfrak{Q}}_L(0) = \mathfrak{Q}_L(0) \oplus \mathrm{span}\{\widehat{G}_L\}. \tag{5.57}$$

However, in the current setting we need to understand the relationship between $\mathrm{ev}_0^-\,G_L^l$ and the space

$$\mathrm{Range}\,\pi^-_{Q_L(0)} \subset C([-1,0],\mathbb{C}). \tag{5.58}$$

The key ingredient that we will exploit in order to understand this relationship is provided by the following technical result.

Lemma 5.4. *Consider a scalar version of the MFDE (2.10) for which the characteristic equation $\Delta_L(z) = 0$ admits no roots with $\mathrm{Re}\,z = 0$. Choose any $z_0 > 0$, write $p(z) = (z - z_0)$ and let (p, L_-, L_+) be a Wiener-Hopf triplet for L, which we recall implies that*

$$(z - z_0)\Delta_L(z) = \Delta_{L_-}(z)\Delta_{L_+}(z). \tag{5.59}$$

Suppose furthermore that this Wiener-Hopf triplet has the special property that $\Delta_{L_+}(z) = 0$ admits no roots with $\mathrm{Re}\,z \le 0$.

Then for any z_* that has $\mathrm{Re}\,z_* > 0$, we have the identity

$$[\Pi^{\mathrm{sp}}_{L_-}(z_*)\,\mathrm{ev}_0^-\,\widehat{G}_L](\theta) = -\mathrm{Res}_{z=z_*}e^{z\theta}\Delta_{L_-}(z)^{-1}. \tag{5.60}$$

Proof. Using $\mathrm{ev}_0^- \widehat{G}_L = \mathrm{ev}_0^- G_L^l$ and writing μ_- for the measure associated to L_-, we recall the definition

$$[\Pi^{\mathrm{sp}}_{L_-}(z_*) \,\mathrm{ev}_0^- \,\widehat{G}_L](\theta) = \mathrm{Res}_{z=z_*} e^{z\theta} \Delta_{L_-}(z)^{-1} \big[G_L^l(0) + \int_{-1}^0 d\mu_-(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} G_L^l(\tau) d\tau\big] \qquad (5.61)$$

and start by studying the term

$$\mathcal{T}_1(z) = \int_{-1}^0 d\mu_-(\sigma) e^{z\sigma} \int_\sigma^0 e^{-z\tau} G_L^l(\tau) d\tau. \qquad (5.62)$$

Mimicking the computation (5.37), we see that we have, for any $\alpha > 0$,

$$\begin{array}{lcl} \mathcal{T}_1(z) & = & \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\nu - z} [z - i\nu + \Delta_{L_-}(i\nu) - \Delta_{L_-}(z)][\Delta_L^{-1}(i\nu) - \frac{1}{\alpha + i\nu}] \, d\nu \\ & = & -\frac{1}{2\pi} \int_{-\infty}^{\infty} [\Delta_L^{-1}(i\nu) - \frac{1}{\alpha+i\nu}] \, d\nu \\ & & \quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\nu - z} [\Delta_{L_-}(i\nu) - \Delta_{L_-}(z)][\Delta_L^{-1}(i\nu) - \frac{1}{\alpha+i\nu}] \, d\nu \\ & = & -G_L^l(0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(i\nu - z)(\alpha + i\nu)} [(\alpha + i\nu) \Delta_{L_-}(i\nu) \Delta_L^{-1}(i\nu) - \Delta_{L_-}(i\nu)] \, d\nu \\ & & \quad - \frac{1}{2\pi} \Delta_{L_-}(z) \int_{-\infty}^{\infty} \frac{1}{i\nu - z} [\Delta_L^{-1}(i\nu) - \frac{1}{\alpha+i\nu}] \, d\nu \\ & = & -G_L^l(0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(i\nu - z)(\alpha + i\nu)} [(\alpha + i\nu) - \Delta_{L_-}(i\nu)] \, d\nu \\ & & \quad - \frac{1}{2\pi} \Delta_{L_-}(z) \int_{-\infty}^{\infty} \frac{1}{i\nu - z} [\Delta_L^{-1}(i\nu) - \frac{1}{\alpha+i\nu}] \, d\nu \\ & & \quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\nu - z} [\Delta_{L_-}(i\nu) \Delta_L^{-1}(i\nu) - 1] \, d\nu. \end{array} \qquad (5.63)$$

Comparing this to the expression (5.37) for $\mathcal{T}_1(z)$ computed in the proof of Lemma 5.3, we find that it now suffices to show that

$$\int_{-\infty}^{\infty} \frac{1}{i\nu - z} [\Delta_{L_-}(i\nu) \Delta_L^{-1}(i\nu) - 1] \, d\nu = 0 \qquad (5.64)$$

for all z in a small neighbourhood of z_*. To see this, observe that

$$\begin{array}{lcl} \frac{1}{i\nu - z} [\Delta_{L_-}(i\nu) \Delta_L^{-1}(i\nu) - 1] & = & \frac{1}{i\nu - z} [(z - z_0) \Delta_{L_+}^{-1}(i\nu) - 1] \\ & = & [\Delta_{L_+}^{-1}(i\nu) - \frac{1}{i\nu - z}] + (z - z_0) \frac{1}{i\nu - z} \Delta_{L_+}^{-1}(i\nu). \end{array} \qquad (5.65)$$

Remembering that $\mathrm{Re}\, z > 0$, we recall the special function H_z^- from (5.7) that has Fourier transform

$$\mathcal{F}^+(H_z^-)(\nu) = \frac{1}{z - i\nu}. \qquad (5.66)$$

Applying arguments similar to those in the proof of Proposition 5.1, we can introduce the Green's function

$$\widehat{G}_{L_+} = (G_{L_+}^l, G_{L_+}^r) \in C((-\infty, 0], \mathbb{C}) \times C([0, \infty), \mathbb{C}) \qquad (5.67)$$

for the advanced equation

$$x'(\xi) = L_+ \,\mathrm{ev}_\xi^+ \, x, \qquad (5.68)$$

where the evaluation operator ev_ξ^+ now maps into $C([0, 1], \mathbb{C})$. We note that $G_{L_+}^r$ satisfies (5.68) for all $\xi \geq 0$. In addition, we have the usual jump condition

$$G_{L_+}^r(0) - G_{L_+}^l(0) = 1. \qquad (5.69)$$

In particular, $\widehat{G}_{L_+} + H_z^-$ is a continuous function on $\mathbb{R}$, which shows that

$$\int_{-\infty}^{\infty} \frac{1}{i\nu - z}[\Delta_{L_+}^{-1}(i\nu) - \frac{1}{i\nu - z}]\, d\nu = G_{L_+}^r(0). \tag{5.70}$$

On the other hand, the Fourier convolution theorem implies that

$$\begin{array}{rcl} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{i\nu - z}\Delta_{L_+}^{-1}(i\nu)\, d\nu & = & [H_z^- * \widehat{G}_{L_+}](0) \\ & = & \int_{-\infty}^{\infty} H_z^-(\xi)\widehat{G}_{L_+}(-\xi)\, d\xi \\ & = & \int_{-\infty}^{0} H_z^-(\xi) G_{L_+}^r(-\xi)\, d\xi. \end{array} \tag{5.71}$$

In particular, in order to establish (5.64) it suffices to show that $G_{L+}^r(\xi) = 0$ for all $\xi \geq 0$. But this follows from the differential equation (5.68), which implies that $G_{L_+}^r \in \mathfrak{Q}_{L_+}(0)$. Indeed, in view of our assumption that $\Delta_{L_+}(z) = 0$ admits no roots with $\operatorname{Re} z \leq 0$, Proposition 2.1 implies that $\mathfrak{Q}_{L_+}(0) = \{0\}$. □

Proof of Theorem 3.10. Without loss of generality, we will focus on the case that $\eta = 0$. Suppose that we have a Wiener-Hopf factorization

$$(z - z_0)\Delta_L(z) = \Delta_{L_-}(z)\Delta_{L_+}(z). \tag{5.72}$$

By rearranging roots according to the procedure described in [29, Lem. 5.7], we can ensure that $\operatorname{Re} z_- \leq \operatorname{Re} z_+$ for any pair $z_-, z_+ \in \mathbb{C}$ that has $\Delta_{L_-}(z_-) = 0$ and $\Delta_{L_+}(z_+)$. In addition, we can ensure that $\operatorname{Re} z_0 > 0$ and $\Delta_+(z_0) = 0$.

Let us first consider the situation that $\Delta_{L_-}(z) = 0$ admits at least one root with $\operatorname{Re} z > 0$. Notice that this implies that all roots of $\Delta_{L_+}(z) = 0$ have $\operatorname{Re} z > 0$. Remembering that $\operatorname{Re} z_0 > 0$ and recalling the quantities (2.26), we may compute

$$n_L^{\#}(0) = n_{L_+}^-(0) - n_{L_-}^+(0) + n_{z-z_0}^+(0) = -n_{L_-}^+(0) + 1 \leq 0, \tag{5.73}$$

since in the current situation $n_{L_-}^+(0) \geq 1$. In particular, we need to establish the identities

$$\dim \operatorname{Ker} \pi^-_{\widehat{Q}_L(0)} = 0, \qquad \dim \operatorname{Ker} \widehat{\pi}^-_{\widehat{Q}_L(0)} = 0, \tag{5.74}$$

together with

$$\operatorname{codim} \operatorname{Range} \pi^-_{\widehat{Q}_L(0)} = n_{L_-}^+(0) - 1, \qquad \operatorname{codim} \operatorname{Range} \widehat{\pi}^-_{\widehat{Q}_L(0)} = n_{L_-}^+(0). \tag{5.75}$$

The key ingredients we will use here are the results [29, Thms. 5.3-5.4][3], which allow us to conclude that

$$\mathfrak{Q}_L(0) = \mathfrak{Q}_{L_-}(0). \tag{5.76}$$

In particular, any $\psi \in C([-1,1], \mathbb{C})$ satisfies $\psi \in Q_L(0)$ if and only if $\mathrm{ev}_0^- \psi \in Q_{L_-}(0)$, which can be rewritten as

$$\Pi_{L_-}^{\mathrm{sp}}(\Sigma_{L_-}^+)\, \mathrm{ev}_0^- \psi = \sum_{\ell=1}^{n_{L_-}^+(0)} \psi_\ell \widetilde{M}_\ell \mathrm{ev}_0^- \psi = 0, \tag{5.77}$$

upon recalling the projection (5.27) and the operators (5.30).

[3]The results in [16] show how the proofs of these theorems can be adjusted for our weaker condition (HL).

Turning our attention to the Green's function $\widehat{G}_L$, we remark that there exists a set

$$\widetilde{\alpha}_\ell \in \mathbb{C}, \qquad 1 \le \ell \le n^+_{L_-}(0), \tag{5.78}$$

such that we can write

$$\Pi^{\rm sp}_{L_-}(\Sigma^+_{L_-})\,{\rm ev}^-_0\,\widehat{G}_L = \sum_{\ell=1}^{n^+_{L_-}(0)} \widetilde{\alpha}_\ell \psi_\ell. \tag{5.79}$$

Lemma 5.4 guarantees that, possibly after reordering the eigenfunctions $\{\psi_\ell\}$, we have $\widetilde{\alpha}_1 \neq 0$. In particular, ${\rm ev}^-_0\,\widehat{G}_L \notin Q_{L_-}(0)$.

To see the statements (5.74) concerning the kernels of $\widehat{\pi}^-_{\widehat{Q}_L(0)}$ and $\pi^-_{\widehat{Q}_L(0)}$, let us suppose that $\widehat{x} \in \widehat{\mathfrak{Q}}_L(0)$ has $\widehat{\rm ev}^-_0 \widehat{x} = (0, v)$ for some $v \in \mathbb{C}$. Then we have

$$\widehat{x} - \widehat{G}_L v \in \mathfrak{Q}_L(0) = \mathfrak{Q}_{L_-}(0) \tag{5.80}$$

and hence

$${\rm ev}^-_0 \widehat{x} - {\rm ev}^-_0 \widehat{G}_L v = -{\rm ev}^-_0 \widehat{G}_L v \in Q_{L_-}(0). \tag{5.81}$$

Our discussion above implies that $v = 0$ and hence $\widehat{x} = 0$.

Moving on to the identities (5.75), notice that $(\phi, v) \in C([-1,0],\mathbb{C}) \times \mathbb{C}$ satisfies $(\phi, v) \in {\rm Range}\,\widehat{\pi}^-_{\widehat{Q}_L(0)}$ if and only if

$$\phi - {\rm ev}^-_0\,\widehat{G}_L\big(v - \phi(0)\big) \in Q_{L_-}(0), \tag{5.82}$$

which implies

$$\Pi^{\rm sp}_{L_-}(\Sigma^+_{L_-})\,\phi = \Pi^{\rm sp}_{L_-}(\Sigma^+_{L_-})\,{\rm ev}^-_0\,\widehat{G}_L\big(v - \phi(0)\big) \tag{5.83}$$

and hence

$$\widetilde{M}_\ell \phi = \widetilde{\alpha}_\ell\big(v - \phi(0)\big), \qquad 1 \le \ell \le n^+_{L_-}(0). \tag{5.84}$$

This in turn can be rewritten as the system

$$\begin{array}{lcl} v & = & \phi(0) + \widetilde{\alpha}_1^{-1}\widetilde{M}_1\phi \\ \widetilde{M}_\ell\phi & = & \widetilde{\alpha}_\ell\widetilde{\alpha}_1^{-1}\widetilde{M}_1\phi, \qquad 2 \le \ell \le n^+_{L_-}(0), \end{array} \tag{5.85}$$

which consists of $n^+_{L_-}(0)$ linearly independent conditions on the pair (ϕ, v). Alternatively, this system can be viewed as $n^+_{L_-}(0) - 1$ conditions on ϕ, with v depending explicitly on ϕ. This establishes the identities (5.75).

We now proceed to discuss the alternative situation in which all roots of $\Delta_{L_-}(z) = 0$ have ${\rm Re}\, z < 0$. Again remembering that ${\rm Re}\, z_0 > 0$, we now compute

$$n^\#_L(0) = n^-_{L_+}(0) - n^+_{L_-}(0) + n^+_{z - z_0}(0) = n^-_{L_+}(0) + 1 \ge 1. \tag{5.86}$$

In particular, we need to establish the identities

$$\dim {\rm Ker}\, \pi^-_{\widehat{Q}_L(0)} = n^\#_L(0) = 1 + n^-_{L_+}(0), \qquad \dim {\rm Ker}\, \widehat{\pi}^-_{\widehat{Q}_L(0)} = n^\#_L(0) - 1 = n^-_{L_+}(0), \tag{5.87}$$

together with

$${\rm codim}\,{\rm Range}\, \pi^-_{\widehat{Q}_L(0)} = 0, \qquad {\rm codim}\,{\rm Range}\, \widehat{\pi}^-_{\widehat{Q}_L(0)} = 0. \tag{5.88}$$

In this situation, [29, Thm. 5.3] gives us the inclusion

$$\mathfrak{Q}_{L_-}(0) \subset \mathfrak{Q}_L(0). \tag{5.89}$$

In addition, since $n^+_{L_-}(0) = 0$ we have $Q_{L_-}(0) = C([-1,0],\mathbb{C})$. In particular, for any pair $(\phi, v) \in C([-1,0],\mathbb{C}) \times \mathbb{C}$, we can define $\widehat{x} \in \widehat{BC}_0^{\oplus}$ by way of

$$\widehat{x} = E_{L_-}[\phi - \mathrm{ev}_0^- \widehat{G}_L(v - \phi(0))] + \widehat{G}_L(v - \phi(0)). \tag{5.90}$$

It is easy to see that $\widehat{x} \in \widehat{\mathfrak{Q}}_L(0)$ with $\widehat{\mathrm{ev}}_0^-\,\widehat{x} = (\phi, v)$, which establishes the identities (5.88).

To see (5.87), let us consider any $\widehat{x} \in \widehat{BC}_0^{\oplus}$ that has $\widehat{\mathrm{ev}}_0^-\,\widehat{x} = (0, v)$ for some $v \in \mathbb{C}$. Then we have $\widehat{x} \in \widehat{\mathfrak{Q}}_L(0)$ if and only if

$$y = \widehat{x} - \widehat{G}_L v + E_{L_-} \mathrm{ev}_0^-\, \widehat{G}_L v \tag{5.91}$$

satisfies $y \in \mathfrak{Q}_L(0)$. Since $\mathrm{ev}_0^- y = 0$, this condition is equivalent to requiring

$$\mathrm{ev}_0\, y \in \mathrm{Ker}\, \pi^-_{Q_L(0)}, \tag{5.92}$$

which is an $(n^{\#}_L(0) - 1)$-dimensional space according to Proposition 2.5. $\square$

5.3 Differential-Algebraic Equations

In this section we study the mixed differential-algebraic system

$$\mathcal{I}x'(\xi) = M\,\widehat{\mathrm{ev}}_\xi\, x \tag{5.93}$$

and set out to prove Theorem 3.16. We recall that $\mathcal{I}$ is a diagonal $(n \times n)$-matrix that has $\mathcal{I}^2 = \mathcal{I}$ and that there exists a measure $\mu \in NBV\big([-1,1],\mathbb{C}^{n\times n}\big)$ such that

$$M\psi = \int_{-1}^{1} d\mu(\sigma)\psi(\sigma) \tag{5.94}$$

for any $\psi \in C([-1,1],\mathbb{C}^n)$, while M is extended to $\widehat{X}_\alpha$ for $-1 < \alpha \le 0$ as in §3.2. Finally, remember that the characteristic function for (5.93) is given by

$$\delta_{\mathcal{I},M}(z) = \mathcal{I}z - Me^{z\cdot}I. \tag{5.95}$$

Our arguments in this section will roughly follow the approach developed in [16, §5]. However, the relevant computations need to be generalized to account for discontinuities and the fact that the condition (HM) is weaker than its counterpart in [16].

We start by studying the condition (HM) in more detail. In particular, let us write

$$\ell_* = \max\{\ell_1, \ldots, \ell_n\} \tag{5.96}$$

and define $n \times n$ diagonal matrices J_i for $0 \le i \le \ell_*$ such that

$$\mathcal{J}_\alpha(z) = J_0 + J_1(z - \alpha) + \ldots + J_{\ell_*}(z - \alpha)^{\ell_*}. \tag{5.97}$$

Notice that $J_0 = \mathcal{I}$, $J_i^2 = J_i$ for all $0 \le i \le n$ and $J_iJ_j = 0$ whenever $i \ne j$. In addition, we have

$$J_0 + J_1 + \ldots + J_{\ell_*} = I. \tag{5.98}$$

Lemma 5.5. *Consider the system (5.93) and suppose that (HM) is satisfied. Recall the function μ featured in (5.94). Then there exists a measure $\zeta \in NBV([-1,1], \mathbb{C}^{n\times n})$ such that following holds true.*

(hμ1) For any $1 \le \ell \le \ell_*$, we have $J_\ell \mu \in W^{\ell-1}_{\mathrm{loc}}([-1,1], \mathbb{C}^{n\times n})$, with

$$(-1)^{\ell-1}[D^{\ell-1}J_\ell\mu](\sigma) = -J_\ell H(\sigma) + \int_{-1}^{\sigma} J_\ell \zeta(\tau)d\tau, \qquad -1 \le \sigma \le 1, \tag{5.99}$$

in which H denotes the Heaviside function, which has $H(\sigma) = 1$ for all $\xi \ge 1$ and $H(\sigma) = 0$ for all $\xi < 0$.

(hμ2) For any $2 \le \ell \le \ell_*$, we have

$$D^s J_\ell \mu(\pm 1) = 0, \qquad 1 \le s \le \ell - 1. \tag{5.100}$$

(hμ3) We have

$$J_0\zeta = -J_0\mu. \tag{5.101}$$

Proof. Let us pick any $1 \le \ell \le \ell_*$. We can compute

$$J_\ell \Delta_L(z) = J_\ell \mathcal{J}_\alpha(z)\delta_{\mathcal{I},M}(z) = (z-\alpha)^\ell(-J_\ell M e^{z\cdot} I). \tag{5.102}$$

By filling the zero-rows of $J_\ell M e^{z\cdot}I$ and $J_\ell \Delta_L(z)$ by shuffled versions of the non-zero rows, we can build operators $\widetilde{M}$ and $\widetilde{L}$ that have $J_\ell\widetilde{M} = M$ and $J_\ell \widetilde{L} = L$ and

$$\Delta_{\widetilde{L}}(z) = (z-\alpha)^\ell(-J_\ell M e^{z\cdot} I) = \delta_{0,\widetilde{M}}(z). \tag{5.103}$$

This allows us to apply [17, Prop. 3.1] and conclude that all the statements in (hμ1) and (hμ2) hold upon replacing J_ℓ replaced by I and μ by $\widetilde{\mu}$. Left multiplying by J_ℓ and exploiting that $J_\ell\widetilde{\mu} = J_\ell\mu$ subsequently yields the desired original properties (hμ1) and (hμ2). The remaining property (hμ3) can be satisfied by observing that $J_0\zeta$ can be freely chosen without interfering with (hμ1) and (hμ2). □

A first consequence of this result is that we can now establish Lemma 3.15, which states that the portion of the right hand side of (5.93) associated with purely algebraic equations is continuous on $\mathbb{R}_+$ for all $\widehat{x} \in \widehat{BC}^{\oplus}_\eta$.

Proof of Lemma 3.15. The property (hμ1) above implies that for $\psi \in C([-1,1], \mathbb{C}^n)$ we may write

$$(I - J_0)M\psi = -J_1\psi(0) + \int_{-1}^{1} \widetilde{\zeta}(\sigma)\psi(\sigma)d\sigma, \tag{5.104}$$

for some $\widetilde{\zeta} \in NBV([-1,1], \mathbb{C}^{n\times n})$. In particular, the only term that can lead to discontinuities is the $-J_1\psi(0)$ term. However, our choice (3.33) implies that we interpret $\psi(0)$ as $\psi(0+)$ whenever there is ambiguity. More precisely, let us choose any $-1 < \alpha \le 1$ and consider a function

$$\widehat{\psi} = (\psi^l, \psi^r) \in \widehat{X}_\alpha. \tag{5.105}$$

We then have

$$(I - J_0)M\widehat{\psi} = -J_1\psi^r(0) + \int_{-1}^{\alpha} \widetilde{\zeta}(\sigma)\psi^l(\sigma)d\sigma + \int_{\alpha}^{1} \widetilde{\zeta}(\sigma)\psi^r(\sigma)d\sigma, \tag{5.106}$$

which completes our proof. □

Using the function ζ appearing in Lemma 5.5, we introduce a new bounded linear operator

$$L_* : C([-1,1],\mathbb{C}^n) \to \mathbb{C}^n \tag{5.107}$$

by writing

$$L_* \psi = (I - J_0)\zeta(1)\psi(\xi+1) - \int_{-1}^{1} d\zeta(\sigma)\psi(\xi+\sigma). \tag{5.108}$$

We write μ_* for the usual measure associated to L_*. Notice that

$$\Delta_{L_*}(z) = z + \int_{-1}^{1} d\zeta(\sigma)e^{z\sigma} - (I - J_0)\zeta(1)e^{z}. \tag{5.109}$$

In addition, for $1 \le \ell \le \ell_*$ we introduce the measures

$$\mu_\ell(\sigma) = \sum_{i=\ell}^{\ell_*} (-1)^{i-\ell} J_i D^{i-\ell} \mu(\sigma) \tag{5.110}$$

and the characteristic functions

$$\delta_\ell(z) = - \int_{-1}^{1} d\mu_\ell(\sigma) e^{z\sigma}. \tag{5.111}$$

Notice that for every $1 \le \ell \le \ell_*$ we have the identity

$$J_\ell \delta_\ell(z) = - \int_{-1}^{1} J_\ell d\mu(\sigma) e^{z\sigma} = J_\ell \delta_{\mathcal{I},M}(z). \tag{5.112}$$

We also introduce the operators $M_\ell : C([-1,1],\mathbb{C}^n) \to \mathbb{C}^n$ via

$$M_\ell \psi = \int_{-1}^{1} d\mu_\ell(\sigma)\psi \tag{5.113}$$

and extend these operators in the usual fashion to the spaces $\widehat{X}_\alpha$. In particular, a short calculation shows that for any $\widehat{\psi} = (\psi^l, \psi^r) \in \widehat{X}_0$, we have

$$\begin{array}{lcl} M_1 \widehat{\psi} & = & \sum_{i=1}^{\ell_*} J_i \big[-\psi^r(0) + \int_{-1}^{1} \zeta(\sigma)\widehat{\psi}(\sigma) d\sigma \big] \\ & = & (I - J_0) \big[-\psi^r(0) + \int_{-1}^{1} \zeta(\sigma)\widehat{\psi}(\sigma) d\sigma \big]. \end{array} \tag{5.114}$$

There is an intricate relation between the algebraic system (5.93) and the MFDE

$$x'(\xi) = L_* \, \widehat{\mathrm{ev}}_\xi \, x. \tag{5.115}$$

Our next two results serve as a preparation to help us exploit this relationship.

Lemma 5.6. *For any $1 \le \ell \le \ell_*$, we have the identity*

$$\Delta_{L_*}(z) = (\sum_{i=0}^{\ell-1} J_i z^i)\delta_{\mathcal{I},M}(z) + z^\ell \delta_\ell(z). \tag{5.116}$$

In particular, we have

$$\Delta_{L_*}(z) = \mathcal{J}_0(z)\delta_{\mathcal{I},M}(z). \tag{5.117}$$

Proof. Note first that the choice (hμ3) implies that

$$J_0\Delta_{L_*}(z) = J_0 z - J_0 \int_{-1}^{1} d\mu(\sigma)e^{z\sigma} = J_0\delta_{\mathcal{I},M}(z). \tag{5.118}$$

In addition, property (hμ1) implies

$$\begin{array}{lcl} \delta_1(z) & = & -\sum_{i=1}^{\ell_*}(-1)^{i-1}J_i\int_{-1}^{1} d[D^{i-1}\mu](\sigma)e^{z\sigma} \\ & = & -\sum_{i=1}^{\ell_*}\big[-J_i + J_i\int_{-1}^{1}\zeta(\sigma)e^{z\sigma}d\sigma\big] \\ & = & (I-J_0)\big[I - \int_{-1}^{1}\zeta(\sigma)e^{z\sigma}d\sigma\big]. \end{array} \tag{5.119}$$

Turning our attention to $\Delta_{L_*}(z)$, we integrate by parts to find

$$\int_{-1}^{1} e^{z\sigma}d\zeta(\sigma) = \zeta(1)e^z - z\int_{-1}^{1} e^{z\sigma}\zeta(\sigma)d\sigma, \tag{5.120}$$

which shows that

$$\begin{array}{lcl} (I-J_0)\Delta_{L_*}(z) & = & (I-J_0)z + (I-J_0)[\zeta(1)e^z - z\int_{-1}^{1}e^{z\sigma}\zeta(\sigma)d\sigma\big] \\ & & \quad -(I-J_0)\zeta(1)e^z \\ & = & (I-J_0)z\big[I - \int_{-1}^{1}e^{z\sigma}\zeta(\sigma)d\sigma\big] \\ & = & z\delta_1(z) \end{array} \tag{5.121}$$

and establishes (5.116) for $\ell = 1$. We claim that it now suffices to show that

$$(I-J_\ell)\delta_\ell(z) = z\delta_{\ell+1}(z) \tag{5.122}$$

holds for all $1 \le \ell \le \ell_* - 1$. Indeed, assuming this is the case, we can use (5.112) to write

$$\begin{array}{lcl} \delta_\ell(z) & = & J_\ell\delta_\ell(z) + z\delta_{\ell+1}(z) \\ & = & J_\ell\delta_{\mathcal{I},M}(z) + z\delta_{\ell+1}(z). \end{array} \tag{5.123}$$

In order to show (5.122), we compute

$$\begin{array}{lcl} (I-J_\ell)\delta_\ell(z) & = & -(I-J_\ell)\sum_{i=\ell}^{\ell_*}(-1)^{i-\ell}J_i\int_{-1}^{1}d[D^{i-\ell}\mu](\sigma)e^{z\sigma} \\ & = & -\sum_{i=\ell+1}^{\ell_*}(-1)^{i-\ell}J_i\int_{-1}^{1}d[D^{i-\ell}\mu](\sigma)e^{z\sigma} \\ & = & -\sum_{i=\ell+1}^{\ell_*}(-1)^{i-\ell}J_i\big[D^{i-\ell}\mu(1)e^z - D^{i-\ell}\mu(-1)e^{-z} \\ & & \quad -z\int_{-1}^{1}D^{i-\ell}\mu(\sigma)e^{z\sigma}d\sigma\big] \\ & = & z\sum_{i=\ell+1}^{\ell_*}(-1)^{i-\ell}J_i\int_{-1}^{1}D^{i-\ell}\mu(\sigma)e^{z\sigma}d\sigma \end{array} \tag{5.124}$$

on account of (hμ2). Continuing our computation, we find

$$\begin{array}{lcl} (I-J_\ell)\delta_\ell(z) & = & z\sum_{i=\ell+1}^{\ell_*}(-1)^{i-\ell}J_i\int_{-1}^{1}d[D^{i-\ell-1}\mu](\sigma)e^{z\sigma} \\ & = & -z\sum_{i=\ell+1}^{\ell_*}(-1)^{i-(\ell+1)}J_i\int_{-1}^{1}d[D^{i-(\ell+1)}\mu](\sigma)e^{z\sigma} \\ & = & z\delta_{\ell+1}(z), \end{array} \tag{5.125}$$

as desired. The final identity (5.117) follows from the observation that $J_\ell\mu_\ell = \mu_\ell$. $\square$

Lemma 5.7. *For any $\widehat{\psi} = (\psi^l, \psi^r) \in \widehat{X}_0$, we have the identity*

$$\begin{array}{lcl} z\int_{-1}^{1} d\mu_1(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)d\tau & = & (I - J_0)\int_{-1}^{1} d\mu_*(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)d\tau \\ & & +(I - J_0)\psi^r(0) + M_1\widehat{\psi}. \end{array} \tag{5.126}$$

In addition, for any integer $1 \le \ell \le \ell_* - 1$, we have

$$\begin{array}{lcl} z\int_{-1}^{1} d\mu_{\ell+1}(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)d\tau & = & (I - J_\ell)\int_{-1}^{1} d\mu_\ell(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)d\tau \\ & & +M_{\ell+1}\widehat{\psi}. \end{array} \tag{5.127}$$

Proof. Setting out to establish (5.126), we observe that

$$\int_{-1}^{1} d\mu_1(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)d\tau \;\; = \;\; (I - J_0)\int_{-1}^{1} \zeta(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)d\tau d\sigma. \tag{5.128}$$

An integration by parts shows that

$$\begin{array}{lcl} (I - J_0)\int_{-1}^{1} \zeta(\sigma)\widehat{\psi}(\sigma)d\sigma & = & -(I - J_0)\int_{-1}^{1} d\mu_*(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)d\tau \\ & & +z(I - J_0)\int_{-1}^{1} \zeta(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)d\tau d\sigma. \end{array} \tag{5.129}$$

Recalling the identity (5.114) completes the proof of (5.126).

Let us now pick $1 \le \ell \le \ell_* - 1$ and set out to establish (5.127) by writing

$$\mathcal{T}_\ell(z) = (I - J_\ell)\int_{-1}^{1} d\mu_\ell(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)d\tau. \tag{5.130}$$

Using (hμ2), we proceed as in (5.124) - (5.125) to compute

$$\begin{array}{lcl} \mathcal{T}_\ell(z) & = & (I - J_\ell)\sum_{i=\ell}^{\ell_*}(-1)^{i-\ell}J_i\int_{-1}^{1} d[D^{i-\ell}\mu](\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)\,d\tau \\ & = & \sum_{i=\ell+1}^{\ell_*}(-1)^{i-\ell}J_i\int_{-1}^{1} d[D^{i-\ell}\mu](\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)\,d\tau \\ & = & -z\sum_{i=\ell+1}^{\ell_*}(-1)^{i-\ell}J_i\int_{-1}^{1} D^{i-\ell}\mu(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)\,d\tau\,d\sigma \\ & & \quad +\sum_{i=\ell+1}^{\ell_*}(-1)^{i-\ell}J_i\int_{-1}^{1} D^{i-\ell}\mu(\sigma)\widehat{\psi}(\sigma)\,d\sigma \\ & = & z\int_{-1}^{1} d\mu_{\ell+1}(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)\,d\tau - M_{\ell+1}\widehat{\psi}, \end{array} \tag{5.131}$$

which establishes (5.127). $\square$

For any $\widehat{\psi} = (\psi^l, \psi^r) \in \widehat{X}_0$, we remark that a repeated application of Lemma 5.7 yields the identity

$$\begin{array}{lcl} \sum_{\ell=1}^{\ell_*} z^{\ell-1}M_\ell\widehat{\psi} & = & -(I - J_0)\psi^r(0) - \int_{-1}^{1} d\mu_*(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)\,d\tau \\ & & +\mathcal{J}_0(z)\int_{-1}^{1} d\mu(\sigma)e^{z\sigma}\int_{\sigma}^{0} e^{-z\tau}\widehat{\psi}(\tau)\,d\tau. \end{array} \tag{5.132}$$

This identity can be used to study the relation between the algebraic equation (5.93) and the differential equation (5.115).

Lemma 5.8. *Consider any $\eta \in \mathbb{R}$ and a function $\widehat{x} \in \widehat{BC}_\eta^{\oplus}$ that has*

$$J_0\,\widehat{x} \in W_\eta^{1,\infty}([0,\infty), \mathbb{C}^n). \tag{5.133}$$

Then x solves the differential-algebraic equation (5.93) for all $\xi \ge 0$ if and only if $\widehat{x}$ solves the differential equation (5.115) for $\xi \ge 0$ and in addition satisfies the identities

$$M_\ell\,\widehat{\mathrm{ev}}_0\,\widehat{x} = 0 \tag{5.134}$$

for all integers $1 \le i \le \ell$.

Proof. Let us consider any $\widehat{x} \in \widehat{BC}_{\eta}^{\oplus}$ that satisfies (5.133) and define the function $v \in L^{\infty}_{\eta}([0,\infty), \mathbb{C}^n)$ via

$$v(\xi) = J_0\, \widehat{x}'(\xi) - M\, \widehat{\mathrm{ev}}_{\xi}\, \widehat{x}. \tag{5.135}$$

For any z with $\mathrm{Re}\, z > \eta$, the Laplace transforms $\mathcal{L}[v](z)$ and $\mathcal{L}[\widehat{x}](z)$ are well-defined and related via

$$\begin{array}{lcl} \mathcal{L}[v](z) & := & \int_0^{\infty} e^{-z\xi} v(\xi) d\xi \\ & = & J_0\big(z\mathcal{L}[\widehat{x}](z) - \widehat{x}(0+)\big) - \int_{-1}^{1} d\mu(\sigma) \int_0^{\infty} e^{-z\xi} \widehat{x}(\xi+\sigma) d\xi \\ & = & J_0\big(z\mathcal{L}[\widehat{x}](z) - \widehat{x}(0+)\big) \\ & & \qquad - \int_{-1}^{1} d\mu(\sigma) e^{z\sigma} \big(\mathcal{L}[\widehat{x}](z) + \int_{\sigma}^{0} e^{-z\tau} \widehat{x}(\tau) d\tau\big) \\ & = & \delta_{\mathcal{I},M}(z)\mathcal{L}[\widehat{x}](z) - J_0 \widehat{x}(0+) - \int_{-1}^{1} d\mu(\sigma) e^{z\sigma} \int_{\sigma}^{0} e^{-z\tau} \widehat{x}(\tau) d\tau, \end{array} \tag{5.136}$$

in which we have used Fubini's theorem to change the order of integration. Similarly, if $\widehat{x} \in \widehat{BC}_{\eta}^{\oplus}$ and $\widehat{x}' \in BC_{\eta}^{+}$, then we may write

$$w(\xi) = \widehat{x}'(\xi) - \int_{-1}^{1} d\mu_*(\sigma) \widehat{x}(\xi + \sigma) \tag{5.137}$$

and compute the Laplace transform $\mathcal{L}[w](z)$ for any z with $\mathrm{Re}\, z > \eta$. A similar computation as above and an application of (5.132) yields

$$\begin{array}{lcl} \mathcal{L}[w](z) & = & \Delta_{L_*}(z)\mathcal{L}[\widehat{x}](z) - \widehat{x}(0+) - \int_{-1}^{1} d\mu_*(\sigma) e^{z\sigma} \int_{\sigma}^{0} e^{-z\tau} \widehat{x}(\tau) d\tau \\ & = & \Delta_{L_*}(z)\mathcal{L}[\widehat{x}](z) - J_0 \widehat{x}(0+) - \mathcal{J}_0(z) \int_{-1}^{1} d\mu(\sigma) e^{z\sigma} \int_{\sigma}^{0} e^{-z\tau} \widehat{x}(\tau) d\tau \\ & & \qquad + \sum_{\ell=1}^{\ell_*} z^{\ell-1} M_{\ell}\, \widehat{\mathrm{ev}}_0 \widehat{x}. \end{array} \tag{5.138}$$

Now, suppose that $\widehat{x} \in \widehat{BC}_{\eta}^{\oplus}$ satisfies the differential-algebraic equation (5.93). The identities (5.134) can be easily verified by differentiating (5.93) and subsequently using integration by parts together with the boundary condition (hμ2). Using [17, Prop. 4.2(iii)], we may conclude that $\widehat{x}' \in L^{\infty}_{\eta}([0,\infty), \mathbb{C}^n)$. This means that the Laplace transform $\mathcal{L}[w](z)$ is well-defined for $\mathrm{Re}\, z > \eta$. Comparing (5.136) and (5.138), noting that $\mathcal{L}[v](z) = 0$ and using (5.134), we see that also $\mathcal{L}[w](z) = 0$, which implies that $\widehat{x}$ satisfies the differential equation (5.115). The converse statement can be easily established by inspection of (5.134), (5.136) and (5.138). □

In order to relate the criteria (5.134) back to a spectral projection as in [16], we need to introduce the extended spectral projection

$$\widehat{\Pi}^{\mathrm{sp}}_{L_*}(z_*) : \widehat{X}_0 \to C([-1,1], \mathbb{C}^n) \tag{5.139}$$

that acts as follows on $\widehat{\psi} = (\psi^l, \psi^r) \in \widehat{X}_0$,

$$[\widehat{\Pi}^{\mathrm{sp}}_{L_*}(z_*)\widehat{\psi}](\theta) \;=\; \mathrm{Res}_{z=z_*}\, e^{z\theta} \Delta_{L_*}(z)^{-1} \big[\psi^r(0) + \int_{-1}^{1} d\mu_*(\sigma) e^{z\sigma} \int_{\sigma}^{0} e^{-z\tau} \widehat{\psi}(\tau) d\tau\big] \tag{5.140}$$

Lemma 5.9. *Suppose that* $\det \delta_{\mathcal{I},M}(0) \neq 0$. *Then any* $\widehat{\psi} = (\psi^l, \psi^r) \in \widehat{X}_0$ *satisfies* $\widehat{\Pi}^{\mathrm{sp}}_{L_*}(0)\widehat{\psi} = 0$ *if and only if*

$$M_{\ell}\, \widehat{\psi} = 0 \tag{5.141}$$

holds for all integers $1 \leq \ell \leq \ell_*$.

Proof. Since $\Delta_{L_*}(z) = \mathcal{J}_0(z)\delta_{\mathcal{I},M}(z)$, we find that $\Delta_{L_*}(z)^{-1}$ can be written as

$$\Delta_{L_*}(z)^{-1} = (A_0 + A_1 z + \ldots + A_{\ell_*-1}z^{\ell_*-1})\mathcal{J}_0(z)^{-1} + O(1) \tag{5.142}$$

as $z \to 0$, with $\det A_0 \neq 0$. Applying the identity (5.132), we find

$$\begin{array}{rcl} [\widehat{\Pi}^{\mathrm{sp}}_{L_*}(0)\,\phi](\theta) & = & \mathrm{Res}_{z=0}\, e^{z\theta}[\sum_{k=0}^{\ell_*-1} A_k z^k]\mathcal{J}_0(z)^{-1}\big[J_0\psi^r(0) - \sum_{\ell=1}^{\ell_*} z^{\ell-1}M_\ell\widehat{\psi} \\ & & \qquad + \mathcal{J}_0(z)\int_{-1}^1 d\mu(\sigma)e^{z\sigma}\int_\sigma^0 e^{-z\tau}\widehat{\psi}(\tau)d\tau\big]. \end{array} \tag{5.143}$$

Since $\mathcal{J}_0(z)^{-1}J_0$ is analytic at $z=0$, we obtain

$$\begin{array}{rcl} -[\widehat{\Pi}^{\mathrm{sp}}_{L_*}(0)\,\phi](\theta) & = & \mathrm{Res}_{z=0}\,[\sum_{j=0}^{\ell_*-1}\frac{1}{j!}z^j\theta^j][\sum_{k=0}^{\ell_*-1}A_k z^k][\sum_{i=1}^{\ell_*} z^{-i}J_i][\sum_{\ell=1}^{\ell_*} z^{\ell-1}M_\ell\widehat{\psi}] \\ & = & \sum_{j=0}^{\ell_*-1} b_j\theta^j \end{array} \tag{5.144}$$

for some set $\{b_0, \ldots, b_{\ell_*-1}\} \subset \mathbb{C}^n$. Matching powers and exploiting the fact that A_0 is invertible shows that the condition $b_0 = b_1 = \ldots = b_{\ell_*-1} = 0$ is equivalent to the requirement that

$$J_i\, M_\ell\, \widehat{\psi} = 0 \tag{5.145}$$

for all pairs (i,ℓ) that have $1 \le \ell \le i \le \ell_*$. This in turn is equivalent to the condition (5.141). $\square$

Comparing the conditions in Lemma 5.8 and 5.9 and applying exponential shifts, we arrive at the following result.

Lemma 5.10. *Consider the setting of Theorem 3.16. Pick any $\eta < \eta_*$ and consider a function $\widehat{x} \in \widehat{BC}^{\oplus}_{\eta}$. Then $\widehat{x}$ solves the differential-algebraic equation (5.93) for all $\xi \ge 0$ if and only if $\widehat{x}$ solves the differential equation (5.115) for $\xi \ge 0$ and in addition satisfies the identity*

$$\widehat{\Pi}^{\mathrm{sp}}_{L'}(\eta_*)\,\widehat{\mathrm{ev}}_0\,\widehat{x} = 0. \tag{5.146}$$

The final ingredient we need to prove Theorem 3.16 is to compute the modified spectral projection of the Green's function associated to an MFDE.

Lemma 5.11. *Consider any bounded linear operator $L : C([-1,1],\mathbb{C}^n) \to \mathbb{C}^n$. Then for any $z_* \in \mathbb{C}$ that has $\mathrm{Re}\, z_* > 0$, we have the identity*

$$\widehat{\Pi}^{\mathrm{sp}}_{L}(z_*)\,\widehat{\mathrm{ev}}_0\,\widehat{G}_L = 0. \tag{5.147}$$

Proof. Let us pick an arbitrary $\alpha > 0$ and write

$$F_L(\xi) = \widehat{G}_L(\xi) - H^+_\alpha(\xi)I, \tag{5.148}$$

which is a continuous function that has Fourier transform

$$\mathcal{F}^+(F_L)(\nu) = \Delta_L^{-1}(i\nu) - \frac{1}{\alpha + i\nu}I. \tag{5.149}$$

In particular, we can repeat the calculation in the proof of Lemma 5.3 up to the first line of (5.40) and write

$$[\widehat{\Pi}^{\mathrm{sp}}_{L}(z_*)\widehat{\mathrm{ev}}_0\,F_L](\theta) \quad = \quad \mathrm{Res}_{z=z_*}e^{z\theta}\Delta_L(z)^{-1}\mathcal{T}_1(z) \tag{5.150}$$

in which we have

$$\begin{array}{rcl} \mathcal{T}_1(z) & = & -\frac{\alpha}{z+\alpha}H_{z,\alpha}(0)I - \frac{1}{z+\alpha}\int_{-1}^1 d\mu_L(\sigma)H_{z,\alpha}(\sigma)I \\ & = & -\frac{\alpha}{z+\alpha}I - \frac{1}{z+\alpha}\int_{-1}^0 d\mu_L(\sigma)e^{z\sigma}I - \frac{1}{z+\alpha}\int_0^1 d\mu_L(\sigma)e^{-\alpha\sigma}I, \end{array} \tag{5.151}$$

where μ_L here denotes the usual measure associated to L via (3.30). On the other hand, we can write

$$[\widehat{\Pi}^{\mathrm{sp}}_L(z_*)\,\widehat{\mathrm{ev}}_0\,H^+_\alpha I](\theta) \quad = \quad \mathrm{Res}_{z=z_*} e^{z\theta}\Delta_L(z)^{-1}\mathcal{T}_2(z) \tag{5.152}$$

and directly use the definition (5.140) to compute

$$\begin{array}{rcl} \mathcal{T}_2(z) & = & I + \int_0^1 d\mu_L(\sigma)e^{z\sigma}\int_\sigma^0 e^{-(z+\alpha)\tau}d\tau \\ & = & I - \frac{1}{z+\alpha}\int_0^1 d\mu_L(\sigma)e^{z\sigma} + \frac{1}{z+\alpha}\int_0^1 d\mu_L(\sigma)e^{-\alpha\sigma} \\ & = & I - \frac{1}{z+\alpha}\int_{-1}^1 d\mu_L(\sigma)e^{z\sigma} \\ & & \quad + \frac{1}{z+\alpha}\int_{-1}^0 d\mu_L(\sigma)e^{z\sigma} + \frac{1}{z+\alpha}\int_0^1 d\mu_L(\sigma)e^{-\alpha\sigma} \\ & = & \frac{\alpha}{z+\alpha} + \frac{1}{z+\alpha}\Delta_L(z) \\ & & \quad + \frac{1}{z+\alpha}\int_{-1}^0 d\mu_L(\sigma)e^{z\sigma} + \frac{1}{z+\alpha}\int_0^1 d\mu_L(\sigma)e^{-\alpha\sigma}. \end{array} \tag{5.153}$$

Recalling that the term involving $\Delta_L(z)$ in $\mathcal{T}_2(z)$ can be neglected in the residue computation, the proof can be completed by comparing the expressions (5.151) and (5.153). □

Proof of Theorem 3.16. Without loss of generality we will suppose that $\eta_* > 0$ and that $\eta = 0$. In view of Lemma 5.10, we may use the characterization [16, Lem. 5.4] to conclude that

$$\mathfrak{q}_{\mathcal{I},M}(0) = \mathfrak{Q}_{L'}(0). \tag{5.154}$$

We have already seen in §5.2 that

$$\widehat{\mathfrak{Q}}_{L'}(0) = \mathfrak{Q}_{L'}(0) \oplus \mathrm{span}_{\mathbb{C}^n}\{\widehat{G}_{L'}\}. \tag{5.155}$$

Furthermore, Lemma's 5.10 and 5.11 together imply that $\widehat{G}_{L'}$ in fact solves the differential-algebraic equation (5.93), which implies the identity

$$\widehat{\mathfrak{q}}_{\mathcal{I},M}(0) = \mathfrak{q}_{\mathcal{I},M}(0) \oplus \mathrm{span}_{\mathbb{C}^n}\{\widehat{G}_{L'}\}. \tag{5.156}$$

In particular, we conclude that $\widehat{\mathfrak{Q}}_{L'}(0) = \widehat{\mathfrak{q}}_{\mathcal{I},M}(0)$, which concludes our proof. □

5.4 Nonlinear Equations

We are now ready to study the local stable manifold of the nonlinear equation

$$\mathcal{I}\widehat{x}'(\xi) = M\,\widehat{\mathrm{ev}}_\xi\,\widehat{x} + \mathcal{M}\big(\widehat{\mathrm{ev}}_\xi\,\widehat{x}\big) \tag{5.157}$$

near the zero equilibrium. We employ the classic Lyapunov-Perron fixed point method to construct the stable manifold.

As a first step, we need to study the linear inhomogeneous equation

$$\mathcal{I}x'(\xi) = M\,\mathrm{ev}_\xi\,x + f(\xi). \tag{5.158}$$

Recalling the function $\mathcal{J}_\alpha$ and the integers $\ell_1, \ldots, \ell_n$ from (HM), we introduce the product spaces

$$\mathcal{W}^{\mathcal{J}}_\eta(\mathbb{R},\mathbb{C}^n) = W^{\ell_1,\infty}_\eta(\mathbb{R},\mathbb{C}^n) \times W^{\ell_2,\infty}_\eta(\mathbb{R},\mathbb{C}) \times \ldots \times W^{\ell_n,\infty}_\eta(\mathbb{R},\mathbb{C}), \tag{5.159}$$

where as usual we make the interpretation $W^{0,\infty}_\eta(\mathbb{R},\mathbb{C}) = L^\infty_\eta(\mathbb{R},\mathbb{C})$. On account of the structure of the differential-algebraic system, we will only need to solve (5.158) for inhomogeneities $f \in \mathcal{W}^{\mathcal{J}}_\eta(\mathbb{R},\mathbb{C}^n)$.

For any $\eta \in \mathbb{R}$, we define the linear operator

$$\Lambda_{\mathcal{I},M} : W^{1,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n) \to \mathcal{W}^{\mathcal{J}}_{\eta}(\mathbb{R}, \mathbb{C}^n) \tag{5.160}$$

that is associated to (5.158) and defined by

$$\Lambda_{\mathcal{I},M} x = \mathcal{I} x'(\xi) - M \mathrm{ev}_{\xi} x. \tag{5.161}$$

We note that it is not immediately clear that this operator indeed maps into $\mathcal{W}^{\mathcal{J}}_{\eta}(\mathbb{R}, \mathbb{C}^n)$. In order to clarify this, we recall the operator L appearing in (HM) and write

$$\Lambda_L : W^{1,\infty}(\mathbb{R}, \mathbb{C}^n) \to L^{\infty}(\mathbb{R}, \mathbb{C}^n) \tag{5.162}$$

for the operator defined by

$$\Lambda_L y = y'(\xi) - L \,\mathrm{ev}_{\xi}\, y. \tag{5.163}$$

In addition, for any $f \in \mathcal{W}^{\mathcal{J}}_{\eta}(\mathbb{R}, \mathbb{C}^n)$ we introduce the notation

$$\mathcal{J}_{\alpha}(D) = \big((D - \alpha)^{\ell_1} f_1, \dots , (D - \alpha)^{\ell_n} f_n \big) \in L^{\infty}(\mathbb{R}, \mathbb{C}^n) \tag{5.164}$$

where D is the differentiation operator. We now have the following result.

Proposition 5.12. *Consider the system (5.158) and suppose that (HM) is satisfied. Suppose furthermore that the accompanying characteristic equation*

$$\det \Delta_L(z) = \det \mathcal{J}_{\alpha}(z) \delta_{\mathcal{I},M}(z) = 0 \tag{5.165}$$

admits no roots with $\operatorname{Re} z = \eta$. *Then the operator* $\Lambda_{\mathcal{I},M}$ *is an isomorphism from* $W^{1,\infty}_{\eta}(\mathbb{R}, \mathbb{C}^n)$ *into* $\mathcal{W}^{\mathcal{J}}_{\eta}(\mathbb{R}, \mathbb{C}^n)$, *with inverse given by*

$$\Lambda^{-1}_{\mathcal{I},M} f = \Lambda^{-1}_L \mathcal{J}_{\alpha}(D) f. \tag{5.166}$$

Proof. The arguments used to establish [17, Prop. 6.2] can be copied almost verbatim. □

From now on, we will assume that the quantity α appearing in (HM) satisfies $\alpha > 0$, which implies that for all $\eta \geq 0$ we have

$$\mathfrak{q}_{\mathcal{I},M}(-\eta) = \mathfrak{Q}_L(-\eta). \tag{5.167}$$

This allows us to define a normalized solution operator for (5.158) posed on the half line $\mathbb{R}_+$. We have the following result.

Lemma 5.13. *Consider the system (5.158) and suppose that (HM) is satisfied and that* $\alpha > 0$. *Then for every* $\eta \leq 0$ *there exists a bounded linear operator*

$$\mathcal{K}_{\eta} : \mathcal{W}^{\mathcal{J}}_{\eta}\big([0, \infty), \mathbb{C}^n\big) \to W^{1,\infty}_{\eta}\big([0, \infty), \mathbb{C}^n\big) \cap BC^{\oplus}_{\eta} \tag{5.168}$$

that satisfies the following properties.

(i) For any $f \in \mathcal{W}^{\mathcal{J}}_{\eta}\big([0, \infty), \mathbb{C}^n\big)$, *the function* $x = \mathcal{K}_{\eta} f$ *satisfies (5.158) for all* $\xi \geq 0$.

(ii) For any $f \in \mathcal{W}^{\mathcal{J}}_{\eta}\big([0, \infty), \mathbb{C}^n\big)$, *we have*

$$\Pi_{Q_L(\eta)} \,\mathrm{ev}_0\, \mathcal{K}_{\eta} f = 0, \tag{5.169}$$

where $\Pi_{Q_L(\eta)}$ *projects* X *onto* $Q_L(\eta)$ *along* $P_L(\eta)$; *see (2.19).*

Proof. First, we note that we can proceed as in [17, Eqs. (6.17)-(6.18)] to define a bounded extension operator

$$\mathcal{E} : \mathcal{W}^{\mathcal{J}}_{\eta}\big([0,\infty),\mathbb{C}^n\big) \to \mathcal{W}^{\mathcal{J}}_{\eta}\big(\mathbb{R},\mathbb{C}^n\big) \tag{5.170}$$

such that $(\mathcal{E}f)(\xi) = f(\xi)$ for all $\xi \geq 0$. We then write

$$\mathcal{K}_{\eta} f = \Lambda^{-1}_{\mathcal{I},M}\mathcal{E}f - E_L \Pi_{Q_L(\eta)}\mathrm{ev}_0 \Lambda^{-1}_{\mathcal{I},M}\mathcal{E}f, \tag{5.171}$$

from which the properties (i) and (ii) follow easily, exploiting the identification (5.167). □

Let us now turn our attention to the nonlinearity $\mathcal{M}$. For any $\eta \in \mathbb{R}$, we introduce the function space

$$\widehat{Y}_{\eta} = \widehat{BC}^{\oplus}_{\eta} \cap W^{1,\infty}_{\eta}([0,\infty),\mathbb{C}^n) \tag{5.172}$$

that has the norm

$$\|\widehat{y}\|_{\widehat{Y}_{\eta}} = \|\widehat{y}\|_{\widehat{BC}^{\oplus}_{\eta}} + \|\widehat{y}\|_{W^{1,\infty}_{\eta}}. \tag{5.173}$$

In addition, for any $\delta > 0$ we introduce the open ball

$$\widehat{Y}^{\delta}_{\eta} = \{\widehat{y} \in \widehat{Y}_{\eta} \mid \|\widehat{y}\|_{\widehat{Y}_{\eta}} < \delta\}. \tag{5.174}$$

Lemma 5.14. *Suppose that (H$\mathcal{M}$) is satisfied and recall the integer $k \geq 1$ appearing in this condition. Then the nonlinearity $\mathcal{M}$ is locally Lipschitz continuous as a map from $\widehat{Y}_0$ into $\mathcal{W}^{\mathcal{J}}_0([0,\infty),\mathbb{C}^n)$. In addition, $\mathcal{M}$ is a C^k-smooth map from*

$$\widehat{Y}^{\delta}_{-\eta} \to \mathcal{W}^{\mathcal{J}}_{-\eta}([0,\infty),\mathbb{C}^n) \tag{5.175}$$

for any $\eta > 0$ and $\delta > 0$.

Proof. The statements follow directly from (H$\mathcal{M}$) and item (iv) of [11, Lem. App.IV.1.1]. □

We can now set up the fixed point problem that will yield our desired stable manifold. In particular, let us pick $\eta > 0$ in such a way that $\det \delta_{\mathcal{I},M}(z) = 0$ admits no roots with $-\eta \leq \mathrm{Re}\, z \leq 0$. For any $\widehat{\psi} \in \widehat{q}_{\mathcal{I},M}(0) = \widehat{q}_{\mathcal{I},M}(-\eta)$, we note that any solution to the fixed point problem

$$\widehat{u} = \widehat{E}_{\mathcal{I},M}\widehat{\psi} + \mathcal{K}_{-\eta}\mathcal{M}(\widehat{u}) \tag{5.176}$$

posed on the space $\widehat{Y}_{-\eta}$ yields a solution to (5.157).

Proof of Theorem 3.17. The projection operator $\widehat{\Pi}_{\widehat{q}_{\mathcal{I},M}(0)}$ can be defined as

$$\widehat{\Pi}_{\widehat{q}_{\mathcal{I},M}(0)}\widehat{\psi} = \Pi_{Q_L(0)}\big[\widehat{\psi} - \widehat{\mathrm{ev}}_0\, \widehat{G}_L(\widehat{\psi}(0+) - \widehat{\psi}(0-))\big] + \widehat{\mathrm{ev}}_0\, \widehat{G}_L\big(\widehat{\psi}(0+) - \widehat{\psi}(0-)\big). \tag{5.177}$$

The statements (i) through (iv) can be established in a standard fashion by noting that the fixed point problem (5.176) has a unique solution $\widehat{u} = \widehat{u}^*(\widehat{\psi}) \in \widehat{Y}_0$ for all sufficiently small $\widehat{\psi} \in \widehat{q}_{\mathcal{I},M}(0)$. The smoothness of the function $\widehat{u}^*$ follows from the implicit function theorem together with Lemma 5.14. □

Proof of Corollaries 3.19 - 3.29. Without loss of generality, we will assume $\eta = 0$. We will only concern ourselves with the statements concerning $\pi^-_{\widehat{q}_{\mathcal{I},M}(0)}$. First of all, let us write

$$R = \mathrm{Range}\, \pi^-_{\widehat{q}_{\mathcal{I},M}(0)} \subset C([-1,0],\mathbb{C}^n), \qquad \widehat{K} = \mathrm{Ker}\, \pi^-_{\widehat{q}_{\mathcal{I},M}(0)} \subset \widehat{q}_{\mathcal{I},M}(0) \tag{5.178}$$

and note that R is closed and has finite codimension in $C([-1,0],\mathbb{C}^n)$, while $\widehat{K}$ is finite dimensional. In particular, we can can find a finite dimensional space $R_\perp$ and a space $\widehat{K}_\perp$ such that

$$C([-1,0],\mathbb{C}^n) = R \oplus R_\perp, \qquad \widehat{q}_{\mathcal{I},M}(0) = \widehat{K} \oplus \widehat{K}_\perp, \tag{5.179}$$

together with a projection Π_R that maps $C([-1,0],\mathbb{C}^n)$ onto R along $R_\perp$ and a projection $\widehat{\Pi}_{\widehat{K}}$ that maps $\widehat{X}_0$ onto $\widehat{K}$ along $\widehat{K}_\perp$. In the special case that $R = C([-1,0],\mathbb{C}^n)$, we simply use $\Pi_R = I$. We note that

$$\pi^-_{\widehat{q}_{\mathcal{I},M}(0)} : \widehat{K}_\perp \to R \tag{5.180}$$

is an isomorphism, which allows us to define a bounded inverse

$$[\pi^-_{\widehat{q}_{\mathcal{I},M}(0)}]^{-1} : R \to \widehat{K}_\perp. \tag{5.181}$$

This in turn allows us to introduce a new solution operator $\mathcal{K}^{(2)}$ to (5.158) that has the properties

$$\Pi_R \,\mathrm{ev}_0^- \,\mathcal{K}^{(2)} = 0, \qquad \widehat{\Pi}_{\widehat{K}} \,\widehat{\mathrm{ev}}_0 \,\mathcal{K}^{(2)} = 0. \tag{5.182}$$

Indeed, we may write

$$\begin{array}{rcl} \mathcal{K}^{(1)} f & = & \mathcal{K}_0 f - \widehat{E}_{\mathcal{I},M} [\pi^-_{\widehat{q}_{\mathcal{I},M}(0)}]^{-1} \Pi_R \,\mathrm{ev}_0^- \,\mathcal{K}_0 f, \\ \mathcal{K}^{(2)} f & = & \mathcal{K}^{(1)} f - \widehat{E}_{\mathcal{I},M} \,\widehat{\Pi}_{\widehat{K}} \widehat{\mathrm{ev}}_0 \,\mathcal{K}^{(1)} f. \end{array} \tag{5.183}$$

For any sufficiently small $\phi \in R$ and $\widehat{\psi} \in \widehat{K}$, we can now study the modified fixed point problem

$$\widehat{u} = \widehat{E}_{\mathcal{I},M} [\pi^-_{\widehat{q}_{\mathcal{I},M}(0)}]^{-1} \phi + \widehat{E}_{\mathcal{I},M} \widehat{\psi} + \mathcal{K}^{(2)} \mathcal{M}(\widehat{u}), \tag{5.184}$$

which has a unique solution $\widehat{u} = \widehat{u}^*(\phi, \widehat{\psi})$. By construction, we have

$$\Pi_R \,\mathrm{ev}_0^- \,\widehat{u}^*(\phi, \widehat{\psi}) = \phi, \qquad \widehat{\Pi}_{\widehat{K}} \,\widehat{\mathrm{ev}}_0 \,\widehat{u}^*(\phi, \widehat{\psi}) = \widehat{\psi}. \tag{5.185}$$

All the statements in Corollaries 3.19 - 3.23 follow immediately from this characterization. The statements in Corollaries 3.25 - 3.29 can be established in exactly the same fashion upon studying the operator $\widehat{\pi}^-_{\widehat{q}_{\mathcal{I},M}(0)}$. □

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