

Nonlinear stability of pulse solutions for the discrete FitzHugh-Nagumo equation with infinite-range interactions

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Abstract

We establish the existence and nonlinear stability of travelling pulse solutions for the discrete FitzHugh-Nagumo equation with infinite-range interactions close to the continuum limit. For the verification of the spectral properties, we need to study a functional differential equation of mixed type (MFDE) with unbounded shifts. We avoid the use of exponential dichotomies and phase spaces, by building on a technique developed by Bates, Chen and Chmaj for the discrete Nagumo equation. This allows us to transfer several crucial Fredholm properties from the PDE setting to our discrete setting.

1 Introduction

The FitzHugh-Nagumo partial differential equation (PDE) is given by

$$\begin{aligned}u_t &= u_{xx} + g(u; r_0) - w \\w_t &= \rho(u - \gamma w),\end{aligned}\tag{1.1}$$

where $g(\cdot; r_0)$ is the cubic bistable nonlinearity given by

$$g(u; r_0) = u(1 - u)(u - r_0)\tag{1.2}$$

and ρ, γ are positive constants. This PDE is commonly used as a simplification of the Hodgkin-Huxley equations, which describe the propagation of signals through nerve fibres. The spatially homogeneous version of this equation was first stated by FitzHugh in 1961 [14] in order to describe the potential felt at a single point along a nerve axon as a signal travels by. A few years later [15], the diffusion term in (1.1) was added to describe the dynamics of the full nerve axon instead of just a single point. As early as 1968 [16], FitzHugh released a computer animation based on numerical simulations of (1.1). This videoclip clearly shows that (1.1) admits isolated pulse solutions resembling the spike signals that were measured by Hodgkin and Huxley in the nerve fibres of giant squids [19].

As a consequence of this rich behaviour and the relative simplicity of its structure, (1.1) has served as a prototype for several similar systems. For example, memory devices have been designed using a planar version of (1.1), which supports stable stationary radially symmetric spot patterns [27]. In addition, gas discharges have been described using a three-component FitzHugh-Nagumo system [33, 40], for which it is possible to find stable travelling spots [42].

Mathematically, it turned out to be a major challenge to control the interplay between the excitation and recovery dynamics and rigorously construct the travelling pulses visualized by FitzHugh in [16]. Such pulse solutions have the form

$$(u, w)(x, t) = (\bar{u}_0, \bar{w}_0)(x + c_0 t), \quad (1.3)$$

in which c_0 is the wavespeed and the wave profile (\bar{u}_0, \bar{w}_0) satisfies the limits

$$\lim_{|\xi| \rightarrow \infty} (\bar{u}_0, \bar{w}_0)(\xi) = 0. \quad (1.4)$$

Plugging this Ansatz into (1.1) and writing $\xi = x + c_0 t$, we see that the profiles are homoclinic solutions to the travelling wave ordinary differential equation (ODE)

$$\begin{aligned} c_0 \bar{u}'_0(\xi) &= \bar{u}''_0(\xi) + g(\bar{u}_0(\xi); r_0) - \bar{w}_0(\xi) \\ c_0 \bar{w}'_0(\xi) &= \rho[\bar{u}_0(\xi) - \gamma \bar{w}_0(\xi)]. \end{aligned} \quad (1.5)$$

The analysis of this equation in the singular limit $\rho \downarrow 0$ led to the birth of geometric singular perturbation theory, see for example [25] for an interesting overview. Indeed, the early works [8, 18, 24, 26] used geometric techniques such as the Conley index, exchange lemmas and differential forms to construct pulses and analyse their stability. A more analytic approach was later developed in [30], where Lin's method was used in the $r_0 \approx \frac{1}{2}$ regime to connect a branch of so-called slow-pulse solutions to (1.5) to a branch of fast-pulse solutions. This equation is still under active investigation, see for example [9, 10], where the birth of oscillating tails for the pulse solutions is described as the unstable root r_0 of the nonlinearity g moves towards the stable root at zero.

Many physical, chemical and biological systems have an inherent discrete structure that strongly influences their dynamical behaviour. In such settings *lattice differential equations* (LDEs), i.e. differential equations where the spacial variable can only take values on a lattice such as \mathbb{Z}^n , are the natural replacements for PDEs, see for example [1, 20, 32]. Although mathematically it is a relatively young field of interest, LDEs have already appeared frequently in the more applied literature. For example, they have been used to describe phase transitions in Ising models [1], crystal growth in materials [7] and phase mixing in martensitic structures [43].

To illustrate these points, let us return to the nerve axon described above and reconsider the propagation of electrical signals through nerve fibres. It is well known that electrical signals can only travel at adequate speeds if the nerve fibre is insulated by a myeline coating. This coating admits regularly spaced gaps at the so-called nodes of Ranvier [35]. Through a process called saltatory conduction, it turns out that excitations of nerves effectively jump from one node to the next [31]. Exploiting this fact, it is possible [29] to model this jumping process with the discrete FitzHugh-Nagumo LDE

$$\begin{aligned} \dot{u}_j &= \frac{1}{h^2}(u_{j+1} + u_{j-1} - 2u_j) + g(u_j; r_0) - w_j \\ \dot{w}_j &= \rho[u_j - \gamma w_j]. \end{aligned} \quad (1.6)$$

The variable u_j now represents the potential at the j^{th} node, while the variable w_j denotes a recovery component. The nonlinearity g describes the ionic interactions. Note that this equation arises directly from the FitzHugh-Nagumo PDE upon taking the nearest-neighbour discretisation of the

Laplacian on a grid with spacing $h > 0$.

Inspired by the procedure for partial differential equations, one can substitute a travelling pulse Ansatz $(u_j, w_j)(t) = (\bar{u}_h, \bar{w}_h)(hj + c_h t)$ into (1.6). Instead of an ODE, we obtain the system

$$\begin{aligned} c_h \bar{u}'_h(\xi) &= \frac{1}{h^2} [\bar{u}_h(\xi + h) + \bar{u}_h(\xi - h) - 2\bar{u}_h(\xi)] + g(\bar{u}_h(\xi); r_0) - \bar{w}_h(\xi) \\ c_h \bar{w}'_h(\xi) &= \rho[\bar{u}_h(\xi) - \gamma \bar{w}_h(\xi)] \end{aligned} \quad (1.7)$$

in which $\xi = hj + c_h t$. Such equations are called functional differential equations of mixed type (MFDEs).

In [20, 21], Hupkes and Sandstede studied (1.6) and showed that for small values of ρ and r_0 sufficiently far from $\frac{1}{2}$, there exists a locally unique travelling pulse solution of this system and that it is asymptotically stable with an asymptotic phase shift. No restrictions were required on the discretisation distance h , but the results relied heavily on the existence of exponential dichotomies for MFDEs. As a consequence, the techniques developed in [20, 21] can only be used if the discretisation involves finitely many neighbours. Such discretisation schemes are said to have finite range.

Recently, an active interest has arisen in non-local equations that feature infinite-range interactions. For example, Ising models have been used to describe the infinite-range interactions between magnetic spins arranged on a grid [1]. In addition, many physical systems, such as amorphous semi-conductors [17] and liquid crystals [12], feature non-standard diffusion processes, which are generated by fractional Laplacians. Such operators are intrinsically non-local and hence automatically require infinite-range discretisation schemes [11].

Our primary interest here, however, comes from so-called neural field models, which aim to describe the dynamics of large networks of neurons. These neurons interact with each other by exchanging signals across long distances through their interconnecting nerve axons [4–6, 34, 41]. It is of course a major challenge to find effective equations to describe such complex interactions. One model that has been proposed [4, Eq. (3.31)] features a FitzHugh-Nagumo type system with infinite-range interactions.

Motivated by the above, we consider a class of infinite-range FitzHugh-Nagumo LDEs that includes the prototype

$$\begin{aligned} \dot{u}_j &= \frac{\kappa}{h^2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2} [u_{j+k} + u_{j-k} - 2u_j] + g(u_j; r_0) - w_j \\ \dot{w}_j &= \rho[u_j - \gamma w_j], \end{aligned} \quad (1.8)$$

in which $\kappa > 0$ is a normalisation constant. In [39], Scheel and Faye studied equations such as (1.8) for discretisations with infinite-range interactions featuring exponential decay in the coupling strength. They circumvented the need to use a state space as in [21], which enabled them to construct pulses to (1.8) for arbitrary discretisation distance h . Very recently [38], they developed a center manifold approach that allows bifurcation results to be obtained for neural field equations.

In this paper, we also construct pulse solutions to equations such as (1.8), but under weaker assumptions on the decay rate of the couplings. Moreover, we will establish the nonlinear stability of these pulse solutions, provided the coupling strength decays exponentially. However, both results do require the discretisation distance h to be very small.

In particular, we will be working in the continuum limit. The pulses we construct can be seen as perturbations of the travelling pulse solution of the FitzHugh-Nagumo PDE. However, we will see

that the travelling wave equations are highly singular perturbations of (1.5), which poses a significant mathematical challenge. On the other hand, we do not need to use exponential dichotomies directly in our non-local setting as in [20]. Instead we are able to exploit the detailed knowledge that has been obtained using these techniques for the pulses in the PDE setting.

Our approach to tackle the difficulties arising from this singular perturbation is strongly inspired by the work of Bates, Chen and Chmaj. Indeed, in their excellent paper [1], they study a class of systems that includes the infinite-range discrete Nagumo equation

$$\dot{u}_j = \frac{\kappa}{h^2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2} [u_{j+k} + u_{j-k} - 2u_j] + g(u_j; r_0), \quad (1.9)$$

in which $\kappa > 0$ is a normalisation constant. This equation can be seen as a discretisation of the Nagumo PDE

$$u_t = u_{xx} + g(u; r_0). \quad (1.10)$$

The authors show that, under some natural assumptions, these systems admit travelling pulse solutions for h small enough.

In the remainder of this introduction we outline their approach and discuss our modifications, which significantly broaden the application range of these methods. We discuss these modifications for the prototype (1.8), but naturally they can be applied to a broad class of systems.

Transfer of Fredholm properties: scalar case.

An important role in [1] is reserved for the operator $\mathcal{L}_{h; \bar{u}_{0:\text{sc}}; c_{0:\text{sc}}}$ given by

$$\begin{aligned} \mathcal{L}_{h; \bar{u}_{0:\text{sc}}; c_{0:\text{sc}}} v(\xi) &= c_{0:\text{sc}} v'(\xi) - \frac{\kappa}{h^2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2} [v(\xi + hk) + v(\xi - hk) - 2v(\xi)] \\ &\quad - g_u(\bar{u}_{0:\text{sc}}(\xi); r_0) v(\xi), \end{aligned} \quad (1.11)$$

where $\bar{u}_{0:\text{sc}}$ is the wave solution of the scalar Nagumo PDE (1.10) with wavespeed $c_{0:\text{sc}}$. This operator arises as the linearisation of the scalar Nagumo MFDE

$$c_{0:\text{sc}} u'(\xi) = \frac{\kappa}{h^2} \sum_{k \in \mathbb{Z}_{>0}} e^{-k^2} [v(\xi + hk) + v(\xi - hk) - 2v(\xi)] + g_u(\bar{u}_{0:\text{sc}}(\xi); r_0) v(\xi), \quad (1.12)$$

around the wave solution $\bar{u}_{0:\text{sc}}$ of the scalar Nagumo PDE (1.10). This operator should be compared to

$$\mathcal{L}_{0; \bar{u}_{0:\text{sc}}; c_{0:\text{sc}}} v(\xi) = c_{0:\text{sc}} v'(\xi) - v''(\xi) - g_u(\bar{u}_{0:\text{sc}}(\xi); r_0) v(\xi), \quad (1.13)$$

the linearisation of the scalar Nagumo PDE around its wave solution.

The key contribution in [1] is that the authors fix a constant $\delta > 0$ and use the invertibility of $\mathcal{L}_{0; \bar{u}_{0:\text{sc}}; c_{0:\text{sc}}} + \delta$ to show that also $\mathcal{L}_{h; \bar{u}_{0:\text{sc}}; c_{0:\text{sc}}} + \delta$ is invertible. In particular, they consider weakly-converging sequences $\{v_j\}$ and $\{w_j\}$ with $(\mathcal{L}_{h; \bar{u}_{0:\text{sc}}; c_{0:\text{sc}}} + \delta)v_j = w_j$ and try to find a uniform (in δ and h) upper bound for the L^2 -norm of v_j' in terms of the L^2 -norm of w_j . Such a bound is required to rule out the limitless transfer of energy into oscillatory modes, a key complication when taking weak limits. To obtain this bound, the authors exploit the bistable structure of the nonlinearity g to control the behaviour at $\pm\infty$. This allows the local L^2 -norm of v_j on a compact set to be uniformly bounded away from zero. Since the operator $\mathcal{L}_{h; \bar{u}_{0:\text{sc}}; c_{0:\text{sc}}} + \delta$ is not self-adjoint, this procedure must be repeated for the adjoint operator.

Transfer of Fredholm properties: system case.

Plugging the travelling pulse Ansatz

$$(u, w)_j(t) = (\bar{u}_h, \bar{w}_h)(hj + c_h t) \quad (1.14)$$

into (1.8) and writing $\xi = hj + c_h t$, we see that the profiles are homoclinic solutions to the equation

$$\begin{aligned} c_h \bar{u}'_h(\xi) &= \frac{\kappa}{h^2} \sum_{k>0} e^{-k^2} \left[\bar{u}_h(\xi + kh) + \bar{u}_h(\xi - kh) - 2\bar{u}_h(\xi) \right] + g(\bar{u}_h(\xi); r_0) - \bar{w}_h(\xi) \\ c_h \bar{w}'_h(\xi) &= \rho(\bar{u}_h(\xi) - \gamma \bar{w}_h(\xi)). \end{aligned} \quad (1.15)$$

We start by considering the linearised operator $\mathcal{K}_{h;\bar{u}_0;c_0}$ of the system (1.15) around the pulse solution (\bar{u}_0, \bar{w}_0) of the FitzHugh-Nagumo PDE with wavespeed c_0 . This operator is given by

$$\mathcal{K}_{h;\bar{u}_0;c_0} \begin{pmatrix} v \\ w \end{pmatrix}(\xi) = \begin{pmatrix} \mathcal{L}_{h;\bar{u}_0;c_0} v(\xi) + w(\xi) \\ c_0 w'(\xi) - \rho v(\xi) + \rho \gamma w(\xi) \end{pmatrix}, \quad (1.16)$$

where $\mathcal{L}_{h;\bar{u}_0;c_0}$ is given by equation (1.11), but with $\bar{u}_{0;sc}$ replaced by \bar{u}_0 and $c_{0;sc}$ by c_0 . We use a Fredholm alternative as described above to show the invertibility of $\mathcal{K}_{h;\bar{u}_0;c_0} + \delta$ for fixed $\delta > 0$. However, the transition from a scalar equation to a system is far from trivial. When transferring the Fredholm properties there are multiple cross terms that need to be controlled. We are aided here by the relative simplicity of the equation for \bar{w}'_h , which guarantees that the second component of the linearisation (1.16) has constant coefficients. This will be the content of §3.

Construction of pulses.

Using these results for $\mathcal{K}_{h;\bar{u}_0;c_0}$ we develop a fixed point argument to show that the system (1.8) has a locally unique travelling pulse solution $(\bar{U}_h(t))_j = (\bar{u}_h, \bar{w}_h)(hj + c_h t)$ for h small enough, which converges to a travelling pulse solution of the FitzHugh-Nagumo PDE as $h \downarrow 0$. This is more or less straightforward and is very similar to the arguments used in [1]. The details can be found in §4.

Spectral stability.

The natural next step is to study the linear operator $\mathcal{K}_{h;\bar{u}_h;c_h}$ that arises after linearising the system (1.8) around its new-found pulse solution. This operator is given by

$$\mathcal{K}_{h;\bar{u}_h;c_h} \begin{pmatrix} v \\ w \end{pmatrix}(\xi) = \begin{pmatrix} \mathcal{L}_{h;\bar{u}_h;c_h} v(\xi) + w(\xi) \\ c_h w'(\xi) - \rho v(\xi) + \rho \gamma w(\xi) \end{pmatrix}, \quad (1.17)$$

where $\mathcal{L}_{h;\bar{u}_h}$ is given by equation (1.11), but with $\bar{u}_{0;sc}$ replaced by \bar{u}_h and $c_{0;sc}$ by c_h . The procedure above can be repeated to show that for fixed $\delta > 0$, it also holds that $\mathcal{K}_{h;\bar{u}_h;c_h} + \delta$ is invertible for h small enough. However, to understand the spectral stability of the pulse, we need to consider the eigenvalue problem

$$\mathcal{K}_{h;\bar{u}_h;c_h} v + \lambda v = 0 \quad (1.18)$$

for fixed values of h and λ ranging throughout a half-plane. Switching between these two points of view turns out to be a delicate task.

We start in §5 by showing that $\mathcal{K}_{h;\bar{u}_h;c_h}$ and its adjoint $\mathcal{K}_{h;\bar{u}_h;c_h}^*$ are Fredholm operators with one-dimensional kernels. This is achieved by explicitly constructing a kernel element for $\mathcal{K}_{h;\bar{u}_h;c_h}^*$ that converges to a kernel element of the adjoint of the operator corresponding to the linearised PDE. An abstract perturbation argument then yields the result.

In particular, we see that $\lambda = 0$ is a simple eigenvalue of $\mathcal{K}_{h;\bar{u}_h;c_h}$. In §6 we establish that in a suitable half-plane, the spectrum of this operator consists precisely of the points $\{k2\pi i c_h \frac{1}{h} : k \in \mathbb{Z}\}$, which are all simple eigenvalues. We do this by first showing that the spectrum is invariant under the operation $\lambda \mapsto \lambda + \frac{2\pi i c_h}{h}$, which allows us to restrict ourselves to values of λ with imaginary part in between $-\frac{\pi c_h}{h}$ and $\frac{\pi c_h}{h}$. Note that the period of the spectrum is dependent on h and grows to infinity as $h \downarrow 0$. This is not too surprising, since the spectrum of the linearisation of the PDE around its pulse solution is not periodic. However, this means that we cannot restrict ourselves to

a fixed compact subset of the complex plane for all values of h at the same time. In fact, it takes quite some effort to keep the part of the spectrum with large imaginary part under control.

It turns out to be convenient to partition our 'half-strip' into four parts and to calculate the spectrum in each part using different methods. Values close to 0 are analyzed using the Fredholm properties of $\mathcal{K}_{h;\bar{u}_h;c_h}$ exploiting many of the results from §5; values with a large real part are considered using standard norm estimates, but values with a large imaginary part are treated using a Fourier transform. The final set to consider is a compact set that is independent of h and bounded away from the origin. This allows us to apply a modified version of the procedure described above that exploits the absence of spectrum in this region for the FitzHugh-Nagumo PDE.

Nonlinear stability.

The final step in our program is to leverage the spectral stability results to obtain a nonlinear stability result. To do so, we follow [20] and derive a formula that links the pointwise Green's function of our general problem (1.8) to resolvents of the operator $\mathcal{K}_{h;\bar{u}_h;c_h}$ in §7. Since we have already analyzed the latter operator in detail, we readily obtain a spectral decomposition of this Green's function into an explicit neutral part and a residual that decays exponentially in time and space. Therefore we obtain detailed estimates on the decay rate of the Green's function for the general problem. These Green's functions allow us in §8 to use multiple fixed point arguments to eventually show the nonlinear stability of the family of travelling pulse solutions \bar{U}_h . To be more precise, for each initial condition close to $\bar{U}_h(0)$, we show that the solution with that initial condition converges at an exponential rate to the solution $\bar{U}_h(\cdot + \tilde{\theta})$ for a small (and unique) phase shift $\tilde{\theta}$. The main ideas behind these arguments can already be found in [2, 20]. However, a lot of the ingredients used there are not readily available, so we have to verify that certain constructions can be extended to systems with infinite range.

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2 Main results

We consider the following system of equations

$$\begin{aligned}\dot{u}_j &= \frac{1}{h^2} \sum_{k>0} \alpha_k [u_{j+k} + u_{j-k} - 2u_j] + g(u_j) - w_j \\ \dot{w}_j &= \rho[u_j - \gamma w_j],\end{aligned}\tag{2.1}$$

which we refer to as the (spatially) discrete FitzHugh-Nagumo equation with infinite-range interactions. Often, for example in [20, 21], it is assumed that only finitely many of these coefficients α_k are non-zero. However, we will impose the following much weaker conditions here.

Assumption (H α 1). The coefficients $\{\alpha_k\}_{k \in \mathbb{Z}_{>0}}$ satisfy the bound

$$\sum_{k>0} |\alpha_k| k^2 < \infty\tag{2.2}$$

as well as the identity

$$\sum_{k>0} \alpha_k k^2 = 1.\tag{2.3}$$

Finally, the inequality

$$A(z) := \sum_{k>0} \alpha_k (1 - \cos(kz)) > 0\tag{2.4}$$

holds for all $z \in (0, 2\pi)$.

We note that (2.4) is automatically satisfied if $\alpha_1 > 0$ and $\alpha_k \geq 0$ for all $k \in \mathbb{Z}_{>1}$. The conditions in (H α 1) ensure that for $\phi \in L^\infty(\mathbb{R})$ with $\phi'' \in L^2(\mathbb{R})$, we have the limit

$$\lim_{h \downarrow 0} \left\| \frac{1}{h^2} \sum_{k>0} \alpha_k [\phi(\cdot + hk) + \phi(\cdot - hk) - 2\phi(\cdot)] - \phi'' \right\|_{L^2} = 0,\tag{2.5}$$

see Lemma 3.5. In particular, we can see (2.1) as the discretisation of the FitzHugh-Nagumo PDE (1.1) on a grid with distance h . Additional remarks concerning the assumption (H α 1) can be found in [1, §1].

Throughout this paper, we impose the following standard assumptions on the remaining parameters in (2.1). The last condition on γ in (HS) ensures that the origin is the only j -independent equilibrium of (2.1).

Assumption (HS). The nonlinearity g is given by $g(u) = u(1-u)(u-r_0)$, where $0 < r_0 < 1$. In addition, we have $0 < \rho < 1$ and $0 < \gamma < 4(1-r_0)^{-2}$.

Without explicitly mentioning it, we will allow all constants in this work to depend on r_0, ρ and γ . Dependence on h will always be mentioned explicitly. We will mainly work on the Sobolev spaces

$$\begin{aligned}H^1(\mathbb{R}) &= \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f, f' \in L^2(\mathbb{R})\}, \\ H^2(\mathbb{R}) &= \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f, f', f'' \in L^2(\mathbb{R})\},\end{aligned}\tag{2.6}$$

with their standard norms

$$\begin{aligned}\|f\|_{H^1(\mathbb{R})} &= \left(\|f\|_{L^2(\mathbb{R})}^2 + \|f'\|_{L^2(\mathbb{R})}^2 \right)^{\frac{1}{2}}, \\ \|f\|_{H^2(\mathbb{R})} &= \left(\|f\|_{L^2(\mathbb{R})}^2 + \|f'\|_{L^2(\mathbb{R})}^2 + \|f''\|_{L^2(\mathbb{R})}^2 \right)^{\frac{1}{2}}.\end{aligned}\tag{2.7}$$

Our goal is to construct pulse solutions of (2.1) as small perturbations to the travelling pulse solutions of the FitzHugh-Nagumo PDE. These latter pulses satisfy the system

$$\begin{aligned} c_0 \bar{u}'_0 &= \bar{u}''_0 + g(\bar{u}_0) - \bar{w}_0 \\ c_0 \bar{w}'_0 &= \rho(\bar{u}_0 - \gamma \bar{w}_0) \end{aligned} \quad (2.8)$$

with the boundary conditions

$$\lim_{|\xi| \rightarrow \infty} (\bar{u}_0, \bar{w}_0)(\xi) = (0, 0). \quad (2.9)$$

If (\bar{u}_0, \bar{w}_0) is a solution of (2.8) with wavespeed c_0 , then the linearisation of (2.8) around this solution is characterized by the operator $\mathcal{L}_0 : H^2(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R})$, that acts as

$$\mathcal{L}_0 \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} c_0 \frac{d}{dx} - \frac{d^2}{dx^2} - g_u(\bar{u}_0) & 1 \\ -\rho & c_0 \frac{d}{dx} + \gamma \rho \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}. \quad (2.10)$$

The existence of such pulse solutions for the case when ρ is close to 0 is established in [25, §5.3]. Here, we do not require $\rho > 0$ to be small, but we simply impose the following conditions.

Assumption (HP1). There exists a solution (\bar{u}_0, \bar{w}_0) of (2.8) that satisfies the conditions (2.9) and has wavespeed $c_0 \neq 0$. Furthermore the operator \mathcal{L}_0 is Fredholm with index zero and it has a simple eigenvalue in zero.

Recall that an eigenvalue λ of a Fredholm operator L is said to be *simple* if the kernel of $L - \lambda$ is spanned by one vector v and the equation $(L - \lambda)w = v$ does not have a solution w . Note that if L has a formal adjoint L^* , this is equivalent to the condition that $\langle v, w \rangle \neq 0$ for all non-trivial $w \in \ker(L^* - \bar{\lambda})$.

We note that the conditions on \mathcal{L}_0 formulated in (HP1) were established in [26] for small $\rho > 0$. In addition, these conditions imply that \bar{u}'_0 and \bar{w}'_0 decay exponentially.

In order to find travelling pulse solutions of (2.1), we substitute the Ansatz

$$(u, w)_j(t) = (\bar{u}_h, \bar{w}_h)(hj + c_h t), \quad (2.11)$$

into (2.1) to obtain the system

$$\begin{aligned} c_h \bar{u}'_h(\xi) &= \frac{1}{h^2} \sum_{k>0} \alpha_k \left[\bar{u}_h(\xi + hk) + \bar{u}_h(\xi - hk) - 2\bar{u}_h(\xi) \right] + g(\bar{u}_h(\xi)) - \bar{w}_h(\xi) \\ c_h \bar{w}'_h(\xi) &= \rho [\bar{u}_h(\xi) - \gamma \bar{w}_h(\xi)], \end{aligned} \quad (2.12)$$

in which $\xi = hj + c_h t$. The boundary conditions are given by

$$\lim_{|\xi| \rightarrow \infty} (\bar{u}_h, \bar{w}_h)(\xi) = (0, 0). \quad (2.13)$$

The existence of such solutions is established in our first main theorem.

Theorem 2.1 (see §4). *Assume that (HP1), (HS) and (H α 1) are satisfied. There exists a positive constant h_* such that for all $h \in (0, h_*)$, the problem (2.12) with boundary conditions (2.13) admits at least one solution $(c_h, \bar{u}_h, \bar{w}_h)$, which is locally unique in $\mathbb{R} \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$ up to translation and which has the property that*

$$\lim_{h \downarrow 0} (c_h - c_0, \bar{u}_h - \bar{u}_0, \bar{w}_h - \bar{w}_0) = (0, 0, 0) \quad \text{in } \mathbb{R} \times H^1(\mathbb{R}) \times H^1(\mathbb{R}) \quad (2.14)$$

Note that this result is very similar to [39, Corollary 2.1]. However, Scheel and Faye impose an extra assumption, similar to (H α 2) below, which we do not need in our proof. This is a direct consequence of the strength of the method from [1] that we described in §1.

Building on the existence of the travelling pulse solution, the natural next step is to show that our new-found pulse is asymptotically stable. However, we now do need to impose an extra condition on the coefficients $\{\alpha_k\}_{k>0}$.

Assumption (H α 2). The coefficients $\{\alpha_k\}_{k>0}$ satisfy the bound

$$\sum_{k>0} |\alpha_k| e^{k\nu} < \infty \quad (2.15)$$

for some $\nu > 0$.

Note that the prototype equation (1.8) indeed satisfies both assumptions (H α 1) and (H α 2). An example of a system which satisfies (H α 1), but not (H α 2) is given by

$$\begin{aligned} \dot{u}_j &= \frac{\kappa}{h^2} \sum_{k>0} \frac{1}{k^4} [u_{j+k} + u_{j-k} - 2u_j] + g(u_j) - w_j \\ \dot{w}_j &= \rho[u_j - \gamma w_j], \end{aligned} \quad (2.16)$$

in which $\kappa = \frac{6}{\pi^2}$ is the normalisation constant.

Moreover we need to impose an extra condition on the operator \mathcal{L}_0 given by (2.10). This spectral stability condition is established in [13, Theorem 2] together with [44, Theorem 3.1] for the case where ρ is close to 0.

Assumption (HP2). There exists a constant $\lambda_* > 0$ such that for each $\lambda \in \mathbb{C}$ with $\text{Re } \lambda \geq -\lambda_*$ and $\lambda \neq 0$, the operator

$$\mathcal{L}_0 + \lambda : H^2(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R}) \quad (2.17)$$

is invertible.

To determine if the pulse solution described in Theorem 2.1 is nonlinearly stable, we must first linearise (2.12) around this pulse and determine the spectral stability. The linearised operator now takes the form

$$L_h \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} c_h \frac{d}{dx} - \Delta_h - g_u(\bar{u}_h) & 1 \\ -\rho & c_h \frac{d}{dx} + \gamma\rho \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}. \quad (2.18)$$

Here the operator Δ_h is given by

$$\Delta_h \phi(\xi) = \frac{1}{h^2} \sum_{k>0} \alpha_k [\phi(\xi + hk) + \phi(\xi - hk) - 2\phi(\xi)]. \quad (2.19)$$

As usual, we define the *spectrum*, $\sigma(L)$, of a bounded linear operator $L : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R})$, as

$$\sigma(L) = \{\lambda \in \mathbb{C} : L - \lambda \text{ is not invertible}\}. \quad (2.20)$$

Our second main theorem describes the spectrum of this operator L_h , or rather of $-L_h$, in a suitable half-plane.

Theorem 2.2 (see §6). *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. There exist constants $\lambda_3 > 0$ and $h_{**} > 0$ such that for all $h \in (0, h_{**})$, the spectrum of the operator $-L_h$ in the half-plane $\{z \in \mathbb{C} : \text{Re } z \geq -\lambda_3\}$ consists precisely of the points $k2\pi ic_h \frac{1}{h}$ for $k \in \mathbb{Z}$, which are all simple eigenvalues of L_h .*

We emphasize that λ_3 does not depend on h . The translational invariance of (2.12) guarantees that $\lambda = 0$ is an eigenvalue of $-L_h$. In Lemma 6.1 we show that the spectrum of the operator L_h is periodic with period $2\pi ic_h \frac{1}{h}$, which means that the eigenvalues $k2\pi ic_h \frac{1}{h}$ for $k \in \mathbb{Z}$ all have the same properties as the zero eigenvalue.

Our final result concerns the nonlinear stability of our pulse solution, which we represent with the shorthand

$$\left[\bar{U}_h(t)\right]_j = (\bar{u}_h, \bar{w}_h)(hj + c_h t). \quad (2.21)$$

The perturbations are measured in the spaces ℓ^p , which are defined by

$$\ell^p = \{V \in (\mathbb{R}^2)^{\mathbb{Z}} : \|V\|_{\ell^p} := \left[\sum_{j \in \mathbb{Z}} |V_j|^p\right]^{\frac{1}{p}} < \infty\} \quad (2.22)$$

for $1 \leq p < \infty$ and

$$\ell^\infty = \{V \in (\mathbb{R}^2)^{\mathbb{Z}} : \|V\|_{\ell^\infty} := \sup_{j \in \mathbb{Z}} |V_j| < \infty\}. \quad (2.23)$$

Theorem 2.3 (see §8). *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Fix $0 < h \leq h_{**}$ and $1 \leq p \leq \infty$. Then there exist constants $\delta > 0$, $C > 0$ and $\beta > 0$, which may depend on h but not on p , such that for all initial conditions $U^0 \in \ell^p$ with $\|U^0 - \bar{U}_h(0)\|_{\ell^p} < \delta$, there exists an asymptotic phase shift $\tilde{\theta} \in \mathbb{R}$ such that the solution $U = (u, w)$ of (2.1) with $U(0) = U^0$ satisfies the bound*

$$\|U(t) - \bar{U}_h(t + \tilde{\theta})\|_{\ell^p} \leq C e^{-\beta t} \|U^0 - \bar{U}_h(0)\|_{\ell^p} \quad (2.24)$$

for all $t > 0$.

3 The singular perturbation

The main difficulty in analysing the travelling wave MFDE (2.12) is that it is a singular perturbation of the ODE (2.8). Indeed, the second derivative in (2.8) is replaced by the linear operator $\Delta_h : H^1(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ that acts as

$$\Delta_h \phi(\xi) = \frac{1}{h^2} \sum_{k>0} \alpha_k \left(\phi(\xi + hk) + \phi(\xi - hk) - 2\phi(\xi) \right). \quad (3.1)$$

We will see in Lemma 3.5 that for all $\phi \in L^\infty(\mathbb{R})$ with $\phi'' \in L^2(\mathbb{R})$, we have that $\lim_{h \downarrow 0} \|\Delta_h \phi - \phi''\|_{L^2} = 0$. Hence the bounded operator Δ_h converges pointwise on a dense subset of $H^1(\mathbb{R})$ to an unbounded operator on that same dense subset. In particular, the norm of the operator Δ_h grows to infinity as $h \downarrow 0$.

Since there are no second derivatives involved in (2.12), we have to view it as an equation posed on the space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, while the ODE (2.8) is posed on the space $H^2(\mathbb{R}) \times H^1(\mathbb{R})$. From now on we write

$$\begin{aligned} \mathbf{H}^1 &:= H^1(\mathbb{R}) \times H^1(\mathbb{R}), \\ \mathbf{L}^2 &:= L^2(\mathbb{R}) \times L^2(\mathbb{R}). \end{aligned} \quad (3.2)$$

The main results in this section will be used in several different settings. In order to accommodate this, we introduce the following conditions.

Assumption (hFam). For each $h > 0$ there is a pair $(\tilde{u}_h, \tilde{w}_h) \in \mathbf{H}^1$ and a constant \tilde{c}_h such that $(\tilde{u}_h, \tilde{w}_h) - (\bar{u}_0, \bar{w}_0) \rightarrow 0$ in \mathbf{H}^1 and $\tilde{c}_h \rightarrow c_0$ as $h \downarrow 0$.

Assumption (hM). The set $M \subset \mathbb{C}$ is compact with $0 \notin M$. In addition, recalling the constant λ_* appearing in (HP2), we have $\operatorname{Re} z \geq -\lambda_*$ for all $z \in M$.

In §4 we choose $(\tilde{u}_h, \tilde{w}_h)$ and \tilde{c}_0 to be (\bar{u}_0, \bar{w}_0) and c_0 for all values of h . However, in §5 we let $(\tilde{u}_h, \tilde{w}_h)$ be the travelling pulse (\bar{u}_h, \bar{w}_h) from Theorem 2.1 and we let \tilde{c}_h be its wave speed c_h .

If (hFam) is satisfied, then for $\delta > 0$ and $h > 0$ we define the operators

$$\overline{\mathcal{L}}_{h,\delta}^+ = \begin{pmatrix} \tilde{c}_h \frac{d}{dx} - \Delta_h - g_u(\tilde{u}_h) + \delta & 1 \\ -\rho & \tilde{c}_h \frac{d}{dx} + \gamma\rho + \delta \end{pmatrix} \quad (3.3)$$

and

$$\overline{\mathcal{L}}_{h,\delta}^- = \begin{pmatrix} -\tilde{c}_h \frac{d}{dx} - \Delta_h - g_u(\tilde{u}_h) + \delta & -\rho \\ 1 & -\tilde{c}_h \frac{d}{dx} + \gamma\rho + \delta \end{pmatrix}. \quad (3.4)$$

These operators are bounded linear functions from \mathbf{H}^1 to \mathbf{L}^2 . We see that $\overline{\mathcal{L}}_{h,\delta}^-$ is the adjoint operator of $\overline{\mathcal{L}}_{h,\delta}^+$, in the sense that

$$\langle (\phi, \psi), \overline{\mathcal{L}}_{h,\delta}^+(\theta, \chi) \rangle = \langle \overline{\mathcal{L}}_{h,\delta}^-(\phi, \psi), (\theta, \chi) \rangle \quad (3.5)$$

holds for all $(\phi, \psi), (\theta, \chi) \in \mathbf{L}^2$. Here we have introduced the notation

$$\begin{aligned} \langle (\phi, \psi), (\theta, \chi) \rangle &= \langle \phi, \theta \rangle + \langle \psi, \chi \rangle \\ &= \int_{-\infty}^{\infty} \left(\phi(x)\theta(x) + \psi(x)\chi(x) \right) dx \end{aligned} \quad (3.6)$$

for $(\phi, \psi), (\theta, \chi) \in \mathbf{L}^2$.

Since at some point we want to consider complex-valued functions, we also work in the spaces $H_{\mathbb{C}}^2(\mathbb{R})$, $H_{\mathbb{C}}^1(\mathbb{R})$ and $L_{\mathbb{C}}^2(\mathbb{R})$, which are given by

$$\begin{aligned} H_{\mathbb{C}}^2(\mathbb{R}) &= \{f + gi \mid f, g \in H^2(\mathbb{R})\}, \\ H_{\mathbb{C}}^1(\mathbb{R}) &= \{f + gi \mid f, g \in H^1(\mathbb{R})\}, \\ L_{\mathbb{C}}^2(\mathbb{R}) &= \{f + gi \mid f, g \in L^2(\mathbb{R})\}. \end{aligned} \tag{3.7}$$

These spaces are equipped with the inner product

$$\langle \phi, \psi \rangle = \int (f_1(x) + ig_1(x))(f_2(x) - ig_2(x)) dx \tag{3.8}$$

for $\phi = f_1 + ig_1$, $\psi = f_2 + ig_2$. As before, we write

$$\begin{aligned} \mathbf{H}_{\mathbb{C}}^1 &= H_{\mathbb{C}}^1(\mathbb{R}) \times H_{\mathbb{C}}^1(\mathbb{R}) \\ \mathbf{L}_{\mathbb{C}}^2 &= L_{\mathbb{C}}^2(\mathbb{R}) \times L_{\mathbb{C}}^2(\mathbb{R}). \end{aligned} \tag{3.9}$$

Each operator L from \mathbf{H}^1 to \mathbf{L}^2 can be extended to an operator from $\mathbf{H}_{\mathbb{C}}^1$ to $\mathbf{L}_{\mathbb{C}}^2$ by writing

$$L(f + ig) = Lf + iLg. \tag{3.10}$$

It is well-known that this complexification preserves adjoints, invertibility, inverses, injectivity, surjectivity and boundedness, see for example [36]. If $\lambda \in \mathbb{C}$ then the operators $\overline{\mathcal{L}}_{h,\lambda}^{\pm}$ are defined analogously to their real counterparts, but now we view them as operators from $H_{\mathbb{C}}^1(\mathbb{R}) \times H_{\mathbb{C}}^1(\mathbb{R})$ to $L_{\mathbb{C}}^2(\mathbb{R}) \times L_{\mathbb{C}}^2(\mathbb{R})$. Whenever it is clear that we are working in the complex setting we drop the subscript \mathbb{C} from the spaces $\mathbf{H}_{\mathbb{C}}^1$ and $\mathbf{L}_{\mathbb{C}}^2$ and simply write \mathbf{H}^1 and \mathbf{L}^2 .

We also introduce the operators $\mathcal{L}_0^{\pm} : H^2(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R})$, that act as

$$\mathcal{L}_0^+ = \begin{pmatrix} c_0 \frac{d}{dx} - \frac{d^2}{dx^2} - g_u(\overline{u}_0) & 1 \\ -\rho & c_0 \frac{d}{dx} + \gamma\rho \end{pmatrix} \tag{3.11}$$

and

$$\mathcal{L}_0^- = \begin{pmatrix} -c_0 \frac{d}{dx} - \frac{d^2}{dx^2} - g_u(\overline{u}_0) & -\rho \\ 1 & -c_0 \frac{d}{dx} + \gamma\rho \end{pmatrix}. \tag{3.12}$$

These operators can be viewed as the formal $h \downarrow 0$ limits of the operators $\overline{\mathcal{L}}_{h,0}^{\pm}$. Upon introducing the notation

$$(\phi_0^+, \psi_0^+) = \frac{(\overline{u}'_0, \overline{w}'_0)}{\|(\overline{u}'_0, \overline{w}'_0)\|_{\mathbf{L}^2}}, \tag{3.13}$$

we see that $\mathcal{L}_0^+(\phi_0^+, \psi_0^+) = 0$ by differentiating (2.8).

To set the stage, we summarize several basic properties of \mathcal{L}_0^{\pm} . For completeness, we provide a proof of these facts later on in this section.

Lemma 3.1. *Assume that (HP1), (HS) and (H α 1) are satisfied. The following results hold.*

1. *We have that $(\phi_0^+, \psi_0^+) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and $\ker(\mathcal{L}_0^+) = \text{span}\{(\phi_0^+, \psi_0^+)\}$.*
2. *There exist $(\phi_0^-, \psi_0^-) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ with $\|(\phi_0^-, \psi_0^-)\|_{\mathbf{L}^2} = 1$, with $\langle (\overline{u}'_0, \overline{w}'_0), (\phi_0^-, \psi_0^-) \rangle > 0$ and $\ker(\mathcal{L}_0^-) = \text{span}\{(\phi_0^-, \psi_0^-)\}$.*
3. *For every $(\theta, \chi) \in \mathbf{L}^2$ the problem $\mathcal{L}_0^{\pm}(\phi, \psi) = (\theta, \chi)$ with $(\phi, \psi) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and $\langle (\phi, \psi), (\phi_0^{\pm}, \psi_0^{\pm}) \rangle = 0$ has a unique solution (ϕ, ψ) if and only if $\langle (\theta, \chi), (\phi_0^{\mp}, \psi_0^{\mp}) \rangle = 0$.*

4. There exists a positive constant C_1 such that

$$\|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \leq C_1 \|\mathcal{L}_0^\pm(\phi, \psi)\|_{\mathbf{L}^2} \quad (3.14)$$

for all $(\phi, \psi) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ with $\langle(\phi, \psi), (\phi_0^\pm, \psi_0^\pm)\rangle = 0$.

5. There exists a positive constant C_2 and a small constant $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ we have

$$\|(\mathcal{L}_0^\pm + \delta)^{-1}(\theta, \chi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \leq C_2 \left[\|(\theta, \chi)\|_{L^2(\mathbb{R}) \times L^2(\mathbb{R})} + \frac{1}{\delta} |\langle(\theta, \chi), (\phi_0^\mp, \psi_0^\mp)\rangle| \right] \quad (3.15)$$

for all $(\theta, \chi) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

6. If (HP2) is also satisfied, then for each $M \subset \mathbb{C}$ that satisfies (hM), there exists a constant $C_3 > 0$ such that the uniform bound

$$\|(\mathcal{L}_0^\pm + \lambda)^{-1}(\theta, \chi)\|_{H_{\mathbb{C}}^2(\mathbb{R}) \times H_{\mathbb{C}}^1(\mathbb{R})} \leq C_3 \|(\theta, \chi)\|_{L_{\mathbb{C}}^2(\mathbb{R}) \times L_{\mathbb{C}}^2(\mathbb{R})} \quad (3.16)$$

holds for all $(\theta, \chi) \in L_{\mathbb{C}}^2(\mathbb{R}) \times L_{\mathbb{C}}^2(\mathbb{R})$ and all $\lambda \in M$.

The main goal of this section is to prove the following two propositions, which transfer parts (5) and (6) of Lemma 3.1 to the discrete setting.

Proposition 3.2. *Assume that (hFam), (HP1), (HS) and (H α 1) are satisfied. There exists a positive constant C'_0 and a positive function $h'_0(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, depending only on the choice of $(\tilde{u}_h, \tilde{w}_h)$ and \tilde{c}_h , such that for every $0 < \delta < \delta_0$ and every $h \in (0, h'_0(\delta))$, the operators $\overline{\mathcal{L}}_{h,\delta}^\pm$ are homeomorphisms from \mathbf{H}^1 to \mathbf{L}^2 that satisfy the bounds*

$$\|(\overline{\mathcal{L}}_{h,\delta}^\pm)^{-1}(\theta, \chi)\|_{\mathbf{H}^1} \leq C'_0 \left[\|(\theta, \chi)\|_{\mathbf{L}^2} + \frac{1}{\delta} |\langle(\theta, \chi), (\phi_0^\mp, \psi_0^\mp)\rangle| \right] \quad (3.17)$$

for all $(\theta, \chi) \in \mathbf{L}^2$.

Proposition 3.3. *Assume that (hFam), (HP1), (HP2), (HS) and (H α 1) are satisfied. Let $M \subset \mathbb{C}$ satisfy (hM). Then there exists a constant $h_M > 0$, depending only on M and the choice of $(\tilde{u}_h, \tilde{w}_h)$ and \tilde{c}_h , such that for all $0 < h \leq h_M$ and all $\lambda \in M$ the operator $\overline{\mathcal{L}}_{h,\lambda}^\pm$ is a homeomorphism from \mathbf{H}^1 to \mathbf{L}^2 .*

Our techniques here are inspired strongly by the approach developed in [1, §2-4]. Indeed, Proposition 3.2 and Proposition 3.4 are the equivalents of [1, Theorem 4] and [1, Lemma 6] respectively. The difference between our results and those in [1] is that Bates, Chen and Chmaj study the discrete Nagumo equation, which can be seen as the one-dimensional fast component of the FitzHugh-Nagumo equation by setting $\rho = 0$ in (2.1). In addition, the results in [1] are restricted to $\lambda \in \mathbb{R}$, while we allow $\lambda \in \mathbb{C}$ in Proposition 3.3.

Recall the constant $\delta_0 > 0$ appearing in Lemma 3.1. For $0 < \delta < \delta_0$ and $h > 0$ we define the quantities

$$\overline{\Lambda}^\pm(h, \delta) = \inf_{\|(\phi, \psi)\|_{\mathbf{H}^1} = 1} \left[\|\overline{\mathcal{L}}_{h,\delta}^\pm(\phi, \psi)\|_{\mathbf{L}^2} + \frac{1}{\delta} \left| \langle \overline{\mathcal{L}}_{h,\delta}^\pm(\phi, \psi), (\phi_0^\mp, \psi_0^\mp) \rangle \right| \right], \quad (3.18)$$

together with

$$\overline{\Lambda}^\pm(\delta) = \liminf_{h \downarrow 0} \overline{\Lambda}^\pm(h, \delta). \quad (3.19)$$

Similarly for $M \subset \mathbb{C}$ that satisfies (hM) and $h > 0$ we define

$$\overline{\Lambda}^\pm(h, M) = \inf_{\|(\phi, \psi)\|_{\mathbf{H}^1} = 1, \lambda \in M} \left[\|\overline{\mathcal{L}}_{h,\lambda}^\pm(\phi, \psi)\|_{\mathbf{L}^2} \right], \quad (3.20)$$

together with

$$\bar{\Lambda}^\pm(M) = \liminf_{h \downarrow 0} \bar{\Lambda}^\pm(h, M). \quad (3.21)$$

The key ingredients that we need to establish Propositions 3.2 and 3.3 are lower bounds on the quantities $\bar{\Lambda}^\pm(\delta)$ and $\bar{\Lambda}^\pm(M)$. These are provided in the result below, which we consider to be the technical heart of this section.

Proposition 3.4. *Assume that (hFam), (HP1), (HS) and (H α 1) are satisfied. There exists a positive constant C'_0 , depending only on our choice of $(\tilde{u}_h, \tilde{w}_h)$ and \tilde{c}_h , such that $\bar{\Lambda}^\pm(\delta) > \frac{2}{C'_0}$ for all $0 < \delta < \delta_0$. Similarly if $M \subset \mathbb{C}$ satisfies (hM), then there exists a positive constant C'_M , depending only on M and our choice of $(\tilde{u}_h, \tilde{w}_h)$ and \tilde{c}_h , such that $\bar{\Lambda}^\pm(M) > \frac{2}{C'_M}$.*

Proof of Lemma 3.1. Parts (1), (2) and (3) follow directly from (HP1). Indeed, this condition states that \mathcal{L}_0^\pm is a Fredholm operator with Fredholm index 0 and that the identities

$$\begin{aligned} \ker \mathcal{L}_0^\pm &= \text{span}\{(\phi_0^\pm, \psi_0^\pm)\}, \\ \text{Range } \mathcal{L}_0^\pm &= \text{span}\{(\phi_0^\mp, \psi_0^\mp)\}_c \\ &:= \{(\phi, \psi) \in \mathbf{L}^2 : \langle (\phi, \psi), (\phi_0^\mp, \psi_0^\mp) \rangle = 0\} \end{aligned} \quad (3.22)$$

hold. In particular, we see that

$$\mathcal{L}_0^\pm : \left(\text{span}\{(\phi_0^\pm, \psi_0^\pm)\}_c \right) \cap \left(H^2(\mathbb{R}) \times H^1(\mathbb{R}) \right) \rightarrow \text{span}\{(\phi_0^\mp, \psi_0^\mp)\}_c \quad (3.23)$$

is a continuous bijection. By the Banach isomorphism theorem its inverse is then also continuous, which implies (4).

The proof of part (5) follows the lines of [1, Lemma 5]. We pick $0 < \delta < \delta_0$, where δ_0 will be defined later. Let $(\phi, \psi) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ be arbitrary and set $(\theta, \chi) = \mathcal{L}_0^\pm(\phi, \psi) + \delta(\phi, \psi)$. We decompose

$$(\phi, \psi) = \langle (\phi, \psi), (\phi_0^\pm, \psi_0^\pm) \rangle (\phi_0^\pm, \psi_0^\pm) + (\phi^\perp, \psi^\perp). \quad (3.24)$$

Then $\langle (\phi^\perp, \psi^\perp), (\phi_0^\pm, \psi_0^\pm) \rangle = 0$ and $\mathcal{L}_0^\pm(\phi^\perp, \psi^\perp) = (\theta, \chi) - \delta(\phi, \psi)$, so with part (4) we obtain that

$$\|(\phi^\perp, \psi^\perp)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \leq C_1 [\|(\theta, \chi)\|_{\mathbf{L}^2} + \delta \|(\phi, \psi)\|_{\mathbf{L}^2}]. \quad (3.25)$$

From part (2) we also obtain that $\langle (\theta, \chi) - \delta(\phi, \psi), (\phi_0^\mp, \psi_0^\mp) \rangle = 0$. Hence we must have that

$$\begin{aligned} \langle (\theta, \chi), (\phi_0^\mp, \psi_0^\mp) \rangle &= \delta \langle (\phi, \psi), (\phi_0^\mp, \psi_0^\mp) \rangle \\ &= \delta \langle (\phi, \psi), (\phi_0^\pm, \psi_0^\pm) \rangle \langle (\phi_0^\pm, \psi_0^\pm), (\phi_0^\mp, \psi_0^\mp) \rangle + \delta \langle (\phi^\perp, \psi^\perp), (\phi_0^\mp, \psi_0^\mp) \rangle. \end{aligned} \quad (3.26)$$

Denote $\sigma = \langle (\phi_0^\pm, \psi_0^\pm), (\phi_0^\mp, \psi_0^\mp) \rangle$. Then by (2) we know that $0 < \sigma \leq 1$. Thus we obtain

$$\sigma |\langle (\phi, \psi), (\phi_0^\pm, \psi_0^\pm) \rangle| \leq \|(\phi^\perp, \psi^\perp)\|_{\mathbf{L}^2} + \frac{1}{\sigma} |\langle (\theta, \chi), (\phi_0^\mp, \psi_0^\mp) \rangle|. \quad (3.27)$$

Adding equation (3.25) twice gives

$$\begin{aligned} \sigma |\langle (\phi, \psi), (\phi_0^\pm, \psi_0^\pm) \rangle| + \|(\phi^\perp, \psi^\perp)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} &\leq \frac{1}{\sigma} |\langle (\theta, \chi), (\phi_0^\mp, \psi_0^\mp) \rangle| \\ &\quad + 2C_1 [\|(\theta, \chi)\|_{\mathbf{L}^2} + \delta \|(\phi, \psi)\|_{\mathbf{L}^2}] \\ &\leq \frac{1}{\sigma} |\langle (\theta, \chi), (\phi_0^\mp, \psi_0^\mp) \rangle| \\ &\quad + 2C_1 [\|(\theta, \chi)\|_{\mathbf{L}^2} + \delta \|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})}]. \end{aligned} \quad (3.28)$$

Since $\sigma \leq 1$ and

$$\|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \leq \|(\phi^\perp, \psi^\perp)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} + |\langle(\phi, \psi), (\phi_0^\pm, \psi_0^\pm)\rangle| \quad (3.29)$$

this gives

$$\begin{aligned} (\sigma - 2\delta C_1)\|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} &\leq \sigma|\langle(\phi, \psi), (\phi_0^\pm, \psi_0^\pm)\rangle| + \|(\phi^\perp, \psi^\perp)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \\ &\quad - 2\delta C_1\|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \\ &\leq \frac{1}{\delta}|\langle(\theta, \chi), (\phi_0^\mp, \psi_0^\mp)\rangle| + 2C_1\|(\theta, \chi)\|_{\mathbf{L}^2}. \end{aligned} \quad (3.30)$$

Hence if we take $\delta_0 = \frac{\sigma}{4C_1}$, we obtain the inequality

$$\|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} \leq \frac{4C_1}{\sigma} \left[\|(\theta, \chi)\|_{\mathbf{L}^2} + \frac{1}{\delta} |\langle(\theta, \chi), (\phi_0^\mp, \psi_0^\mp)\rangle| \right] \quad (3.31)$$

for all $\delta \in (0, \delta_0)$, as desired.

Finally we prove part (6). Let $M \subset \mathbb{C}$ satisfy (hM). The invertibility of $\mathcal{L}_0^\pm + \lambda$ for $\lambda \in M$ follows directly from (HP2). Since M is compact and the map

$$\lambda \mapsto \|(\mathcal{L}_0^\pm + \lambda)^{-1}\| \quad (3.32)$$

is continuous, we immediately see that there exists $C_3 > 0$ such that

$$\|(\mathcal{L}_0^\pm + \lambda)^{-1}(\theta, \chi)\|_{H_C^2(\mathbb{R}) \times H_C^1(\mathbb{R})} \leq C_3\|(\theta, \chi)\|_{L_C^2(\mathbb{R}) \times L_C^2(\mathbb{R})} \quad (3.33)$$

for all $(\theta, \chi) \in L_C^2(\mathbb{R}) \times L_C^2(\mathbb{R})$ and all $\lambda \in M$. ■

We now establish some basic facts concerning the operator Δ_h . In particular, we extend the real-valued results from [1] to complex-valued functions. We emphasize that the inequalities in Lemma 3.6 in general do not hold for the imaginary parts of these inner products.

Lemma 3.5. *(see [1, Lemma 3]) Assume that (H α 1) is satisfied. The following three properties hold.*

1. For all $\phi \in L^\infty(\mathbb{R})$ with $\phi'' \in L^2(\mathbb{R})$ we have $\lim_{h \downarrow 0} \|\Delta_h \phi - \phi''\|_{L^2} = 0$.
2. For all $\phi \in H^1(\mathbb{R})$ and $h > 0$ we have $\langle \Delta_h \phi, \phi' \rangle = 0$.
3. For all $\phi, \psi \in L^2(\mathbb{R})$ and $h > 0$ we have $\langle \Delta_h \phi, \psi \rangle = \langle \phi, \Delta_h \psi \rangle$ and $\langle \Delta_h \phi, \phi \rangle \leq 0$.

Lemma 3.6. *Assume that (H α 1) is satisfied and pick $f \in H_C^1(\mathbb{R})$. Then the following properties hold.*

1. For all $h > 0$ we have $\operatorname{Re} \langle -\Delta_h f, f \rangle \geq 0$.
2. For all $h > 0$ we have $\operatorname{Re} \langle \Delta_h f, f' \rangle = 0$.
3. We have $\operatorname{Re} \langle f, f' \rangle = 0$.
4. For all $\lambda \in \mathbb{C}$ we have $\operatorname{Re} \langle \lambda f, f' \rangle = 2 (\operatorname{Im} \lambda) \langle \operatorname{Re} f, \operatorname{Im} f' \rangle$.

Proof. Write $f = \phi + i\psi$ with $\phi, \psi \in H^1(\mathbb{R})$. Lemma 3.5 implies that

$$\begin{aligned} \operatorname{Re} \langle -\Delta_h f, f \rangle &= \operatorname{Re} \int \left(-\Delta_h \phi - i\Delta_h \psi \right)(x) \left(\phi - i\psi \right)(x) dx \\ &= \int (-\Delta_h \phi)(x) \phi(x) + (-\Delta_h \psi)(x) \psi(x) dx \\ &= \langle -\Delta_h \phi, \phi \rangle + \langle -\Delta_h \psi, \psi \rangle \\ &\geq 0. \end{aligned} \quad (3.34)$$

Similarly we have

$$\begin{aligned}\operatorname{Re} \langle \Delta_h f, f' \rangle &= \langle -\Delta_h \phi, \phi' \rangle + \langle -\Delta_h \psi, \psi' \rangle \\ &= 0.\end{aligned}\tag{3.35}$$

For $\lambda \in \mathbb{C}$ we may compute

$$\begin{aligned}\operatorname{Re} \langle \lambda f, f' \rangle &= \operatorname{Re} \int \left(\lambda \phi(x) + \lambda i \psi(x) \right) \left(\phi'(x) - i \psi'(x) \right) dx \\ &= (\operatorname{Re} \lambda) \langle \phi, \phi' \rangle + (\operatorname{Im} \lambda) \langle \phi, \psi' \rangle - (\operatorname{Im} \lambda) \langle \psi, \phi' \rangle + (\operatorname{Re} \lambda) \langle \psi, \psi' \rangle \\ &= 0 + 2 (\operatorname{Im} \lambda) \langle \phi, \psi' \rangle + 0 \\ &= 2 (\operatorname{Im} \lambda) \langle \phi, \psi' \rangle.\end{aligned}\tag{3.36}$$

Taking $\lambda = 1$ gives the third property. ■

We now set out to prove Proposition 3.4. In Lemmas 3.7 and 3.8 we construct weakly converging sequences that realize the infima in (3.18)-(3.21). In Lemmas 3.9-3.11 we exploit the structure of our operators (3.3) and (3.4) to recover bounds on the derivatives of these sequences that are typically lost when taking weak limits. Recall the constant $C_2 > 0$ defined in Lemma 3.1, which does not depend on $\delta > 0$.

Lemma 3.7. *Assume that (hFam), (HP1), (HS) and (H α 1) are satisfied. Consider the setting of Proposition 3.4 and fix $0 < \delta < \delta_0$. Then there exists a sequence $\{(h_j, \phi_j, \psi_j)\}_{j \geq 0}$ in $(0, 1) \times \mathbf{H}^1$ with the following properties.*

1. We have $\lim_{j \rightarrow \infty} h_j = 0$ and $\|(\phi_j, \psi_j)\|_{\mathbf{H}^1} = 1$ for all $j \geq 0$.
2. The sequence $(\theta_j, \chi_j) = \overline{\mathcal{L}}_{h_j, \delta}^+(\phi_j, \psi_j)$ satisfies

$$\lim_{j \rightarrow \infty} \left[\|(\theta_j, \chi_j)\|_{\mathbf{L}^2} + \frac{1}{\delta} |\langle (\theta_j, \chi_j), (\phi_0^-, \psi_0^-) \rangle| \right] = \overline{\Lambda}^+(\delta).\tag{3.37}$$

3. There exist $(\phi, \psi) \in \mathbf{H}^1$ and $(\theta, \chi) \in \mathbf{L}^2$ such that $(\phi_j, \psi_j) \rightharpoonup (\phi, \psi)$ weakly in \mathbf{H}^1 and such that $(\theta_j, \chi_j) \rightharpoonup (\theta, \chi)$ weakly in \mathbf{L}^2 as $j \rightarrow \infty$.
4. We have $(\phi_j, \psi_j) \rightarrow (\phi, \psi)$ in $L_{\text{loc}}^2(\mathbb{R}) \times L_{\text{loc}}^2(\mathbb{R})$ as $j \rightarrow \infty$.
5. The pair (ϕ, ψ) is a weak solution to $(\overline{\mathcal{L}}_0^+ + \delta)(\phi, \psi) = (\theta, \chi)$.
6. We have the bound

$$\begin{aligned}\|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} &\leq C_2 \left[\|(\theta, \chi)\|_{\mathbf{L}^2} + \frac{1}{\delta} |\langle (\theta, \chi), (\phi_0^-, \psi_0^-) \rangle| \right] \\ &\leq C_2 \overline{\Lambda}^+(\delta).\end{aligned}\tag{3.38}$$

The same statements hold upon replacing $\overline{\mathcal{L}}_{h, \delta}^+$, $\overline{\Lambda}^+$ and $\overline{\mathcal{L}}_0^+$ by $\overline{\mathcal{L}}_{h, \delta}^-$, $\overline{\Lambda}^-$ and $\overline{\mathcal{L}}_0^-$.

Proof. Let $0 < \delta < \delta_0$ be fixed. By definition of $\overline{\Lambda}^+(\delta)$ there exists a sequence $\{(h_j, \phi_j, \psi_j)\}$ in $(0, 1) \times \mathbf{H}^1$ such that (1) and (2) hold. Taking a subsequence if necessary, we may assume that there exist $(\phi, \psi) \in \mathbf{H}^1$ and $(\theta, \chi) \in \mathbf{L}^2$ such that $(\phi_j, \psi_j) \rightarrow (\phi, \psi)$ in $L_{\text{loc}}^2(\mathbb{R}) \times L_{\text{loc}}^2(\mathbb{R})$ and weakly in \mathbf{H}^1 as $j \rightarrow \infty$ and such that $(\theta_j, \chi_j) \rightharpoonup (\theta, \chi)$ weakly in \mathbf{L}^2 . By the weak lower-semicontinuity of the \mathbf{L}^2 -norm we obtain

$$\|(\theta, \chi)\|_{\mathbf{L}^2} + \frac{1}{\delta} |\langle (\theta, \chi), (\phi_0^-, \psi_0^-) \rangle| \leq \overline{\Lambda}^+(\delta).\tag{3.39}$$

For any pair of test functions $(\zeta_1, \zeta_2) \in C_c^\infty(\mathbb{R}) \times C_c^\infty(\mathbb{R})$ we have

$$\begin{aligned} \langle (\theta_j, \chi_j), (\zeta_1, \zeta_2) \rangle &= \langle \overline{\mathcal{L}}_{h_j, \delta}^+(\phi_j, \psi_j), (\zeta_1, \zeta_2) \rangle \\ &= \langle (\phi_j, \psi_j), \overline{\mathcal{L}}_{h_j, \delta}^-(\zeta_1, \zeta_2) \rangle. \end{aligned} \quad (3.40)$$

Since \bar{u}_0 is a bounded function, the limit $\tilde{u}_h - \bar{u}_0 \rightarrow 0$ in H^1 implies that also $\tilde{u}_h \rightarrow \bar{u}_0$ in L^∞ . In particular, we can choose $h' > 0$ and $M > 0$ in such a way that $|\tilde{u}_h| \leq M$ and $|\bar{u}_0| \leq M$ for all $0 < h \leq h'$. Since g_u is Lipschitz continuous on $[-M, M]$, there is a constant $K > 0$ such that $|g_u(x) - g_u(y)| \leq K|x - y|$ for all $x, y \in [-M, M]$. We obtain

$$\begin{aligned} \lim_{h \downarrow 0} \|g_u(\tilde{u}_h) - g_u(\bar{u}_0)\|_{L^2}^2 &= \lim_{h \downarrow 0} \int (g_u(\tilde{u}_h) - g_u(\bar{u}_0))^2 dx \\ &\leq \lim_{h \downarrow 0} \int K^2 (\tilde{u}_h - \bar{u}_0)^2 dx \\ &\leq \lim_{h \downarrow 0} K^2 \|\tilde{u}_h - \bar{u}_0\|_{L^2}^2 \\ &= 0, \end{aligned} \quad (3.41)$$

together with

$$\begin{aligned} \lim_{h \downarrow 0} \|g_u(\tilde{u}_h)\zeta_1 - g_u(\bar{u}_0)\zeta_1\|_{L^2} &\leq \lim_{h \downarrow 0} \|\zeta_1\|_\infty \|g_u(\tilde{u}_h) - g_u(\bar{u}_0)\|_{L^2} \\ &= 0. \end{aligned} \quad (3.42)$$

Furthermore we know that $\tilde{c}_h \rightarrow c_0$ as $h \downarrow 0$, which gives

$$\begin{aligned} \lim_{h \downarrow 0} \|\tilde{c}_h \zeta_1' - c_0 \zeta_1'\|_{L^2} &= \lim_{h \downarrow 0} \|\tilde{c}_h \zeta_2' - c_0 \zeta_2'\|_{L^2} \\ &= 0. \end{aligned} \quad (3.43)$$

Finally, Lemma 3.5 implies

$$\lim_{h \downarrow 0} \|\Delta_h \zeta_1 - \zeta_1''\|_{L^2} = 0, \quad (3.44)$$

which means that

$$\|\overline{\mathcal{L}}_{h_j, \delta}^-(\zeta_1, \zeta_2) - (\overline{\mathcal{L}}_0^- + \delta)(\zeta_1, \zeta_2)\|_{L^2} \rightarrow 0 \quad (3.45)$$

as $j \rightarrow \infty$. Sending $j \rightarrow \infty$ in (3.40), this yields

$$\langle (\theta, \chi), (\zeta_1, \zeta_2) \rangle = \langle (\phi, \psi), (\overline{\mathcal{L}}_0^- + \delta)(\zeta_1, \zeta_2) \rangle. \quad (3.46)$$

In particular, we see that (ϕ, ψ) is a weak solution to $(\overline{\mathcal{L}}_0^- + \delta)(\phi, \psi) = (\theta, \chi)$. Since $\phi \in H^1$, $\psi \in L^2$ and

$$\phi'' = c_0 \phi' - g_u(\bar{u}_0)\phi + \delta\phi + \psi, \quad (3.47)$$

we get $\phi'' \in L^2$ and hence $\phi \in H^2$. Since we already know that $\psi \in H^1$, we may apply Lemma 3.1 and (3.39) to obtain

$$\begin{aligned} \|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} &\leq C_2 [\|(\theta, \chi)\|_{L^2} + \frac{1}{\delta} |\langle (\theta, \chi), (\phi_0^-, \psi_0^-) \rangle|] \\ &\leq C_2 \overline{\Lambda}^+(\delta). \end{aligned} \quad (3.48)$$

■

Recall the constant $C_3 > 0$ from Lemma 3.1, which only depends on the choice of the set $M \subset \mathbb{C}$ that satisfies (hM).

Lemma 3.8. *Assume that (HP1), (HP2), (HS) and (H α 1) are satisfied. Let $M \subset \mathbb{C}$ satisfy (hM). There exists a sequence $\{(\lambda_j, h_j, \phi_j, \psi_j)\}$ in $M \times (0, 1) \times \mathbf{H}^1$ with the following properties.*

1. We have $\lim_{j \rightarrow \infty} h_j = 0$, $\lim_{j \rightarrow \infty} \lambda_j = \lambda$ for some $\lambda \in M$ and $\|(\phi_j, \psi_j)\|_{\mathbf{H}^1} = 1$ for all j .

2. The pair $(\theta_j, \chi_j) = \overline{\mathcal{L}}_{h_j, \lambda}^+(\phi_j, \psi_j)$ satisfies

$$\lim_{j \rightarrow \infty} \|(\theta_j, \chi_j)\|_{\mathbf{L}^2} = \overline{\Lambda}^+(M). \quad (3.49)$$

3. There exist $(\phi, \psi) \in \mathbf{H}^1$ and $(\theta, \chi) \in \mathbf{L}^2$ such that as $j \rightarrow \infty$ $(\phi_j, \psi_j) \rightarrow (\phi, \psi)$ in $L_{\text{loc}}^2(\mathbb{R}) \times L_{\text{loc}}^2(\mathbb{R})$ and weakly in \mathbf{H}^1 and such that $(\theta_j, \chi_j) \rightharpoonup (\theta, \chi)$ weakly in \mathbf{L}^2 .

4. The pair (ϕ, ψ) is a weak solution to $(\overline{\mathcal{L}}_0^+ + \lambda)(\phi, \psi) = (\theta, \chi)$.

5. We have the bound

$$\begin{aligned} \|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})} &\leq C_3 \|(\theta, \chi)\|_{\mathbf{L}^2} \\ &\leq C_3 \overline{\Lambda}^+(M), \end{aligned} \quad (3.50)$$

where the constant C_3 is defined in Lemma 3.1.

The same statements hold upon replacing $\overline{\mathcal{L}}_{h, \delta}^+$, $\overline{\Lambda}^+(M)$ and $\overline{\mathcal{L}}_0^+$ by $\overline{\mathcal{L}}_{h, \delta}^-$, $\overline{\Lambda}^-$ and $\overline{\mathcal{L}}_0^-$.

The proof of this result is almost identical to the one of Lemma 3.7 and will be omitted. In our arguments below, we often consider the sequences $\{(h_j, \phi_j, \psi_j)\}$ and $\{(\lambda_j, h_j, \phi_j, \psi_j)\}$ defined in Lemmas 3.7 and 3.8 in a similar fashion. To streamline our notation, we simply write $\{(\lambda_j, h_j, \phi_j, \psi_j)\}$ for all these sequences, with the understanding that $\lambda_j = \delta$ when referring to Lemma 3.7.

As argued in the proof of Lemma 3.7, it is possible to choose $\bar{h} > 0$ in such a way that

$$\begin{aligned} c_* &:= \inf_{0 \leq h \leq \bar{h}} |\tilde{c}_h| &> 0, \\ g_* &:= \sup_{0 \leq h \leq \bar{h}} \|g_u(\tilde{u}_h)\|_{\infty} &< \infty. \end{aligned} \quad (3.51)$$

By taking a subsequence if necessary, we assume from now on that $h_j < \bar{h}$ for all j . It remains to find a positive lower bound for $\|(\phi, \psi)\|_{\mathbf{L}^2}$.

Lemma 3.9. *Assume that (hFam), (HP1), (HS) and (H α 1) are satisfied. Consider the setting of Proposition 3.4 and Lemma 3.7 or Lemma 3.8. Then there exists a constant $B > 0$, such that for all j we have the bound*

$$B \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \geq c_*^2 \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - 4 \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2. \quad (3.52)$$

Proof. We first consider the sequence for $\overline{\Lambda}^+$. Using $\overline{\mathcal{L}}_{h_j, \lambda_j}^+(\phi_j, \psi_j) = (\theta_j, \chi_j)$ and $\text{Re} \langle \Delta_{h_j} \phi_j, \phi'_j \rangle = 0 = \text{Re} \langle \phi_j, \phi'_j \rangle = \text{Re} \langle \psi_j, \psi'_j \rangle$, which follow from Lemma 3.6, we obtain

$$\begin{aligned} \text{Re} \langle (\theta_j, \chi_j), (\phi'_j, \psi'_j) \rangle &= \text{Re} \langle \overline{\mathcal{L}}_{h_j, \lambda}^+(\phi_j, \psi_j), (\phi'_j, \psi'_j) \rangle \\ &= \text{Re} \langle \tilde{c}_{h_j} \phi'_j - \Delta_{h_j} \phi_j - g_u(\tilde{u}_{h_j}) \phi_j + \lambda_j \phi_j + \psi_j, \phi'_j \rangle \\ &\quad + \text{Re} \langle -\rho \phi_j + \tilde{c}_{h_j} \psi'_j + \gamma \rho \psi_j + \lambda_j \psi_j, \psi'_j \rangle \\ &= \tilde{c}_{h_j} \|\phi'_j\|_{\mathbf{L}^2}^2 - \text{Re} \langle g_u(\tilde{u}_{h_j}) \phi_j, \phi'_j \rangle + \text{Re} \langle \psi_j, \phi'_j \rangle + \text{Re} \langle \lambda_j \phi_j, \phi'_j \rangle \\ &\quad - \rho \text{Re} \langle \phi_j, \psi'_j \rangle + \tilde{c}_{h_j} \|\psi'_j\|_{\mathbf{L}^2}^2 + \text{Re} \langle \lambda_j \psi_j, \psi'_j \rangle \\ &= \tilde{c}_{h_j} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - \text{Re} \langle g_u(\tilde{u}_{h_j}) \phi_j, \phi'_j \rangle + (1 + \rho) \langle \psi_j, \phi'_j \rangle \\ &\quad + \text{Re} \langle \lambda_j (\phi_j, \psi_j), (\phi'_j, \psi'_j) \rangle. \end{aligned} \quad (3.53)$$

We write $\lambda_{\max} = \delta_0$ in the setting of Lemma 3.7 or $\lambda_{\max} = \max\{|z| : z \in M\}$ in the setting of Lemma 3.8. We write

$$G = \lambda_{\max} \|(\phi_j, \psi_j)\|_{\mathbf{L}^2} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} + g_* \|\phi_j\|_{L^2} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}. \quad (3.54)$$

Using the Cauchy-Schwartz inequality we now obtain

$$\begin{aligned} G &\geq \lambda_{\max} \|(\phi_j, \psi_j)\|_{\mathbf{L}^2} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} + \|g_u(\tilde{u}_{h_j})\|_{L^\infty} \|\phi_j\|_{L^2} \|\phi'_j\|_{L^2} \\ &\geq \text{sign}(\tilde{c}_{h_j}) \left(-\text{Re} \langle \lambda_j(\phi_j, \psi_j), (\phi'_j, \psi'_j) \rangle + \text{Re} \langle g_u(\tilde{u}_{h_j})\phi_j, \phi'_j \rangle \right) \\ &= \text{sign}(\tilde{c}_{h_j}) \left(\tilde{c}_{h_j} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 + (1 + \rho) \text{Re} \langle \psi_j, \phi'_j \rangle - \text{Re} \langle (\theta_j, \chi_j), (\phi'_j, \psi'_j) \rangle \right) \\ &\geq |\tilde{c}_{h_j}| \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - (1 + \rho) \|\psi_j\|_{L^2} \|\phi'_j\|_{L^2} - \|(\theta_j, \chi_j)\|_{\mathbf{L}^2} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} \\ &\geq c_* \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - (1 + \rho) \|\psi_j\|_{L^2} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} - \|(\theta_j, \chi_j)\|_{\mathbf{L}^2} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}. \end{aligned} \quad (3.55)$$

This implies

$$c_* \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} \leq g_* \|\phi_j\|_{L^2} + (1 + \rho) \|\psi_j\|_{L^2} + \|(\theta_j, \chi_j)\|_{\mathbf{L}^2} + \lambda_{\max} \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}. \quad (3.56)$$

Squaring this equation and using the standard inequality $2\mu\omega \leq \mu^2 + \omega^2$, this implies that

$$\begin{aligned} c_*^2 \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 &\leq 4g_*^2 \|\phi_j\|_{L^2}^2 + 4(1 + \rho)^2 \|\psi_j\|_{L^2}^2 \\ &\quad + 4\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 + 4\lambda_{\max}^2 \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2. \end{aligned} \quad (3.57)$$

In particular, we see

$$4 \left(\max\{g_*^2, (1 + \rho)^2\} + \lambda_{\max}^2 \right) \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \geq c_*^2 \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - 4\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2. \quad (3.58)$$

We now look at the sequence for $\bar{\Lambda}^-$. Using $\bar{\mathcal{L}}_{h_j, \lambda_j}(\phi_j, \psi_j) = (\theta_j, \chi_j)$ and $\text{Re} \langle \Delta_{h_j} \phi_j, \phi'_j \rangle = 0 = \text{Re} \langle \phi_j, \phi'_j \rangle = \text{Re} \langle \psi_j, \psi'_j \rangle$, which follow from Lemma 3.6, we obtain

$$\begin{aligned} \text{Re} \langle (\theta_j, \chi_j), (\phi'_j, \psi'_j) \rangle &= \text{Re} \langle \bar{\mathcal{L}}_{h_j, \lambda_j}(\phi_j, \psi_j), (\phi'_j, \psi'_j) \rangle \\ &= \text{Re} \langle -\tilde{c}_{h_j} \phi'_j - \Delta_{h_j} \phi_j - g_u(\tilde{u}_h) \phi_j + \lambda_j \phi_j - \rho \psi_j, \phi'_j \rangle \\ &\quad + \text{Re} \langle \phi_j - \tilde{c}_h \psi'_j + \gamma \rho \psi_j + \lambda_j \psi_j, \psi'_j \rangle \\ &= -\tilde{c}_{h_j} \|\phi'_j\|_{L^2}^2 - \text{Re} \langle g_u(\tilde{u}_h) \phi_j, \phi'_j \rangle - \rho \text{Re} \langle \psi_j, \phi'_j \rangle + \text{Re} \langle \lambda_j \phi_j, \phi'_j \rangle \\ &\quad + \text{Re} \langle \phi_j, \psi'_j \rangle - \tilde{c}_{h_j} \|\psi'_j\|_{L^2}^2 + \text{Re} \langle \lambda_j \psi_j, \psi'_j \rangle \\ &= -\tilde{c}_{h_j} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - \text{Re} \langle g_u(\tilde{u}_h) \phi_j, \phi'_j \rangle + (1 + \rho) \langle \psi_j, \phi'_j \rangle \\ &\quad + \text{Re} \langle \lambda_j(\phi_j, \psi_j), (\phi'_j, \psi'_j) \rangle. \end{aligned} \quad (3.59)$$

We write

$$G = \lambda_{\max} \|(\phi_j, \psi_j)\|_{\mathbf{L}^2} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} + g_* \|\phi_j\|_{L^2} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}. \quad (3.60)$$

Using the Cauchy-Schwartz inequality we now obtain

$$\begin{aligned} G &\geq \lambda_{\max} \|(\phi_j, \psi_j)\|_{\mathbf{L}^2} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} + \|g_u(\tilde{u}_{h_j})\|_{L^\infty} \|\phi_j\|_{L^2} \|\phi'_j\|_{L^2} \\ &\geq -\text{sign}(\tilde{c}_{h_j}) \left(-\text{Re} \langle \lambda_j(\phi_j, \psi_j), (\phi'_j, \psi'_j) \rangle + \text{Re} \langle g_u(\tilde{u}_{h_j})\phi_j, \phi'_j \rangle \right) \\ &= -\text{sign}(\tilde{c}_{h_j}) \left(-\tilde{c}_{h_j} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - (1 + \rho) \text{Re} \langle \psi_j, \phi'_j \rangle - \text{Re} \langle (\theta_j, \chi_j), (\phi'_j, \psi'_j) \rangle \right) \\ &\geq |\tilde{c}_{h_j}| \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - (1 + \rho) \|\psi_j\|_{L^2} \|\phi'_j\|_{L^2} - \|(\theta_j, \chi_j)\|_{\mathbf{L}^2} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} \\ &\geq c_* \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - (1 + \rho) \|\psi_j\|_{L^2} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2} - \|(\theta_j, \chi_j)\|_{\mathbf{L}^2} \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}. \end{aligned} \quad (3.61)$$

This is the same equation that we derived for $\bar{\Lambda}^+$. Hence we again obtain

$$B\|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \geq c_*^2\|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - 4\|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2, \quad (3.62)$$

where

$$B = 4\left(\max\{g_*^2, (1+\rho)^2\} + \lambda_{\max}^2\right). \quad (3.63)$$

■

Lemma 3.10. *Assume that (hFam), (HP1), (HS) and (H α 1) are satisfied. Consider the setting of Proposition 3.4 and Lemma 3.7 or Lemma 3.8. There exist positive constants a and m , depending only on our choice of $(\tilde{u}_h, \tilde{w}_h)$, such that we have the following inequality for all j*

$$(a + g_*) \int_{|x| \leq m} |\phi_j(x)|^2 dx \geq \left(\frac{1}{2} \min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min}\right) \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 - \frac{1}{2 \min\{a, \frac{1}{2}\rho\gamma\}} \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 - \frac{1-\rho}{\rho} \beta \|\chi_j\|_{\mathbf{L}^2}^2. \quad (3.64)$$

Here we write $\lambda_{\min} = 0$ in the setting of Lemma 3.7 or $\lambda_{\min} = \min\{\operatorname{Re} \lambda : \lambda \in M\}$ in the setting of Lemma 3.8, together with

$$\beta = \frac{1-\rho}{\rho} \frac{1}{4\left(\frac{\rho}{1-\rho} \frac{1}{2}\gamma\rho + \gamma\rho + \lambda_{\min}\right)}. \quad (3.65)$$

Proof. Again we first look at the sequence for $\bar{\Lambda}^+$. We know that $\tilde{u}_h - \bar{u}_0 \rightarrow 0$ in H^1 as $h \downarrow 0$. Hence it follows that $\tilde{u}_h - \bar{u}_0 \rightarrow 0$ in L^∞ and therefore also $g_u(\tilde{u}_h) - g_u(\bar{u}_0) \rightarrow 0$ in L^∞ as $h \downarrow 0$. Thus we can choose m to be a positive constant such that for all $h \in [0, \bar{h}]$ (by making \bar{h} smaller if necessary)

$$\min_{|x| \geq m} [-g_u(\tilde{u}_h(x))] \geq a := \frac{1}{2}r_0 > 0. \quad (3.66)$$

Here r_0 is the constant appearing in the choice of our function g in (HS). Then we obtain, using $\operatorname{Re} \langle \phi'_j, \phi_j \rangle = \operatorname{Re} \langle \psi'_j, \psi_j \rangle = 0$ and $\operatorname{Re} \langle -\Delta_{h_j} \phi_j, \phi_j \rangle \geq 0$, which we know from Lemma 3.6, that

$$\begin{aligned} \operatorname{Re} \langle (\theta_j, \chi_j), (\phi_j, \psi_j) \rangle &= \operatorname{Re} \langle \bar{\mathcal{L}}_{h_j, \lambda_j}^+ (\phi_j, \psi_j), (\phi_j, \psi_j) \rangle \\ &\geq \operatorname{Re} \langle -g_u(\tilde{u}_{h_j}) \phi_j, \phi_j \rangle + \operatorname{Re} \langle \psi_j, \phi_j \rangle \\ &\quad - \rho \operatorname{Re} \langle \psi_j, \phi_j \rangle + \gamma \rho \|\psi_j\|_{L^2}^2 + \lambda_{\min} \|(\phi_j, \psi_j)\|^2 \\ &\geq \min_{|x| \geq m} \{-g_u(\tilde{u}_{h_j}(x))\} \int_{|x| \geq m} |\phi_j(x)|^2 dx \\ &\quad - \|g_u(\tilde{u}_{h_j})\|_{L^\infty} \int_{|x| \leq m} |\phi_j(x)|^2 dx + (1-\rho) \operatorname{Re} \langle \psi_j, \phi_j \rangle \\ &\quad + \gamma \rho \|\psi_j\|_{L^2}^2 + \lambda_{\min} \|(\phi_j, \psi_j)\|^2 \\ &\geq a \|\phi_j\|_{L^2}^2 - (a + g_*) \int_{|x| \leq m} |\phi_j(x)|^2 dx + (1-\rho) \operatorname{Re} \langle \psi_j, \phi_j \rangle \\ &\quad + \gamma \rho \|\psi_j\|_{L^2}^2 + \lambda_{\min} \|(\phi_j, \psi_j)\|^2. \end{aligned} \quad (3.67)$$

We assumed that $0 < \rho < 1$ so we see that $\frac{1-\rho}{-\rho} < 0$. We set

$$\beta_j^+ = \frac{1}{4\left(\frac{\rho}{1-\rho} \frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j\right)}. \quad (3.68)$$

Now we obtain

$$\begin{aligned} \operatorname{Re} \langle \chi_j, \psi_j \rangle &\leq \|\chi_j\|_{L^2} \|\psi_j\|_{L^2} \\ &= \frac{1}{\sqrt{2\left(\frac{\rho}{1-\rho} \frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j\right)}} \|\chi_j\|_{L^2} \sqrt{2\left(\frac{\rho}{1-\rho} \frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j\right)} \|\psi_j\|_{L^2} \\ &\leq \frac{1}{4\left(\frac{\rho}{1-\rho} \frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j\right)} \|\chi_j\|_{L^2}^2 + \left(\frac{\rho}{1-\rho} \frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j\right) \|\psi_j\|_{L^2}^2 \\ &= \beta_j^+ \|\chi_j\|_{L^2}^2 + \left(\frac{\rho}{1-\rho} \frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j\right) \|\psi_j\|_{L^2}^2. \end{aligned} \quad (3.69)$$

Note that the denominator $4(\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j)$ is never zero since we can assume that λ_* is small enough to have $\operatorname{Re} \lambda_j \geq -\lambda_* > -\gamma\rho$. Using the identity

$$\chi_j = -\rho\phi_j + \tilde{c}_{h_j}\psi'_j + \gamma\rho\psi_j + \lambda_j\psi_j \quad (3.70)$$

and the fact that $\operatorname{Re} \langle \psi'_j, \psi_j \rangle = 0$, we also have

$$\operatorname{Re} \langle \chi_j, \psi_j \rangle = -\rho \operatorname{Re} \langle \phi_j, \psi_j \rangle + (\gamma\rho + \operatorname{Re} \lambda_j) \|\psi_j\|_{L^2}^2. \quad (3.71)$$

Hence we must have that

$$\begin{aligned} (1-\rho) \operatorname{Re} \langle \phi_j, \psi_j \rangle &= \frac{1-\rho}{\rho} \left(-\operatorname{Re} \langle \chi_j, \psi_j \rangle + (\gamma\rho + \operatorname{Re} \lambda_j) \|\psi_j\|_{L^2}^2 \right) \\ &\geq \frac{1-\rho}{\rho} \left(-\beta_j^+ \|\chi_j\|_{L^2}^2 - (\frac{\rho}{1-\rho}\frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j) \|\psi_j\|_{L^2}^2 \right. \\ &\quad \left. + (\gamma\rho + \operatorname{Re} \lambda_j) \|\psi_j\|_{L^2}^2 \right) \\ &= -\frac{1-\rho}{\rho} \beta_j^+ \|\chi_j\|_{L^2}^2 - \frac{1}{2}\gamma\rho \|\psi_j\|_{L^2}^2. \end{aligned} \quad (3.72)$$

We now look at the sequence for $\bar{\Lambda}^-$. Let m and a be as before. Then we obtain, using $\bar{\mathcal{L}}_{h_j, \lambda_j}^-(\phi_j, \psi_j) = (\theta_j, \chi_j)$, $\operatorname{Re} \langle \phi'_j, \phi_j \rangle = \operatorname{Re} \langle \psi'_j, \psi_j \rangle = 0$ and $\operatorname{Re} \langle -\Delta_{h_j} \phi_j, \phi_j \rangle \geq 0$ that

$$\begin{aligned} \operatorname{Re} \langle (\theta_j, \chi_j), (\phi_j, \psi_j) \rangle &= \operatorname{Re} \langle \bar{\mathcal{L}}_{h_j, \delta}^-(\phi_j, \psi_j), (\phi_j, \psi_j) \rangle \\ &\geq \operatorname{Re} \langle -g_u(\tilde{u}_h)\phi_j, \phi_j \rangle - \rho \operatorname{Re} \langle \psi_j, \phi_j \rangle \\ &\quad + \operatorname{Re} \langle \psi_j, \phi_j \rangle + \gamma\rho \|\psi_j\|_{L^2}^2 + \lambda_{\min} \|(\phi_j, \psi_j)\|^2 \\ &\geq \min_{|x| \geq m} \{-g_u(\tilde{u}_h(x))\} \int_{|x| \geq m} |\phi_j|^2 dx \\ &\quad - \|g_u(\tilde{u}_h)\|_{L^\infty} \int_{|x| \leq m} |\phi_j|^2 dx + (1-\rho) \langle \psi_j, \phi_j \rangle \\ &\quad + \gamma\rho \|\psi_j\|_{L^2}^2 + \lambda_{\min} \|(\phi_j, \psi_j)\|^2 \\ &\geq a \|\phi_j\|_{L^2}^2 - (a + g_*) \int_{|x| \leq m} |\phi_j|^2 dx + (1-\rho) \langle \psi_j, \phi_j \rangle \\ &\quad + \gamma\rho \|\psi_j\|_{L^2}^2 + \lambda_{\min} \|(\phi_j, \psi_j)\|^2. \end{aligned} \quad (3.73)$$

We set

$$\beta_j^- = \frac{1}{4(\frac{1}{1-\rho}\frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j)}. \quad (3.74)$$

Arguing as in (3.69) with slightly different constants, we obtain

$$\begin{aligned} \operatorname{Re} \langle \chi_j, \psi_j \rangle &\geq -\|\chi_j\|_{L^2} \|\psi_j\|_{L^2} \\ &\geq -\frac{1}{4(\frac{1}{1-\rho}\frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j)} \|\chi_j\|_{L^2}^2 - (\frac{1}{1-\rho}\frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j) \|\psi_j\|_{L^2}^2 \\ &= -\beta_j^- \|\chi_j\|_{L^2}^2 - (\frac{1}{1-\rho}\frac{1}{2}\gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j) \|\psi_j\|_{L^2}^2. \end{aligned} \quad (3.75)$$

Using the identity

$$\chi_j = \phi_j - \tilde{c}_{h_j}\psi'_j + \gamma\rho\psi_j + \lambda_j\psi_j \quad (3.76)$$

and the fact that $\operatorname{Re} \langle \psi'_j, \psi_j \rangle = 0$, we also have

$$\operatorname{Re} \langle \chi_j, \psi_j \rangle = \operatorname{Re} \langle \phi_j, \psi_j \rangle + (\gamma\rho + \operatorname{Re} \lambda_j) \|\psi_j\|_{L^2}^2. \quad (3.77)$$

Hence we must have that

$$\begin{aligned}
(1 - \rho) \operatorname{Re} \langle \phi_j, \psi_j \rangle &= (1 - \rho) \left(\operatorname{Re} \langle \chi_j, \psi_j \rangle - (\gamma\rho + \operatorname{Re} \lambda_j) \|\psi_j\|_{L^2}^2 \right) \\
&\geq (1 - \rho) \left(-\beta_j^- \|\chi_j\|_{L^2}^2 + \left(\frac{1}{1-\rho} \frac{1}{2} \gamma\rho + \gamma\rho + \operatorname{Re} \lambda_j \right) \|\psi_j\|_{L^2}^2 \right. \\
&\quad \left. - (\gamma\rho + \operatorname{Re} \lambda_j) \|\psi_j\|_{L^2}^2 \right) \\
&= -(1 - \rho) \beta_j^- \|\chi_j\|_{L^2}^2 - \frac{1}{2} \gamma\rho \|\psi_j\|_{L^2}^2 \\
&\geq -\frac{(1-\rho)}{\rho} \beta_j^- \|\chi_j\|_{L^2}^2 - \frac{1}{2} \gamma\rho \|\psi_j\|_{L^2}^2.
\end{aligned} \tag{3.78}$$

Upon setting

$$\beta = \frac{1}{4\left(\frac{\rho}{1-\rho} \frac{1}{2} \gamma\rho + \gamma\rho + \lambda_{\min}\right)}, \tag{3.79}$$

we note that $\beta_j^+ \leq \beta$ and $\beta_j^- \leq \beta$ for all j since $\rho < 1$ and since β_j^+ is maximal for $\operatorname{Re} \lambda = \lambda_{\min}$. Therefore, in both cases, we obtain

$$\begin{aligned}
(a + g_*) \int_{|x| \leq m} |\phi_j(x)|^2 dx &\geq a \|\phi_j\|_{L^2}^2 + \rho\gamma \|\psi_j\|_{L^2}^2 - \operatorname{Re} \langle (\theta_j, \chi_j), (\phi_j, \psi_j) \rangle \\
&\quad + (1 - \rho) \operatorname{Re} \langle \psi_j, \phi_j \rangle + \lambda_{\min} \|(\phi_j, \psi_j)\|^2 \\
&\geq a \|\phi_j\|_{L^2}^2 + \frac{1}{2} \rho\gamma \|\psi_j\|_{L^2}^2 - \langle (\theta_j, \chi_j), (\phi_j, \psi_j) \rangle \\
&\quad - \frac{1-\rho}{\rho} \beta \|\chi_j\|_{L^2}^2 + \lambda_{\min} \|(\phi_j, \psi_j)\|^2 \\
&\geq \left(\min\{a, \frac{1}{2} \rho\gamma\} + \lambda_{\min} \right) \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
&\quad - \frac{1}{\sqrt{\min\{a, \frac{1}{2} \rho\gamma\}}} \|(\theta_j, \chi_j)\|_{\mathbf{L}^2} \sqrt{\min\{a, \frac{1}{2} \rho\gamma\}} \|(\phi_j, \psi_j)\|_{\mathbf{L}^2} \\
&\quad - \frac{1-\rho}{\rho} \beta \|\chi_j\|_{L^2}^2
\end{aligned} \tag{3.80}$$

and thus, again using the inequality $2\mu\omega \leq \mu^2 + \omega^2$ for $\mu, \omega \in \mathbb{R}$,

$$\begin{aligned}
(a + g_*) \int_{|x| \leq m} |\phi_j(x)|^2 dx &\geq \left(\min\{a, \frac{1}{2} \rho\gamma\} + \lambda_{\min} \right) \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
&\quad - \frac{1}{2} \left(\frac{1}{\min\{a, \frac{1}{2} \rho\gamma\}} \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 + \min\{a, \frac{1}{2} \rho\gamma\} \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \right) \\
&\quad - \beta \|\chi_j\|_{L^2}^2 \\
&= \left(\frac{1}{2} \min\{a, \frac{1}{2} \rho\gamma\} + \lambda_{\min} \right) \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
&\quad - \frac{1}{2 \min\{a, \frac{1}{2} \rho\gamma\}} \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 - \frac{1-\rho}{\rho} \beta \|\chi_j\|_{L^2}^2,
\end{aligned} \tag{3.81}$$

as desired. \blacksquare

Lemma 3.11. *Assume that (hFam), (HP1), (HS) and (H α 1) are satisfied. Consider the setting of Proposition 3.4 and Lemma 3.7 or Lemma 3.8. There exist positive constants C_4 and C_5 , depending only on our choice of $(\tilde{u}_h, \tilde{w}_h)$, such that for all j we have*

$$(a + g_*) \int_{|x| \leq m} |\phi_j^2(x)| dx \geq C_4 - C_5 \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2. \tag{3.82}$$

Proof. Without loss of generality we assume that $\frac{1}{2} \min\{a, \frac{1}{2} \rho\gamma\} + \lambda_{\min} > 0$. Write

$$\mu = \frac{\frac{1}{2} \min\{a, \frac{1}{2} \rho\gamma\} + \lambda_{\min}}{c_4^2 + B}. \tag{3.83}$$

Adding μ times equation (3.52) to equation (3.64) gives

$$\begin{aligned}
(a + g_*) \int_{|x| \leq m} |\phi_j(x)|^2 dx + B\mu \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 &\geq \mu c_*^2 \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - 3\mu \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 \\
&\quad + \frac{1}{2}(\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min}) \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
&\quad - \frac{1}{2(\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min})} \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 \\
&\quad - \frac{1-\rho}{\rho} \beta \|\chi_j\|_{L^2}^2 \\
&\geq \mu c_*^2 \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 - 3\mu \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 \\
&\quad + \frac{1}{2}(\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min}) \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
&\quad - \frac{1}{2(\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min})} \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 \\
&\quad - \frac{1-\rho}{\rho} \beta \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{3.84}$$

We hence obtain

$$\begin{aligned}
(a + g_*) \int_{|x| \leq m} |\phi_j(x)|^2 dx &\geq -C_5 \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 + \mu c_*^2 \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 \\
&\quad + \frac{1}{2}(\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min}) \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 - B\mu \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2,
\end{aligned} \tag{3.85}$$

where

$$\begin{aligned}
C_5 &= 3\mu + \frac{1}{2(\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min})} + \beta \\
&> 0.
\end{aligned} \tag{3.86}$$

This allows us to compute

$$\begin{aligned}
(a + g_*) \int_{|x| \leq m} |\phi_j(x)|^2 dx &\geq -C_5 \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 + \mu c_*^2 \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 \\
&\quad + \frac{1}{2}(\min\{a, \frac{1}{2}\rho\gamma\} + \lambda_{\min}) \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 - B\mu \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
&= -C_5 \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 + \mu c_*^2 \|(\phi'_j, \psi'_j)\|_{\mathbf{L}^2}^2 \\
&\quad + (\mu(c_*^2 + B) - B\mu) \|(\phi_j, \psi_j)\|_{\mathbf{L}^2}^2 \\
&= \mu c_*^2 \|(\phi_j, \psi_j)\|_{\mathbf{H}^1}^2 - C_5 \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 \\
&= C_4 - C_5 \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2,
\end{aligned} \tag{3.87}$$

where $C_4 = \mu c_*^2 > 0$. ■

Proof of Proposition 3.4. We first choose $0 < \delta < \delta_0$ and consider the setting of Lemma 3.7. Sending $j \rightarrow \infty$ in (3.82), Lemma 3.7 implies

$$\begin{aligned}
C_4 - C_5 \bar{\Lambda}^\pm(\delta) &\leq C_4 - C_5 \lim_{j \rightarrow \infty} \|(\theta_j, \chi_j)\|_{\mathbf{L}^2}^2 \\
&\leq (a + g_*) \int_{|x| \leq m} |\phi|^2 dx \\
&\leq (a + g_*) \|(\phi, \psi)\|_{H^2(\mathbb{R}) \times H^1(\mathbb{R})}^2 \\
&\leq (a + g_*) C_2^2 \bar{\Lambda}^+(\delta)^2.
\end{aligned} \tag{3.88}$$

Solving this quadratic inequality, we obtain

$$\begin{aligned}
\bar{\Lambda}^\pm(\delta) &\geq \frac{-C_5 + \sqrt{C_5^2 + 4(a+g_*)C_2^2 C_4}}{2(a+g_*)C_2^2} \\
&:= \frac{2}{C_0'}.
\end{aligned} \tag{3.89}$$

The analogous computation in the setting of Lemma 3.8 yields

$$\begin{aligned}\bar{\Lambda}^+(M) &\geq \frac{-C_5 + \sqrt{C_5^2 + 4(a+g_*)C_3^2 C_4}}{2(a+g_*)C_3^2} \\ &:= \frac{2}{C'_M}.\end{aligned}\tag{3.90}$$

■

Proof of Proposition 3.2. Let $\delta > 0$ be fixed. Since $\bar{\Lambda}^\pm(\delta) \geq \frac{2}{C'_0}$, the definition (3.19) implies that there exists $h'_0(\delta)$ such that $\bar{\Lambda}(h, \delta) \geq \frac{1}{C'_0}$ for all $h \in (0, h'_0(\delta)]$. Now pick $h \in (0, h'_0(\delta)]$.

First of all, $\bar{\mathcal{L}}_{h,\delta}^\pm$ is a bounded operator from \mathbf{H}^1 to \mathbf{L}^2 . Since $\bar{\Lambda}^\pm(h, \delta)$ is strictly positive, this implies that $\bar{\mathcal{L}}_{h,\delta}^\pm$ is a homeomorphism from \mathbf{H}^1 to its image $\bar{\mathcal{L}}_{h,\delta}^\pm(\mathbf{H}^1)$. Furthermore, the norm of the inverse $(\bar{\mathcal{L}}_{h,\delta}^\pm)^{-1}$ from $\bar{\mathcal{L}}_{h,\delta}^\pm(\mathbf{H}^1) \subset \mathbf{L}^2$ is bounded by $\frac{1}{\bar{\Lambda}^\pm(h,\delta)} \leq C'_0$. Since $\bar{\mathcal{L}}_{h,\delta}^\pm$ is bounded, it follows that $\bar{\mathcal{L}}_{h,\delta}^\pm(\mathbf{H}^1)$ is closed in \mathbf{L}^2 .

For the remainder of this proof, we only consider the operators $\bar{\mathcal{L}}_{h,\delta}^+$, noting that their counterparts $\bar{\mathcal{L}}_{h,\delta}^-$ can be treated in an identical fashion.

Seeking a contradiction, let us assume that $\bar{\mathcal{L}}_{h,\delta}^+(\mathbf{H}^1) \neq \mathbf{L}^2$, which implies that there exists a non-zero $(\theta, \chi) \in \mathbf{L}^2$ orthogonal to $\bar{\mathcal{L}}_{h,\delta}^+(\mathbf{H}^1)$. For any $\phi \in C_c^\infty(\mathbb{R})$, we hence obtain

$$\begin{aligned}0 &= \langle (\bar{\mathcal{L}}_{h,\delta}^+(\phi, 0), (\theta, \chi)) \rangle \\ &= \langle \tilde{c}_h \phi' - \Delta_h \phi - g_u(\tilde{u}_h)\phi + \delta\phi, \theta \rangle + \langle -\rho\phi, \chi \rangle \\ &= \tilde{c}_h \langle \phi', \theta \rangle + \langle \phi, -\Delta_h \theta - g_u(\tilde{u}_h)\theta + \delta\theta - \rho\chi \rangle.\end{aligned}\tag{3.91}$$

By definition this implies that θ has a weak derivative and that $\tilde{c}_h \theta' = -\Delta_h \theta - g_u(\tilde{u}_h)\theta + \delta\theta - \rho\chi \in L^2(\mathbb{R})$. In particular, we see that $\theta \in H^1(\mathbb{R})$.

For any $\psi \in C_c^\infty(\mathbb{R})$ a similar computation yields

$$\begin{aligned}0 &= \langle (\bar{\mathcal{L}}_{h,\delta}^+(0, \psi), (\theta, \chi)) \rangle \\ &= \langle \psi, \theta \rangle + \langle \tilde{c}_h \psi' + (\gamma\rho + \delta)\psi, \chi \rangle \\ &= \tilde{c}_h \langle \psi', \chi \rangle + \langle \psi, \theta + (\gamma\rho + \delta)\chi \rangle.\end{aligned}\tag{3.92}$$

Again, this means that χ has a weak derivative and in fact $\tilde{c}_h \chi' = \theta + (\gamma\rho + \delta)\chi$. In particular it follows that $\chi \in H^1(\mathbb{R})$.

We therefore conclude that

$$\begin{aligned}0 &= \langle (\bar{\mathcal{L}}_{h,\delta}^+(\phi, \psi), (\theta, \chi)) \rangle \\ &= \langle (\phi, \psi), (\bar{\mathcal{L}}_{h,\delta}^-(\theta, \chi)) \rangle\end{aligned}\tag{3.93}$$

holds for all $(\phi, \psi) \in \mathbf{H}^1$. Since \mathbf{H}^1 is dense in \mathbf{L}^2 this implies that $\bar{\mathcal{L}}_{h,\delta}^-(\theta, \chi) = 0$. Since we already know that $\bar{\mathcal{L}}_{h,\delta}^-$ is injective, this means that $(\theta, \chi) = 0$, which gives a contradiction. Hence we must have $\bar{\mathcal{L}}_{h,\delta}^+(\mathbf{H}^1) = \mathbf{L}^2$, as desired. ■

Proof of Proposition 3.3. The result follows in the same fashion as outlined in the proof of Proposition 3.2 above. ■

4 Existence of pulse solutions

In this section we prove our first main result, Theorem 2.1. In particular, we construct solutions to (2.12) by writing

$$(\bar{u}_h, \bar{w}_h) = (\bar{u}_0, \bar{w}_0) + (\phi_h, \psi_h) \quad (4.1)$$

and exploiting the linear results of §3. Here (\bar{u}_0, \bar{w}_0) is the pulse solution of the PDE (1.1).

In particular, fix a small $\delta > 0$ that will be determined later, and consider the linear operator

$$\mathcal{L}_{h,\delta}^+ : \mathbf{H}^1 \rightarrow \mathbf{L}^2, \quad (4.2)$$

defined by

$$\mathcal{L}_{h,\delta}^+ = \begin{pmatrix} c_0 \frac{d}{dx} - \Delta_h - g_u(\bar{u}_0) + \delta & 1 \\ -\rho & c_0 \frac{d}{dx} + \gamma\rho + \delta \end{pmatrix}. \quad (4.3)$$

This operator arises as the linearisation of (2.1) around the pulse solution (\bar{u}_0, \bar{w}_0) of (1.1). A short computation shows that our travelling wave triplet $(c_h, \phi_h, \psi_h) \in \mathbb{R} \times \mathbf{H}^1$ must satisfy

$$\mathcal{L}_{h,\delta}^+(c_h, \psi_h) = \mathcal{R}(c_h, \phi_h, \psi_h), \quad (4.4)$$

where

$$\mathcal{R}(c, \phi, \psi) = \left((c_0 - c)(\bar{u}_0' + \phi') + (\Delta_h - \frac{d^2}{dx^2})\bar{u}_0 + \delta\phi + \mathcal{N}(\bar{u}_0, \phi), (c_0 - c)(\bar{w}_0' + \psi') \right). \quad (4.5)$$

Here we have introduced the nonlinearity

$$\mathcal{N}(\bar{u}_0, \phi) = g(\bar{u}_0 + \phi) - g(\bar{u}_0) - g_u(\bar{u}_0)\phi. \quad (4.6)$$

Corollary 4.1. *There exists a positive constant C_0 and a positive function $h_0(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for all $\delta > 0$ and all $h \in (0, h_0(\delta))$, the operator $\mathcal{L}_{h,\delta}^+$ is a homeomorphism for which we have the bound*

$$\|(\mathcal{L}_{h,\delta}^+)^{-1}(\theta, \chi)\|_{\mathbf{H}^1} \leq C_0 \|(\theta, \chi)\|_{\mathbf{L}^2} \quad (4.7)$$

for all $(\theta, \chi) \in \mathbf{L}^2$ that satisfy $\langle(\theta, \chi), (\phi_0^-, \psi_0^-)\rangle = 0$.

Proof. This is immediate by choosing $(\tilde{u}_h, \tilde{w}_h) = (\bar{u}_0, \bar{w}_0)$ and $\tilde{c}_h = c_0$ for all h in (hFam) and applying Proposition 3.2. \blacksquare

Let η be a small positive constant to be determined later. We define

$$X_\eta = \{(\phi, \psi) \in \mathbf{H}^1 : \|(\phi, \psi)\|_{\mathbf{H}^1} \leq \eta\}. \quad (4.8)$$

For every $(\phi, \psi) \in X_\eta$, we define $c_h = c_h(\phi, \psi)$ to be the constant

$$c_h(\phi, \psi) = c_0 + \frac{\langle \Delta_h \bar{u}_0 - \bar{u}_0'', \phi_0^- \rangle + \delta \langle \phi, \phi_0^- \rangle + \langle \mathcal{N}(\bar{u}_0, \phi), \phi_0^- \rangle}{\langle \bar{u}_0', \phi_0^- \rangle + \langle \phi', \phi_0^- \rangle + \langle \bar{w}_0', \psi_0^- \rangle + \langle \psi', \psi_0^- \rangle}. \quad (4.9)$$

When this expression is well-defined, this choice ensures that

$$\langle \mathcal{R}(c_h(\phi, \psi), \phi, \psi), (\phi_0^-, \psi_0^-) \rangle = 0. \quad (4.10)$$

We define $T : X_\eta \subset \mathbf{H}^1 \rightarrow \mathbf{H}^1$ by

$$T(\phi, \psi) = (\mathcal{L}_{h,\delta}^+)^{-1} \mathcal{R}(c_h(\phi, \psi), \phi, \psi). \quad (4.11)$$

Our goal is to show T maps X_η into itself and is a contraction, since then the fixed point (ϕ_h, ψ_h) leads to a travelling pulse solution of (2.12) via (4.1) and (4.9).

Exploiting (4.10), Corollary 4.1 implies that there exists a constant $C_0 > 0$ such that for all $\Psi = (\phi, \psi) \in X_\eta$ we have the bound

$$\|T(\Psi)\|_{\mathbf{H}^1} \leq C_0 \|\mathcal{R}(c_h(\Psi), \Psi)\|_{\mathbf{L}^2}, \quad (4.12)$$

while for all $\Psi_1 = (\phi_1, \psi_1), \Psi_2 = (\phi_2, \psi_2) \in X_\eta$ we have the bound

$$\|T(\Psi_1) - T(\Psi_2)\|_{\mathbf{H}^1} \leq C_0 \|\mathcal{R}(c_h(\Psi_1), \Psi_1) - \mathcal{R}(c_h(\Psi_2), \Psi_2)\|_{\mathbf{L}^2}. \quad (4.13)$$

In the remainder of this section we therefore set out to estimate the right-hand sides of (4.12) and (4.13). We start by estimating the nonlinear term $\mathcal{N}(\bar{u}_0, \cdot)$.

Lemma 4.2. *Assume that (HP1), (HS) and (H α 1) are satisfied. Then there exists a constant $M > 0$ such that for all $0 < \eta \leq 1$, $(\phi, \psi) \in X_\eta, (\phi_1, \psi_1) \in X_\eta$ and $(\phi_2, \psi_2) \in X_\eta$ we have the pointwise inequalities*

$$\begin{aligned} |\mathcal{N}(\bar{u}_0, \phi)| &\leq M\eta|\phi|, \\ |\mathcal{N}(\bar{u}_0, \phi_1) - \mathcal{N}(\bar{u}_0, \phi_2)| &\leq \eta M|\phi_1 - \phi_2|. \end{aligned} \quad (4.14)$$

Proof. To estimate the nonlinear term $\mathcal{N}(\bar{u}_0, \phi)$, we first recall the embedding $\|\phi\|_{L^\infty} \leq \|\phi\|_{H^1} \leq \eta \leq 1$ for every $(\phi, \psi) \in X_\eta$. Setting $M = \max\{6, \sup_{|s| \leq \|\bar{u}_0\|_\infty} |g_{uu}(s)|\}$, a Taylor expansion around \bar{u}_0 allows us to obtain the pointwise inequalities

$$\begin{aligned} |\mathcal{N}(\bar{u}_0, \phi)| &= |-g(\bar{u}_0 + \phi) + g(\bar{u}_0) + g_u(\bar{u}_0)| \\ &= |-g(\bar{u}_0) - \phi g_u(\bar{u}_0) - \frac{1}{2}\phi^2 g_{uu}(\xi) + g(\bar{u}_0) + g_u(\bar{u}_0)| \\ &= |-\frac{1}{2}\phi^2 g_{uu}(\xi)| \\ &\leq \frac{1}{2}M\eta|\phi| \\ &\leq M\eta|\phi|, \end{aligned} \quad (4.15)$$

where ξ is between \bar{u}_0 and $\bar{u}_0 + \phi$. Note that $g_{uuu} = 6$ is constant. Furthermore, for $(\phi_1, \psi_1) \in X_\eta$ and $(\phi_2, \psi_2) \in X_\eta$, a Taylor expansion around \bar{u}_0 yields the pointwise inequalities

$$\begin{aligned} |\mathcal{N}(\bar{u}_0, \phi_1) - \mathcal{N}(\bar{u}_0, \phi_2)| &= \left| -g(\bar{u}_0 + \phi_1) + g(\bar{u}_0) + g_u(\bar{u}_0)\phi_1 \right. \\ &\quad \left. + g(\bar{u}_0 + \phi_2) - g(\bar{u}_0) - g_u(\bar{u}_0)\phi_2 \right| \\ &= \left| -\frac{1}{2}g_{uu}(\bar{u}_0)\phi_1^2 + \frac{1}{2}g_{uu}(\bar{u}_0)\phi_2^2 - \frac{1}{6}g_{uuu}(\xi_1)\phi_1^3 + \frac{1}{6}g_{uuu}(\xi_2)\phi_2^3 \right| \\ &\leq \frac{1}{2}|g_{uu}(\bar{u}_0)||\phi_1^2 - \phi_2^2| + \frac{1}{6}6|\phi_1^3 - \phi_2^3| \\ &\leq \frac{1}{2}|g_{uu}(\bar{u}_0)|\left[|\phi_1||\phi_1 - \phi_2| + |\phi_2||\phi_1 - \phi_2|\right] + |\phi_1||\phi_1^2 - \phi_2^2| + |\phi_1 - \phi_2||\phi_2^2| \\ &\leq \frac{1}{2}M\left[2\eta|\phi_1 - \phi_2|\right] + \eta\left[2\eta|\phi_1 - \phi_2|\right] + \eta^2|\phi_1 - \phi_2| \\ &\leq \eta M|\phi_1 - \phi_2|, \end{aligned} \quad (4.16)$$

where ξ_1 is between \bar{u}_0 and $\bar{u}_0 + \phi_1$ and ξ_2 is between \bar{u}_0 and $\bar{u}_0 + \phi_2$. \blacksquare

Pick $(\phi, \psi) \in X_\eta$. Recall that we chose (ϕ_0^-, ψ_0^-) so that $\langle (\phi_0^-, \psi_0^-), (\bar{u}'_0, \bar{w}'_0) \rangle > 0$. Let s be defined as

$$s = 2 \frac{1}{\langle (\phi_0^-, \psi_0^-), (\bar{u}'_0, \bar{w}'_0) \rangle} > 0. \quad (4.17)$$

For notational compactness, we write

$$\sigma(\phi, \psi) = \langle \bar{u}'_0, \phi_0^- \rangle + \langle \phi', \phi_0^- \rangle + \langle \bar{w}'_0, \psi_0^- \rangle + \langle \psi', \psi_0^- \rangle \quad (4.18)$$

for $(\phi, \psi) \in X_\eta$. We also write

$$\eta_0 = \min\{1, s^{-1}\}. \quad (4.19)$$

Lemma 4.3. *Assume that (HP1), (HS) and (H α 1) are satisfied. Fix $0 < \eta \leq \eta_0$. Then for all $\Psi = (\phi, \psi) \in X_\eta, \Psi_1 = (\phi_1, \psi_1) \in X_\eta$ and $\Psi_2 = (\phi_2, \psi_2) \in X_\eta$ we have the bounds*

$$0 < [\sigma(\Psi)]^{-1} \leq s, \quad (4.20)$$

together with

$$|\sigma(\Psi_1) - \sigma(\Psi_2)| \leq \|\Psi'_1 - \Psi'_2\|_{L^2}. \quad (4.21)$$

Proof. Using Cauchy-Schwartz, we obtain that

$$\begin{aligned} \sigma(\phi, \psi) &= \langle \bar{u}'_0, \phi_0^- \rangle + \langle \phi', \phi_0^- \rangle + \langle \bar{w}'_0, \psi_0^- \rangle + \langle \psi', \psi_0^- \rangle \\ &\geq 2s^{-1} + \langle (\phi', \psi'), (\phi_0^-, \psi_0^-) \rangle \\ &\geq 2s^{-1} - \eta \\ &\geq s^{-1}, \end{aligned} \quad (4.22)$$

which yields (4.20). In particular we see that

$$\frac{1}{\sigma(\Psi)} \leq s. \quad (4.23)$$

The remaining inequality (4.21) follows immediately from Cauchy-Schwarz. \blacksquare

Lemma 4.4. *Assume that (HP1), (HS) and (H α 1) are satisfied. Recall the constant M from Lemma 4.2 and the constant s from (4.17). Then for all $0 < \eta \leq \eta_0, \Psi \in X_\eta, \Psi_1 \in X_\eta$ and $\Psi_2 \in X_\eta$ we have the inequality*

$$|c_h(\Psi) - c_0| \leq s \left(\|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + \delta\eta + M\eta^2 \right), \quad (4.24)$$

together with

$$|c_h(\Psi_1) - c_h(\Psi_2)| \leq s \|\Psi_1 - \Psi_2\|_{H^1} \left(s \|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + 2(\delta + M\eta) \right). \quad (4.25)$$

Proof. By (4.20) we have that $[\sigma(\Psi)]^{-1} \leq s$ for all $\Psi \in X_\eta$. By definition of $c_h(\Psi)$ and Lemma 4.2, we obtain for all $\Psi = (\phi, \psi) \in X_\eta$ that

$$\begin{aligned} |c_h(\Psi) - c_0| &= \left| \frac{\langle \Delta_h \bar{u}_0 - \bar{u}_0'', \phi_0^- \rangle + \delta \langle \phi, \phi_0^- \rangle + \langle \mathcal{N}(\bar{u}_0, \phi), \phi_0^- \rangle}{\sigma(\Psi)} \right| \\ &\leq s \left(\langle \Delta_h \bar{u}_0 - \bar{u}_0'', \phi_0^- \rangle + \delta \langle \phi, \phi_0^- \rangle + \langle \mathcal{N}(\bar{u}_0, \phi), \phi_0^- \rangle \right) \\ &\leq s \left(\|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} \|\phi_0^-\|_{L^2} + \delta \|\phi\|_{L^2} \|\phi_0^-\|_{L^2} \right) + sM\eta \|\phi\|_{L^2} \\ &\leq s \left(\|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} \|(\phi_0^-, \psi_0^-)\|_{\mathbf{L}^2} + \delta \|\phi\|_{L^2} \|(\phi_0^-, \psi_0^-)\|_{\mathbf{L}^2} \right) + sM\eta \|\phi\|_{L^2} \\ &= s \left(\|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + \delta \|\phi\|_{L^2} + M\eta \|\phi\|_{L^2} \right) \\ &\leq s \left(\|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + (\delta + M\eta)\eta \right). \end{aligned} \quad (4.26)$$

For notational compactness we write

$$d(\Psi) = \langle \Delta_h \bar{u}_0 - \bar{u}_0'', \phi_0^- \rangle + \delta \langle \phi, \phi_0^- \rangle + \langle \mathcal{N}(\bar{u}_0, \phi), \phi_0^- \rangle. \quad (4.27)$$

Then we obtain with (4.20) that for all $\Psi_1 = (\phi_1, \psi_1) \in X_\eta$ and $\Psi_2 = (\phi_2, \psi_2) \in X_\eta$

$$\begin{aligned}
|c_h(\Psi_1) - c_h(\Psi_2)| &= \left| \frac{d(\Psi_1)}{\sigma(\Psi_1)} - \frac{d(\Psi_2)}{\sigma(\Psi_2)} \right| \\
&= \left| \frac{d(\Psi_1)\sigma(\Psi_2) - d(\Psi_2)\sigma(\Psi_1)}{\sigma(\Psi_1)\sigma(\Psi_2)} \right| \\
&\leq \frac{\left[|d(\Psi_2)|\sigma(\Psi_2) - \sigma(\Psi_1)| + |d(\Psi_1) - d(\Psi_2)|\sigma(\Psi_2) \right]}{|\sigma(\Psi_1)|\sigma(\Psi_2)} \\
&\leq s^2 |d(\Psi_2)|\sigma(\Psi_2) - \sigma(\Psi_1)| + s|d(\Psi_1) - d(\Psi_2)|.
\end{aligned} \tag{4.28}$$

Observe using Lemma 4.2 that

$$\begin{aligned}
|d(\Psi_2)| &\leq \|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + \delta \|\phi_2\|_{L^2} + \|\mathcal{N}(\bar{u}_0, \phi_2)\|_{L^2} \\
&\leq \|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + \delta \eta + M\eta \|\phi_2\|_{L^2} \\
&\leq \|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + \delta \eta + M\eta^2
\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}
|d(\Psi_1) - d(\Psi_2)| &\leq \delta \|\phi_1 - \phi_2\|_{L^2} + \|\mathcal{N}(\bar{u}_0, \phi_1) - \mathcal{N}(\bar{u}_0, \phi_2)\|_{L^2} \\
&\leq \delta \|\phi_1 - \phi_2\|_{L^2} + \eta M \|\phi_1 - \phi_2\|_{L^2} \\
&\leq (\delta + M\eta) \|\phi_1 - \phi_2\|_{L^2}.
\end{aligned} \tag{4.30}$$

Using Lemma 4.3, we hence see that

$$\begin{aligned}
|c_h(\Psi_1) - c_h(\Psi_2)| &\leq s^2 |d(\Psi_2)|\sigma(\Psi_2) - \sigma(\Psi_1)| + s|d(\Psi_1) - d(\Psi_2)| \\
&\leq s^2 \left(\|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + \delta \eta + M\eta^2 \right) |\sigma(\Psi_2) - \sigma(\Psi_1)| \\
&\quad + s(\delta + M\eta) \|\phi_1 - \phi_2\|_{L^2} \\
&\leq s^2 \left(\|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + \delta \eta + M\eta^2 \right) \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \\
&\quad + s(\delta + M\eta) \|\phi_1 - \phi_2\|_{L^2} \\
&\leq s \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \left(s \|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + (\delta + M\eta)(1 + s\eta) \right) \\
&\leq s \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \left(s \|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + 2(\delta + M\eta) \right).
\end{aligned} \tag{4.31}$$

■

Lemma 4.5. *Assume that (HP1), (HS) and (H α 1) are satisfied. Recall the constant M from Lemma 4.2 and the constant s from (4.17). Then for all $0 < \eta \leq \eta_0$, $\Psi \in X_\eta$, $\Psi_1 \in X_\eta$ and $\Psi_2 \in X_\eta$ we have the inequality*

$$\|\mathcal{R}(c_h(\Psi), \Psi)\|_{L^2} \leq \left[1 + s \left(\|\bar{u}_0'\|_{L^2} + \|\bar{w}_0'\|_{L^2} + \eta \right) \right] \left[\|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + \delta \eta + M\eta^2 \right], \tag{4.32}$$

together with

$$\begin{aligned}
\|\mathcal{R}(c_h(\Psi_1), \Psi_1) - \mathcal{R}(c_h(\Psi_2), \Psi_2)\|_{L^2} &\leq \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \left(2 + 2s\eta + s \|\bar{u}_0'\|_{L^2} + s \|\bar{w}_0'\|_{L^2} \right) \\
&\quad \times \left(s \|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + 2(\delta + M\eta) \right).
\end{aligned} \tag{4.33}$$

Proof. For any $\Psi = (\phi, \psi) \in X_\eta$, Lemma 4.4 together with the definition of $\mathcal{R}(c_h(\Psi), \Psi)$ allows us to estimate

$$\begin{aligned}
\|\mathcal{R}(c_h(\Psi), \Psi)\|_{\mathbf{L}^2} &\leq |c_0 - c_h(\Psi)| \left(\|\bar{u}'_0\|_{L^2} + \|\phi'\|_{L^2} \right) + \|\Delta_h \bar{u}_0 - \bar{u}''_0\|_{L^2} + \delta\eta + M\eta^2 \\
&\quad + |c_0 - c_h(\Psi)| \left(\|\bar{w}'_0\|_{L^2} + \|\psi'\|_{L^2} \right) \\
&\leq s \left(\|\Delta_h \bar{u}_0 - \bar{u}''_0\|_{L^2} + \delta\eta + M\eta \right) \left(\|\bar{u}'_0\|_{L^2} + \|\bar{w}'_0\|_{L^2} + \eta \right) \\
&\quad + \|\Delta_h \bar{u}_0 - \bar{u}''_0\|_{L^2} + \delta\eta + M\eta^2 \\
&= \left[1 + s \left(\|\bar{u}'_0\|_{L^2} + \|\bar{w}'_0\|_{L^2} + \eta \right) \right] \left[\|\Delta_h \bar{u}_0 - \bar{u}''_0\|_{L^2} + \delta\eta + M\eta^2 \right].
\end{aligned} \tag{4.34}$$

For $\Psi_1 = (\phi_1, \psi_1) \in X_\eta$ and $\Psi_2 = (\phi_2, \psi_2) \in X_\eta$ we write

$$d(\Psi_1, \Psi_2) := \|\mathcal{R}(c_h(\Psi_1), \Psi_1) - \mathcal{R}(c_h(\Psi_2), \Psi_2)\|_{\mathbf{L}^2}. \tag{4.35}$$

Substituting (4.5), we compute

$$\begin{aligned}
d(\Psi_1, \Psi_2) &\leq \left\| \left((c_0 - c_h(\Psi_1))(\phi'_1 - \phi'_2) + (c_h(\Psi_1) - c_h(\Psi_2))(\bar{u}'_0 - \phi'_2) \right. \right. \\
&\quad \left. \left. + \delta(\phi_1 - \phi_2) + (\mathcal{N}(\bar{u}_0, \phi_2) - \mathcal{N}(\bar{u}_0, \phi_1)) \right) \right\|_{L^2} \\
&\quad + \left\| (c_0 - c_h(\Psi_1))(\psi'_1 - \psi'_2) + (c_h(\Psi_1) - c_h(\Psi_2))(\bar{w}'_0 - \psi'_2) \right\|_{L^2} \\
&\leq \left(|c_h(\Psi_1) - c_0| + \delta + M\eta \right) \|\phi_1 - \phi_2\|_{H^1} + \left(\|\bar{u}'_0\|_{L^2} + \eta \right) |c_h(\Psi_1) - c_h(\Psi_2)| \\
&\quad + |c_h(\Psi_1) - c_0| \|\psi_1 - \psi_2\|_{H^1} + \left(\|\bar{w}'_0\|_{L^2} + \eta \right) |c_h(\Psi_1) - c_h(\Psi_2)|.
\end{aligned} \tag{4.36}$$

Another application of Lemma 4.4 yields the desired bound

$$\begin{aligned}
d(\Psi_1, \Psi_2) &\leq \left(s \left(\|\Delta_h \bar{u}_0 - \bar{u}''_0\|_{L^2} + (\delta + M\eta) \right) + \delta + M\eta \right) \|\phi_1 - \phi_2\|_{H^1} \\
&\quad + \left(\|\bar{w}'_0\|_{L^2} + \eta \right) s \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \left(s \|\Delta_h \bar{u}_0 - \bar{u}''_0\|_{L^2} + 2(\delta + M\eta) \right) \\
&\quad + s \left(\|\Delta_h \bar{u}_0 - \bar{u}''_0\|_{L^2} + (\delta + M\eta) \eta \right) \|\psi_1 - \psi_2\|_{H^1} \\
&\quad + \left(\|\bar{w}'_0\|_{L^2} + \eta \right) s^2 \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \left(\|\Delta_h \bar{u}_0 - \bar{u}''_0\|_{L^2} + 2\delta(\delta + M\eta) \right) \\
&\leq \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \left(2 + 2s\eta + s\|\bar{u}'_0\|_{L^2} + s\|\bar{w}'_0\|_{L^2} \right) \left(s\|\Delta_h \bar{u}_0 - \bar{u}''_0\|_{L^2} + 2(\delta + M\eta) \right).
\end{aligned} \tag{4.37}$$

With these estimates in hand, we can choose our parameters δ and η to ensure that the map T maps X_η into itself and is a contraction. This allows us to prove our first main theorem.

Proof of Theorem 2.1. We let

$$C_6 = \max \left\{ C_0 \left(1 + s\|\bar{u}'_0\|_{L^2} + s\|\bar{w}'_0\|_{L^2} + s \right), C_0 \left(4 + s\|\bar{u}'_0\|_{L^2} + s\|\bar{w}'_0\|_{L^2} \right) \right\}, \tag{4.38}$$

which is independent of δ, h and $\eta \in (0, s^{-1}]$. Using Lemma 4.5 together with (4.12) and (4.13), we see that for all $0 < \eta \leq \eta_0$, $\Psi = (\phi, \psi) \in X_\eta$, $\Psi_1 = (\phi_1, \psi_1) \in X_\eta$ and $\Psi_2 = (\phi_2, \psi_2) \in X_\eta$ we have

$$\|T(\Psi)\|_{H^1} \leq C_6 \left(\|\Delta_h \bar{u}_0 - \bar{u}''_0\|_{L^2} + \delta\eta + M\eta^2 \right) \tag{4.39}$$

and

$$\|T(\Psi_1) - T(\Psi_2)\|_{\mathbf{H}^1} \leq C_6 \left(s\|\Delta_h \bar{u}_0 - \bar{u}''_0\| + 2(\delta + M\eta) \right) \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}. \tag{4.40}$$

We fix

$$\begin{aligned}\delta &= \frac{1}{8C_6} \\ \eta &= \min\{\eta_0, \frac{1}{8MC_6}\},\end{aligned}\tag{4.41}$$

so that indeed $\eta \leq \eta_0$. Using the notation from Corollary 4.1, we pick $0 < h_* \leq h_0(\delta)$ in such a way that

$$\sup_{h \in (0, h_*)} \|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} \leq \frac{\eta}{8C_6}.\tag{4.42}$$

Then we see for $h \in (0, h_*)$ that

$$\begin{aligned}\|T(\Phi)\|_{\mathbf{H}^1} &\leq C_6 \left(\|\Delta_h \bar{u}_0 - \bar{u}_0''\|_{L^2} + \delta\eta + M\eta^2 \right) \\ &\leq C_6 \left(\frac{\eta}{8C_6} + \frac{1}{8C_6}\eta + M\frac{1}{8MC_6}\eta \right) \\ &\leq \eta\end{aligned}\tag{4.43}$$

and

$$\begin{aligned}\|T(\Psi_1) - T(\Psi_2)\|_{\mathbf{H}^1} &\leq C_6 \left(s\|\Delta_h \bar{u}_0 - \bar{u}_0''\| + 2(\delta + M\eta) \right) \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \\ &\leq C_6 \left(s\frac{\eta}{8C_6} + 2\left(\frac{1}{8C_6} + M\frac{1}{8MC_6}\right) \right) \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \\ &\leq \frac{3}{4} \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}.\end{aligned}\tag{4.44}$$

In particular, T maps X_η into itself and is a contraction. The local uniqueness of the family $(c_h, \bar{u}_h, \bar{w}_h)$ follows directly from the uniqueness of solutions to fixed point problems. This completes the proof. \blacksquare

5 The point and essential spectrum

In this section we discuss several properties of the operator that arises after linearising the traveling pulse MFDE (2.12) around our wave solution (\bar{u}_h, \bar{w}_h) . The main goals are to determine the Fredholm properties of this operator. In particular we show that both the linearised operator and its adjoint have Fredholm index 0 and that they both have a one-dimensional kernel. Moreover, we construct a family of kernel elements of the adjoint operator that converges to (ϕ_0^-, ψ_0^-) , the kernel element of the operator \mathcal{L}_0^- .

Pick $0 < h < \min\{h_*, \bar{h}\}$, where h_* is given in Theorem 2.1 and \bar{h} is characterized by (3.51). We recall the operator $L_h : \mathbf{H}^1 \rightarrow \mathbf{L}^2$, introduced in §2, which acts as

$$L_h = \begin{pmatrix} c_h \frac{d}{dx} - \Delta_h - g_u(\bar{u}_h) & 1 \\ -\rho & c_h \frac{d}{dx} + \gamma\rho \end{pmatrix}. \quad (5.1)$$

In addition, we write $L_h^* : \mathbf{H}^1 \rightarrow \mathbf{L}^2$ for the formal adjoint of L_h , which is given by

$$L_h^* = \begin{pmatrix} -c_h \frac{d}{dx} - \Delta_h - g_u(\bar{u}_h) & -\rho \\ 1 & -c_h \frac{d}{dx} + \gamma\rho \end{pmatrix}. \quad (5.2)$$

We emphasize that L_h and L_h^* correspond to the operators $\bar{\mathcal{L}}_{h,0}^+$ and $\bar{\mathcal{L}}_{h,0}^-$ defined in §3 respectively upon writing

$$\begin{aligned} (\tilde{u}_h, \tilde{w}_h) &= (\bar{u}_h, \bar{w}_h), \\ \tilde{c}_h &= c_h \end{aligned} \quad (5.3)$$

for the family featuring in (hFam). Finally, we introduce the notation

$$\begin{aligned} \Phi_h^+ &= (\phi_h^+, \psi_h^+) \\ &= \frac{1}{\|(\bar{u}'_h, \bar{w}'_h)\|_{\mathbf{L}^2}} (\bar{u}'_h, \bar{w}'_h) \\ \Phi_0^+ &= (\phi_0^+, \psi_0^+) \\ \Phi_0^- &= (\phi_0^-, \psi_0^-). \end{aligned} \quad (5.4)$$

The results of this section should be seen as a bridge between the singular perturbation theory developed in §3 and the spectral analysis performed in §6. Indeed, one might be tempted to think that most of the work required for the spectral analysis of the operator L_h can already be found in Proposition 3.2 and Proposition 3.3. However, the problem is that we have no control over the δ -dependence of the interval $(0, h'_0(\delta))$, which contains all values of h for which $L_h + \delta = \bar{\mathcal{L}}_{h,\delta}^+$ is invertible. In particular, for fixed $h > 0$ we cannot directly conclude that $\bar{\mathcal{L}}_{h,\delta}^+$ is invertible for all δ in a subset of the positive real axis.

Our main task in this section is therefore to remove the δ -dependence and study L_h and L_h^* directly. The main conclusions are summarized in the results below.

Proposition 5.1. *Assume that (HP1), (HS), (H α 1) and (H α 2) are satisfied. Then there exists a constant $\tilde{\lambda} > 0$ such that for all $\lambda \in \mathbb{C}$ with $\text{Re } \lambda > -\tilde{\lambda}$ and all $0 < h < \min\{h_*, \bar{h}\}$ the operator $L_h + \lambda$ is Fredholm with index 0.*

Proposition 5.2. *Assume that (HP1), (HS), (H α 1) and (H α 2) are satisfied. Then there exists a constant $h_{**} > 0$, together with a family $\Phi_h^- = (\phi_h^-, \psi_h^-) \in \mathbf{H}^1$, defined for $0 < h < h_{**}$, such that the following properties hold.*

1. For each $0 < h < h_{**}$ we have the identities

$$\begin{aligned}\ker(L_h) &= \text{span}\{\Phi_h^+\} \\ &= \{\Psi \in \mathbf{L}^2 : \langle \Psi, \Theta \rangle = 0 \text{ for all } \Theta \in \text{Range}(L_h^*)\}\end{aligned}\quad (5.5)$$

and

$$\begin{aligned}\ker(L_h^*) &= \text{span}\{\Phi_h^-\} \\ &= \{\Psi \in \mathbf{L}^2 : \langle \Psi, \Theta \rangle = 0 \text{ for all } \Theta \in \text{Range}(L_h)\}.\end{aligned}\quad (5.6)$$

2. The family Φ_h^- converges to Φ_0^- in \mathbf{H}^1 as $h \downarrow 0$.

3. The functions Φ_h^+ and Φ_h^- together with their derivatives decay exponentially for each $0 < h < h_{**}$.

4. Upon introducing the spaces

$$X_h = \{\Theta \in \mathbf{H}^1 : \langle \Phi_h^-, \Theta \rangle = 0\} \quad (5.7)$$

and

$$Y_h = \{\Theta \in \mathbf{L}^2 : \langle \Phi_h^-, \Theta \rangle = 0\}, \quad (5.8)$$

the operator $L_h : X_h \rightarrow Y_h$ is invertible and there exists a constant $C_{\text{unif}} > 0$ such that for each $0 < h < h_{**}$ we have the uniform bound

$$\|L_h^{-1}\|_{\mathcal{B}(Y_h, X_h)} \leq C_{\text{unif}}. \quad (5.9)$$

A direct consequence of these results is that the zero eigenvalue of L_h is simple. In addition, these results allow us to construct a quasi-inverse for L_h that we use heavily in §6 and §7.

Corollary 5.3. *Assume that (HP1), (HS), (H α 1) and (H α 2) are satisfied. Then for any $0 < h < h_{**}$ the zero eigenvalue of L_h is simple.*

Proof. We can assume that $\langle \Phi_h^-, \Phi_h^+ \rangle \neq 0$ for all $0 < h < h_{**}$, since by Proposition 5.2 $\langle \Phi_h^-, \Phi_h^+ \rangle \rightarrow \langle \Phi_0^-, \Phi_0^+ \rangle \neq 0$. Equation (5.6) now implies that $\Phi_h^+ \notin \text{Range}(L_h)$, which together with (5.5) completes the proof. \blacksquare

Corollary 5.4. *Assume that (HP1), (HS), (H α 1) and (H α 2) are satisfied. There exist linear maps*

$$\begin{aligned}\gamma_h : \mathbf{L}^2 &\rightarrow \mathbb{R} \\ L_h^{\text{qinv}} : \mathbf{L}^2 &\rightarrow \mathbf{H}^1,\end{aligned}\quad (5.10)$$

such that for all $\Theta \in \mathbf{L}^2$ and each $0 < h < h_{**}$ the pair

$$(\gamma, \Psi) = (\gamma_h \Theta, L_h^{\text{qinv}} \Theta) \quad (5.11)$$

is the unique solution to the problem

$$L_h \Psi = \Theta + \gamma \Phi_h^+ \quad (5.12)$$

that satisfies the normalisation condition

$$\langle \Phi_h^-, \Psi \rangle = 0. \quad (5.13)$$

Proof. Fix $0 < h < h_{**}$ and $\Theta \in \mathbf{L}^2$. Upon defining

$$\gamma_h[\Theta] = -\frac{\langle \Phi_h^-, \Theta \rangle}{\langle \Phi_h^-, \Phi_h^+ \rangle}, \quad (5.14)$$

we see that $\Theta + \gamma_h[\Theta]\Phi_h^+ \in Y_h$. In particular, Proposition 5.2 implies that the problem

$$L_h \Psi = \Theta + \gamma_h[\Theta]\Phi_h^+ \quad (5.15)$$

has a unique solution $\Psi \in X_h$, which we refer to as $L_h^{\text{qinv}}\Theta$. \blacksquare

The results in [3, 39] allow us to read off the Fredholm properties of L_h from the behaviour of this operator in the limits $\xi \rightarrow \pm\infty$. In particular, we let $L_{h,\infty}$ be the operator defined by

$$\begin{aligned} L_{h,\infty} &= \begin{pmatrix} c_h \frac{d}{dx} - \Delta_h - \lim_{\xi \rightarrow \infty} g_u(\bar{u}_h(\xi)) & 1 \\ -\rho & c_h \frac{d}{dx} + \gamma\rho \end{pmatrix} \\ &= \begin{pmatrix} c_h \frac{d}{dx} - \Delta_h - g_u(0) & 1 \\ -\rho & c_h \frac{d}{dx} + \gamma\rho \end{pmatrix}. \end{aligned} \quad (5.16)$$

This system has constant coefficients. For $\lambda \in \mathbb{C}$ we introduce the notation

$$L_{h,\infty;\lambda} = L_{h,\infty} + \lambda. \quad (5.17)$$

We show that for λ in a suitable right half-plane the operator $L_{h,\infty;\lambda}$ is hyperbolic in the sense of [3], i.e. we write

$$\begin{aligned} \Delta_{L_{h,\infty;\lambda}}(z) &= [L_{h,\infty;\lambda} e^{z\xi}](0) \\ &= \begin{pmatrix} c_h z - \frac{1}{h^2} \left[\sum_{k>0} \alpha_k (e^{khz} + e^{-khz} - 2) \right] - g_u(0) + \lambda & 1 \\ -\rho & c_h z + \gamma\rho + \lambda \end{pmatrix} \end{aligned} \quad (5.18)$$

and show that $\det(\Delta_{L_{h,\infty;\lambda}}(iy)) \neq 0$ for all $y \in \mathbb{R}$. In the terminology of [3, 39], this means that $L_h + \lambda$ is asymptotically hyperbolic. This allows us to compute the Fredholm index of $L_h + \lambda$.

Remark 5.5. From this section onward we assume that (H α 2) is satisfied. This is done for technical reasons, allowing us to apply the results from [3]. In particular, this condition implies that the function $\Delta_{L_{h,\infty;\lambda}}(z)$ defined in (5.18) is well-defined in a vertical strip $|\text{Re}(z)| < \nu$.

Lemma 5.6. *Assume that (HP1), (HS), (H α 1) and (H α 2) are satisfied. There exists a constant $\tilde{\lambda} > 0$ such that for all $0 < h < \min\{h_*, \bar{h}\}$ and all $\lambda \in \mathbb{C}$ with $\text{Re } \lambda > -\tilde{\lambda}$ the operator $L_{h,\infty;\lambda}$ is hyperbolic and thus the operator $L_h + \lambda$ is asymptotically hyperbolic.*

Proof. Remembering that $-g_u(0) = r_0 > 0$ and picking $y \in \mathbb{R}$, we compute

$$\begin{aligned} \Delta_{L_{h,\infty;\lambda}}(iy) &= \begin{pmatrix} c_h iy + \frac{1}{h^2} \left[\sum_{k>0} \alpha_k (2 - 2 \cos(khy)) \right] + r_0 + \lambda & 1 \\ -\rho & c_h iy + \gamma\rho \end{pmatrix} \\ &= \begin{pmatrix} c_h iy + \frac{1}{h^2} A(hy) + r_0 + \lambda & 1 \\ -\rho & c_h iy + \gamma\rho + \lambda \end{pmatrix}, \end{aligned} \quad (5.19)$$

where $A(hy) \geq 0$ is defined in (H α 1). We hence see

$$\det(\Delta_{L_{h,\infty;\lambda}}(iy)) = \left(c_h iy + \frac{1}{h^2} A(hy) + r_0 + \lambda \right) (c_h iy + \gamma\rho + \lambda) + \rho. \quad (5.20)$$

Let $\tilde{\lambda} = \frac{1}{4} \min\{\gamma\rho, r_0\}$ and assume that $\operatorname{Re} \lambda > -\tilde{\lambda}$. If $y \neq -\frac{\operatorname{Im} \lambda}{c_h}$ then we obtain

$$\begin{aligned} \operatorname{Im} \left(\det(\Delta_{L_{h,\infty;\lambda}}(iy)) \right) &= (c_h y + \operatorname{Im} \lambda)(\gamma\rho + \operatorname{Re} \lambda) + \left(\frac{1}{h^2} A(hy) + r_0 + \operatorname{Re} \lambda\right)(c_h y + \operatorname{Im} \lambda) \\ &= (c_h y + \operatorname{Im} \lambda)(\gamma\rho + \frac{1}{h^2} A(hy) + r_0 + 2 \operatorname{Re} \lambda) \\ &\neq 0, \end{aligned} \tag{5.21}$$

since $\gamma\rho + \frac{1}{h^2} A(hy) + r_0 + \operatorname{Re} \lambda > 0$. For $y = -\frac{\operatorname{Im} \lambda}{c_h}$ we obtain

$$\begin{aligned} \operatorname{Re} \left(\det(\Delta_{L_{h,\infty;\lambda}}(y)) \right) &= \left(\frac{1}{h^2} A(hy) + r_0 + \operatorname{Re} \lambda\right)(\gamma\rho + \operatorname{Re} \lambda) + \rho \\ &> \rho \\ &> 0. \end{aligned} \tag{5.22}$$

In particular, we see that $\det(\Delta_{L_{h,\infty;\lambda}}(iy)) \neq 0$ for all $y \in \mathbb{R}$, as desired. \blacksquare

Before we consider the Fredholm properties of $L_h + \lambda$, we establish a technical estimate for the function $\Delta_{L_{h,\infty;\lambda}}$, which we need in §7 later on.

Lemma 5.7. *Assume that (HP1), (HS), (H α 1) and (H α 2) are satisfied. Fix $0 < h < \min\{h_*, \bar{h}\}$ and $S \subset \mathbb{C}$ compact in such a way that $\operatorname{Re} \lambda > -\tilde{\lambda}$ for all $\lambda \in S$. Then there exist constants $\kappa > 0$ and $\Gamma > 0$, possibly depending on h and S , such that for all $z = x + iy \in \mathbb{C}$ with $|x| \leq \kappa$ and all $\lambda \in S$ we have the bound*

$$|\det(\Delta_{L_{h,\infty;\lambda}}(z))| \geq \frac{1}{\Gamma}. \tag{5.23}$$

Proof. Using assumption (H α 2) we can pick $\kappa_1 > 0$ and $\Gamma_1 > 0$ in such a way that the bound

$$\begin{aligned} \left| \frac{1}{h^2} A(hz) \right| &:= \left| \frac{1}{h^2} \left[\sum_{k>0} \alpha_k (2 - e^{khz} - e^{-khz}) \right] \right| \\ &\leq \frac{1}{h^2} \sum_{k>0} |\alpha_k| (e^{hk|x|} + 3) \\ &\leq \Gamma_1 \end{aligned} \tag{5.24}$$

holds for all $z = x + iy \in \mathbb{C}$ with $|x| \leq \kappa_1$.

Observe that for $z = x + iy \in \mathbb{C}$ and $\lambda \in S$ we have

$$\begin{aligned} \operatorname{Re} \left(\det(\Delta_{L_{h,\infty;\lambda}}(z)) \right) &= \left(c_h x + \frac{1}{h^2} \operatorname{Re} A(hz) + r_0 + \operatorname{Re} \lambda \right) (c_h x + \gamma\rho + \operatorname{Re} \lambda) \\ &\quad - (c_h y + \operatorname{Im} \lambda)^2 - (c_h y + \operatorname{Im} \lambda) \frac{1}{h^2} (\operatorname{Im} A(y)) + \rho. \end{aligned} \tag{5.25}$$

Since S is compact we can find $Y > 0$ such that for all $z = x + iy \in \mathbb{C}$ with $|y| \geq Y$ and $|x| \leq \kappa_1$ and all $\lambda \in S$ we have

$$\begin{aligned} \left| \operatorname{Re} \left(\det(\Delta_{L_{h,\infty;\lambda}}(z)) \right) \right| &\geq \frac{1}{2} c_h^2 y^2 \\ &\geq \frac{1}{2} c_h^2 Y^2. \end{aligned} \tag{5.26}$$

Seeking a contradiction, let us assume that for each $0 < \kappa \leq \kappa_1$ and each $\Gamma > 0$ there exist $\lambda \in S$ and $z = x + iy \in \mathbb{C}$ with $|x| \leq \kappa$ and $|y| \leq Y$ for which

$$|\det(\Delta_{L_{h,\infty;\lambda}}(z))| < \frac{1}{\Gamma}. \tag{5.27}$$

Then we can construct a sequence $\{\kappa_n, z_n, \lambda_n\}$ with $0 < \kappa_n \leq \kappa_1$ for each n , $\kappa_n \rightarrow 0$, $\lambda_n \in S$ for each n and $z_n = x_n + iy_n \in \mathbb{C}$ with $|x_n| \leq \kappa_n$ and $|y_n| \leq Y$ in such a way that $|\det(\Delta_{L_{h,\infty;\lambda_n}}(z_n))| < \frac{1}{n}$ for each n . By taking a subsequence if necessary we see that $\lambda_n \rightarrow \lambda$ for some $\lambda \in S$ and $z_n \rightarrow iy$

for some $y \in \mathbb{R}$ with $|y| \leq Y$. Since $\det(\Delta_{L_h, \infty; \lambda}(z))$ is continuous as a function of λ and z it follows that

$$\begin{aligned} \det(\Delta_{L_h, \infty; \lambda}(iy)) &= \lim_{n \rightarrow \infty} \det(\Delta_{L_h, \infty; \lambda_n}(z_n)) \\ &= 0, \end{aligned} \tag{5.28}$$

which contradicts Lemma 5.6. Hence we can find $\kappa > 0$ and $\Gamma > 0$ as desired. \blacksquare

Proof of Proposition 5.1. We have already seen in Lemma 5.6 that $L_h + \lambda$ is asymptotically hyperbolic in the sense of [3]. Now according to [3, Theorem 1.4], which is a variation of [37, Theorem 2], we obtain that $L_h + \lambda$ is a Fredholm operator and that the following identities hold

$$\begin{aligned} \dim(\ker(L_h + \lambda)) &= \operatorname{codim}(\operatorname{Range}(L_h^* + \bar{\lambda})), \\ \operatorname{codim}(\operatorname{Range}(L_h + \lambda)) &= \dim(\ker(L_h^* + \bar{\lambda})), \\ \operatorname{ind}(L_h + \lambda) &= -\operatorname{ind}(L_h^* + \bar{\lambda}), \end{aligned} \tag{5.29}$$

where

$$\operatorname{ind}(L_h + \lambda) = \dim(\ker(L_h + \lambda)) - \operatorname{codim}(\operatorname{Range}(L_h + \lambda)) \tag{5.30}$$

is the Fredholm index of $L_h + \lambda$.

We follow the proof of [32, Theorem B]. For $0 \leq \vartheta \leq 1$, we let the operator $L^\vartheta(h)$ be defined by

$$L^\vartheta(h) = (1 - \vartheta)(L_h + \lambda) + \vartheta(L_{h, \infty} + \lambda). \tag{5.31}$$

Note that the operator $L^\vartheta(h)$ is asymptotically hyperbolic for each ϑ and thus [3, Theorem 1.4] implies that these operators $L^\vartheta(h)$ are Fredholm. Moreover, the family $L^\vartheta(h)$ varies continuously with ϑ in $\mathcal{B}(\mathbf{H}^1, \mathbf{L}^2)$, which means the Fredholm index is constant. In particular, we see that

$$\begin{aligned} \operatorname{ind}(L_h + \lambda) &= \operatorname{ind}(L_{h, \infty} + \lambda) \\ &= 0, \end{aligned} \tag{5.32}$$

where the last equality follows from [3, Theorem 1.4]. \blacksquare

We can now concentrate on the kernel of L_h . The goal is to develop a Liapunov-Schmidt argument to exclude kernel elements other than Φ_h^+ . In order to accomplish this, we construct a quasi-inverse for L_h by mimicking the approach of [23, Proposition 3.2]. As a preparation, we obtain the following technical result.

Lemma 5.8. *Assume that (HP1), (HS) and (H α 1) are satisfied. Recall the constant δ_0 from Lemma 3.1. Let $0 < \lambda < \min\{\frac{1}{2}, \delta_0\}$ be given. Then there exists constant $0 < h_1^* \leq \min\{h_*, \bar{h}\}$ and $\kappa > 0$ such that for all $0 < h \leq h_1^*$ we have*

$$\begin{aligned} \langle \Phi_0^-, (L_h + \lambda)^{-1} \Phi_0^+ \rangle &> \frac{1}{2} \lambda^{-1} \langle \Phi_0^-, \Phi_h^+ \rangle \\ &> \frac{1}{2} \lambda^{-1} \kappa \\ &> 0. \end{aligned} \tag{5.33}$$

Proof. We know from Lemma 3.1 that $\langle \Phi_0^-, \Phi_0^+ \rangle > 0$. Since Φ_h^+ converges to Φ_0^+ in \mathbf{L}^2 , it follows that $\langle \Phi_0^-, \Phi_h^+ \rangle$ converges to $\langle \Phi_0^-, \Phi_0^+ \rangle > 0$. Fix $h_1^* \leq \min\{h_*, \bar{h}, h'_0(\lambda)\}$ in such a way that

$$\|\Phi_0^+ - \Phi_h^+\|_{\mathbf{L}^2} < \frac{1}{2} \frac{\langle \Phi_0^-, \Phi_h^+ \rangle}{2C_{\text{unif}}} \tag{5.34}$$

holds for all $0 \leq h \leq h_1^*$, where

$$C_{\text{unif}} = 4C'_0 \quad (5.35)$$

and C'_0 is defined in Proposition 3.2. The factor 4 in the definition is for technical reasons in a later proof. We assume from now on that $0 < h \leq h_1^*$. Using $L_h \Phi_h^+ = 0$ we readily see

$$(L_h + \lambda)^{-1} \Phi_h^+ = \lambda^{-1} \Phi_h^+. \quad (5.36)$$

Recall that $\|\Phi_0^-\|_{\mathbf{L}^2} = 1$. Since $1 < \lambda^{-1}$, we may use Proposition 5.2 to obtain

$$\begin{aligned} \|(L_h + \lambda)^{-1} \Phi_0^+ - \lambda^{-1} \Phi_h^+\|_{\mathbf{L}^2} &= \|(L_h + \lambda)^{-1} [\Phi_0^+ - \Phi_h^+]\|_{\mathbf{L}^2} \\ &\leq C_{\text{unif}} \left[\|\Phi_h^+ - \Phi_0^+\|_{\mathbf{L}^2} + \lambda^{-1} |\langle \Phi_h^+ - \Phi_0^+, \Phi_0^- \rangle| \right] \\ &< C_{\text{unif}} \lambda^{-1} \|\Phi_0^+ - \Phi_h^+\|_{\mathbf{L}^2} (1 + \|\Phi_0^-\|_{\mathbf{L}^2}) \\ &= 2C_{\text{unif}} \lambda^{-1} \|\Phi_0^+ - \Phi_h^+\|_{\mathbf{L}^2}. \end{aligned} \quad (5.37)$$

Remembering $\langle \Phi_0^-, \Phi_h^+ \rangle > 0$ and using Cauchy-Schwarz, we see that

$$\begin{aligned} \left| \left\langle \frac{\Phi_0^-}{\langle \Phi_0^-, \Phi_h^+ \rangle}, (L_h + \lambda)^{-1} \Phi_0^+ - \lambda^{-1} \Phi_h^+ \right\rangle \right| &= \left| \left\langle \frac{\Phi_0^-}{\langle \Phi_0^-, \Phi_h^+ \rangle}, (L_h + \lambda)^{-1} \Phi_0^+ - \lambda^{-1} \Phi_h^+ \right\rangle \right| \\ &< \frac{\|\Phi_0^-\|_{\mathbf{L}^2}}{\langle \Phi_0^-, \Phi_h^+ \rangle} 2C_{\text{unif}} \lambda^{-1} \|\Phi_0^+ - \Phi_h^+\|_{\mathbf{L}^2} \\ &\leq \frac{1}{\langle \Phi_0^-, \Phi_h^+ \rangle} 2C_{\text{unif}} \lambda^{-1} \frac{1}{2} \frac{\langle \Phi_0^-, \Phi_h^+ \rangle}{2C_{\text{unif}}} \\ &= \frac{1}{2} \lambda^{-1}. \end{aligned} \quad (5.38)$$

Hence we must have

$$\langle \Phi_0^-, (L_h + \lambda)^{-1} \Phi_0^+ \rangle > \frac{1}{2} \lambda^{-1} \langle \Phi_0^-, \Phi_h^+ \rangle > 0. \quad (5.39)$$

■

Lemma 5.9. *Assume that (HP1), (HS) and (H α 1) are satisfied. There exists $0 < h_{**} \leq \min\{h_*, \bar{h}\}$ together with linear maps*

$$\begin{aligned} \tilde{\gamma}_h^+ : \mathbf{L}^2 &\rightarrow \mathbb{R} \\ \tilde{L}_h^{\text{qinv}} : \mathbf{L}^2 &\rightarrow \mathbf{H}^1, \end{aligned} \quad (5.40)$$

defined for all $0 < h < h_{**}$, such that for all $\Theta \in \mathbf{L}^2$ the pair

$$(\gamma, \Psi) = (\tilde{\gamma}_h^+ \Theta, \tilde{L}_h^{\text{qinv}} \Theta) \quad (5.41)$$

is the unique solution to the problem

$$L_h \Psi = \Theta + \gamma \Phi_0^+ \quad (5.42)$$

that satisfies the normalisation condition

$$\langle \Phi_0^-, \Psi \rangle = 0. \quad (5.43)$$

In addition, there exists $C > 0$ such that for all $0 < h < h_{**}$ and all $\Theta \in \mathbf{L}^2$ we have the bound

$$|\tilde{\gamma}_h^+ \Theta| + \|\tilde{L}_h^{\text{qinv}} \Theta\|_{\mathbf{H}^1} \leq C \|\Theta\|_{\mathbf{L}^2}. \quad (5.44)$$

Proof. Fix $0 < \lambda < \min\{\frac{1}{2}, \delta_0\}$ and let $0 < h \leq \min\{h_*, \bar{h}, h'_0(\lambda)\}$ be given, where $h'_0(\lambda)$ is defined in Proposition 3.2. We define the set

$$Z^1 = \{\Psi \in \mathbf{H}^1 : \langle \Phi_0^-, \Psi \rangle = 0\}. \quad (5.45)$$

Pick $\Theta \in \mathbf{L}^2$. We look for a solution $(\gamma, \Psi) \in \mathbb{R} \times Z^1$ of the problem

$$\Psi = (L_h + \lambda)^{-1}[\Theta + \gamma\Phi_0^+ + \lambda\Psi]. \quad (5.46)$$

By Lemma 5.8 we have $\langle \Phi_0^-, (L_h + \lambda)^{-1}\Phi_0^+ \rangle \neq 0$. Hence for given $\Theta \in \mathbf{L}^2, \Psi \in Z^1, h, \lambda$, we may write

$$\gamma(\Psi, \Theta, h, \lambda) = -\frac{\langle \Phi_0^-, (L_h + \lambda)^{-1}(\Theta + \lambda\Psi) \rangle}{\langle \Phi_0^-, (L_h + \lambda)^{-1}\Phi_0^+ \rangle}, \quad (5.47)$$

which is the unique value for γ for which

$$(L_h + \lambda)^{-1}[\Theta + \gamma\Phi_0^+ + \lambda\Psi] \in Z^1. \quad (5.48)$$

Recall the constant C_{unif} from (5.35). With Proposition 3.2 we obtain

$$\begin{aligned} |\langle \Phi_0^-, (L_h + \lambda)^{-1}(\Theta + \lambda\Psi) \rangle| &\leq \|\Phi_0^-\|_{\mathbf{L}^2} C_{\text{unif}} \left[\|\Theta + \lambda\Psi\|_{\mathbf{L}^2} + \frac{1}{\lambda} |\langle \Theta + \lambda\Psi, \Phi_0^- \rangle| \right] \\ &\leq \|\Phi_0^-\|_{\mathbf{L}^2} C_{\text{unif}} \left[\left(1 + \frac{1}{\lambda}\right) \|\Theta\|_{\mathbf{L}^2} + \lambda \|\Psi\|_{\mathbf{L}^2} \right] \\ &\leq C_1 \left[\lambda^{-1} \|\Theta\|_{\mathbf{L}^2} + \lambda \|\Psi\|_{\mathbf{L}^2} \right] \end{aligned} \quad (5.49)$$

for some C_1 that is independent of h, λ . Here we used that $\lambda < 1$ and thus $1 + \frac{1}{\lambda} < \frac{2}{\lambda}$. Exploiting $\lambda < \frac{1}{2}$ and applying Lemma 5.8, we see that

$$\begin{aligned} |\gamma(\Psi, \Theta, h, \lambda)| &= |\langle \Phi_0^-, (L_h + \lambda)^{-1}(\Theta + \lambda\Psi) \rangle| \frac{1}{|\langle \Phi_0^-, (L_h + \lambda)^{-1}\Phi_0^+ \rangle|} \\ &\leq C_1 \left[\lambda^{-1} \|\Theta\|_{\mathbf{L}^2} + \lambda \|\Psi\|_{\mathbf{L}^2} \right] \frac{1}{\frac{1}{2}\lambda^{-1} \langle \Phi_0^-, \Phi_h^+ \rangle} \\ &\leq C_1 \left[\|\Theta\|_{\mathbf{L}^2} + \kappa\lambda^2 \|\Psi\|_{\mathbf{L}^2} \right] \\ &\leq C_2 \left[\|\Theta\|_{\mathbf{L}^2} + \lambda^2 \|\Psi\|_{\mathbf{L}^2} \right]. \end{aligned} \quad (5.50)$$

Here we used that $\langle \Phi_0^-, \Phi_h^+ \rangle$ converges to $\langle \Phi_0^-, \Phi_0^+ \rangle > 0$, which means that $\langle \Phi_0^-, \Phi_h^+ \rangle$ can be bounded away from zero. For $\Psi \in Z^1$ we write

$$t(\Psi) = \Theta + \gamma(\Psi, \Theta, h, \lambda)\Phi_0^+ + \lambda\Psi \quad (5.51)$$

and

$$T(\Psi) = (L_h + \lambda)^{-1}t(\Psi). \quad (5.52)$$

For $\Psi \in Z^1$ Proposition 3.2 implies

$$\begin{aligned} \|T(\Psi)\|_{\mathbf{H}^1} &\leq C_{\text{unif}} \left[\|\Theta + \gamma(\Psi, \Theta, h, \lambda)\Phi_0^+ + \lambda\Psi\|_{\mathbf{L}^2} \right. \\ &\quad \left. + \frac{1}{\lambda} |\langle \Theta + \gamma(\Psi, \Theta, h, \lambda)\Phi_0^+ + \lambda\Psi, \Phi_0^- \rangle| \right] \\ &\leq C_3 \left[\frac{1}{\lambda} \|\Theta\|_{\mathbf{L}^2} + \lambda \|\Psi\|_{\mathbf{L}^2} \right] \\ &\leq C_3 \left[\frac{1}{\lambda} \|\Theta\|_{\mathbf{L}^2} + \lambda \|\Psi\|_{\mathbf{H}^1} \right], \end{aligned} \quad (5.53)$$

since $\|\Phi_h^+\|_{\mathbf{L}^2}$ can be uniformly bounded by a constant for $0 < h \leq h_*$. For $\Psi_1, \Psi_2 \in Z^1$, a second application of Proposition 3.2 yields

$$\begin{aligned}
|\gamma(\Psi_1, \Theta, h, \lambda) - \gamma(\Psi_2, \Theta, h, \lambda)| &= \left| \frac{\langle \Phi_0^-, (L_h + \lambda)^{-1} (\lambda \Psi_1 - \lambda \Psi_2) \rangle}{\langle \Phi_0^-, (L_h + \lambda)^{-1} \Phi_0^+ \rangle} \right| \\
&\leq \frac{1}{\langle \Phi_0^-, (L_h + \lambda)^{-1} \Phi_0^+ \rangle} C_{\text{unif}} \left[\lambda \|\Psi_1 - \Psi_2\|_{\mathbf{L}^2} + \frac{1}{\lambda} |\langle \lambda \Psi_1 - \lambda \Psi_2, \Phi_0^- \rangle| \right] \\
&\leq C_4 \lambda \left[\lambda \|\Psi_1 - \Psi_2\|_{\mathbf{L}^2} + 0 \right] \\
&\leq C_4 \lambda^2 \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}.
\end{aligned} \tag{5.54}$$

Applying Proposition 3.2 for the final time, we see

$$\begin{aligned}
\|T(\Psi_1) - T(\Psi_2)\|_{\mathbf{H}^1} &\leq C_{\text{unif}} \left[\|t(\Psi_1) - t(\Psi_2)\|_{\mathbf{L}^2} + \frac{1}{\lambda} |\langle t(\Psi_1) - t(\Psi_2), \Phi_0^- \rangle| \right] \\
&\leq C_{\text{unif}} \left[\|t(\Psi_1) - t(\Psi_2)\|_{\mathbf{L}^2} \right. \\
&\quad \left. + \frac{1}{\lambda} (\langle \gamma(\Psi_1, \Theta, h, \lambda) - \gamma(\Psi_2, \Theta, h, \lambda) \Phi_0^+ + \lambda(\Psi_1, \Psi_2), \Phi_0^- \rangle) \right] \\
&\leq C_{\text{unif}} \left[\|t(\Psi_1) - t(\Psi_2)\|_{\mathbf{L}^2} + \frac{1}{\lambda} (C_4 \lambda^2 \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} + 0) \right] \\
&\leq C_{\text{unif}} C_4 \lambda^2 \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} + C_{\text{unif}} \lambda \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} + C_4 \lambda \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1} \\
&\leq C_5 \lambda \|\Psi_1 - \Psi_2\|_{\mathbf{H}^1}.
\end{aligned} \tag{5.55}$$

In view of these bounds, we pick λ to be small enough to have $C_3 \lambda < \frac{1}{2}$ and $C_5 \lambda < \frac{1}{2}$. In addition, we write $h_{**} = \min\{h_1^*, h'_0(\lambda)\}$ and pick $0 < h < h_{**}$. Then $T : Z^1 \rightarrow Z^1$ is a contraction, so the fixed point theorem implies that there is a unique $\tilde{L}_h^{\text{qinv}}(\Theta) \in Z^1$ for which

$$\tilde{L}_h^{\text{qinv}}(\Theta) = (L_h + \lambda)^{-1} \left[\Theta + \gamma(\tilde{L}_h^{\text{qinv}}(\Theta), \Theta, h, \lambda) \Phi_0^+ + \lambda \tilde{L}_h^{\text{qinv}}(\Theta) \right]. \tag{5.56}$$

Furthermore, we have

$$\begin{aligned}
\frac{1}{2} \|\tilde{L}_h^{\text{qinv}}(\Theta)\|_{\mathbf{H}^1} &\leq (1 - \lambda C_3) \|\tilde{L}_h^{\text{qinv}}(\Theta)\|_{\mathbf{H}^1} \\
&\leq C_3 \lambda^{-1} \|\Theta\|_{\mathbf{L}^2} \\
&\leq C_4 \|\Theta\|_{\mathbf{L}^2}.
\end{aligned} \tag{5.57}$$

Writing $\tilde{\gamma}_h^+(\Theta) = \gamma(\tilde{L}_h^{\text{qinv}}(\Theta), \Theta, h, \lambda)$, we compute

$$\begin{aligned}
|\tilde{\gamma}_h^+(\Theta)| &\leq C_4 [\|\Theta\|_{\mathbf{L}^2} + \lambda^2 \lambda^{-1} \|\Theta\|_{\mathbf{L}^2}] \\
&\leq C_5 \|\Theta\|_{\mathbf{L}^2}.
\end{aligned} \tag{5.58}$$

Finally we see that (5.46) is in fact equivalent to (5.42)-(5.43), so in fact $\tilde{L}_h^{\text{qinv}}(\Theta)$ and $\tilde{\gamma}_h^+(\Theta)$ do not depend on λ . \blacksquare

With this quasi-inverse in hand we are able to characterize the kernel of L_h . Indeed, Lemma 5.9 allows us to generalize the Liapunov-Schmidt argument developed in [22] to our current singularly perturbed setting.

Lemma 5.10. *Assume that (HP1), (HS), (H α 1) and (H α 2) are satisfied. Let $0 < h < h_{**}$ be given. Then we have the inclusion*

$$\begin{aligned}
\text{span}\{\Phi_h^+\} &\subset \ker(L_h) \\
&= \{\Psi \in \mathbf{L}^2 : \langle \Psi, \Theta \rangle = 0 \text{ for all } \Theta \in \text{Range}(L_h^*)\},
\end{aligned} \tag{5.59}$$

where L_h^* is the formal adjoint of L_h .

Proof. By differentiating the differential equation (2.12) we see that $L_h \Phi_h^+ = 0$. We know that $(\bar{u}_h, \bar{w}_h) - (\bar{u}_0, \bar{w}_0) \rightarrow 0 \in \mathbf{H}^1$. Since (\bar{u}'_0, \bar{w}'_0) decays exponentially, we get $(\bar{u}'_0, \bar{w}'_0) \in \mathbf{L}^2$. Hence we can assume that h_{**} is small enough such that $\Phi_h^+ \in \mathbf{L}^2$ for all $0 < h < h_{**}$. Since $L_h \Phi_h^+ = 0$ we obtain from the differential equation that also $(\Phi_h^+)' \in \mathbf{L}^2$. In particular, we see that $\Phi_h^+ \in \mathbf{H}^1$ and hence $\Phi_h^+ \in \ker(L_h)$. \blacksquare

Lemma 5.11. *Assume that (HP1), (HS), (H α 1) and (H α 2) are satisfied. Let $0 < h < h_{**}$ be given. Then we have*

$$\begin{aligned} \ker(L_h) &= \text{span}\{\Phi_h^+\} \\ &= \{\Psi \in \mathbf{L}^2 : \langle \Psi, \Theta \rangle = 0 \text{ for all } \Theta \in \text{Range}(L_h^*)\}, \end{aligned} \quad (5.60)$$

where L_h^* is the formal adjoint of L_h .

Proof. We show that $\dim(\ker(L_h)) = 1$. We follow the steps on page 11 and Proposition 3.3 of [22]. For notational consistency we write

$$\begin{aligned} L_0 &= \mathcal{L}_0^+ \\ L_0^* &= \mathcal{L}_0^-, \end{aligned} \quad (5.61)$$

where the operators \mathcal{L}_0^\pm were defined in §3. We introduce the shorthand

$$\mathcal{R} = \text{Range}(L_0), \quad (5.62)$$

together with

$$\begin{aligned} \mathcal{K} &= \ker(L_0) \\ &= \text{span}\{\Phi_0^+\}. \end{aligned} \quad (5.63)$$

Moreover we define

$$\mathcal{K}_c = \{\Psi \in \mathbf{H}^1 : \langle \Psi, \Phi_0^- \rangle = 0\} \quad (5.64)$$

and

$$\mathcal{R}_c = \text{span}\{\Phi_0^+\}. \quad (5.65)$$

From Lemma 3.1 it follows that we can decompose \mathbf{H}^1 and \mathbf{L}^2 as $\mathbf{H}^1 = \mathcal{K} \oplus \mathcal{K}_c$ and $\mathbf{L}^2 = \mathcal{R} \oplus \mathcal{R}_c$ respectively.

Fix $0 < h < h_{**}$. For $B \in \{\mathcal{K}, \mathcal{K}_c, \mathcal{R}, \mathcal{R}_c\}$ let π_B denote the projection from \mathbf{H}^1 or \mathbf{L}^2 onto B , corresponding to the above decompositions. We first show that

$$\pi_{\mathcal{R}} L_h : \mathcal{K}_c \rightarrow \mathcal{R} \quad (5.66)$$

is invertible.

Consider any $\Psi \in \mathcal{K}_c$ for which $\pi_{\mathcal{R}} L_h \Psi = 0$. This means that $L_h \Psi \in \mathcal{R}_c$ and hence $L_h \Psi = \mu \Phi_0^+$ for some $\mu \in \mathbb{R}$. Since $\Psi \in \mathcal{K}_c$ we must have $\langle \Psi, \Phi_0^- \rangle = 0$. Lemma 5.9 now implies

$$\begin{aligned} \mu &= \gamma_h^+[0] \\ &= 0, \\ \Psi &= \tilde{L}_h^{\text{qinv}}[0] \\ &= 0, \end{aligned} \quad (5.67)$$

by linearity of γ_h^+ and $\tilde{L}_h^{\text{qinv}}$. In particular, we must have $\Psi = 0$, which means that $\pi_{\mathcal{R}} L_h$ is injective.

Pick $\Theta \in \mathcal{R}$. Lemma 5.9 implies that we can choose $\Psi \in \mathcal{K}_c$ and $\gamma \in \mathbb{R}$ for which

$$L_h \Psi = \Theta + \gamma \Phi_0^+. \quad (5.68)$$

Since $\gamma \Phi_0^+ \in \mathcal{R}_c$ we have $\pi_{\mathcal{R}}(\gamma \Phi_0^+) = 0$. This allows us to compute

$$\begin{aligned} \pi_{\mathcal{R}}[L_h \Psi] &= \pi_{\mathcal{R}}[L_h \Psi - \gamma \Phi_0^+] \\ &= \pi_{\mathcal{R}}[\Theta] \\ &= \Theta, \end{aligned} \quad (5.69)$$

since $\Theta \in \mathcal{R}$. Hence $\pi_{\mathcal{R}} L_h$ is also surjective and hence invertible as a map from \mathcal{K}_c to \mathcal{R} .

We now consider the map

$$\begin{aligned} \mu(h) : \mathcal{K} &\rightarrow \mathcal{K}_c, \\ B &\mapsto -[\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} L_h B. \end{aligned} \quad (5.70)$$

We claim that looking for a solution Ψ of $L_h \Psi = \Theta$ is equivalent to finding a solution $B \in \mathcal{K}$ of the problem

$$\pi_{\mathcal{R}_c} L_h (B + \mu(h)B) = \pi_{\mathcal{R}_c} \Theta - \pi_{\mathcal{R}_c} L_h [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} \Theta. \quad (5.71)$$

Indeed, let $B \in \mathcal{K}$ be a solution of (5.71) and write

$$\Psi = [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} \Theta + B + \mu(h)B. \quad (5.72)$$

By substitution it is clear that

$$\begin{aligned} \pi_{\mathcal{R}_c} L_h \Psi &= \pi_{\mathcal{R}_c} L_h [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} \Theta + \pi_{\mathcal{R}_c} L_h (B + \mu(h)B) \\ &= \pi_{\mathcal{R}_c} \Theta. \end{aligned} \quad (5.73)$$

Furthermore we see that

$$\begin{aligned} \pi_{\mathcal{R}} L_h \Psi &= \pi_{\mathcal{R}} L_h [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} \Theta + \pi_{\mathcal{R}} L_h B + \pi_{\mathcal{R}} L_h \mu(h)B \\ &= \pi_{\mathcal{R}} \Theta + \pi_{\mathcal{R}} L_h B - \pi_{\mathcal{R}} L_h [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} L_h B \\ &= \pi_{\mathcal{R}} \Theta \end{aligned} \quad (5.74)$$

by our choice of $\mu(h)$, which gives $L_h \Psi = \Theta$. Clearly each different choice of B gives a different choice of Ψ .

Conversely, let Ψ be a solution of $L_h \Psi = \Theta$. Writing $B = \pi_{\mathcal{K}} \Psi \in \mathcal{K}$, we compute

$$\begin{aligned} B + \mu(h)B &= \pi_{\mathcal{K}} \Psi - [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} L_h [\pi_{\mathcal{K}} \Psi] \\ &= \pi_{\mathcal{K}} \Psi - [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} \Theta + [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} L_h [\pi_{\mathcal{K}_c} \Psi] \\ &= \pi_{\mathcal{K}} \Psi - [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} \Theta + \pi_{\mathcal{K}_c} \Psi \\ &= x - [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} \Theta, \end{aligned} \quad (5.75)$$

which shows that B satisfies (5.71). Finally, upon defining

$$\tilde{\Psi} = [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} \Theta + B + \mu(h)B, \quad (5.76)$$

we see that

$$\begin{aligned} \tilde{\Psi} &= [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} \Theta + B + \mu(h)B \\ &= [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} \Theta + \Psi - [\pi_{\mathcal{R}} L_h]^{-1} \pi_{\mathcal{R}} \Theta \\ &= \Psi. \end{aligned} \quad (5.77)$$

In particular, there is a bijection between the set of solutions Ψ of $L_h\Psi = \Theta$ and the solutions of equation (5.71) and this bijection is linear if $\Theta = 0$. From this we can derive

$$\begin{aligned} \dim(\ker(L_h)) &= \dim\left(\ker\left(\pi_{\mathcal{R}_c}[L_h](I + \mu(h))\right)\right) \\ &= \dim(\mathcal{K}) - \dim\left(\text{Range}\left(\pi_{\mathcal{R}_c}[L_h](I + \mu(h))\right)\right) \\ &\leq \dim(\mathcal{K}) \\ &= 1. \end{aligned} \tag{5.78}$$

Since we already know that $\Phi_h^+ \in \ker(L_h)$, we must have $\dim(\ker(L_h)) = 1$, which completes the proof. \blacksquare

The remaining major goal of this section is to find a family of elements $\Phi_h^- \in \ker(L_h^*)$ which satisfies $\Phi_h^- \rightarrow \Phi_0^-$ as $h \downarrow 0$. To establish this, we repeat part of the process above for the adjoint operator L_h^* . The key difference is that we must construct the family Φ_h^- by hand. This requires a significant refinement of the Liapunov-Schmidt argument used above to characterize $\ker(L_h^*)$. First we need a technical result, similar to Lemma 5.8.

Lemma 5.12. *Assume that (HP1), (HS) and (H α 1) are satisfied. Fix $0 < \lambda < \frac{1}{2}$ and $0 \leq h \leq \min\{h_{**}, h'_0(\lambda)\}$, where $h'_0(\lambda)$ is defined in Proposition 3.2. Then we have*

$$\langle \Phi_0^+, (L_h^* + \lambda)^{-1}\Phi_0^- \rangle > \frac{\langle \Phi_0^+, \Phi_0^- \rangle}{2} \lambda^{-1}. \tag{5.79}$$

Proof. Lemma 3.1 implies that $\langle \Phi_0^+, \Phi_0^- \rangle > 0$. Remembering that

$$L_h^* - L_0^* = \begin{pmatrix} (c_0 + c_h)\frac{d}{dx} + (\Delta_h - \frac{d^2}{dx^2}) & 0 \\ 0 & (c_0 - c_h)\frac{d}{dx} \end{pmatrix} \tag{5.80}$$

and that $L_0^*\Phi_0^- = 0$, we obtain

$$\begin{aligned} (L_h^* + \lambda)\left[(L_h^* + \lambda)^{-1}\Phi_0^- - (L_0^* + \lambda)^{-1}\Phi_0^-\right] &= \Phi_0^- - \Phi_0^- + (L_h^* - L_0^*)(L_0^* + \lambda)^{-1}\Phi_0^- \\ &= (L_h^* - L_0^*)\lambda^{-1}\Phi_0^-. \end{aligned} \tag{5.81}$$

Recall the constant C_{unif} from (5.35). Proposition 3.2 yields

$$\begin{aligned} \|(L_h^* + \lambda)^{-1}\Phi_0^- - (L_0^* + \lambda)^{-1}\Phi_0^-\|_{\mathbf{L}^2} &\leq C_{\text{unif}}\left[\|(L_h^* - L_0^*)\lambda^{-1}\Phi_0^-\|_{\mathbf{L}^2} + |\langle (L_h^* - L_0^*)\lambda^{-1}\Phi_0^-, \Phi_0^+ \rangle|\right] \\ &\leq C_{\text{unif}}(1 + \lambda^{-1})\|(L_h^* - L_0^*)\lambda^{-1}\Phi_0^-\|_{\mathbf{L}^2}. \end{aligned} \tag{5.82}$$

Using Lemma 3.5 and the fact that c_h converges to c_0 , it follows that

$$C_{\text{unif}}(1 + \lambda^{-1})\|(L_h^* - L_0^*)\lambda^{-1}\Phi_0^-\|_{\mathbf{L}^2} \rightarrow 0 \tag{5.83}$$

as $h \downarrow 0$. Possibly after decreasing $h_{**} > 0$, we hence see that

$$\begin{aligned} \langle \Phi_0^+, (L_h^* + \lambda)^{-1}\Phi_0^- \rangle &= \langle \Phi_0^+, (L_0^* + \lambda)^{-1}\Phi_0^- \rangle + \langle \Phi_0^+, (L_h^* + \lambda)^{-1}\Phi_0^- - (L_0^* + \lambda)^{-1}\Phi_0^- \rangle \\ &= \lambda^{-1}\langle \Phi_0^+, \Phi_0^- \rangle + \langle \Phi_0^+, (L_h^* + \lambda)^{-1}\Phi_0^- - (L_0^* + \lambda)^{-1}\Phi_0^- \rangle \\ &> \frac{\langle \Phi_0^+, \Phi_0^- \rangle}{2} \lambda^{-1} \end{aligned} \tag{5.84}$$

holds for all $0 < h < h_{**}$. \blacksquare

Lemma 5.13. *Assume that (HP1), (HS), (H α 1) and (H α 2) are satisfied. Fix $0 < h < h_{**}$. There exist linear maps*

$$\begin{aligned}\tilde{\gamma}_h^- : \mathbf{L}^2 &\rightarrow \mathbb{R}, \\ \tilde{L}_h^{*,\text{qinv}} : \mathbf{L}^2 &\rightarrow \mathbf{H}^1\end{aligned}\tag{5.85}$$

such that for all $\Theta \in \mathbf{L}^2$ the pair

$$(\gamma, \Psi) = (\tilde{\gamma}_h^- \Theta, \tilde{L}_h^{*,\text{qinv}} \Theta)\tag{5.86}$$

is the unique solution to the problem

$$L_h^* \Psi = \Theta + \gamma \Phi_0^-\tag{5.87}$$

that satisfies the normalisation condition

$$\langle \Phi_0^+, \Psi \rangle = 0.\tag{5.88}$$

Furthermore, there exists $C^* > 0$, such that for all $0 < h < h_{**}$ and all $\Theta \in \mathbf{L}^2$ we have the bound

$$|\tilde{\gamma}_h^- \Theta| + \|\tilde{L}_h^{*,\text{qinv}} \Theta\|_{\mathbf{H}^1} \leq C^* \|\Theta\|_{\mathbf{L}^2}.\tag{5.89}$$

Proof. We define the set

$$Z^1 = \{\Psi \in \mathbf{H}^1 : \langle \Phi_0^+, \Psi \rangle = 0\}.\tag{5.90}$$

Pick $\Theta \in \mathbf{L}^2$. We look for a solution $(\gamma, \Psi) \in \mathbb{R} \times Z^1$ of the problem

$$\Psi = (L_h^* + \lambda)^{-1}[\Theta + \gamma \Phi_0^- + \lambda \Psi].\tag{5.91}$$

Lemma 5.12 implies that $\langle \Phi_0^+, (L_h^* + \lambda)^{-1} \Phi_0^- \rangle \neq 0$. Hence for given $\Theta \in \mathbf{L}^2, \Psi \in Z^1, h, \lambda$, we may write

$$\gamma(\Psi, \Theta, h, \lambda) = -\frac{\langle \Phi_0^+, (L_h^* + \lambda)^{-1} (\Theta + \lambda \Psi) \rangle}{\langle \Phi_0^+, (L_h^* + \lambda)^{-1} \Phi_0^- \rangle},\tag{5.92}$$

which is the unique value for γ for which

$$(L_h^* + \lambda)^{-1}[\Theta + \gamma \Phi_0^- + \lambda \Psi] \in Z^1.\tag{5.93}$$

From now on the proof is identical to that of Lemma 5.9, so we omit it. \blacksquare

Lemma 5.14. *Assume that (HP1), (HS), (H α 1) and (H α 2) are satisfied. For each $0 < h < h_{**}$ there exists an element $\Phi_h^- \in \ker(L_h^*)$ such that the family Φ_h^- converges to Φ_0^- in \mathbf{H}^1 as $h \downarrow 0$.*

Proof. We repeat some of the steps of the proof of Lemma 5.11, but now for L_h^* .

Fix $0 < h < h_{**}$. We let

$$\mathcal{R}^* = \text{Range}(L_0^*)\tag{5.94}$$

and we let

$$\begin{aligned}\mathcal{K}^* &= \ker(L_0^*) \\ &= \text{span}\{\Phi_0^-\}.\end{aligned}\tag{5.95}$$

Moreover we define

$$\mathcal{K}_c^* = \{\Psi \in \mathbf{H}^1 : \langle \Psi, \Phi_0^+ \rangle = 0\}\tag{5.96}$$

and

$$\mathcal{R}_c^* = \text{span}\{\Phi_0^-\}.\tag{5.97}$$

From Lemma 3.1 it follows that we can decompose \mathbf{H}^1 and \mathbf{L}^2 as $\mathbf{H}^1 = \mathcal{K}^* \oplus \mathcal{K}_c^*$ and $\mathbf{L}^2 = \mathcal{R}^* \oplus \mathcal{R}_c^*$ respectively.

For $S \in \{\mathcal{K}^*, \mathcal{K}_c^*, \mathcal{R}^*, \mathcal{R}_c^*\}$ let π_S denote the projection from \mathbf{H}^1 or \mathbf{L}^2 onto S , corresponding to the above decompositions. Following the same steps as in the proof of Lemma 5.11 we obtain that

$$\pi_{\mathcal{R}^*} L_h^* : \mathcal{K}_c^* \rightarrow \mathcal{R}^* \quad (5.98)$$

is invertible and that

$$[\pi_{\mathcal{R}^*} L_h^*]^{-1} = L_h^{*, \text{qinv}}. \quad (5.99)$$

We now write

$$\begin{aligned} \mu^*(h) : \mathcal{K}^* &\rightarrow \mathcal{K}_c^*, \\ B &\mapsto -[\pi_{\mathcal{R}^*} L_h^*]^{-1} \pi_{\mathcal{R}^*} L_h^* B. \end{aligned} \quad (5.100)$$

Lemma 5.11 together with (5.29) and (5.30) implies that

$$\dim(\ker(L_h^*)) = 1. \quad (5.101)$$

Arguing as before, we see that

$$\begin{aligned} 1 &= \dim(\ker(L_h^*)) \\ &= \dim(\ker(\pi_{\mathcal{R}_c^*}[L_h^*](I + \mu^*(h)))) \\ &= \dim(\mathcal{K}^*) - \dim(\text{Range}(\pi_{\mathcal{R}_c^*}[L_h^*](I + \mu^*(h)))) \\ &\leq \dim(\mathcal{K}^*) \\ &= 1. \end{aligned} \quad (5.102)$$

In particular, $\pi_{\mathcal{R}_c^*}[L_h^*](I + \mu^*(h))$ must be the zero map on \mathcal{K}^* . Upon defining

$$\Phi_h^- = \Phi_0^- + \mu^*(h)\Phi_0^-, \quad (5.103)$$

we hence see that $\Phi_h^- \in \ker(L_h^*)$. In addition, Lemma 5.11 implies that

$$\begin{aligned} \|\Phi_h^- - \Phi_0^-\|_{\mathbf{H}^1} &= \|\mu^*(h)\Phi_0^-\|_{\mathbf{H}^1} \\ &= \|[\pi_{\mathcal{R}^*} L_h^*]^{-1} \pi_{\mathcal{R}^*} L_h^* \Phi_0^-\|_{\mathbf{H}^1} \\ &= \|L_h^{*, \text{qinv}} \pi_{\mathcal{R}^*} L_h^* \Phi_0^-\|_{\mathbf{H}^1} \\ &\leq C^* \|\pi_{\mathcal{R}^*} L_h^* \Phi_0^-\|_{\mathbf{L}^2}. \end{aligned} \quad (5.104)$$

The final term goes to 0 as $h \downarrow 0$, since we know that $L_h^* \Psi \rightarrow L_0^* \Psi$ whenever $\Psi \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$. So we obtain that $\Phi_h^- \rightarrow \Phi_0^-$ strongly in \mathbf{H}^1 as $h \downarrow 0$. \blacksquare

In the final part of this section we establish items (3) and (4) of Proposition 5.2. To this end, we recall the spaces

$$X_h = \{\Theta \in \mathbf{H}^1 : \langle \Phi_h^-, \Theta \rangle = 0\} \quad (5.105)$$

and

$$Y_h = \{\Theta \in \mathbf{L}^2 : \langle \Phi_h^-, \Theta \rangle = 0\}, \quad (5.106)$$

together with the constant C_{unif} from (5.35).

Lemma 5.15. *Assume that (HP1), (HS), (H α 1) and (H α 2) are satisfied. For each $0 < h < h_{**}$ we have that $L_h : X_h \rightarrow Y_h$ is invertible and we have the uniform bound*

$$\|L_h^{-1}\| \leq C_{\text{unif}}. \quad (5.107)$$

Proof. Fix $0 < h < h_{**}$. Clearly $L_h : X_h \rightarrow Y_h$ is a bounded bijective linear map, so the Banach isomorphism theorem implies that $L_h^{-1} : Y_h \rightarrow X_h$ is bounded. Now let $\delta > 0$ be a small constant such that $\delta C_{\text{unif}} < 1$. Without loss of generality we assume that $0 < h_{**} \leq h'_0(\delta)$ and that $\|\Phi_h^- - \Phi_0^-\|_{\mathbf{H}^1} \leq \delta$ for all $0 < h < h_{**}$. This is possible by Lemma 5.14.

Pick any $\Psi \in X_h$. Remembering that $\langle \Psi, \Phi_h^- \rangle = 0$ and $\langle L_h \Psi, \Phi_h^- \rangle = 0$, we obtain the estimate

$$\begin{aligned} \frac{1}{\delta} |\langle (L_h + \delta)\Psi, \Phi_0^- \rangle| &= \frac{1}{\delta} |\langle (L_h + \delta)\Psi, \Phi_0^- - \Phi_h^- \rangle| \\ &\leq \frac{1}{\delta} \|(L_h + \delta)\Psi\|_{\mathbf{L}^2} \delta \\ &\leq \|L_h \Psi\|_{\mathbf{L}^2} + \delta \|\Psi\|_{\mathbf{H}^1}. \end{aligned} \quad (5.108)$$

Applying Proposition 3.2, we hence see

$$\begin{aligned} \|\Psi\|_{\mathbf{H}^1} &\leq \frac{1}{4} C_{\text{unif}} [\|(L_h + \delta)\Psi\|_{\mathbf{L}^2} + \frac{1}{\delta} |\langle (L_h + \delta)\Psi, \Phi_0^- \rangle|] \\ &\leq \frac{1}{4} C_{\text{unif}} [2\|L_h \Psi\|_{\mathbf{L}^2} + 2\delta \|\Psi\|_{\mathbf{H}^1}] \\ &\leq \frac{1}{2} C_{\text{unif}} \|L_h \Psi\|_{\mathbf{L}^2} + \frac{1}{2} \|\Psi\|_{\mathbf{H}^1}. \end{aligned} \quad (5.109)$$

We therefore get the bound

$$\|\Psi\|_{\mathbf{H}^1} \leq C_{\text{unif}} \|L_h \Psi\|_{\mathbf{L}^2}, \quad (5.110)$$

which yields the desired estimate $\|L_h^{-1}\| \leq C_{\text{unif}}$. \blacksquare

Lemma 5.16. *Fix $0 < h < h_{**}$. Then there exist constants $K_1 > 0, K_2 > 0, \beta > 0$ and $\tilde{\beta} > 0$, possibly depending on h , such that*

$$\begin{aligned} |\Phi_h^+(\xi)| &\leq K_1 e^{-\beta|\xi|} \|\Phi_h^+\|_{\infty}, \\ |\Phi_h^-(\xi)| &\leq K_2 e^{-\tilde{\beta}|\xi|} \|\Phi_h^-\|_{\infty} \end{aligned} \quad (5.111)$$

for all $\xi \in \mathbb{R}$.

Proof. Recall from Lemma 5.6 that L_h is asymptotically hyperbolic. Hence we obtain from [3, Lemma 4.3] that there are constants $\beta > 0$ and $K_1 > 0$ for which

$$|\Psi(\xi)| \leq K_1 e^{-\beta|\xi|} \|\Psi\|_{\infty} + K_1 \int_{-\infty}^{\infty} e^{-\beta|\xi-\eta|} |\Theta(\eta)| d\eta \quad (5.112)$$

holds for each $\Psi \in \mathbf{H}^1$, where $\Theta = L_h \Psi$. Since $L_h \Phi_h^+ = 0$ we conclude that

$$|\Phi_h^+(\xi)| \leq K_1 e^{-\beta|\xi|} \|\Phi_h^+\|_{\infty} \quad (5.113)$$

for all ξ . As stated on [3, Page 33] the operator L_h^* is also asymptotically hyperbolic. Hence there are $\tilde{\beta} > 0$ and $K_2 > 0$ for which

$$|\Psi(\xi)| \leq K_2 e^{-\tilde{\beta}|\xi|} \|\Psi\|_{\infty} + K_2 \int_{-\infty}^{\infty} e^{-\tilde{\beta}|\xi-\eta|} |\Theta(\eta)| d\eta \quad (5.114)$$

holds for each $\Psi \in \mathbf{H}^1$, where $\Theta = L_h^* \Psi$. Since $L_h^* \Phi_h^- = 0$ we obtain that

$$|\Phi_h^-(\xi)| \leq K_2 e^{-\tilde{\beta}|\xi|} \|\Phi_h^-\|_{\infty} \quad (5.115)$$

for all ξ . \blacksquare

Lemma 5.17. *Assume that (HP1),(HP2), (HS), (H α 1) and (H α 2) are satisfied. Then there exist constants $K_3 > 0$ and $\beta > 0$, possibly depending on h , such that*

$$|(\Phi_h^\pm)'(\xi)| \leq K_3 e^{-\beta|\xi|} \quad (5.116)$$

for all $\xi \in \mathbb{R}$.

Proof. Lemma 5.16 implies that

$$\begin{aligned} |\Delta_h \phi_h^+(\xi)| &\leq \frac{1}{h^2} K_1 \sum_{k>0} |\alpha_k| (e^{-\beta|\xi+hk|} + e^{-\beta|\xi-hk|} + 2e^{-\beta|\xi|}) \\ &\leq K_1 e^{-\beta|\xi|} \left(\frac{1}{h^2} \sum_{k>0} |\alpha_k| (2e^{\beta hk} + 2) \right), \end{aligned} \quad (5.117)$$

where the last sum converges by (H α 2), possibly after decreasing $\beta > 0$. Using the fact that

$$(\Phi_h^+)' = \frac{1}{c_h} \left(\frac{\Delta_h \phi_h^+ + g_u(\bar{u}_h) \phi_h^+ - \psi_h^+}{\rho \phi_h^+ - \rho \gamma \psi_h^+} \right) \quad (5.118)$$

we hence see that there exists a constant $K_3 > 0$ such that

$$|(\Phi_h^+)'(\xi)| \leq K_3 e^{-\beta|\xi|}. \quad (5.119)$$

The proof for the bound on $(\Phi_h^-)'$ is identical. ■

Proof of Proposition 5.2. This result follows directly from Lemmas 5.11, 5.14, 5.15, 5.16 and 5.17. ■

6 The resolvent set

In this section we prove Theorem 2.2 by explicitly determining the spectrum of the operator $-L_h$ defined in (2.18). Our approach hinges on the periodicity of this spectrum, which we describe in our first result.

Lemma 6.1. *Assume that (HP1), (HS), (H α 1) and (H α 2) are satisfied. Fix $0 < h < h_{**}$. Then the spectrum of L_h is invariant under the operation $\lambda \mapsto \lambda + 2\pi ic_h \frac{1}{h}$.*

In particular, we can restrict our attention to values with imaginary part in between $-\frac{\pi c_h}{h}$ and $\frac{\pi c_h}{h}$. We divide our 'half-strip' into four regions and in each region we calculate the spectrum. Values close to 0 (region R_1) will be treated in Proposition 6.2; values with a large real part (region R_2) in Proposition 6.3 and values with a large imaginary part (region R_3) in Proposition 6.6. In Corollary 6.7 we discuss the remaining intermediate subset (region R_4), which is compact and independent of h . The regions are illustrated in Figure 1 below.

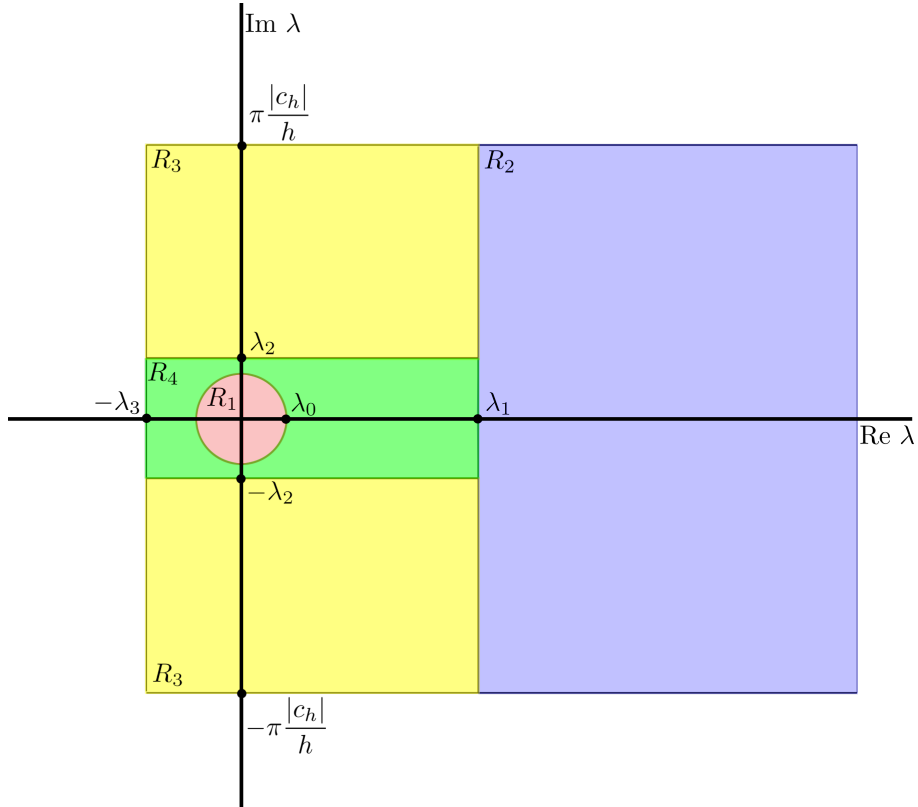


Figure 1: Illustration of the regions R_1, R_2, R_3 and R_4 . Note that the regions R_2 and R_3 grow when h decreases, while the regions R_1 and R_4 are independent of h .

From this section onward we need to assume that (HP2) is satisfied. Indeed, this allows us to lift the invertibility of $L_0 + \lambda$ to $L_h + \lambda$ simultaneously for all λ in appropriate compact sets.

Proof of Lemma 6.1. Fix $k \in \mathbb{Z}$ and write $p = 2\pi ik \frac{1}{h}$. We define the exponential shift operator e_ω by

$$[e_\omega V](x) = e^{\omega x} V(x). \quad (6.1)$$

For any $\lambda \in \mathbb{C}$, $\Psi = (\phi, \psi) \in \mathbf{H}^1$ and $x \in \mathbb{R}$ we obtain

$$\begin{aligned}
(e_{-p}\Delta_h e_p)\phi(x) &= e^{-px}\Delta_h(e_p\phi)(x) \\
&= \frac{1}{h^2} \sum_{l>0} \alpha_l (e^{plh}\phi(x+lh) + e^{-plh}\phi(x-lh) - 2\phi(x)) \\
&= \frac{1}{h^2} \sum_{l>0} \alpha_l (\phi(x+lh) + \phi(x-lh) - 2\phi(x)) \\
&= \Delta_h\phi(x),
\end{aligned} \tag{6.2}$$

since $plh \in 2\pi i\mathbb{Z}$ for all $l > 0$. In particular, we can compute

$$\begin{aligned}
[e_{-p}(L_h - \lambda)e_p\Psi](x) &= e^{-px}[(L_h - \lambda)e_p\Psi](x) \\
&= e^{-px} \begin{pmatrix} c_h \frac{d}{dx}(e^{px}\phi(x)) - \Delta_h(e_p\phi)(x) \\ -\rho e^{px} + c_h \frac{d}{dx}(e^{px}\psi(x)) \end{pmatrix} \\
&\quad + e^{-px} \begin{pmatrix} -g_u(\bar{u}_h)e^{px}\phi(x) + e^{px}\psi(x) - \lambda e^{px}\phi(x) \\ +\gamma\rho e^{px}\psi(x) - \lambda e^{px}\psi(x) \end{pmatrix} \\
&= \begin{pmatrix} pc_h\phi(x) + c_h\phi'(x) - g_u(\bar{u}_h)\phi(x) + \psi(x) \\ -\rho\phi(x) + pc_h\psi(x) + c_h\psi'(x) + \gamma\rho\psi(x) - \lambda\psi(x) \end{pmatrix} \\
&\quad + \begin{pmatrix} -\Delta_h\phi(x) - \lambda\phi(x) \\ 0 \end{pmatrix} \\
&= (L_h - \lambda + pc_h)\Psi(x).
\end{aligned} \tag{6.3}$$

Since e_p and e_{-p} are invertible operators on \mathbf{H}^1 and \mathbf{L}^2 respectively, we know that the spectrum of L_h equals that of $e_{-p}L_h e_p$ and thus that of $L_h + pc_h$. \blacksquare

Region R_1 .

Since L_h has a simple eigenvalue at zero, it is relatively straightforward to construct a small neighbourhood around the origin that contains no other part of the spectrum. Exploiting the results from §5, it is possible to control the size of this neighbourhood as $h \downarrow 0$.

Proposition 6.2. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. There exists a constant $\lambda_0 > 0$ such that for all $0 < h < h_{**}$ the operator $L_h + \lambda : \mathbf{H}^1 \rightarrow \mathbf{L}^2$ is invertible for all $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \lambda_0$.*

Proof. Fix $0 < h < h_{**}$ and $\Theta \in \mathbf{L}^2$. We recall the notation $(\gamma_h[\Theta], L_h^{\text{qinv}}\Theta)$ from Corollary 5.4 for the unique solution (γ, Ψ) of the equation

$$L_h\Psi = \Theta + \gamma\Phi_h^+ \tag{6.4}$$

in the space

$$X_h = \{\Theta \in \mathbf{H}^1 : \langle \Phi_h^-, \Theta \rangle = 0\}. \tag{6.5}$$

Also recall the space

$$Y_h = \{\Theta \in \mathbf{L}^2 : \langle \Phi_h^-, \Theta \rangle = 0\}. \tag{6.6}$$

Now for $\lambda \in \mathbb{C}$ with $|\lambda|$ small enough, but $\lambda \neq 0$, we want to solve the equation $L_h\Psi = \lambda\Psi + \Theta$. Upon writing

$$\Psi = L_h^{\text{qinv}}\Theta + \lambda^{-1}\gamma_h[\Theta]\Phi_h^+ + \tilde{\Psi}, \tag{6.7}$$

with $\tilde{\Psi} \in X_h$, we see that

$$\begin{aligned}
(L_h - \lambda)\Psi &= (L_h - \lambda)L_h^{\text{qinv}}\Theta + \lambda^{-1}(L_h - \lambda)\gamma_h[\Theta]\Phi_h^+ + (L_h - \lambda)\tilde{\Psi} \\
&= \Theta + \gamma_h[\Theta]\Phi_h^+ - \lambda L_h^{\text{qinv}}\Theta - \gamma_h[\Theta]\Phi_h^+ + (L_h - \lambda)\tilde{\Psi}.
\end{aligned} \tag{6.8}$$

In particular, we must find a solution $\tilde{\Psi} \in X_h$ for the equation

$$L_h \tilde{\Psi} = \lambda \tilde{\Psi} + \lambda L_h^{\text{qinv}} \Theta, \quad (6.9)$$

which we can rewrite as

$$[I - \lambda L_h^{-1}] \tilde{\Psi} = \lambda L_h^{-1} L_h^{\text{qinv}} \Theta. \quad (6.10)$$

Note that $L_h^{-1} : X_h \rightarrow X_h$ is also a bounded operator since $X_h \subset Y_h$. Since

$$\begin{aligned} \|L_h^{-1} \Psi\|_{\mathbf{H}^1} &\leq C_{\text{unif}} \|\Psi\|_{\mathbf{L}^2} \\ &\leq C_{\text{unif}} \|\Psi\|_{\mathbf{H}^1}, \end{aligned} \quad (6.11)$$

we obtain

$$\|L_h^{-1}\|_{\mathcal{B}(X_h, X_h)} \leq C_{\text{unif}}. \quad (6.12)$$

We choose λ_0 in such a way that $0 < \lambda_0 C_{\text{unif}} < 1$. Then it is well-known that $I - \lambda L_h^{-1}$ is invertible as an operator on X_h for $0 < |\lambda| < \lambda_0$. Since $\lambda L_h^{-1} L_h^{\text{qinv}} \Theta \in X_h$, we see that (6.10) indeed has a unique solution $\tilde{\Psi} \in X_h$. Hence the equation $(L_h - \lambda)\Psi = \Theta$ always has a unique solution. Proposition 5.1 states that $L_h - \lambda$ is Fredholm with index 0, which now implies that $L_h - \lambda$ is indeed invertible. \blacksquare

Region R_2 .

We now show that in an appropriate right half-plane, which can be chosen independently of h , the spectrum of $-L_h$ is empty. The proof proceeds via a relatively direct estimate that is strongly inspired by [1, Lemma 5].

Proposition 6.3. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. There exists a constant $\lambda_1 > 0$ such that for all $\lambda \in \mathbb{C}$ with $\text{Re } \lambda \geq \lambda_1$ and all $0 < h < h_{**}$ the operator $L_h + \lambda$ is invertible.*

Proof. Write

$$\lambda_1 = 1 + g_* + \frac{1}{2}(1 - \rho), \quad (6.13)$$

where g_* is defined in Lemma 3.9. Pick any $\lambda \in \mathbb{C}$ with $\text{Re } \lambda \geq \lambda_1$ and any $0 < h < h_{**}$. Let $\Psi = (\phi, \psi) \in \mathbf{H}^1$ be arbitrary and set $\Theta = L_h \Psi + \lambda \Psi$. Then we see that

$$\begin{aligned} \|\Psi\|_{\mathbf{L}^2} \|\Theta\|_{\mathbf{L}^2} &\geq \text{Re} \langle L_h \Psi + \lambda \Psi, \Psi \rangle \\ &\geq \text{Re} \langle -\Delta_h \phi, \phi \rangle - \|g_u(\bar{u}_h)\|_{L^\infty} \|\phi\|_{L^2}^2 \\ &\quad - (1 - \rho) |\text{Re} \langle \phi, \psi \rangle| + \gamma \rho \|\psi\|_{L^2}^2 + \text{Re } \lambda \|\Psi\|_{\mathbf{L}^2}^2 \\ &\geq -g_* \|\phi\|_{L^2}^2 - (1 - \rho) |\text{Re} \langle \phi, \psi \rangle| \\ &\quad + \gamma \rho \|\psi\|_{L^2}^2 + \text{Re } \lambda \|\Psi\|_{\mathbf{L}^2}^2 \\ &\geq -g_* \|\phi\|_{L^2}^2 - (1 - \rho) \|\phi\|_{L^2} \|\psi\|_{L^2} \\ &\quad + \gamma \rho \|\psi\|_{L^2}^2 + \text{Re } \lambda \|\Psi\|_{\mathbf{L}^2}^2 \\ &\geq -(g_* + \frac{1}{2}(1 - \rho)) \|\Psi\|_{\mathbf{L}^2}^2 + \text{Re } \lambda \|\Psi\|_{\mathbf{L}^2}^2. \end{aligned} \quad (6.14)$$

Hence we obtain

$$\left(\text{Re } \lambda - (g_* + \frac{1}{2}(1 - \rho)) \right) \|\Psi\|_{\mathbf{L}^2} \leq \|\Theta\|_{\mathbf{L}^2}. \quad (6.15)$$

Since $\text{Re } \lambda \geq 1 + g_* + \frac{1}{2}(1 - \rho)$, we obtain the bound $\|\Psi\|_{\mathbf{L}^2} \leq \|\Theta\|_{\mathbf{L}^2}$.

In particular, if $\Theta = 0$ then we necessarily have $\Psi = 0$, which implies that $L_h + \lambda$ is injective. Since also $\text{ind}(L_h + \lambda) = 0$ by Proposition 5.1, this means that $L_h + \lambda$ is invertible. \blacksquare

Region R_3 .

We now use a Fourier transform argument to determine the spectrum of $-L_h$ in the region R_3 where the imaginary part of λ is large.

Pick $\lambda \in \mathbb{C}$ with $\lambda_0 < |\text{Im } \lambda| \leq \frac{|c_h|}{h} \pi$ and write

$$\lambda = \lambda_r + i\lambda_{\text{im}}. \quad (6.16)$$

Introducing the new variable $\tau = \text{Im } \lambda \xi$, we can write the eigenvalue problem $(L_h + \lambda)(v, w) = 0$ in the form

$$\begin{aligned} c_h v_\tau(\tau) &= \frac{1}{\lambda_{\text{im}} h^2} \sum_{k>0} \alpha_k \left[v(\tau + kh\lambda_{\text{im}}) + v(\tau - kh\lambda_{\text{im}}) - 2v(\tau) \right] \\ &\quad + \frac{1}{\lambda_{\text{im}}} g_u(\bar{u}_h(\tau)) v(\tau) - iv(\tau) - \frac{1}{\lambda_{\text{im}}} \lambda_r v(\tau) - \frac{1}{\lambda_{\text{im}}} w(\tau), \\ c_h w_\tau(\tau) &= \frac{1}{\lambda_{\text{im}}} \left(\rho v(\tau) - \rho \gamma w(\tau) + \lambda w(\tau) \right). \end{aligned} \quad (6.17)$$

Our computations below show that the leading order terms in the appropriate $|\lambda_{\text{im}}| \rightarrow \infty$ limit are encoded by the 'homogeneous operator' $H_{h,\lambda}$ that acts as

$$H_{h,\lambda} v(\tau) = c_h v_\tau(\tau) + iv(\tau) - \frac{1}{\lambda_{\text{im}} h^2} \sum_{k>0} \alpha_k \left[v(\tau + kh\lambda) + v(\tau - kh\lambda) - 2v(\tau) \right]. \quad (6.18)$$

Writing $\mathcal{H}_{h,\lambda}$ for the Fourier symbol associated to $H_{h,\lambda}$, we see that

$$\begin{aligned} \mathcal{H}_{h,\lambda}(i\omega) &= c_h i\omega + i - \frac{1}{\lambda_{\text{im}} h^2} \sum_{k>0} \alpha_k \left[\exp(ihk\lambda_{\text{im}}\omega) + \exp(-ihk\lambda_{\text{im}}\omega) - 2 \right] \\ &= c_h i\omega + i - \frac{2}{\lambda_{\text{im}} h^2} \sum_{k>0} \alpha_k \left[\cos(hk\lambda_{\text{im}}\omega) - 1 \right]. \end{aligned} \quad (6.19)$$

Lemma 6.4. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. There exist small constants $\varepsilon > 0$, $h_* > 0$ and $\omega_0 > 0$ so that for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, all $0 < h < h_*$ and all $\omega \in \mathbb{R}$, the inequality*

$$|\text{Im } \mathcal{H}_{h,\lambda}(i\omega)| < \varepsilon \quad (6.20)$$

can only be satisfied if the inequalities

$$\begin{aligned} |c_h \omega| &\leq \frac{3}{2} \\ |\omega| &\geq \omega_0 \end{aligned} \quad (6.21)$$

both hold.

Proof. Note that

$$|\text{Im } \mathcal{H}_{h,\lambda}(i\omega)| = |c_h \omega + 1|. \quad (6.22)$$

In particular, upon choosing $\varepsilon = \frac{1}{4}$, we see that

$$|\text{Im } \mathcal{H}_{h,\lambda}(i\omega)| < \varepsilon \quad (6.23)$$

implies

$$||c_h \omega| - 1| \leq |c_h \omega + 1| < \varepsilon \quad (6.24)$$

and hence

$$\frac{1}{2} < 1 - \varepsilon \leq |c_h \omega| \leq 1 + \varepsilon < \frac{3}{2}. \quad (6.25)$$

Since $c_h \rightarrow c_0 \neq 0$ as $h \downarrow 0$, the desired inequalities (6.21) follow. \blacksquare

Lemma 6.5. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Then there exists a constant $C > 0$ such that for all $\omega \in \mathbb{R}$ and $0 < h < h_{**}$ and all $\lambda \in \mathbb{C}$ with $|\lambda| > \lambda_0$ and $|\operatorname{Im} \lambda| \leq \frac{|c_h|}{h} \pi$, we have the inequality*

$$|\mathcal{H}_{h,\lambda}(i\omega)| \geq \frac{1}{C}. \quad (6.26)$$

Proof. We show that $\mathcal{H}_{h,\lambda}(i\omega)$ is bounded away from 0, uniformly in h, λ and ω . To do so, we show that the real part of $\mathcal{H}_{h,\lambda}(i\omega)$ can be bounded away from zero, whenever the imaginary part is small, i.e. when (6.21) holds.

Recall the function $A(y) = \sum_{k>0} \alpha_k [1 - \cos(ky)]$ defined in Assumption (H α 1), which satisfies $A(y) > 0$ for $y \in (0, 2\pi)$. A direct calculation shows that $A'(0) = 0$ and

$$\begin{aligned} A''(0) &= \sum_{k>0} \alpha_k k^2 \\ &= 1. \end{aligned} \quad (6.27)$$

Hence we can pick $d_0 > 0$ in such a way that

$$\frac{1}{y^2} A(y) > d_0 \quad (6.28)$$

holds for all $0 < |y| \leq \frac{3}{2}\pi$.

Writing $\mu = h\lambda_{\operatorname{im}}\omega$, we see

$$\begin{aligned} \operatorname{Re} \mathcal{H}_{h,\lambda}(i\omega) &= \frac{2\omega^2 \lambda_{\operatorname{im}}}{\mu^2} \sum_{k>0} \alpha_k [1 - \cos(k\mu)] \\ &= \frac{2\omega^2 \lambda_{\operatorname{im}}}{\mu^2} A(\mu). \end{aligned} \quad (6.29)$$

Now fix ω, h, λ for which $|\operatorname{Im} \mathcal{H}_{h,\lambda}(i\omega)| < \varepsilon$. The conditions (6.21) now imply that $|\omega| \geq \omega_0$ and $|\mu| \leq h \frac{|c_h|}{h} \pi |\omega| \leq \frac{3}{2}\pi$. Using (6.27), we hence see that

$$\begin{aligned} |\operatorname{Re} \mathcal{H}_{h,\lambda}(i\omega)| &= \left| \frac{2\omega^2 \lambda_{\operatorname{im}}}{\mu^2} A(\mu) \right| \\ &\geq 2|\lambda_{\operatorname{im}}| \omega^2 d_0 \\ &\geq 2\lambda_0 \omega_0^2 d_0, \end{aligned} \quad (6.30)$$

which shows that $\mathcal{H}_{h,\lambda}(i\omega)$ can indeed be uniformly bounded away from zero. \blacksquare

Proposition 6.6. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. There exist constants $\lambda_2 > 0$ and $\lambda_3 > 0$ such that for all $\lambda \in \mathbb{C}$ with $\lambda_2 \leq |\operatorname{Im} \lambda| \leq \frac{|c_h|}{2h} 2\pi$ and $-\lambda_3 \leq \operatorname{Re} \lambda \leq \lambda_1$ and all $0 < h < h_{**}$ the operator $L_h + \lambda$ is invertible.*

Proof. Since Proposition 5.1 implies that $L_h + \lambda$ is Fredholm with index zero, it suffices to prove that $L_h + \lambda$ is injective.

Let $\lambda_3 = \min\{\frac{1}{2}\rho\gamma, \lambda_*, \tilde{\lambda}\}$, where λ_* is defined in (HP2) and $\tilde{\lambda}$ is defined in Proposition 5.1. Pick $\lambda \in \mathbb{C}$ with $\lambda_0 \leq |\operatorname{Im} \lambda| \leq \frac{|c_h|}{2h}2\pi$ and $-\lambda_3 \leq |\operatorname{Re} \lambda| \leq \lambda_1$. Write $\lambda = \lambda_r + i\lambda_{\operatorname{im}}$ as before. Suppose $\Psi = (v, w)$ satisfies $(L_h + \lambda)\Psi = 0$.

Write \hat{v} and \hat{w} for the Fourier transforms of v and w respectively. For $f \in L^2$ with Fourier transform \hat{f} , the identity

$$H_{h,\lambda}v = f \quad (6.31)$$

implies that

$$\mathcal{H}_{h,\lambda}(i\omega)\hat{v}(i\omega) = \hat{f}(i\omega). \quad (6.32)$$

In particular, we obtain

$$\hat{v}(i\omega) = \frac{1}{\mathcal{H}_{h,\lambda}(i\omega)}\hat{f}(i\omega), \quad (6.33)$$

which using Lemma 6.5 implies that

$$\|v\|_{L^2} \leq C\|f\|_{L^2} \quad (6.34)$$

for some constant $C > 0$ that is independent of h, λ and ω .

Since Ψ is an eigenfunction, (6.17) hence yields

$$\|v\|_{L^2} \leq C\frac{1}{|\lambda_{\operatorname{im}}|}(g_* + |\lambda_r|)\|v\|_{L^2} + C\frac{1}{|\lambda_{\operatorname{im}}|}\|w\|_{L^2}. \quad (6.35)$$

Furthermore applying a Fourier Transform to the second line of (6.17), we find

$$\lambda_{\operatorname{im}}c_h i\omega\hat{w}(i\omega) = \rho\hat{v}(i\omega) - \rho\gamma\hat{w}(i\omega) + \lambda\hat{w}(i\omega). \quad (6.36)$$

Our choice $\lambda_3 \leq \frac{1}{2}\rho\gamma$ implies that $-\rho\gamma + \lambda_r$ is bounded away from 0. We may hence write

$$\hat{w}(i\omega) = \frac{1}{\rho\gamma - \lambda_r + i(\omega\lambda_{\operatorname{im}}c_h - \lambda_{\operatorname{im}})}\rho\hat{v}(i\omega), \quad (6.37)$$

which yields the bound

$$\|w\|_{L^2} \leq C'\|v\|_{L^2} \quad (6.38)$$

for some constant $C' > 0$. Therefore we obtain that

$$\|v\|_{L^2} \leq C''\frac{1}{|\lambda_{\operatorname{im}}|}\|v\|_{L^2} \quad (6.39)$$

for some constant C'' , which is independent of λ, h and v . Clearly this is impossible for $v \neq 0$ if

$$|\lambda_{\operatorname{im}}| \geq \lambda_2 := 2C''. \quad (6.40)$$

Furthermore if $v = 0$, then clearly also $w = 0$. Therefore $\Psi = 0$, allowing us to conclude that $L_h + \lambda$ is invertible. \blacksquare

Region R_4 .

We conclude our spectral analysis by considering the remaining region R_4 . This region is compact and bounded away from the origin, allowing us to directly apply the theory developed in §3.

Corollary 6.7. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. For all $\lambda \in \mathbb{C}$ with $|\lambda| \geq \lambda_0$, $-\lambda_3 \leq |\operatorname{Re} \lambda| \leq \lambda_1$ and $|\operatorname{Im} \lambda| \leq \lambda_2$ and all $0 < h < h_{**}$ the operator $L_h + \lambda$ is invertible.*

Proof. The statement follows by applying Proposition 3.3 with the choices $(\tilde{u}_h, \tilde{w}_h) = (\bar{u}_h, \bar{w}_h)$, $\tilde{c}_h = c_h$ and $M = R_4$. \blacksquare

Proof of Theorem 2.2. The result follows directly from Lemma 6.1, Proposition 6.2, Proposition 6.3, Proposition 6.6 and Corollary 6.7. \blacksquare

7 Green's functions

In order to establish the nonlinear stability of the pulse solution (\bar{u}_h, \bar{w}_h) , we need to consider two types of Green's functions. In particular, we first study $G_{h,\lambda}(\xi, \xi_0)$, which can roughly be seen as a solution of the equation

$$\left[(L_h + \lambda)G_{h,\lambda}(\cdot, \xi_0) \right](\xi) = \delta(\xi - \xi_0), \quad (7.1)$$

where δ is the Dirac delta-distribution. We then use these functions to build a Green's function \mathcal{G} for the linearisation of the LDE (2.1) around the travelling pulse solution.

We will loosely follow §2 of [20]. We will also base quite a few results on [2]. However, we emphasize that both [20] and [2] only consider finite range interactions.

In order to proceed, let us consider the linearisation of the original LDE (2.1) around the travelling pulse solution $\bar{U}_h(t)$ given by (2.21). In particular, we introduce the Hilbert space

$$\mathbb{L}^2 := \left\{ V \in (\text{Mat}_2(\mathbb{R}))^{\mathbb{Z}} : \sum_{j \in \mathbb{Z}} |V(j)|^2 < \infty \right\}, \quad (7.2)$$

in which $\text{Mat}_2(\mathbb{R})$ is the space of 2×2 -matrices with real coefficients which we equip with the maximum-norm $|\cdot|$. For any $\mathcal{V} \in \mathbb{L}^2$, we often write $\mathcal{V} = \begin{pmatrix} \mathcal{V}^{(1,1)} & \mathcal{V}^{(1,2)} \\ \mathcal{V}^{(2,1)} & \mathcal{V}^{(2,2)} \end{pmatrix}$, when we need to refer to the component sequences $\mathcal{V}^{(i,j)} \in \ell^2(\mathbb{Z}; \mathbb{R})$. For any $t \in \mathbb{R}$ we now introduce the linear operator $\mathcal{A}_h(t) : \mathbb{L}^2 \rightarrow \mathbb{L}^2$ that acts as

$$\mathcal{A}_h(t) \cdot \mathcal{V} = \frac{1}{c_h} \begin{pmatrix} A^{(1,1)}(t) & A^{(1,2)}(t) \\ A^{(2,1)}(t) & A^{(2,2)}(t) \end{pmatrix} \begin{pmatrix} \mathcal{V}^{(1,1)} & \mathcal{V}^{(1,2)} \\ \mathcal{V}^{(2,1)} & \mathcal{V}^{(2,2)} \end{pmatrix}, \quad (7.3)$$

where

$$\begin{aligned} (A^{(1,1)}(t)v)_j &= \frac{1}{h^2} \sum_{k>0} \alpha_k [v_{j+k} + v_{j-k} - 2v_j] + g_u(\bar{u}_h(hj + c_h t))v_j \\ (A^{(1,2)}(t)w)_j &= -w_j \\ (A^{(2,1)}(t)v)_j &= \rho v_j \\ (A^{(2,2)}(t)v)_j &= -\rho \gamma w_j \end{aligned} \quad (7.4)$$

for $v \in \ell^2(\mathbb{Z}; \mathbb{R})$ and $w \in \ell^2(\mathbb{Z}; \mathbb{R})$. With all this notation in hand, we can write the desired linearisation as the ODE

$$\frac{d}{dt} \mathcal{V}(t) = \mathcal{A}_h(t) \cdot \mathcal{V}(t) \quad (7.5)$$

posed on \mathbb{L}^2 .

Fix $t_0 \in \mathbb{R}$ and $j_0 \in \mathbb{Z}$. Consider the function

$$\mathbb{R} \ni t \mapsto \mathcal{G}^{j_0}(t, t_0, h) = \{ \mathcal{G}_j^{j_0}(t, t_0, h) \}_{j \in \mathbb{Z}} \in \mathbb{L}^2 \quad (7.6)$$

that is uniquely determined by the initial value problem

$$\begin{cases} \frac{d}{dt} \mathcal{G}^{j_0}(t, t_0, h) = \mathcal{A}_h(t) \cdot \mathcal{G}^{j_0}(t, t_0, h) \\ \mathcal{G}_j^{j_0}(t_0, t_0, h) = \delta_j^{j_0} I. \end{cases} \quad (7.7)$$

Here we have introduced

$$\delta_j^{j_0} = \begin{cases} 1 & \text{if } j = j_0 \\ 0 & \text{else,} \end{cases} \quad (7.8)$$

where $I \in \text{Mat}_2(\mathbb{R})$ is the identity matrix. We remark that $\mathcal{G}_j^{j_0}(t, t_0, h)$ is an element of $\text{Mat}_2(\mathbb{R})$ for each $j \in \mathbb{Z}$.

This function \mathcal{G} is called the Green's function for the linearisation around our travelling pulse. Indeed, the general solution of the inhomogeneous equation

$$\begin{cases} \frac{dV}{dt} &= \mathcal{A}_h(t) \cdot V(t) + F(t) \\ V(0) &= V^0, \end{cases} \quad (7.9)$$

where now $V(t) \in \ell^2(\mathbb{Z}; \mathbb{R}^2) \cong \ell^2(\mathbb{Z}; \mathbb{R}^{2 \times 1})$ and $F(t) \in \ell^2(\mathbb{Z}; \mathbb{R}^2) \cong \ell^2(\mathbb{Z}; \mathbb{R}^{2 \times 1})$, is given by

$$V_j(t) = \sum_{j_0 \in \mathbb{Z}} \mathcal{G}_j^{j_0}(t, 0, h) V_{j_0}^0 + \int_0^t \sum_{j_0 \in \mathbb{Z}} \mathcal{G}_j^{j_0}(t, t_0, h) F_{j_0}(t_0) dt_0. \quad (7.10)$$

Introducing the standard convolution operator $*$, this can be written in the abbreviated form

$$V = \mathcal{G}(t, 0, h) * V^0 + \int_0^t \mathcal{G}(t, t_0, h) * F(t_0) dt_0. \quad (7.11)$$

The main result of this section is the following proposition, which shows that we can decompose the Green's function \mathcal{G} into a part that decays exponentially and a neutral part associated with translation along the family of travelling pulses.

Proposition 7.1. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. For any pair $t \geq t_0$ and any $j, j_0 \in \mathbb{Z}$, we have the representation*

$$\mathcal{G}_j^{j_0}(t, t_0, h) = \mathcal{E}_j^{j_0}(t, t_0, h) + \tilde{\mathcal{G}}_j^{j_0}(t, t_0, h), \quad (7.12)$$

in which

$$\mathcal{E}_j^{j_0}(t, t_0, h) = \frac{h}{M} \begin{pmatrix} \phi_h^-(hj_0 + c_h t_0) \phi_h^+(hj + c_h t) & \psi_h^-(hj_0 + c_h t_0) \phi_h^+(hj + c_h t) \\ \phi_h^-(hj_0 + c_h t_0) \psi_h^+(hj + c_h t) & \psi_h^-(hj_0 + c_h t_0) \psi_h^+(hj + c_h t) \end{pmatrix}, \quad (7.13)$$

while $\tilde{\mathcal{G}}$ satisfies the bound

$$|\tilde{\mathcal{G}}_j^{j_0}(t, t_0, h)| \leq K e^{-\tilde{\beta}(t-t_0)} e^{-\tilde{\beta}|hj+c_h t-hj_0-c_h t_0|} \quad (7.14)$$

for some $K > 0$ and $\tilde{\beta} > 0$ which may depend on h . The constant $M > 0$ is given by

$$M = \langle \Phi_h^-, \Phi_h^+ \rangle. \quad (7.15)$$

Furthermore for any $t \geq t_0$ we have the representation

$$\mathcal{G}_j^{j_0}(t, t_0, h) = \sum_{i \in \mathbb{Z}} \left[\mathcal{E}_j^i(t, t_0, h) \mathcal{E}_i^{j_0}(t_0, t_0, h) + \tilde{\mathcal{G}}_j^i(t, t_0, h) (\delta_i^{j_0} I - \mathcal{E}_i^{j_0}(t_0, t_0, h)) \right], \quad (7.16)$$

which can be abbreviated as

$$\mathcal{G}(t, t_0, h) = \mathcal{E}(t, t_0, h) * \mathcal{E}(t_0, t_0, h) + \tilde{\mathcal{G}}(t, t_0, h) * (I - \mathcal{E}(t_0, t_0, h)). \quad (7.17)$$

Our first task is to define $G_{h,\lambda}$ in a more rigorous fashion. Recall the operator $L_{h,\infty;\lambda}$ and the function $\Delta_{L_{h,\infty;\lambda}}$ from Lemma 5.6. We will show that $L_{h,\infty;\lambda}$ has a Green's function which takes values in the space $\text{Mat}_2(\mathbb{R})$ and that this function has some nice properties. Recall the constant $\tilde{\lambda}$ from Lemma 5.6. For each $\lambda \in \mathbb{C}$ with $\text{Re } \lambda \geq -\frac{\tilde{\lambda}}{2}$ and each $0 < h < h_{**}$ we define $G_{h,\infty;\lambda} : \mathbb{R} \rightarrow \text{Mat}_2(\mathbb{R})$ by

$$G_{h,\infty;\lambda}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\eta\xi} (\Delta_{L_{h,\infty;\lambda}}(i\eta))^{-1} d\eta. \quad (7.18)$$

We also introduce the notation

$$G_{h,\infty} = G_{h,\infty;0}. \quad (7.19)$$

Here (H α 2) is essential to ensure that that these Green's functions decay exponentially.

Lemma 7.2. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Fix $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq -\frac{\bar{\lambda}}{2}$ and $0 < h < h_{**}$. The function $G_{h,\infty;\lambda}$ is bounded and continuous on $\mathbb{R} \setminus \{0\}$ and C^1 -smooth on $\mathbb{R} \setminus h\mathbb{Z}$. Furthermore $(L_{h,\infty} + \lambda)G_{h,\infty;\lambda}(\cdot - \xi_0)$ is constantly zero except at $\xi = \xi_0$ and satisfies the identity*

$$\int_{-\infty}^{\infty} \left[(L_{h,\infty} + \lambda)G_{h,\infty;\lambda}(\cdot - \xi_0) \right] (\xi) f(\xi) d\xi = f(\xi_0) \quad (7.20)$$

for all $\xi \in \mathbb{R}$ and all $f \in \mathbf{H}^1$.

Finally for each $\chi > 0$ there exist constants $K_* > 0$ and $\beta_* > 0$, which may depend on χ and h , such that for each $\lambda \in \mathbb{C}$ with $-\frac{\bar{\lambda}}{2} \leq \operatorname{Re} \lambda \leq \chi$ and $|\operatorname{Im} \lambda| \leq \frac{\pi|c_h|}{h}$ we have the bound

$$|G_{h,\infty;\lambda}(\xi - \xi_0)| \leq K_* e^{-\beta_* |\xi - \xi_0|} \quad (7.21)$$

for all $\xi, \xi_0 \in \mathbb{R}$.

Pick $\lambda \in \mathbb{C} \setminus \sigma(-L_h)$ with $\operatorname{Re} \lambda \geq -\frac{\bar{\lambda}}{2}$. Observe that

$$L_h - L_{h,\infty} = \begin{pmatrix} -g_u(\bar{u}_h) + r_0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (7.22)$$

We know that $G_{h,\infty;\lambda}(\cdot - \xi_0) \in L^2(\mathbb{R}, \operatorname{Mat}_2(\mathbb{R}))$ since it decays exponentially. Therefore also

$$[L_h - L_{h,\infty}]G_{h,\infty;\lambda}(\cdot - \xi_0) \in L^2(\mathbb{R}, \operatorname{Mat}_2(\mathbb{C})). \quad (7.23)$$

Hence it is possible to define the function $G_{h,\lambda}$ by

$$G_{h,\lambda}(\xi, \xi_0) = G_{h,\infty;\lambda}(\xi - \xi_0) - \left[(\lambda + L_h)^{-1} [L_h - L_{h,\infty}] G_{h,\infty;\lambda}(\cdot - \xi_0) \right] (\xi). \quad (7.24)$$

The next result shows that $G_{h,\lambda}$ can be interpreted as the Green's function of $L_h + \lambda$. It is based on [20, Lemma 2.6].

Lemma 7.3. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. For $\lambda \in \mathbb{C} \setminus \sigma(-L_h)$ with $\operatorname{Re} \lambda \geq -\frac{\bar{\lambda}}{2}$ we have that $G_{h,\lambda}(\cdot, y)$ is continuous on $\mathbb{R} \setminus \{y\}$ and C^1 -smooth on $\mathbb{R} \setminus \{y + kh : k \in \mathbb{Z}\}$. Furthermore it satisfies*

$$\int_{-\infty}^{\infty} \left[(\lambda + L_h)G_{h,\lambda}(\cdot, \xi_0) \right] (\xi) f(\xi) d\xi = f(\xi_0) \quad (7.25)$$

for all $\xi \in \mathbb{R}$ and all $f \in \mathbf{H}^1$.

The link between our two types of Green's functions is provided by the following key result. It is based on [2, Theorem 4.2], where it was used to study one-sided spatial discretisation schemes for systems with conservation laws.

Proposition 7.4. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Let $\chi > \lambda_{\text{unif}}$ be given, where λ_{unif} is as in Lemma 7.5. For all $t \geq t_0$ the Green's function $\mathcal{G}_j^{j_0}(t, t_0, h)$ of (7.7) is given by*

$$\mathcal{G}_j^{j_0}(t, t_0, h) = -\frac{h}{2\pi i} \int_{\chi - \frac{i\pi c_h}{h}}^{\chi + \frac{i\pi c_h}{h}} e^{\lambda(t-t_0)} G_{h,\lambda}(hj + c_h t, hj_0 + c_h t_0) d\lambda \quad (7.26)$$

where $G_{h,\lambda}$ is the Green's function of $\lambda + L_h$ as defined in (7.24).

In the first half of this section we establish several basic facts concerning $G_{h,\lambda}$ that can be used to verify the integral representation (7.26). We then establish a meromorphic expansion for $G_{h,\lambda}$ that allows us to shift the integration path in (7.26) to the left of the imaginary axis. The decomposition (7.12) for the Green's function \mathcal{G} can subsequently be read off from this expression.

Proof of Lemma 7.2. Lemma 5.6 implies that $L_h + \lambda$ is asymptotically hyperbolic. Hence these results except the smoothness properties and the bound (7.21) follow from [3, Theorem 3.7].

We show that $\Delta_h G_{h,\infty;\lambda}(\cdot - \xi_0)$ is continuous outside of $\{\xi_0 + kh : k \in \mathbb{Z}\}$. For convenience we set $\xi_0 = 0$ and $\lambda = 0$. Pick $\xi \in \mathbb{R}$ with $\xi \notin \{kh : k \in \mathbb{Z}\}$. Then $G_{h,\infty}(\cdot)$ is continuous in each point $\xi + kh$ for $k \in \mathbb{Z}$. Choose $\varepsilon > 0$. Since $G_{h,\infty}(\cdot)$ is bounded and $\sum_{j=1}^{\infty} |\alpha_j| < \infty$, we can pick $K > 0$ in such a way that

$$2\|G_{h,\infty}\|_{\infty} \frac{1}{h^2} \sum_{j=K}^{\infty} |\alpha_j| < \frac{\varepsilon}{2}. \quad (7.27)$$

For $j \in \{1, \dots, K-1\}$ we can pick $\delta_j > 0$ in such a way that

$$\frac{1}{h^2} |\alpha_j| \left| G_{h,\infty}(\xi + y + hj) - G_{h,\infty}(\xi + hj) \right| < \frac{\varepsilon}{2^{K+1}} \quad (7.28)$$

for all $y \in \mathbb{R}$ with $|y| < \delta_j$. Let $\delta = \min\{\delta_j : 1 \leq j < K\} > 0$. Then for $y \in \mathbb{R}$ with $|y| < \delta$ we obtain

$$\begin{aligned} |\Delta_h G_{h,\infty}(\xi + y) - \Delta_h G_{h,\infty}(\xi)| &\leq \frac{1}{h^2} \sum_{j=K}^{\infty} |\alpha_j| \left(|G_{h,\infty}(\xi + y + jh)| + |G_{h,\infty}(\xi + jh)| \right) \\ &\quad + \frac{1}{h^2} \sum_{j=1}^{K-1} |\alpha_j| \left| G_{h,\infty}(\xi + y + jh) - G_{h,\infty}(\xi + jh) \right| \\ &\leq \frac{2}{h^2} \sum_{j=K}^{\infty} |\alpha_j| \|G_{h,\infty}\|_{\infty} + \sum_{j=1}^{K-1} \frac{\varepsilon}{2^{K+1}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned} \quad (7.29)$$

So $\Delta_h G_{h,\infty}(\cdot)$ is continuous outside of $\{kh : k \in \mathbb{Z}\}$. Hence by definition of $L_{h,\infty}$, we obtain that $G_{h,\infty}(\cdot)$ is C^1 -smooth outside of $\{kh : k \in \mathbb{Z}\}$.

Pick $\chi > 0$ and set

$$R = \left\{ \lambda \in \mathbb{C} : -\frac{\chi}{2} \leq \operatorname{Re} \lambda \leq \chi \text{ and } |\operatorname{Im} \lambda| \leq \frac{\pi |c_h|}{h} \right\}. \quad (7.30)$$

From [3, Proposition 3.7] it follows that for each $\lambda \in R$ there exist constants $K_\lambda > 0$ and $\beta_\lambda > 0$ for which

$$|G_{h,\infty;\lambda}(\xi - \xi_0)| \leq K_\lambda e^{-\beta_\lambda |\xi - \xi_0|} \quad (7.31)$$

for all $\xi, \xi_0 \in \mathbb{R}$. Lemma 5.7 implies that we can choose $\beta_* > 0$ and $K_* > 0$ in such a way that

$$\|\Delta_{L_{h,\infty;\lambda}}(z)^{-1}\| \leq \frac{K_*}{1 + |\operatorname{Im} z|} \quad (7.32)$$

for all $\lambda \in R$ and all $z \in \mathbb{C}$ with $|\operatorname{Re} z| \leq 2\beta_*$. Shifting the integration path in (7.24) in the standard fashion described in [3], we obtain the bound

$$|G_{h,\infty;\lambda}(\xi - \xi_0)| \leq K_* e^{-\beta_* |\xi - \xi_0|} \quad (7.33)$$

for all $\xi, \xi_0 \in \mathbb{R}$ and $\lambda \in R$. ■

Proof of Lemma 7.3. Pick $\lambda \in \mathbb{C} \setminus \sigma(-L_h)$ and compute

$$\begin{aligned} (\lambda + L_h)G_{h,\lambda}(\cdot, \xi_0) &= (\lambda + L_h)G_{h,\infty;\lambda}(\cdot - \xi_0) - [L_h - L_{h,\infty}]G_{h,\infty;\lambda}(\cdot - \xi_0) \\ &= (\lambda + L_{h,\infty})G_{h,\infty;\lambda}(\cdot - \xi_0). \end{aligned} \quad (7.34)$$

The last statement follows immediately from this identity.

Write

$$\hat{G}_{h,\infty;\lambda}(\cdot - \xi_0) = [L_h - L_{h,\infty}]G_{h,\infty;\lambda}(\cdot - \xi_0). \quad (7.35)$$

We have already seen that $\hat{G}_{h,\infty;\lambda}(\cdot - \xi_0) \in L^2(\mathbb{R}, \text{Mat}_2(\mathbb{C}))$. Hence it follows that

$$(\lambda + L_h)^{-1}\hat{G}_{h,\infty;\lambda}(\cdot - \xi_0) \in H^1(\mathbb{R}, \text{Mat}_2(\mathbb{C})). \quad (7.36)$$

In particular this function is continuous. Together with Lemma 7.2 we obtain that $G_{h,\lambda}(\cdot, \xi_0)$ is continuous on $\mathbb{R} \setminus \{\xi_0\}$.

Set $H = (\lambda + L_h)^{-1}\hat{G}_{h,\infty;\lambda}$ and write $H = \begin{pmatrix} H^{(1,1)} & H^{(1,2)} \\ H^{(2,1)} & H^{(2,2)} \end{pmatrix}$. Using the definition of L_h we see that

$$c_h \frac{d}{dx} H = -\lambda H - \hat{G}_{h,\infty} - \tilde{H}, \quad (7.37)$$

where

$$\tilde{H} = - \begin{pmatrix} -\Delta_h H^{(1,1)} - g_u(\bar{u}_h)H^{(1,1)} + H^{(2,1)} & \Delta_h H^{(1,2)} - g_u(\bar{u}_h)H^{(1,2)} + H^{(2,2)} \\ -\rho H^{(1,1)} + \gamma\rho H^{(2,1)} & -\rho H^{(1,2)} + \gamma\rho H^{(2,2)} \end{pmatrix}. \quad (7.38)$$

Since $\bar{u}'_h \in H^1$ and hence continuous, we must have that \bar{u}_h is continuous. As argued before $\Delta_h H^{(1,1)}$ and $\Delta_h H^{(1,2)}$ are also continuous. Hence we see that $c_h \frac{d}{dx} H$ is continuous on $\mathbb{R} \setminus \{\xi_0\}$ and thus that $\frac{d}{dx} H$ is continuous on $\mathbb{R} \setminus \{\xi_0\}$. Therefore we obtain that $G_{h,\lambda}(\cdot, \xi_0)$ is C^1 -smooth on $\mathbb{R} \setminus \{\xi_0 + kh : k \in \mathbb{Z}\}$. \blacksquare

Now we show that for λ with sufficiently large real part, $G_{h,\lambda}(\cdot, \xi_0)$ is bounded uniformly by a constant. This result is based on [2, Lemma 4.1].

Lemma 7.5. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Then there exist constants K and λ_{unif} , which may both depend on h , so that the Green's function $G_{h,\lambda}$ enjoys the uniform estimate*

$$|G_{h,\lambda}(\xi, \xi_0)| \leq K, \quad (7.39)$$

for all $\xi, \xi_0 \in \mathbb{R}$, whenever $\text{Re } \lambda > \lambda_{\text{unif}}$.

Proof. We write $L_h = c_h \frac{d}{dx} + B$ with

$$B = \begin{pmatrix} -\Delta_h - g_u(\bar{u}_h) & 1 \\ -\rho & \gamma\rho \end{pmatrix}. \quad (7.40)$$

We introduce $G_{h,\lambda}^0$ as the Green's function of $(\lambda + c_h \frac{d}{dx})$ viewed as a map from \mathbf{H}^1 to \mathbf{L}^2 . Luckily, it is well-known that this Green's function admits the estimate

$$|G_{h,\lambda}^0(\xi, \xi_0)| \leq \frac{1}{|c_h|} e^{-\text{Re } \lambda |\xi - \xi_0| / |c_h|}. \quad (7.41)$$

We can look for the Green's function $G_{h,\lambda}$ as the solution of the fixed point problem

$$G_{h,\lambda}(\xi, \xi_0) = G_{h,\lambda}^0(\xi, \xi_0) + \int_{\mathbb{R}} G_{h,\lambda}(\xi, z) (BG_{h,\lambda}^0)(z, \xi_0) dz. \quad (7.42)$$

Since $\lambda + L_h$ is invertible by Theorem 2.2, $G_{h,\lambda}$ must necessarily satisfy the fixed point problem (7.42).

For a matrix $A \in \text{Mat}_2(\mathbb{C})$ we write $A = \begin{pmatrix} A^{(1,1)} & A^{(1,2)} \\ A^{(2,1)} & A^{(2,2)} \end{pmatrix}$. We make the decomposition

$$B = B_0 + B_1, \quad (7.43)$$

where

$$\begin{aligned} B_0 &= \begin{pmatrix} -\Delta_h & 0 \\ 0 & 0 \end{pmatrix}, \\ B_1 &= \begin{pmatrix} -g_u(\bar{u}_h) & 1 \\ -\rho & \gamma\rho \end{pmatrix}. \end{aligned} \quad (7.44)$$

We estimate

$$\begin{aligned} |(B_0 G_{h,\lambda}^0)(\xi, \xi_0)| &= |\Delta_h G_{h,\lambda}^0(\xi, \xi_0)^{(1,1)}| \\ &\leq \sum_{j=1}^{\infty} \left[\frac{1}{h^2} |\alpha_j| \left(|G_{h,\lambda}^0(\xi + hj, \xi_0)^{(1,1)}| + |G_{h,\lambda}^0(\xi - hj, \xi_0)^{(1,1)}| + 2|G_{h,\lambda}^0(\xi, \xi_0)^{(1,1)}| \right) \right] \\ &\leq \frac{1}{|c_h|} \sum_{j=1}^{\infty} \left[\frac{1}{h^2} |\alpha_j| \left(e^{-\text{Re } \lambda |\xi + hj - \xi_0|/|c_h|} + e^{-\text{Re } \lambda |\xi - hj - \xi_0|/|c_h|} + 2e^{-\text{Re } \lambda |\xi - \xi_0|/|c_h|} \right) \right] \end{aligned} \quad (7.45)$$

and observe that

$$\begin{aligned} \int_{\mathbb{R}} |(B_0 G_{h,\lambda}^0)(\xi, \xi_0)| d\xi &\leq \frac{1}{|c_h|} \left(\sum_{j=1}^{\infty} 4 \left[\frac{1}{h^2} |\alpha_j| \frac{1}{\text{Re } \lambda / |c_h|} \right] \right) \\ &= \frac{4}{h^2 \text{Re } \lambda} \sum_{j=1}^{\infty} |\alpha_j|. \end{aligned} \quad (7.46)$$

We now fix $G \in L^\infty(\mathbb{R}^2, \text{Mat}_2(\mathbb{C}))$ and consider the expressions

$$\begin{aligned} \mathcal{I}_0 &= \int_{\mathbb{R}} \left[G(\xi, z) (B_0 G_{h,\lambda}^0)(z, \xi_0) \right]^{(1,1)} dz, \\ \mathcal{I}_1 &= \int_{\mathbb{R}} \left[G(\xi, z) (B_1 G_{h,\lambda}^0)(z, \xi_0) \right]^{(1,1)} dz. \end{aligned} \quad (7.47)$$

Using Fubini's theorem for positive functions to switch the integral and the sum, we obtain the estimates

$$\begin{aligned} |\mathcal{I}_0| &\leq \|G\|_{L^\infty} \int_{\mathbb{R}} |(B_0 G_{h,\lambda}^0)(z, \xi_0)| dz \\ &\leq \|G\|_{L^\infty} \frac{4}{h^2 \text{Re } \lambda} \sum_{j=1}^{\infty} |\alpha_j| \end{aligned} \quad (7.48)$$

and

$$\begin{aligned} |\mathcal{I}_1| &\leq \|G\|_{L^\infty} \int_{\mathbb{R}} \left(|g_u(\bar{u}_h(z))| |G_{h,\lambda}^0(z, \xi_0)^{(1,1)}| + \rho |G_{h,\lambda}^0(z, \xi_0)^{(1,1)}| + (1 + \gamma\rho) |G_{h,\lambda}^0(z, \xi_0)^{(2,1)}| \right) dz \\ &\leq \|G\|_{L^\infty} \frac{1}{|c_h|} \int_{\mathbb{R}} \left((|g_u(\bar{u}_h(z))| + \rho + 1 + \gamma\rho) e^{-\text{Re } \lambda |z - \xi_0|/|c_h|} \right) dz \\ &\leq \|G\|_{L^\infty} \frac{1}{|c_h|} \left(\|g_u(\bar{u}_h)\|_{L^\infty} + \rho + 1 + \gamma\rho \right) \left(\frac{1}{\text{Re } \lambda / |c_h|} \right) \\ &\leq \|G\|_{L^\infty} \left(g_* + \rho + 1 + \gamma\rho \right) \left(\frac{1}{\text{Re } \lambda} \right). \end{aligned} \quad (7.49)$$

Similar estimates hold for the other components of $\int_{\mathbb{R}} G(\xi, z) (B G_{h,\lambda}^0)(z, \xi_0) dz$. Therefore, the mapping $G \mapsto \int_{\mathbb{R}} G(\xi, z) (B G_{h,\lambda}^0)(z, \xi_0) dz$ is a contraction in $L^\infty(\mathbb{R}^2, \text{Mat}_2(\mathbb{C}))$ for $\text{Re } \lambda > \lambda_{\text{unif}}$ for λ_{unif} large enough, with λ_{unif} possibly dependent of $h \in (0, h_{**})$. Hence we get a unique bounded solution

of (7.42), which must be $G_{h,\lambda}$. The desired bound on $G_{h,\lambda}$ is now immediate. \blacksquare

Proof of Proposition 7.4. Fix $j_0 \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$. Since (7.7) is merely a linear ODE in the Banach space \mathbb{L}^2 , it follows from the Cauchy-Lipschitz theorem that (7.7) indeed has a unique solution $\mathcal{V} : [t_0, \infty) \rightarrow \mathbb{L}^2$. For any $Z \in C_c^\infty(\mathbb{R}; \mathbb{L}^2)$, an integration by parts yields

$$\begin{aligned} \int_{t_0}^\infty \sum_{j \in \mathbb{Z}} \left[-\frac{dZ_j}{dt}(t) \mathcal{V}_j(t) - (\mathcal{A}_h(t) \cdot \mathcal{V})_j(t) Z_j(t) \right] dt &= \int_{t_0}^\infty \sum_{j \in \mathbb{Z}} \left[\left(\frac{d\mathcal{V}_j}{dt}(t) - (\mathcal{A}_h(t) \cdot \mathcal{V}(t))_j \right) Z_j(t) \right] dt \\ &\quad - \sum_{j \in \mathbb{Z}} \mathcal{V}_j(t_0) Z_j(t_0) \\ &= -Z_{j_0}(t_0). \end{aligned} \tag{7.50}$$

We want to show that the function $V_j(t) := \mathcal{G}_j^{j_0}(t, t_0, h)$ defined by (7.26) coincides with \mathcal{V} on $[t_0, \infty)$. To accomplish this, we define

$$I = \int_{t_0}^\infty \sum_{j \in \mathbb{Z}} \left[-\frac{dZ_j}{dt}(t) V_j(t) - (\mathcal{A}_h(t) \cdot V(t))_j Z_j(t) \right] dt \tag{7.51}$$

and show that V is a weak solution to (7.7) in the sense that

$$I = -Z_{j_0}(t_0) \tag{7.52}$$

holds for all $Z \in C_c^\infty(\mathbb{R}; \mathbb{L}^2)$. Indeed, the unicity of weak solutions then implies that $V = \mathcal{V}$.

Note first that $V(t) = 0$ for $t < t_0$, which can be seen by using (7.39) and taking $\chi \rightarrow \infty$ in (7.26). We write $y = hj_0 + c_h t_0$, $\chi_- = \chi - \frac{i\pi c_h}{h}$ and $\chi_+ = \chi + \frac{i\pi c_h}{h}$. We see that

$$I = \int_{-\infty}^\infty \sum_{j \in \mathbb{Z}} \left[-\frac{dZ_j}{dt}(t) V_j(t) - (\mathcal{A}_h(t) \cdot V(t))_j Z_j(t) \right] dt, \tag{7.53}$$

since $V(t) = 0$ for $t < t_0$. Moreover we write

$$\mathbb{G}_j(t) = G_{h,\lambda}(hj + c_h t, y). \tag{7.54}$$

Using our definition of $V(t)$, we have

$$I = -\frac{h}{2\pi i} \int_{\chi_-}^{\chi_+} \sum_{j \in \mathbb{Z}} \left[\int_{-\infty}^\infty \mathcal{I}_j(t, \lambda) dt \right] d\lambda, \tag{7.55}$$

where

$$\mathcal{I}_j(t, \lambda) = e^{\lambda(t-t_0)} \left[-\mathbb{G}_j(t) \frac{dZ_j}{dt}(t) - (\mathcal{A}_h(t) \cdot \mathbb{G}(t))_j Z_j(t) \right]. \tag{7.56}$$

The permutation of the summations and integrations is allowed by Lebesgue's theorem because Z and $\frac{dZ}{dt}$ are compactly supported and $G_{h,\lambda}$ is uniformly bounded by (7.39). Fix $\chi_- \leq \lambda \leq \chi_+$ and $j \in \mathbb{Z}$. Using the change of variable $x = hj + c_h t$ we obtain

$$\begin{aligned} \int_{-\infty}^\infty \mathcal{I}_j(t, \lambda) dt &= \frac{1}{c_h} \int_{x=-\infty}^{x=\infty} \left[-c_h \mathbb{G}_j \left(\frac{x-hj}{c_h} \right) \frac{dZ_j}{dx} + \lambda \mathbb{G}_j \left(\frac{x-hj}{c_h} \right) \mathcal{Z}_j(x, \lambda) \right. \\ &\quad \left. - \left(\mathcal{A}_h \left(\frac{x-hj}{c_h} \right) \cdot \mathbb{G} \left(\frac{x-hj}{c_h} \right) \right)_j \mathcal{Z}_j(x, \lambda) \right] dx, \end{aligned} \tag{7.57}$$

where

$$\mathcal{Z}_j(x, \lambda) = e^{\lambda((x-hj)/c_h - t_0)} Z_j \left(\frac{x-hj}{c_h} \right). \tag{7.58}$$

Exploiting the fact that Z_j and therefore \mathcal{Z}_j is compactly supported, (7.25) yields

$$\begin{aligned} \int_{-\infty}^\infty \mathcal{I}_j(t, \lambda) dt &= \frac{1}{c_h} \int_{-\infty}^\infty [(L_h + \lambda) G_{h,\lambda}(x, y) \mathcal{Z}_j(x, \lambda)] dx \\ &= \frac{1}{c_h} \mathcal{Z}_j(y). \end{aligned} \tag{7.59}$$

Now since Z_j is compactly supported, we can exchange sums and integrals in equation (7.55). This allows us to compute

$$\begin{aligned}
I &= -\frac{h}{2\pi i} \frac{1}{c_h} \int_{\chi_-}^{\chi_+} \sum_{j \in \mathbb{Z}} \int_{-\infty}^{\infty} \mathcal{I}_j(t, \lambda) dt d\lambda \\
&= -\frac{h}{2\pi i} \frac{1}{c_h} \int_{\chi_-}^{\chi_+} \sum_{j \in \mathbb{Z}} \mathcal{Z}_j(y, \lambda) d\lambda \\
&= -\frac{h}{2\pi i c_h} \int_{\chi_-}^{\chi_+} \sum_{j \in \mathbb{Z}} e^{\lambda \frac{(hj_0 - hj)}{c_h}} Z_j \left(\frac{(hj_0 - hj)}{c_h} + t_0 \right) d\lambda \\
&= -\frac{h}{2\pi i c_h} \sum_{j \in \mathbb{Z}} \int_{\chi_-}^{\chi_+} e^{\lambda \frac{(hj_0 - hj)}{c_h}} Z_j \left(\frac{(hj_0 - hj)}{c_h} + t_0 \right) d\lambda \\
&= -\frac{h}{2\pi i c_h} \sum_{j \in \mathbb{Z}} 2\pi i c_h \delta_j^{j_0} Z_j \left(\frac{(hj_0 - hj)}{c_h} + t_0 \right) \\
&= -Z_{j_0}(t_0),
\end{aligned} \tag{7.60}$$

as desired. \blacksquare

We recall the spaces

$$\begin{aligned}
X_h &= \{\Theta \in \mathbf{H}^1 : \langle \Phi_h^-, \Theta \rangle = 0\} \\
Y_h &= \{\Theta \in \mathbf{L}^2 : \langle \Phi_h^-, \Theta \rangle = 0\},
\end{aligned} \tag{7.61}$$

together with the operators L_h^{-1} in the spaces $\mathcal{B}(X_h, X_h)$ and in $\mathcal{B}(Y_h, X_h)$ that were defined in Proposition 5.2. We also recall the notation $L_h^{\text{qinv}} \Theta$ that was introduced in Corollary 5.4 for the unique solution Ψ of the equation

$$L_h \Psi = \Theta - \frac{\langle \Phi_h^-, \Theta \rangle}{\langle \Phi_h^-, \Phi_h^+ \rangle} \Phi_h^+ \tag{7.62}$$

in the space X_h , which is given explicitly by

$$L_h^{\text{qinv}} \Theta = L_h^{-1} \left[\Theta - \frac{\langle \Phi_h^-, \Theta \rangle}{\langle \Phi_h^-, \Phi_h^+ \rangle} \Phi_h^+ \right] \tag{7.63}$$

We now exploit these operators to decompose the Green's function of $\lambda + L_h$ into a meromorphic and an analytic part. This result is based on [20, Lemma 2.7].

Lemma 7.6. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. For all $0 < h < h_{**}$ there exists a constant $0 < \lambda_h \leq \lambda_0$ such that for all $0 < |\lambda| < \lambda_h$ we have the representation*

$$G_{h,\lambda}(\xi, \xi_0) = E_{h,\lambda}(\xi, \xi_0) + \tilde{G}_{h,\lambda}(\xi, \xi_0) \tag{7.64}$$

Here the meromorphic (in λ) term can be written as

$$E_{h,\lambda}(\xi, \xi_0) = -\frac{1}{\lambda M} \begin{pmatrix} \phi_h^-(\xi_0) \phi_h^+(\xi) & \psi_h^-(\xi_0) \phi_h^+(\xi) \\ \phi_h^-(\xi_0) \psi_h^+(\xi) & \psi_h^-(\xi_0) \psi_h^+(\xi) \end{pmatrix} \tag{7.65}$$

and the analytic (in λ) term $\tilde{G}_{h,\lambda}$ is given by

$$\begin{aligned}
\tilde{G}_{h,\lambda}(\xi, \xi_0) &= G_{h,\infty;\lambda}(\xi - \xi_0) - \left[[I + \lambda L_h^{-1}]^{-1} L_h^{\text{qinv}} (L_h - L_{h,\infty}) G_{h,\infty,\lambda}(\cdot - \xi_0) \right] (\xi) \\
&\quad - \frac{1}{M} \langle \Phi_h^-, G_{h,\infty}(\cdot - \xi_0) \rangle \Phi_h^+(\xi).
\end{aligned} \tag{7.66}$$

Here we recall the notation

$$M = \langle \Phi_h^-, \Phi_h^+ \rangle. \tag{7.67}$$

Proof. Pick $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \lambda_0$ and $0 < h < h_{**}$. By the proof of Proposition 6.2 we see that

$$(L_h + \lambda)^{-1}\Theta = \lambda^{-1} \frac{\langle \Phi_h^-, \Theta \rangle}{M} \Phi_h^+ + L_h^{\text{qinv}} \Theta - [I + \lambda L_h^{-1}]^{-1} \lambda L_h^{-1} L_h^{\text{qinv}} \Theta. \quad (7.68)$$

We now compute

$$\begin{aligned} \langle \Phi_h^-, (L_h - L_{h,\infty}) G_{h,\infty;\lambda}(\cdot - \xi_0) \rangle &= \langle \Phi_h^-, -L_{h,\infty} G_{h,\infty;\lambda}(\cdot - \xi_0) \rangle \\ &= -\Phi_h^-(\xi_0) + \lambda \langle \Phi_h^-, G_{h,\infty;\lambda}(\cdot - \xi_0) \rangle. \end{aligned} \quad (7.69)$$

In particular, writing

$$\hat{L}_h = L_h - L_{h,\infty}, \quad (7.70)$$

we obtain

$$\begin{aligned} (L_h + \lambda)^{-1} \hat{L}_h G_{h,\infty}(\cdot - \xi_0) &= \frac{1}{\lambda M} \begin{pmatrix} \phi_h^-(\xi_0) \phi_h^+(\xi) & \psi_h^-(\xi_0) \phi_h^+(\xi) \\ \phi_h^-(\xi_0) \psi_h^+(\xi) & \psi_h^-(\xi_0) \psi_h^+(\xi) \end{pmatrix} \\ &\quad + \frac{\langle \Phi_h^-, G_{h,\infty;\lambda}(\cdot - \xi_0) \rangle}{M} \Phi_h^+ + L_h^{\text{qinv}} \hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0) \\ &\quad - [I + \lambda L_h^{-1}]^{-1} \lambda L_h^{-1} L_h^{\text{qinv}} \hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0). \end{aligned} \quad (7.71)$$

We may hence write

$$G_{h,\lambda}(\xi, \xi_0) = E_{h,\lambda}(\xi, \xi_0) + \tilde{G}_{h,\lambda}(\xi, \xi_0) \quad (7.72)$$

with

$$E_{h,\lambda}(\xi, \xi_0) = -\frac{1}{\lambda M} \begin{pmatrix} \phi_h^-(\xi_0) \phi_h^+(\xi) & \psi_h^-(\xi_0) \phi_h^+(\xi) \\ \phi_h^-(\xi_0) \psi_h^+(\xi) & \psi_h^-(\xi_0) \psi_h^+(\xi) \end{pmatrix} \quad (7.73)$$

and

$$\begin{aligned} \tilde{G}_{h,\lambda}(\cdot, \xi_0) &= G_{h,\infty;\lambda}(\cdot - \xi_0) - L_h^{\text{qinv}} \hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0) \\ &\quad + [I + \lambda L_h^{-1}]^{-1} \lambda L_h^{-1} L_h^{\text{qinv}} \hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0) \\ &\quad - \frac{1}{M} \langle \Phi_h^-, G_{h,\infty;\lambda}(\cdot - \xi_0) \rangle \Phi_h^+ \\ &= G_{h,\infty;\lambda}(\cdot - \xi_0) - [I + \lambda L_h^{-1}]^{-1} \lambda L_h^{-1} L_h^{\text{qinv}} \hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0) \\ &\quad - \frac{1}{M} \langle \Phi_h^-, G_{h,\infty}(\cdot - \xi_0) \rangle \Phi_h^+. \end{aligned} \quad (7.74)$$

Clearly $E_{h,\lambda}$ is meromorphic in λ , while $\tilde{G}_{h,\lambda}$ is analytic in λ in the region $|\lambda| < \lambda_0$. \blacksquare

We fix $\chi > \lambda_{\text{unif}}$, where λ_{unif} was defined in Lemma 7.3, and set

$$R = \left\{ \lambda \in \mathbb{C} : -\frac{\tilde{\chi}}{2} \leq \text{Re } \lambda \leq \chi \text{ and } |\text{Im } \lambda| \leq \frac{\pi |c_h|}{h} \right\}. \quad (7.75)$$

We now set out to obtain an estimate on the function $\tilde{G}_{h,\lambda}$ from Lemma 7.6 by exploiting the asymptotic hyperbolicity of L_h . We treat each of the terms in (7.66) separately in the results below.

Lemma 7.7. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. There exist constants $K_4 > 0$ and $\tilde{\chi} > 0$, which may depend on h , such that for all $\lambda \in R$*

$$\left| \langle \Phi_h^-, (L_h - L_{h,\infty}) G_{h,\infty;\lambda}(\cdot - \xi_0) \rangle \right| \leq K_4 e^{-\tilde{\chi} |\xi_0|}. \quad (7.76)$$

Proof. We reuse the notation $\hat{L}_h = L_h - L_{h,\infty}$ from the previous proof. Lemma 7.2 implies that we can pick constants $\beta_* > 0$ and $K_* > 0$ in such a way that

$$|G_{h,\infty;\lambda}(\xi - \xi_0)| \leq K_* e^{-\beta_* |\xi - \xi_0|} \quad (7.77)$$

for all values of ξ, ξ_0 . Recall the constants $K_2, \tilde{\alpha}$ from Lemma 5.16 and set $K_3 = K_2 \|\Phi_h^-\|_\infty$. Then we obtain

$$\begin{aligned}
|\langle \Phi_h^-, \hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0) \rangle| &\leq \int_{-\infty}^{\infty} K_3 e^{-\tilde{\alpha}|\xi|} g_* K_* e^{-\beta_*|\xi - \xi_0|} d\xi \\
&= K_3 g_* K_* \left(\frac{1}{\tilde{\alpha} + \beta_*} (e^{-\tilde{\alpha}|\xi_0|} + e^{-\beta_*|\xi_0|}) + \frac{1}{\beta_* - \tilde{\alpha}} (e^{-\tilde{\alpha}|\xi_0|} - e^{-\beta_*|\xi_0|}) \right) \\
&\leq K_3 g_* K_* \left(\frac{1}{\tilde{\alpha} + \beta_*} 2e^{-\min\{\tilde{\alpha}, \beta_*\}|\xi_0|} + \frac{1}{|\beta_* - \tilde{\alpha}|} 2e^{-\min\{\tilde{\alpha}, \beta_*\}|\xi_0|} \right) \\
&= K_4 e^{-\tilde{\chi}|\xi_0|}
\end{aligned} \tag{7.78}$$

for some $K_4 > 0$ and $\tilde{\chi} > 0$. \blacksquare

Lemma 7.8. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. There exist constants $K_{10} > 0$ and $\tilde{\gamma} > 0$, which may depend on h , such that for all $\lambda \in \mathbb{R}$*

$$\begin{aligned}
\left| \left[L_h^{\text{qinv}}(L_h - L_{h,\infty}) G_{h,\infty;\lambda}(\cdot - \xi_0) \right](\xi) \right| &\leq K_{10} e^{-\tilde{\gamma}|\xi|} e^{-\tilde{\gamma}|\xi_0|} \\
&\leq K_{10} e^{-\tilde{\gamma}|\xi - \xi_0|}.
\end{aligned} \tag{7.79}$$

Proof. We reuse the notation $\hat{L}_h = L_h - L_{h,\infty}$ from the previous proof. Recall the constants K_1, α from Lemma 5.16. Writing

$$H_{\xi_0}(\xi) = \left[L_h^{\text{qinv}} \hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0) \right](\xi), \tag{7.80}$$

we may use [3, Lemma 4.3] to estimate

$$|H_{\xi_0}(\xi)| \leq K_1 e^{-\alpha|\xi|} \|H_{\xi_0}\|_\infty + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi - \eta|} |L_h H_{\xi_0}(\eta)| d\eta. \tag{7.81}$$

Recalling (7.61)-(7.63), we obtain

$$\begin{aligned}
\|H_{\xi_0}\|_\infty &\leq \|H_{\xi_0}\|_{\mathbf{H}^1} \\
&\leq C_{\text{unif}} \left\| \hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0) - \frac{\langle \Phi_h^-, \hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0) \rangle}{M} \Phi_h^+ \right\|_{\mathbf{L}^2} \\
&\leq C_{\text{unif}} \left(1 + \frac{\|\Phi_h^-\|_{\mathbf{L}^2}}{|M|} \|\Phi_h^+\|_{\mathbf{L}^2} \right) \|\hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0)\|_{\mathbf{L}^2} \\
&\leq K_5 \|\hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0)\|_{\mathbf{L}^2}
\end{aligned} \tag{7.82}$$

for some constant $K_5 > 0$.

Using Lemma 5.16 we see that there exists a constant $K_6 > 0$ for which

$$\begin{aligned}
|\bar{u}_h(\xi)| &= \left| \int_{\xi}^{\infty} \bar{u}_h'(\xi') d\xi' \right| \\
&\leq \int_{\xi}^{\infty} K_1 \|(\bar{u}_h', \bar{w}_h')\|_{\mathbf{L}^2} e^{-\alpha|\xi'|} d\xi' \\
&= K_6 e^{-\alpha|\xi|}
\end{aligned} \tag{7.83}$$

holds for all $\xi \in \mathbb{R}$. Recall that

$$\hat{L}_h = \begin{pmatrix} -g_u(\bar{u}_h) + r_0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7.84}$$

Observe that $-g_u(0) + r_0 = 0$. Then we obtain that

$$|-g_u(\bar{u}_h(\xi)) + r_0| \leq K_7 e^{-\alpha|\xi|} \tag{7.85}$$

for all $\xi \in \mathbb{R}$ and for some constant $K_7 > 0$. Lemma 7.2 implies that

$$|G_{h,\infty;\lambda}(\xi - \xi_0)| \leq K_* e^{-\beta_* |\xi - \xi_0|} \quad (7.86)$$

for all $\xi \in \mathbb{R}$. Therefore we must have

$$\begin{aligned} \|\hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0)\|_{\mathbf{L}^2}^2 &\leq \int_{\mathbb{R}} K_7^2 K_*^2 e^{-2\alpha|\xi|} e^{-2\beta_* |\xi - \xi_0|} d\xi \\ &\leq K_8 e^{-2\tilde{\gamma}|\xi_0|} \end{aligned} \quad (7.87)$$

for some constants $K_8 > 0$, $\tilde{\gamma} > 0$ with $\tilde{\gamma} \leq \beta_*$, $\tilde{\gamma} \leq \frac{1}{2}\alpha$ and $\tilde{\gamma} \leq \frac{1}{2}\tilde{\chi}$. In particular we obtain the estimate

$$\|H_{\xi_0}\|_{\infty} \leq K_5 \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|}. \quad (7.88)$$

In a similar fashion, using Lemma 7.7, we see that

$$\begin{aligned} |L_h H_{\xi_0}(\xi)| &\leq \left| \left[\hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0) \right](\xi) - \frac{\langle \Phi_h^-, \hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0) \rangle}{M} \Phi_h^+(\xi) \right| \\ &\leq K_7 K_* e^{-\alpha|\xi|} e^{-\beta_* |\xi - \xi_0|} + \frac{1}{|M|} K_4 e^{-\tilde{\chi}|\xi_0|} K_1 e^{-\alpha|\xi|} \\ &\leq K_9 \left[e^{-2\tilde{\gamma}|\xi|} e^{-\tilde{\gamma}|\xi - \xi_0|} + e^{-\tilde{\gamma}|\xi_0|} e^{-\tilde{\gamma}|\xi|} \right] \end{aligned} \quad (7.89)$$

for all $\xi \in \mathbb{R}$ and some constant $K_9 > 0$. Combining (7.81) with (7.82) and (7.87), we hence obtain

$$\begin{aligned} |H_{\xi_0}(\xi)| &\leq K_1 e^{-\alpha|\xi|} \|H_{\xi_0}\|_{\infty} + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi - \eta|} |L_h H_{\xi_0}(\eta)| d\eta \\ &\leq K_1 e^{-\alpha|\xi|} K_5 \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi - \eta|} K_9 \left[e^{-2\tilde{\gamma}|\eta|} e^{-\tilde{\gamma}|\eta - \xi_0|} + e^{-\tilde{\gamma}|\xi_0|} e^{-\tilde{\gamma}|\eta|} \right] d\eta \\ &\leq K_1 e^{-\alpha|\xi|} K_5 \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi - \eta|} 2K_9 e^{-\tilde{\gamma}|\eta|} e^{-\tilde{\gamma}|\xi_0|} d\eta \\ &\leq K_{10} e^{-\tilde{\gamma}|\xi|} e^{-\tilde{\gamma}|\xi_0|} \\ &\leq K_{10} e^{-\tilde{\gamma}|\xi - \xi_0|} \end{aligned} \quad (7.90)$$

for some constant $K_{10} > 0$. ■

Lemma 7.9. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. There exist constants $K_{13} > 0$ and $\omega > 0$, which may depend on h , such that the function $\tilde{G}_{h,\lambda}$ from Lemma 7.6 satisfies the bound*

$$|\tilde{G}_{h,\lambda}(\xi, \xi_0)| \leq K_{13} e^{-\omega|\xi - \xi_0|} \quad (7.91)$$

for all ξ, ξ_0 and all $0 < |\lambda| < \lambda_h$.

Proof. As before, we write

$$H_{\xi_0}(\xi) = L_h^{\text{qinv}} \hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0)(\xi). \quad (7.92)$$

Using [3, Lemma 4.3], Lemma 7.8 and (7.88) and recalling (7.61)-(7.63), we obtain the estimate

$$\begin{aligned} |L_h^{-1} H_{\xi_0}(\xi)| &\leq K_1 e^{-\alpha|\xi|} \|L_h^{-1} H_{\xi_0}\|_{\infty} + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi - \eta|} |H_{\xi_0}(\eta)| d\eta \\ &\leq K_1 e^{-\alpha|\xi|} C_{\text{unif}} \|H_{\xi_0}\|_{\mathbf{L}^2} + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi - \eta|} K_{10} e^{-\tilde{\gamma}|\eta - \xi_0|} d\eta \\ &\leq K_1 e^{-\alpha|\xi|} C_{\text{unif}} K_5 \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + K_1 \int_{-\infty}^{\infty} e^{-\alpha|\xi - \eta|} K_{10} e^{-\tilde{\gamma}|\eta - \xi_0|} d\eta \\ &\leq K_{10} K_{11} e^{-\tilde{\gamma}|\xi - \xi_0|} \end{aligned} \quad (7.93)$$

for some constant $K_{11} > 0$. Using Proposition 5.2 and (7.88) we obtain that

$$\|(L_h^{-1})^n H_{\xi_0}\|_{\mathbf{H}^1} \leq K_5 \sqrt{K_8} (C_{\text{unif}})^n e^{-\tilde{\gamma}|\xi_0|} \quad (7.94)$$

for all $n \in \mathbb{Z}_{>0}$. Continuing in this fashion, we see that

$$|(L_h^{-1})^n H_{\xi_0}(\xi)| \leq K_{10} K_{11}^n e^{-\tilde{\gamma}|\xi - \xi_0|} \quad (7.95)$$

for all $n \in \mathbb{Z}_{>0}$. If we set

$$\lambda_h = \min\left\{\frac{\tilde{\lambda}}{2}, \lambda_0, \chi, \frac{1}{C_{\text{unif}} K_5 \sqrt{K_8}}, \frac{1}{K_{11}}\right\}, \quad (7.96)$$

then for each $n \in \mathbb{Z}_{>0}$ and each $0 < |\lambda| < \lambda_h$ we have

$$\|(-\lambda)^n (L_h^{-1})^n H_{\xi_0}\|_{\mathbf{H}^1} \leq \frac{1}{2}. \quad (7.97)$$

In particular, it follows that

$$\sum_{n=0}^{\infty} (-\lambda)^n (L_h^{-1})^n H_{\xi_0} \rightarrow [I + \lambda L_h^{-1}]^{-1} H_{\xi_0} \quad (7.98)$$

in \mathbf{H}^1 . Since \mathbf{H}^1 -convergence implies point-wise convergence, we conclude that

$$\begin{aligned} |[I + \lambda L_h^{-1}]^{-1} H_{\xi_0}(\xi)| &= \left| \sum_{n=0}^{\infty} (-\lambda)^n (L_h^{-1})^n H_{\xi_0}(\xi) \right| \\ &\leq \sum_{n=0}^{\infty} \lambda_h^n K_{11} K_{12}^n e^{-\tilde{\gamma}|\xi - \xi_0|} \\ &\leq \frac{K_{11}}{1 - \lambda_h K_{11}} e^{-\tilde{\gamma}|\xi - \xi_0|} \\ &:= K_{12} e^{-\tilde{\gamma}|\xi - \xi_0|} \end{aligned} \quad (7.99)$$

for all $\xi \in \mathbb{R}$ and for some constant $K_{12} > 0$.

Combining this estimate with Lemma 5.16 and Lemma 7.8 yields the desired bound

$$\begin{aligned} |\tilde{G}_{h,\lambda}(\xi, \xi_0)| &= |G_{h,\infty;\lambda}(\xi - \xi_0) - [I + \lambda L_h^{-1}]^{-1} L_h^{\text{qinv}} \hat{L}_h G_{h,\infty;\lambda}(\cdot - \xi_0)](\xi) \\ &\quad - \frac{1}{M} \langle \Phi_h^-, G_{h,\infty}(\cdot - \xi_0) \rangle \Phi_h^+(\xi)| \\ &\leq K_* e^{-\beta_*|\xi - \xi_0|} + K_{12} e^{-\tilde{\gamma}|\xi - \xi_0|} + K_4 \frac{1}{M} e^{-\tilde{\chi}|\xi_0|} K_1 e^{-\alpha|\xi|} \|\Phi_h^+\|_{\infty} \\ &\leq K_{13} e^{-\omega|\xi - \xi_0|} \end{aligned} \quad (7.100)$$

for some constants $K_{13} > 0$ and $\omega > 0$. ■

We write

$$S = \left\{ -\lambda_h + i\omega : \omega \in \left[-\frac{\pi|c_h|}{h}, \frac{\pi|c_h|}{h} \right] \right\}, \quad (7.101)$$

where λ_h is defined in the proof of Lemma 7.9.

Lemma 7.10. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Then there exist constants $K > 0$ and $\tilde{\beta} > 0$, which may depend on h , such that for all $\lambda \in S$ we have the bound*

$$|G_{h,\lambda}(\xi, \xi_0)| \leq K e^{-\tilde{\beta}|\xi - \xi_0|} \quad (7.102)$$

for all ξ, ξ_0 .

Proof. Fix $\lambda_0 \in S$. For $\lambda \in S$ sufficiently close to λ_0 we have

$$\begin{aligned} [L_h + \lambda]^{-1} &= [L_h + \lambda_0 + \lambda - \lambda_0]^{-1} \\ &= \left[(L_h + \lambda_0) \left(I + (L_h + \lambda_0)^{-1}(\lambda - \lambda_0) \right) \right]^{-1} \\ &= \left[I + (L_h + \lambda_0)^{-1}(\lambda - \lambda_0) \right]^{-1} [L_h + \lambda_0]^{-1}. \end{aligned} \quad (7.103)$$

In particular, upon writing

$$H_{\xi_0}(\xi) = \left[[L_h + \lambda_0]^{-1} \hat{L}_h G_{h,\lambda;\infty}(\cdot - \xi_0) \right](\xi), \quad (7.104)$$

we see that

$$G_{h,\lambda}(\xi, \xi_0) - G_{h,\infty;\lambda}(\xi - \xi_0) = \left[[I + (L_h + \lambda_0)^{-1}(\lambda - \lambda_0)]^{-1} H_{\xi_0} \right](\xi). \quad (7.105)$$

Lemma 5.6 implies that $L_h + \lambda_0$ is asymptotically hyperbolic. Using [3, Lemma 4.3] we can pick constants $k_{\lambda_0} > 0$ and $\alpha_{\lambda_0} > 0$ in such a way that

$$|H_{\xi_0}(\xi)| \leq k_{\lambda_0} e^{-\alpha_{\lambda_0}|\xi|} \|H_{\xi_0}\|_{\infty} + k_{\lambda_0} \int_{-\infty}^{\infty} e^{-\alpha_{\lambda_0}|\xi-\eta|} |(L_h + \lambda_0)H_{\xi_0}(\eta)| d\eta. \quad (7.106)$$

Recall the constant C_S appearing in Proposition 3.3. This allows us to estimate

$$\begin{aligned} \|H_{\xi_0}\|_{\infty} &\leq \|H_{\xi_0}\|_{\mathbf{H}^1} \\ &\leq C_S \|\hat{L}_h G_{h,\lambda_0}(\xi, \xi_0)\|_{\mathbf{L}^2} \\ &\leq C_S \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|}. \end{aligned} \quad (7.107)$$

This yields the bound

$$\begin{aligned} |H_{\xi_0}(\xi)| &\leq k_{\lambda_0} e^{-\alpha_{\lambda_0}|\xi|} C_S \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + k_{\lambda_0} \int_{-\infty}^{\infty} e^{-\alpha_{\lambda_0}|\xi-\eta|} |\hat{L}_h G_{h,\lambda;\infty}(\eta, \xi_0)| d\eta \\ &\leq k_{\lambda_0} e^{-\alpha_{\lambda_0}|\xi|} C_S \sqrt{K_8} e^{-\tilde{\gamma}|\xi_0|} + k_{\lambda_0} \int_{-\infty}^{\infty} e^{-\alpha_{\lambda_0}|\xi-\eta|} K_7 K_* e^{-\alpha|\eta|} e^{-2\beta_*|\eta-\xi_0|} d\eta \\ &\leq k_{\lambda_0;2} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|} \end{aligned} \quad (7.108)$$

for some constants $k_{\lambda_0;2}, \alpha_{\lambda_0;2}$, which may depend on λ_0 , but not on λ . Arguing as in (7.93), we obtain

$$\begin{aligned} |[L_h + \lambda_0]^{-1} H_{\xi_0}(\xi)| &\leq k_{\lambda_0} e^{-\alpha_{\lambda_0}|\xi|} \|[L_h + \lambda_0]^{-1} H_{\xi_0}\|_{\infty} + k_{\lambda_0} \int_{-\infty}^{\infty} e^{-\alpha_{\lambda_0}|\xi-\eta|} |H_{\xi_0}(\eta)| d\eta \\ &\leq k_{\lambda_0;2} k_{\lambda_0;3} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|} \end{aligned} \quad (7.109)$$

for some constant $k_{\lambda_0;3} > 0$, which may depend on λ_0 , but not on λ . Following the same steps as the proof of Lemma 7.9 and setting

$$\varepsilon_{\lambda_0} = \min\left\{ \frac{1}{k_{\lambda_0} C_S \sqrt{K_8}}, \frac{1}{k_{\lambda_0;3}} \right\}, \quad (7.110)$$

we conclude that

$$\begin{aligned} |G_{h,\lambda}(\xi, \xi_0) - G_{h,\infty;\lambda}(\xi - \xi_0)| &= \left| \left[[I + [L_h + \lambda_0]^{-1}(\lambda - \lambda_0)]^{-1} H_{\xi_0} \right](\xi) \right| \\ &\leq k_{\lambda_0;4} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|} \end{aligned} \quad (7.111)$$

holds for each $\lambda \in S$ with $|\lambda - \lambda_0| < \varepsilon_{\lambda_0}$, for some constant $k_{\lambda_0;4} > 0$, which may depend on λ_0 . In particular, we obtain that

$$\begin{aligned} |G_{h,\lambda}(\xi, \xi_0)| &\leq k_{\lambda_0;4} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|} + K_* e^{-\beta_*|\xi-\xi_0|} \\ &\leq k_{\lambda_0;5} e^{-\alpha_{\lambda_0;2}|\xi-\xi_0|} \end{aligned} \quad (7.112)$$

holds for each $\lambda \in S$ with $|\lambda - \lambda_0| < \varepsilon_{\lambda_0}$, for some constant $k_{\lambda_0;5} > 0$, which may depend on λ_0 .

Since S is compact we can find $\lambda_1, \dots, \lambda_n \in S$ in such a way that

$$S \subset \bigcup_{i=1}^n \{\lambda \in \mathbb{C} : |\lambda - \lambda_i| < \varepsilon_{\lambda_i}\}. \quad (7.113)$$

Setting

$$\begin{aligned} K &= \max\{k_{\lambda_i;5} : i \in \{1, \dots, n\}\}, \\ \tilde{\beta} &= \min\{\alpha_{\lambda_i;2} : i \in \{1, \dots, n\}\}, \end{aligned} \quad (7.114)$$

we conclude that

$$|G_{h,\lambda}(\xi, \xi_0)| \leq Ke^{-\tilde{\beta}|\xi - \xi_0|} \quad (7.115)$$

holds for all $\lambda \in S$ and all $\xi, \xi_0 \in \mathbb{R}$. ■

We now use the spectral properties of $-L_h$ to also decompose the Green's function $G(t, t_0, h)$ into two parts. We obtain the following result, which is based on [20, Corollary 2.8].

Lemma 7.11. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. For any pair $t \geq t_0$ and any $j, j_0 \in \mathbb{Z}$, we have the representation*

$$\mathcal{G}_j^{j_0}(t, t_0, h) = \mathcal{E}_j^{j_0}(t, t_0, h) + \tilde{\mathcal{G}}_j^{j_0}(t, t_0, h) \quad (7.116)$$

in which

$$\mathcal{E}_j^{j_0}(t, t_0, h) = \frac{h}{M} \begin{pmatrix} \phi_h^-(hj_0 + c_h t_0)\phi_h^+(hj + c_h t) & \psi_h^-(hj_0 + c_h t_0)\phi_h^+(hj + c_h t) \\ \phi_h^-(hj_0 + c_h t_0)\psi_h^+(hj + c_h t) & \psi_h^-(hj_0 + c_h t_0)\psi_h^+(hj + c_h t) \end{pmatrix}, \quad (7.117)$$

while $\tilde{\mathcal{G}}$ satisfies the found

$$|\tilde{\mathcal{G}}_j^{j_0}(t, t_0, h)| \leq Ke^{-\tilde{\beta}(t-t_0)}e^{-\tilde{\beta}|hj + c_h t - hj_0 - c_h t_0|} \quad (7.118)$$

for some $K > 0$ and $\tilde{\beta} > 0$ which may depend on h .

Proof. Recall the representation of $\mathcal{G}_j^{j_0}$ from Proposition 7.4. Note that $G_{h,\lambda}(\xi, \xi_0)$ is meromorphic for λ in the strip $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq -\lambda_3, |\operatorname{Im} \lambda| \leq \frac{c_h \pi}{h}\}$ with a simple pole at $\lambda = 0$ by Lemma 7.6, Lemma 7.3 and Theorem 2.2. Lemma 7.6 also implies that the residue of $G_{h,\lambda}(\xi, \xi_0)$ in $\lambda = 0$ is given by

$$\operatorname{Res}(G_{h,\lambda}(\xi, \xi_0), 0) = -\frac{1}{M} \begin{pmatrix} \phi_h^-(\xi_0)\phi_h^+(\xi) & \psi_h^-(\xi_0)\phi_h^+(\xi) \\ \phi_h^-(\xi_0)\psi_h^+(\xi) & \psi_h^-(\xi_0)\psi_h^+(\xi) \end{pmatrix}. \quad (7.119)$$

We write

$$H(\cdot, \xi_0) = e^{2\pi i \frac{1}{h} k \xi_0} (L_h + \lambda + 2\pi i k \frac{c_h}{h}) e_{-2\pi i \frac{1}{h} k} G_{h,\lambda}(\cdot, \xi_0). \quad (7.120)$$

In a similar fashion as in the proof of Lemma 6.1 we see that for $k \in \mathbb{Z}$ we have

$$(L_h + \lambda + 2\pi i k \frac{c_h}{h}) e_{-2\pi i \frac{1}{h} k} = e_{-2\pi i \frac{1}{h} k} (L_h + \lambda). \quad (7.121)$$

Therefore it follows that

$$\begin{aligned} H(\cdot, \xi_0) &= e^{2\pi i \frac{1}{h} k \xi_0} (L_h + \lambda + 2\pi i k \frac{c_h}{h}) e_{-2\pi i \frac{1}{h} k} G_{h,\lambda}(\cdot, \xi_0) \\ &= e^{2\pi i \frac{1}{h} k \xi_0} e_{-2\pi i \frac{1}{h} k} (L_h + \lambda) G_{h,\lambda}(\cdot, \xi_0). \end{aligned} \quad (7.122)$$

For any $f \in \mathbf{H}^1$ we may hence compute

$$\begin{aligned} \int H(\xi, \xi_0) f(\xi_0) d\xi_0 &= \int e^{2\pi i \frac{1}{h} k \xi_0} e^{-2\pi i \frac{1}{h} k \xi} (L_h + \lambda) G_{h,\lambda}(\cdot, \xi_0)(\xi) f(\xi_0) d\xi_0 \\ &= e^{-2\pi i \frac{1}{h} k \xi} [e^{2\pi i \frac{1}{h} k \xi} f(\xi)] \\ &= f(\xi). \end{aligned} \quad (7.123)$$

Therefore by the invertibility of $L_h + \lambda + 2\pi i k \frac{c_h}{h}$, we must have

$$G_{h,\lambda+2\pi i k \frac{c_h}{h}}(\xi, \xi_0) = e^{2\pi i k \frac{1}{h} (\xi_0 - \xi)} G_{h,\lambda}(\xi, \xi_0). \quad (7.124)$$

Now recall the constants χ, χ_+, χ_- from (the proof of) Proposition 7.4 and define

$$\begin{aligned} \lambda_h^- &= -\frac{\lambda_h}{2} - i \frac{\pi c_h}{h} \\ \lambda_h^+ &= -\frac{\lambda_h}{2} + i \frac{\pi c_h}{h}. \end{aligned} \quad (7.125)$$

Writing $x = hj + c_h t, y = hj_0 + c_h t_0$, we see that

$$\begin{aligned} \int_{\lambda_h^-}^{\lambda_h^+} e^{\lambda(t-t_0)} G_{h,\lambda}(x, y) d\lambda &= \int_{\lambda_h^+}^{\lambda_h^-} e^{(\lambda+2\pi i \frac{c_h}{h})(t-t_0)} e^{-2\pi i \frac{1}{h} (y-x)} G_{h,\lambda}(x, y) d\lambda \\ &= \int_{\lambda_h^+}^{\lambda_h^-} e^{\lambda(t-t_0)} G_{h,\lambda}(x, y) d\lambda. \end{aligned} \quad (7.126)$$

Hence if we integrate the function $e^{\lambda(t-t_0)} G_{h,\lambda}(hj + c_h t, hj_0 + c_h t_0)$ along the rectangle with edges $-\frac{\lambda_h}{2} - i \frac{\pi c_h}{h}, -\frac{\lambda_h}{2} + i \frac{\pi c_h}{h}, \chi - i \frac{\pi c_h}{h}$ and $\chi + i \frac{\pi c_h}{h}$ that the integrals from $\chi - i \frac{\pi c_h}{h}$ to $-\frac{\lambda_h}{2} - i \frac{\pi c_h}{h}$ and from $-\frac{\lambda_h}{2} + i \frac{\pi c_h}{h}$ to $\chi + i \frac{\pi c_h}{h}$ cancel each other out. In particular, again writing $x = hj + c_h t, y = hj_0 + c_h t_0$, the residue theorem implies

$$\begin{aligned} \mathcal{G}_j^{j_0}(t, t_0, h) &= \frac{-h}{2\pi i} \int_{\chi - i \frac{\pi c_h}{h}}^{\chi + i \frac{\pi c_h}{h}} e^{\lambda(t-t_0)} G_{h,\lambda}(x, y) d\lambda \\ &= \frac{h}{2\pi i} \int_{-\frac{\lambda_h}{2} - i \frac{\pi c_h}{h}}^{-\frac{\lambda_h}{2} + i \frac{\pi c_h}{h}} e^{\lambda(t-t_0)} G_{h,\lambda}(x, y) d\lambda + \frac{h}{M} \begin{pmatrix} \phi_h^-(y) \phi_h^+(x) & \psi_h^-(y) \phi_h^+(x) \\ \phi_h^-(y) \psi_h^+(x) & \psi_h^-(y) \psi_h^+(x) \end{pmatrix}. \end{aligned} \quad (7.127)$$

Using Lemma 7.10 we also get the estimate

$$\left| \frac{h}{2\pi i} \int_{-\frac{\lambda_h}{2} - i \frac{\pi c_h}{h}}^{-\frac{\lambda_h}{2} + i \frac{\pi c_h}{h}} e^{\lambda(t-t_0)} G_{h,\lambda}(x, y) d\lambda \right| \leq \frac{h}{2\pi} \frac{2c_h \pi}{h} e^{-\lambda_h(t-t_0)} K e^{-\tilde{\beta}|x-y|}, \quad (7.128)$$

which yields the desired bound (7.118). \blacksquare

For any $t \in \mathbb{R}$, we introduce the suggestive notation

$$\Pi_h^c(t) = \mathcal{E}(t, t, h) \quad (7.129)$$

together with

$$\Pi_h^s(t) = I - \Pi_h^c(t). \quad (7.130)$$

Recalling the notation introduced in (7.11), we set out to show that $\Pi_h^c(t) * \Pi_h^c(t) = \Pi_h^c(t)$ and $\Pi_h^s(t) * \Pi_h^s(t) = \Pi_h^s(t)$. Later on, we will view these operators as projections that correspond to the center and stable parts of the flow induced by \mathcal{G} respectively.

To establish the identity $\Pi_h^c(t) * \Pi_h^c(t) = \Pi_h^c(t)$, it suffices to show that

$$\begin{aligned} \frac{M}{h} \begin{pmatrix} \phi_h^-(y) \phi_h^+(x) & \psi_h^-(y) \phi_h^+(x) \\ \phi_h^-(y) \psi_h^+(x) & \psi_h^-(y) \psi_h^+(x) \end{pmatrix} &= \sum_{i \in \mathbb{Z}} \begin{pmatrix} \phi_h^-(hi + c_h t) \phi_h^+(x) & \psi_h^-(hi + c_h t) \phi_h^+(x) \\ \phi_h^-(hi + c_h t) \psi_h^+(x) & \psi_h^-(hi + c_h t) \psi_h^+(x) \end{pmatrix} \\ &\quad \times \begin{pmatrix} \phi_h^-(y) \phi_h^+(hi + c_h t) & \psi_h^-(y) \phi_h^+(hi + c_h t) \\ \phi_h^-(y) \psi_h^+(hi + c_h t) & \psi_h^-(y) \psi_h^+(hi + c_h t) \end{pmatrix}, \end{aligned} \quad (7.131)$$

in which $x = hj + c_h t$ and $y = hj_0 + c_h t$. We now write our linear operator in the form

$$L_h \Psi(\xi) = c_h \frac{d}{d\xi} \Psi(\xi) + \sum_{j=-\infty}^{\infty} A_j(\xi) \Psi(\xi + jh), \quad (7.132)$$

where

$$A_j(\xi) = \begin{cases} \begin{pmatrix} \frac{1}{h^2} \alpha_{|j|} & 0 \\ 0 & 0 \end{pmatrix} & \text{if } j \neq 0 \\ \begin{pmatrix} -2 \frac{1}{h^2} \sum_{k>0} \alpha_k + g_u(\bar{u}_h(\xi)) & 1 \\ -\rho & \rho\gamma \end{pmatrix} & \text{if } j = 0. \end{cases} \quad (7.133)$$

Before we continue, we first prove a small lemma that will help us to relate discrete inner products with their continuous counterparts.

Lemma 7.12. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. For all $\xi \in \mathbb{R}$ we have the identity*

$$c_h \begin{pmatrix} \phi_h^-(\xi) \phi_h^+(\xi) \\ \psi_h^-(\xi) \psi_h^+(\xi) \end{pmatrix} = \sum_{j=-\infty}^{\infty} \int_0^{hj} B(\xi + \theta - hj) A_j(\xi + \theta - hj) \Phi_h^+(\xi + \theta) d\theta, \quad (7.134)$$

where

$$B(\xi) = \begin{pmatrix} \phi_h^-(\xi) & 0 \\ 0 & \psi_h^-(\xi) \end{pmatrix}. \quad (7.135)$$

Proof. Our strategy is to differentiate both sides of (7.134) and to show their derivatives are equal. Starting with the first component, we pick $N \in \mathbb{Z}_{>0} \cup \{\infty\}$ and write

$$D(N) := \frac{d}{d\xi} \sum_{j=-N}^N \int_0^{hj} \phi_h^-(\xi + \theta - hj) \left[A_j(\xi + \theta - hj) \Phi_h^+(\xi + \theta) \right]^{(1)} d\theta. \quad (7.136)$$

For finite N , we may compute

$$\begin{aligned} D(N) &= \frac{d}{d\xi} \sum_{j=-N}^N \int_{\xi-hj}^{\xi} \phi_h^-(\theta) \left[A_j(\theta) \Phi_h^+(\theta + hj) \right]^{(1)} d\theta \\ &= \sum_{j=-N}^N \phi_h^-(\xi) \left[A_j(\xi) \Phi_h^+(\xi + hj) \right]^{(1)} - \sum_{j=-N}^N \phi_h^-(\xi - hj) \left[A_j(\xi - hj) \Phi_h^+(\xi) \right]^{(1)}. \end{aligned} \quad (7.137)$$

Now for $j > 0$ we have $|A_j(\xi) \Phi_h^+(\xi + hj)| \leq \frac{1}{h^2} |\alpha_j|$, so the partial sums converge uniformly. Hence it follows that

$$\begin{aligned} D(\infty) &= \sum_{j=-\infty}^{\infty} \phi_h^-(\xi) \left[A_j(\xi) \Phi_h^+(\xi + hj) \right]^{(1)} - \sum_{j=-\infty}^{\infty} \phi_h^-(\xi - hj) \left[A_j(\xi - hj) \Phi_h^+(\xi) \right]^{(1)} \\ &= \phi_h^-(\xi) c_h (\phi_h^+)'(\xi) + c_h (\phi_h^-)'(\xi) \phi_h^+(\xi) \\ &= c_h (\phi_h^- \bar{u}_h)'(\xi), \end{aligned} \quad (7.138)$$

since $\Phi_h^+ \in \ker(L_h)$ and $\Phi_h^- \in \ker(L_h^*)$.

We now set out to show that both sides of (7.134) converge to zero as $\xi \rightarrow \infty$. Pick $\varepsilon > 0$ and let $N \in \mathbb{Z}_{>0}$ be large enough to ensure that

$$\sum_{j \geq N} \frac{1}{h^2} j |\alpha_j| \leq \frac{\varepsilon}{4(1 + \|\phi_h^-\|_{\infty}) \|\Phi_h^+\|_{\infty}}. \quad (7.139)$$

In addition, let Ξ be large enough to have

$$|\phi_h^-(\xi)| \leq \frac{\varepsilon}{4(1 + \sum_{j=-N}^N |j\alpha_{|j|}|) \|\Phi_h^+\|_\infty} \quad (7.140)$$

for all $\xi \geq \Xi - N$. This Ξ exists since $\phi_h^- \in H^1$. For such ξ we may estimate

$$\left| \sum_{j=-\infty}^{\infty} \int_0^{hj} \phi_h^-(\xi + \theta - hj) \left[A_j(\xi + \theta - hj) \Phi_h^+(\xi + \theta) \right]^{(1)} d\theta \right| < \varepsilon, \quad (7.141)$$

which allows us to compute

$$\begin{aligned} \lim_{\xi \rightarrow \infty} \sum_{j=-\infty}^{\infty} \int_0^{hj} \phi_h^-(\xi + \theta - hj) \left[A_j(\xi + \theta - hj) \Phi_h^+(\xi + \theta) \right]^{(1)} d\theta &= 0 \\ &= \lim_{\xi \rightarrow \infty} c_h \phi_h^-(\xi) \phi_h^+(\xi). \end{aligned} \quad (7.142)$$

With that we have proved our claim. Furthermore we can repeat the arguments above to obtain

$$c_h \psi_h^-(\xi) \psi_h^+(\xi) = \sum_{j=-\infty}^{\infty} \int_0^{hj} \psi_h^-(\xi + \theta - hj) \left[A_j(\xi + \theta - hj) \Phi_h^+(\xi + \theta) \right]^{(2)} d\theta. \quad (7.143)$$

■

We are now ready to show that $\Pi_h^c(t) * \Pi_h^c(t) = \Pi_h^c(t)$ and $\Pi_h^s(t) * \Pi_h^s(t) = \Pi_h^s(t)$. This result is based on the first part of [20, Lemma 2.9].

Lemma 7.13. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Then $\Pi_h^c(t) * \Pi_h^c(t) = \Pi_h^c(t)$ and $\Pi_h^s(t) * \Pi_h^s(t) = \Pi_h^s(t)$ for all $t \in \mathbb{R}$.*

Proof. Using the results from Lemma 7.12 we may compute

$$\begin{aligned} c_h \sum_{k=-\infty}^{\infty} \phi_h^-(hk + c_h t) \phi_h^+(hk + c_h t) &= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \int_0^{hj} \phi_h^-(hk + c_h t + \theta - hj) \\ &\quad \times \left[A_j(hk + c_h t + \theta - hj) \Phi_h^+(hk + c_h t + \theta) \right]^{(1)} d\theta \\ &= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \int_{hk+c_h t-hj}^{hk+c_h t} \phi_h^-(\theta) \left[A_j(\theta) \Phi_h^+(\theta + hj) \right]^{(1)} d\theta \\ &= \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} j \phi_h^-(\theta) \left[A_j(\theta) \Phi_h^+(\theta + hj) \right]^{(1)} d\theta, \end{aligned} \quad (7.144)$$

where we were allowed to interchange the two infinite sums because

$$\begin{aligned} \left| \sum_{k=-N}^N \int_{hk+c_h t-hj}^{hk+c_h t} \phi_h^-(\theta) \left[A_j(\theta) \Phi_h^+(\theta + hj) \right]^{(1)} d\theta \right| &\leq \left| \int_{-\infty}^{\infty} j \phi_h^-(\theta) \left[A_j(\theta) \Phi_h^+(\theta + hj) \right]^{(1)} d\theta \right| \\ &\leq \|\phi_h^-\|_1 \frac{1}{h^2} |j\alpha_{|j|}| \|\phi_h^+\|_\infty \end{aligned} \quad (7.145)$$

holds for all $N \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}$. This expression is summable over j , allowing us to apply Lebesgue's theorem. On the other hand, we have

$$\begin{aligned} c_h \int_{-\infty}^{\infty} \phi_h^-(\xi) \phi_h^+(\xi) d\xi &= \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \int_0^{hj} \phi_h^-(\xi + \theta - hj) \left[A_j(\xi + \theta - hj) \Phi_h^+(\xi + \theta) \right]^{(1)} d\theta d\xi \\ &= \sum_{j=-\infty}^{\infty} \int_0^{hj} \int_{-\infty}^{\infty} \phi_h^-(\xi + \theta - hj) \left[A_j(\xi + \theta - hj) \Phi_h^+(\xi + \theta) \right]^{(1)} d\xi d\theta \\ &= \sum_{j=-\infty}^{\infty} \int_0^{hj} \int_{-\infty}^{\infty} \phi_h^-(\xi - hj) \left[A_j(\xi - hj) \Phi_h^+(\xi) \right]^{(1)} d\xi d\theta \\ &= \sum_{j=-\infty}^{\infty} hj \int_{-\infty}^{\infty} \phi_h^-(\xi - hj) \left[A_j(\xi - hj) \Phi_h^+(\xi) \right]^{(1)} d\xi. \end{aligned} \quad (7.146)$$

Interchanging the integral with the sum was allowed since ϕ_h^- and ϕ_h^+ decay exponentially, say $|\phi_h^-(x)| \leq \kappa e^{-\alpha|x|}$ and $|\phi_h^+(x)| \leq \kappa e^{-\alpha|x|}$, and thus for each $N \in \mathbb{Z}_{>0}$ and each $\xi \in \mathbb{R}$ we have

$$\left| \sum_{j=-N}^N \int_0^{hj} \phi_h^-(\xi + \theta - hj) \left[A_j(\xi + \theta - hj) \Phi_h^+(\xi + \theta) \right]^{(1)} d\theta \right| \leq \sum_{j=-\infty}^{\infty} h \kappa^2 e^{-\alpha|\xi|} |j \alpha_{|j|}| \|\Phi_h^+\|_{\infty}, \quad (7.147)$$

which is integrable in ξ . Furthermore the interchanging of the two integrals was allowed, since by the exponential decay of ϕ_h^- we also see that for each $j \in \mathbb{Z}, \xi \in \mathbb{R}$ and $\theta \in (0, hj)$ we have

$$|\phi_h^-(\xi + \theta - hj) \left[A_j(\xi + \theta - hj) \Phi_h^+(\xi) \right]^{(1)}| \leq \kappa e^{-\alpha|\xi + \theta - hj|} |\alpha_{|j|}| \|\Phi_h^+\|_{\infty}. \quad (7.148)$$

This is an integrable function for $(\xi, \theta) \in \mathbb{R} \times (0, hj)$, allowing us to apply Fubini's theorem.

In particular, we see that

$$\int_{-\infty}^{\infty} \phi_h^-(\xi) \phi_h^+(\xi) d\xi = h \sum_{k=-\infty}^{\infty} \phi_h^-(hk + c_h t) \phi_h^+(hk + c_h t). \quad (7.149)$$

In the same way we obtain

$$\int_{-\infty}^{\infty} \psi_h^-(\xi) \psi_h^+(\xi) d\xi = h \sum_{k=-\infty}^{\infty} \psi_h^-(hk + c_h t) \psi_h^+(hk + c_h t). \quad (7.150)$$

By writing out the sums it now follows that indeed (7.131) holds. \blacksquare

Proof of Proposition 7.1. The calculations above imply that $\mathcal{E}(t, t_0, h) = \mathcal{E}(t, t_0, h) * \Pi_h^c(t_0)$, which means that we must also have $\mathcal{E}(t, t_0, h) * \Pi_h^s(t_0) = 0$.

Observe that for any $t_0 \in \mathbb{R}$, the function $V_j(t) := \begin{pmatrix} \phi_h^+(hj + c_h t) \\ \psi_h^+(hj + c_h t) \end{pmatrix}$ is the unique solution to (7.5) with $V_j(t_0) = \begin{pmatrix} \phi_h^+(hj + c_h t_0) \\ \psi_h^+(hj + c_h t_0) \end{pmatrix}$. Hence by the definition of the Green's function $\mathcal{G}(t, t_0, h)$ we see that

$$V(t) = \mathcal{G}(t, t_0, h) * V(t_0) \quad (7.151)$$

for all $t \in \mathbb{R}$. Furthermore we recall that

$$\mathcal{E}_j^{j_0}(t_0, t_0, h) = V_j(t_0) \Phi_h^-(hj_0 + c_h t_0). \quad (7.152)$$

For $j, j_0 \in \mathbb{Z}$ we may hence compute

$$\begin{aligned} \left[\mathcal{G}(t, t_0, h) * \Pi_h^c(t_0) \right]_j^{j_0} &= \sum_{i \in \mathbb{Z}} \mathcal{G}_j^i(t, t_0, h) * \mathcal{E}_i^{j_0}(t, t_0, h) \\ &= \frac{h}{M} \sum_{i \in \mathbb{Z}} \mathcal{G}_j^i(t, t_0, h)_i * V_i(t_0) \Phi_h^-(hj_0 + c_h t_0) \\ &= \frac{h}{M} V_j(t) \Phi_h^-(hj_0 + c_h t_0) \\ &= \mathcal{E}_j^{j_0}(t, t_0, h). \end{aligned} \quad (7.153)$$

In particular, we obtain $\mathcal{G}(t, t_0, h) * \Pi_h^c(t_0) = \mathcal{E}(t, t_0, h)$ and thus

$$\begin{aligned} \tilde{\mathcal{G}}(t, t_0, h) * \Pi_h^c(t_0) &= \mathcal{G}(t, t_0, h) * \Pi_h^c(t_0) - \mathcal{E}(t, t_0, h) * \Pi_h^c(t_0) \\ &= \mathcal{E}(t, t_0, h) - \mathcal{E}(t, t_0, h) \\ &= 0. \end{aligned} \quad (7.154)$$

Therefore we must have

$$\begin{aligned}\mathcal{G}(t, t_0, h) &= \mathcal{E}(t, t_0, h) + \tilde{\mathcal{G}}(t, t_0, h) \\ &= \mathcal{E}(t, t_0, h) * \left(\Pi_h^c(t_0) + \Pi_h^s(t_0) \right) + \tilde{\mathcal{G}}(t, t_0, h) * \left(\Pi_h^c(t_0) + \Pi_h^s(t_0) \right) \\ &= \mathcal{E}(t, t_0, h) * \Pi_h^c(t_0) + \tilde{\mathcal{G}}(t, t_0, h) * \Pi_h^s(t_0),\end{aligned}\tag{7.155}$$

which completes the proof. ■

8 Nonlinear stability

In this section we will finally prove Theorem 2.3, along the lines of the approach described in [20]. Recall from §2 that the space ℓ^p is defined by

$$\ell^p = \{V \in (\mathbb{R}^2)^{\mathbb{Z}} : \|V\|_{\ell^p} := \sum_{j \in \mathbb{Z}} |V_j|^p < \infty\} \quad (8.1)$$

for $1 \leq p < \infty$ and

$$\ell^\infty = \{V \in (\mathbb{R}^2)^{\mathbb{Z}} : \|V\|_{\ell^\infty} := \sup_{j \in \mathbb{Z}} |V_j| < \infty\}. \quad (8.2)$$

In addition, we recall the notation $(\bar{U}_h)_j(t) = (\bar{u}_h(hj + c_h t), \bar{w}_h(hj + c_h t))$ and we let $\tilde{\beta} > 0$ be the constant appearing in Proposition 7.1.

Exploiting Proposition 5.2 we see that

$$\|\mathcal{E}_j^{j_0}(t, t_0, h)\| \leq C_1 e^{-\tilde{\beta}|hj + c_h t - hj_0 - c_h t_0|} \quad (8.3)$$

for some constant $C_1 > 0$. Lemma 7.13 hence allows us to define $\Pi_h^c(t) \in \mathcal{B}(\ell^p; \ell^p)$ and $\Pi_h^s(t) \in \mathcal{B}(\ell^p, \ell^p)$ by writing

$$\begin{aligned} \Pi_h^c(t)V &= \mathcal{E}(t, t, h) * V, \\ \Pi_h^s(t)V &= [I - \Pi_h^c(t)]V. \end{aligned} \quad (8.4)$$

The proof of our nonlinear stability result proceeds in two main steps. In particular, we first construct the stable manifolds of the solutions $(\bar{u}_h, \bar{w}_h)(\cdot - \tilde{\theta})$ for each $\tilde{\theta} \in \mathbb{R}$. This result is based on the first half of the proof of [20, Proposition 2.1].

Proposition 8.1. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. There exists a constant $\eta > 0$, independent of p , but possibly depending on h , such that for each $\tilde{\theta} \in \mathbb{R}$ and each $W_s \in \text{Range}(\Pi_h^s(\tilde{\theta}))$ with $\|W_s\|_{\ell^p} < \eta$ there is a unique function $\mathcal{U}_*^{\tilde{\theta}}(W_s) : [0, \infty) \rightarrow \ell^p$ such that*

1. $U(t) = \bar{U}_h(t + \tilde{\theta}) + \mathcal{U}_*^{\tilde{\theta}}(W_s)(t)$ is a solution of (2.1) for all $t \geq 0$,
2. $\mathcal{U}_*^{\tilde{\theta}}(W_s)(t)$ decays exponentially to 0 as $t \rightarrow \infty$,
3. $\Pi_h^s(\tilde{\theta})\mathcal{U}_*^{\tilde{\theta}}(W_s)(0) = W_s$.

In addition, there exist constants $C_6 > 0$ and $C_{13} > 0$, independent of p , but possibly depending on h , such that the estimate

$$\|\Pi_h^c(\tilde{\theta})\mathcal{U}_*^{\tilde{\theta}}(W_s)(0)\|_{\ell^p} \leq C_6 \|W_s\|_{\ell^p}^2 \quad (8.5)$$

holds for all $\tilde{\theta} \in \mathbb{R}$ and each $W_s \in \text{Range}(\Pi_h^s(\tilde{\theta}))$ with $\|W_s\|_{\ell^p} < \eta$, while the estimate

$$\begin{aligned} \|\Pi_h^c(\tilde{\theta}_1)\mathcal{U}_*^{\tilde{\theta}_1}(W_s^1)(0) - \Pi_h^c(\tilde{\theta}_2)\mathcal{U}_*^{\tilde{\theta}_2}(W_s^2)(0)\|_{\ell^p} &\leq C_{13} \left[\|W_s^1\|_{\ell^p} + \|W_s^2\|_{\ell^p} \right] \\ &\quad \times \left[\|W_s^1 - W_s^2\|_{\ell^p} + |\theta_1 - \theta_2| \right] \end{aligned} \quad (8.6)$$

holds for all $W_s^1 \in \text{Range}(\Pi_h^s(\tilde{\theta}_1))$, all $W_s^2 \in \text{Range}(\Pi_h^s(\tilde{\theta}_2))$ and all $\tilde{\theta}_1 \in \mathbb{R}$ and $\tilde{\theta}_2 \in \mathbb{R}$ with $\|W_s^1\|_{\ell^p} < \eta$, $\|W_s^2\|_{\ell^p} < \eta$ and $|\tilde{\theta}_2 - \tilde{\theta}_1| < \eta$.

It then suffices to show that the space around the family of travelling pulse solutions can be completely covered by these stable manifolds. We remark that $\tilde{\theta}$ in the result below will correspond with the asymptotic phase shift.

Proposition 8.2. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Then there exists a constant $\delta > 0$, which may depend on h but not on p , such that for all initial conditions $U^0 \in \ell^p$ with $\|U^0 - \bar{U}_h(0)\|_{\ell^p} < \delta$ there exists $\tilde{\theta} \in \mathbb{R}$ and $W_s \in \text{Range}(\Pi_h^s(\tilde{\theta}))$ such that*

$$U^0 = \bar{U}_h(\tilde{\theta}) + \mathcal{U}_*^{\tilde{\theta}}(W_s) \quad (8.7)$$

We write the LDE (2.1) as

$$\frac{d}{dt}V(t) = \mathcal{F}_h(V(t)), \quad (8.8)$$

where

$$\mathcal{F}_h(V(t))_j = \frac{1}{c_h} \left(\frac{1}{h^2} \sum_{k>0} \alpha_k [V_{j+k}^{(1)}(t) + V_{j-k}^{(1)}(t) - 2V_j^{(1)}(t)] + g(V_j^{(1)}(t)) - V_j^{(2)}(t) \right) - \rho[V_j^{(1)}(t) - \gamma V_j^{(2)}(t)]. \quad (8.9)$$

Then we see that $\mathcal{A}_h(t) = D\mathcal{F}_h(\bar{U}_h(t))$, where $\mathcal{A}_h(t)$ is defined in (7.5). We now write

$$\mathcal{N}_{h;\theta}(t, V(t)) = \mathcal{F}_h(V(t) + \bar{U}_h(t + \theta)) - \mathcal{F}_h(\bar{U}_h(t + \theta)) - D\mathcal{F}_h(\bar{U}_h(t + \theta))V(t) \quad (8.10)$$

and set out to solve the differential equation

$$\frac{d}{dt}V(t) = D\mathcal{F}_h(\bar{U}_h(t + \theta))V(t) + \mathcal{N}_{h;\theta}(t, V(t)). \quad (8.11)$$

Indeed, if V satisfies (8.11), then we see that

$$\begin{aligned} \frac{d}{dt}(\bar{U}_h(t + \theta) + V(t)) &= \mathcal{F}_h(\bar{U}_h(t + \theta) + V(t)) + D\mathcal{F}_h(\bar{U}_h(t + \theta))V(t) + \mathcal{N}_{h;\theta}(t, V(t)) \\ &= \mathcal{F}_h(\bar{U}_h(t + \theta) + V(t)), \end{aligned} \quad (8.12)$$

which means that $\bar{U}_h(\cdot + \theta) + V$ is indeed a solution of (2.1).

Our goal is to construct decaying solutions to (8.11) for multiple values of θ using a single Green's function. To this end, we write

$$\mathcal{M}_h^{\tilde{\theta}}(\theta, t, V) = \mathcal{N}_{h;\theta}(t, V(t)) + D\mathcal{F}_h(\bar{U}_h(t + \theta))V(t) - D\mathcal{F}_h(\bar{U}_h(t + \tilde{\theta}))V(t). \quad (8.13)$$

This allow us to rewrite (8.11) as

$$\frac{d}{dt}V(t) = D\mathcal{F}_h(\bar{U}_h(t + \tilde{\theta}))V(t) + \mathcal{M}_h^{\tilde{\theta}}(\theta, t, V). \quad (8.14)$$

Lemma 8.3. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Then $\mathcal{M}_h^{\tilde{\theta}}(\theta, t, \cdot)$ maps ℓ^p into itself and there exists a constant $C_2 > 0$, independent of p and $\tilde{\theta}$ but possibly depending on h , so that we have the estimate*

$$\|\mathcal{M}_h^{\tilde{\theta}}(\theta, t, V)\|_{\ell^p} \leq C_2\|V\|_{\ell^p}^2 + C_2|\tilde{\theta} - \theta|\|V\|_{\ell^p}, \quad (8.15)$$

for $V \in \ell^p$ with $\|V\|_{\ell^p} \leq 1$ and $\theta \in \mathbb{R}$ with $|\theta - \tilde{\theta}| \leq 1$, together with

$$\begin{aligned} \|\mathcal{M}_h^{\tilde{\theta}}(\theta_1, t, V_1) - \mathcal{M}_h^{\tilde{\theta}}(\theta_2, t, V_2)\|_{\ell^p} &\leq C_2 \left[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p} + |\tilde{\theta} - \theta_1| + |\theta_2 - \tilde{\theta}| \right] \|V_1 - V_2\|_{\ell^p} \\ &\quad + C_2|\tilde{\theta}_1 - \tilde{\theta}_2| \left[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p} \right] \end{aligned} \quad (8.16)$$

for $V_1 \in \ell^p$, $V_2 \in \ell^p$, $\theta_1 \in \mathbb{R}$ and $\theta_2 \in \mathbb{R}$ with $\|V_1\|_{\ell^p} \leq 1$, $\|V_2\|_{\ell^p} \leq 1$, $|\theta_1 - \tilde{\theta}| \leq 1$ and $|\theta_2 - \tilde{\theta}| \leq 1$.

Proof. A Taylor expansion around $\bar{U}_h^{(1)}(t + \tilde{\theta})$ yields the pointwise identity

$$\begin{aligned}
\mathcal{M}_h^{\tilde{\theta}}(\theta, t, V) &= \frac{1}{c_h} \begin{pmatrix} g(V^{(1)} + \bar{U}_h^{(1)}(t + \theta)) - g_u(\bar{U}_h^{(1)}(t + \tilde{\theta}))V^{(1)} - g(\bar{U}_h^{(1)}(t + \theta)) \\ 0 \end{pmatrix} \\
&= \frac{1}{c_h} \begin{pmatrix} \frac{1}{2}g_{uu}(\xi_1)(V^{(1)})^2 + [g_u(\bar{U}_h^{(1)}(t + \theta)) - g_u(\bar{U}_h^{(1)}(t + \tilde{\theta}))]V^{(1)} \\ 0 \end{pmatrix} \\
&= \frac{1}{c_h} \begin{pmatrix} \frac{1}{2}g_{uu}(\xi_1)(V^{(1)})^2 + \frac{1}{2}g_{uu}(\xi_2)[\bar{U}_h^{(1)}(t + \theta) - \bar{U}_h^{(1)}(t + \tilde{\theta})]V^{(1)} \\ 0 \end{pmatrix} \\
&= \frac{1}{c_h} \begin{pmatrix} \frac{1}{2}g_{uu}(\xi_1)(V^{(1)})^2 + \frac{1}{2}g_{uu}(\xi_2)\left[\frac{d}{dt}\bar{U}_h^{(1)}(\xi_3)\right]|\theta - \tilde{\theta}|V^{(1)} \\ 0 \end{pmatrix},
\end{aligned} \tag{8.17}$$

where ξ_1 is between $\bar{U}_h^{(1)}(t + \tilde{\theta})$ and $\bar{U}_h^{(1)}(t + \tilde{\theta}) + V$, ξ_2 is between $\bar{U}_h^{(1)}(t + \tilde{\theta})$ and $\bar{U}_h^{(1)}(t)$ and ξ_3 is between $t + \theta$ and $t + \tilde{\theta}$. For a bounded function f , we have the pointwise bound

$$\begin{aligned}
|g_{uu}(f)| &= |6f + 2r_0 + 2| \\
&\leq 6\|f\|_\infty + 2r_0 + 2.
\end{aligned} \tag{8.18}$$

Therefore we get the pointwise bound

$$\begin{aligned}
|\mathcal{M}_h^{\tilde{\theta}}(\theta, t, V)| &\leq \left| \frac{1}{c_h} \frac{1}{2} \begin{pmatrix} (6\|\bar{u}_h\|_\infty + 2r_0 + 2)\|\bar{u}'_h\|_\infty|\theta - \tilde{\theta}|V^{(1)} \\ 0 \end{pmatrix} \right| \\
&\quad + \left| \frac{1}{c_h} \frac{1}{2} \begin{pmatrix} (6\|V^{(1)}\|_{\ell^\infty} + 6\|\bar{u}_h\|_\infty + 2r_0 + 2)(V^{(1)})^2 \\ 0 \end{pmatrix} \right| \\
&\leq \frac{1}{|c_h|} \frac{1}{2} (6\|\bar{u}_h\|_\infty + 2r_0 + 2)\|\bar{u}'_h\|_\infty|\theta - \tilde{\theta}||V| \\
&\quad + \frac{1}{|c_h|} \frac{1}{2} (6\|V\|_{\ell^\infty} + 6\|\bar{u}_h\|_\infty + 2r_0 + 2)|(V^{(1)})|^2.
\end{aligned} \tag{8.19}$$

Furthermore, for $1 \leq p < \infty$ we see

$$\begin{aligned}
\left\| \begin{pmatrix} |V^{(1)}|^2 \\ 0 \end{pmatrix} \right\|_{\ell^p} &= \left(\sum_{j \in \mathbb{Z}} |V_j^{(1)}|^{2p} \right)^{\frac{1}{p}} \\
&\leq \left(\left(\sum_{j \in \mathbb{Z}} |V_j^{(1)}|^p \right) \left(\sup_{j \in \mathbb{Z}} |V_j^{(1)}|^p \right) \right)^{\frac{1}{p}} \\
&\leq \|V\|_{\ell^p}^2,
\end{aligned} \tag{8.20}$$

which clearly also holds for $p = \infty$ upon skipping the intermediate two steps. We hence obtain the bound

$$\begin{aligned}
\|\mathcal{M}_h^{\tilde{\theta}}(\theta, t, V)\|_{\ell^p} &\leq \frac{1}{|c_h|} \frac{1}{2} (6\|\bar{u}_h\|_\infty + 2r_0 + 2)\|\bar{u}'_h\|_\infty|\theta - \tilde{\theta}|\|V\|_{\ell^p} \\
&\quad + \frac{1}{|c_h|} \frac{1}{2} (6\|\bar{u}_h\|_\infty + 2r_0 + 8)\|V\|_{\ell^p}^2 \\
&\leq C_2\|V\|_{\ell^p}^2 + C_2|\theta - \tilde{\theta}|\|V\|_{\ell^p},
\end{aligned} \tag{8.21}$$

for some constant $C_2 > 0$, which is independent of p and $\tilde{\theta}$.

We now write

$$\begin{aligned}
d\mathcal{M} &= \mathcal{M}_h^{\tilde{\theta}}(\theta_1, t, V_1) - \mathcal{M}_h^{\tilde{\theta}}(\theta_2, t, V_2) \\
&= \frac{1}{c_h} \left(\begin{array}{c} g(V_1^{(1)} + \bar{U}_h^{(1)}(t + \theta_1)) - g_u(\bar{U}_h^{(1)}(t + \tilde{\theta}))V_1^{(1)} - g(\bar{U}_h^{(1)}(t + \theta_1)) \\ 0 \end{array} \right) \\
&\quad + \frac{1}{c_h} \left(\begin{array}{c} -g(V_2^{(1)} + \bar{U}_h^{(1)}(t + \theta_2)) + g_u(\bar{U}_h^{(1)}(t + \tilde{\theta}))V_2^{(1)} + g(\bar{U}_h^{(1)}(t + \theta_2)) \\ 0 \end{array} \right). \tag{8.22}
\end{aligned}$$

Using Taylor expansions around $\bar{U}_h^{(1)}(t + \theta_1)$ and $\bar{U}_h^{(1)}(t + \theta_2)$, we obtain the pointwise identities

$$\begin{aligned}
g(V_1^{(1)} + \bar{U}_h^{(1)}(t + \theta_1)) - g(\bar{U}_h^{(1)}(t + \theta_1)) &= \frac{1}{6}g_{uuu}(\xi_1)(V_1^{(1)})^3 + \frac{1}{2}g_{uu}(\bar{U}_h^{(1)}(t + \theta_1))(V_1^{(1)})^2 \\
&\quad + g_u(\bar{U}_h^{(1)}(t + \theta_1)), \\
g(V_2^{(1)} + \bar{U}_h^{(1)}(t + \theta_2)) - g(\bar{U}_h^{(1)}(t + \theta_2)) &= \frac{1}{6}g_{uuu}(\xi_2)(V_2^{(1)})^3 + \frac{1}{2}g_{uu}(\bar{U}_h^{(1)}(t + \theta_2))(V_2^{(1)})^2 \\
&\quad + g_u(\bar{U}_h^{(1)}(t + \theta_2)), \tag{8.23}
\end{aligned}$$

where ξ_1 is in between $\bar{U}_h^{(1)}(t + \theta_1)$ and $V_1^{(1)} + \bar{U}_h^{(1)}(t + \theta_1)$ and ξ_2 is in between $\bar{U}_h^{(1)}(t + \theta_2)$ and $\bar{U}_h^{(1)}(t + \theta_2) + V_2$. This allows us to collect all terms of the same order together and write

$$d\mathcal{M} = dM_1 + dM_2 + dM_3, \tag{8.24}$$

where

$$\begin{aligned}
dM_1 &= \frac{1}{c_h} \left(\begin{array}{c} \frac{1}{6}g_{uuu}(\xi_1)(V_1^{(1)})^3 - \frac{1}{6}g_{uuu}(\xi_2)(V_2^{(1)})^3 \\ 0 \end{array} \right), \\
dM_2 &= \frac{1}{c_h} \left(\begin{array}{c} \frac{1}{2}g_{uu}(\bar{U}_h^{(1)}(t + \theta_1))(V_1^{(1)})^2 - \frac{1}{2}g_{uu}(\bar{U}_h^{(1)}(t + \theta_2))(V_2^{(1)})^2 \\ 0 \end{array} \right), \\
dM_3 &= \frac{1}{c_h} \left(\begin{array}{c} [g_u(\bar{U}_h^{(1)}(t + \theta_1)) - g_u(\bar{U}_h^{(1)}(t + \tilde{\theta}))]V_1^{(1)} \\ 0 \end{array} \right) \\
&\quad + \frac{1}{c_h} \left(\begin{array}{c} -[g_u(\bar{U}_h^{(1)}(t + \theta_2)) - g_u(\bar{U}_h^{(1)}(t + \tilde{\theta}))]V_2^{(1)} \\ 0 \end{array} \right). \tag{8.25}
\end{aligned}$$

Note that $g_{uuu} = 6$ is constant. A Taylor expansion around $\bar{U}_h^{(1)}(t + \theta_1)$ yields the pointwise identity

$$\begin{aligned}
dM_2 &= \frac{1}{c_h} \left(\begin{array}{c} \frac{1}{2}(g_{uu}(\bar{U}_h^{(1)}(t + \theta_1)) - g_{uu}(\bar{U}_h^{(1)}(t + \theta_2)))(V_1^{(1)})^2 \\ 0 \end{array} \right) \\
&\quad + \frac{1}{c_h} \left(\begin{array}{c} -\frac{1}{2}g_{uu}(\bar{U}_h^{(1)}(t + \theta_2))((V_2^{(1)})^2 - (V_1^{(1)})^2) \\ 0 \end{array} \right) \\
&= \frac{1}{c_h} \left(\begin{array}{c} 3(\bar{U}_h^{(1)}(t + \theta_1) - \bar{U}_h^{(1)}(t + \theta_2))(V_1^{(1)})^2 - \frac{1}{2}g_{uu}(\bar{U}_h^{(1)}(t + \theta_2))((V_2^{(1)})^2 - (V_1^{(1)})^2) \\ 0 \end{array} \right) \\
&= \frac{1}{c_h} \left(\begin{array}{c} 3(\bar{U}_h^{(1)})'(\xi_3)(\theta_1 - \theta_2)(V_1^{(1)})^2 - \frac{1}{2}g_{uu}(\bar{U}_h^{(1)}(t + \theta_2))((V_2^{(1)})^2 - (V_1^{(1)})^2) \\ 0 \end{array} \right), \tag{8.26}
\end{aligned}$$

where ξ_3 is in between $t + \theta_1$ and $t + \theta_2$. Using Taylor expansions around $\bar{U}_h^{(1)}(t + \theta_2)$ and $\bar{U}_h^{(1)}(t + \tilde{\theta})$, we obtain the pointwise identity

$$\begin{aligned} dM_3 &= \frac{1}{c_h} \left(\begin{array}{c} \left[g_u(\bar{U}_h^{(1)}(t + \theta_1)) - g_u(\bar{U}_h^{(1)}(t + \tilde{\theta})) \right] V_1^{(1)} \\ 0 \end{array} \right) \\ &\quad + \frac{1}{c_h} \left(\begin{array}{c} - \left[g_u(\bar{U}_h^{(1)}(t + \theta_2)) - g_u(\bar{U}_h^{(1)}(t + \tilde{\theta})) \right] V_2^{(1)} \\ 0 \end{array} \right) \\ &= \frac{1}{c_h} \left(\begin{array}{c} \left[g_{uu}(\xi_4)(\theta_1 - \theta_2) \right] V_1^{(1)} - \left[g_{uu}(\xi_5)(\theta_2 - \tilde{\theta}) \right] V_2^{(1)} \\ 0 \end{array} \right), \end{aligned} \quad (8.27)$$

where ξ_4 is in between $\bar{U}_h^{(1)}(t + \theta_1)$ and $\bar{U}_h^{(1)}(t + \theta_2)$ and ξ_5 is in between $\bar{U}_h^{(1)}(t)$ and $\bar{U}_h^{(1)}(t + \theta_2)$. We estimate

$$\begin{aligned} \|dM_1\|_{\ell^p} &\leq \frac{1}{|c_h|} \|V_1^3 - V_2^3\|_{\ell^p} \\ &\leq \frac{1}{|c_h|} \left[\|V_1\|_{\ell^\infty} \|V_1^2 - V_2^2\|_{\ell^p} + \|V_1 - V_2\|_{\ell^\infty} \|V_2^2\|_{\ell^p} \right] \\ &\leq \frac{1}{|c_h|} \left[\|V_1\|_{\ell^p} \left[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p} \right] \|V_1 - V_2\|_{\ell^p} + \|V_1 - V_2\|_{\ell^p} \|V_2\|_{\ell^p}^2 \right], \end{aligned} \quad (8.28)$$

together with

$$\begin{aligned} \|dM_2\|_{\ell^p} &\leq \frac{1}{|c_h|} \left[3 \|(\bar{U}_h^{(1)})'(\xi_3)(\theta_1 - \theta_2)\|_{\infty} \|V_1\|_{\ell^p}^2 \right. \\ &\quad \left. + \left(6 \|\bar{u}_h\|_{\infty} + 2r_0 + 2 \right) \left[\|V_1\|_{\ell^\infty} \|V_1 - V_2\|_{\ell^p} + \|V_1 - V_2\|_{\ell^\infty} \|V_2\|_{\ell^p} \right] \right] \\ &\leq \frac{1}{|c_h|} \left[3 \|u_h'\|_{\infty} |\theta_1 - \theta_2| \|V_1\|_{\ell^p}^2 + \left(6 \|\bar{u}_h\|_{\infty} + 2r_0 + 2 \right) \left[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p} \right] \|V_1 - V_2\|_{\ell^p} \right] \end{aligned} \quad (8.29)$$

and

$$\begin{aligned} \|dM_3\|_{\ell^p} &\leq \frac{1}{|c_h|} \left[\|g_{uu}(\xi_4)(\theta_1 - \theta_2)\|_{\infty} \|V_1\|_{\ell^p} + \|g_{uu}(\xi_5)(\theta_2 - \tilde{\theta})\|_{\infty} \|V_1 - V_2\|_{\ell^p} \right] \\ &\leq \frac{1}{|c_h|} \left[\left(6 \|\bar{u}_h\|_{\infty} + 2r_0 + 2 \right) |\theta_1 - \theta_2| \|V_1\|_{\ell^p} + \left(6 \|\bar{u}_h\|_{\infty} + 2r_0 + 2 \right) |\theta_2 - \tilde{\theta}| \|V_1 - V_2\|_{\ell^p} \right]. \end{aligned} \quad (8.30)$$

Combining these estimates yields

$$\begin{aligned} \|dM\|_{\ell^p} &\leq C_2 \left[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p} + |\tilde{\theta} - \theta_1| + |\theta_2 - \tilde{\theta}| \right] \|V_1 - V_2\|_{\ell^p} \\ &\quad + C_2 |\tilde{\theta}_1 - \tilde{\theta}_2| \left[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p} \right]. \end{aligned} \quad (8.31)$$

■

Lemma 8.4. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Then there exists a constant $C_3 > 0$, independent of p but possibly depending on h , such that for $V \in \ell^p$ we have the bound*

$$\|\tilde{\mathcal{G}}(t, t_0, h)V\|_{\ell^p} \leq C_3 e^{-\tilde{\beta}(t-t_0)} \|V\|_{\ell^p}, \quad (8.32)$$

for all $t, t_0 \in \mathbb{R}$.

Proof. We let $f(t, t_0, h)_j = e^{-\tilde{\beta}|c_h t - c_h t_0 + h_j|}$ and write $\tilde{V}_j = |V_j|$ for $V \in \ell^p$. Using Lemma 7.11

and Young's inequality we obtain

$$\begin{aligned}
\|\tilde{\mathcal{G}}(t, t_0, h)V\|_{\ell^p} &\leq 8C e^{-\tilde{\beta}(t-t_0)}\|f(t, t_0, h) * \tilde{V}\|_{\ell^p(\mathbb{Z}, \mathbb{R})} \\
&\leq 8C e^{-\tilde{\beta}(t-t_0)}\|f(t, t_0, h)\|_{\ell^1(\mathbb{Z}, \mathbb{R})}\|\tilde{V}\|_{\ell^p(\mathbb{Z}, \mathbb{R})} \\
&= 8C e^{-\tilde{\beta}(t-t_0)}\|f(t, t_0, h)\|_{\ell^1(\mathbb{Z}, \mathbb{R})}\|V\|_{\ell^p} \\
&\leq C_3 e^{-\tilde{\beta}(t-t_0)}\|V\|_{\ell^p},
\end{aligned} \tag{8.33}$$

where

$$C_3 = 8C \sup_{x \in [0, h]} \sum_{j \in \mathbb{Z}} e^{-\tilde{\beta}|hj+x|} < \infty. \tag{8.34}$$

Note that C_3 is independent of p . ■

From the defining system (7.7), it is clear that for each $\tilde{\theta} \in \mathbb{R}$ the Green's function of the linearisation of (2.1) around the wave $\bar{U}_h(\cdot + \tilde{\theta})$ is given by $\mathcal{G}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h)$.

Fix $W_s \in \text{Range}\left(\Pi_h^s(\tilde{\theta})\right)$ and consider the fixed point problem

$$\begin{aligned}
V(t) &= \tilde{\mathcal{G}}(t + \tilde{\theta}, \tilde{\theta}, h)W_s + \int_0^t \tilde{\mathcal{G}}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h)\Pi_h^s(t_0 + \tilde{\theta})\mathcal{M}_h^{\tilde{\theta}}(\tilde{\theta}, t_0, V(t_0))dt_0 \\
&\quad + \int_\infty^t \mathcal{E}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h)\Pi_h^c(t_0 + \tilde{\theta})\mathcal{M}_h^{\tilde{\theta}}(\tilde{\theta}, t_0, V(t_0))dt_0.
\end{aligned} \tag{8.35}$$

We aim to construct decaying solutions to (8.14) by solving this fixed point problem in the space

$$BC_{-\tilde{\beta}/2}([0, \infty), \ell^p) := \{V \in C([0, \infty), \ell^p) : \|V\|_{-\tilde{\beta}/2} < \infty\}, \tag{8.36}$$

where

$$\|V\|_{-\tilde{\beta}/2} = \sup_{\xi \in [0, \infty)} e^{\frac{\tilde{\beta}}{2}\xi} \|V(\xi)\|_{\ell^p} \tag{8.37}$$

Here the integrals are taking component-wise, but we see that for each $1 \leq p \leq \infty$ this component-wise integral corresponds to the Bochner integral.

Lemma 8.5. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. If the function $V \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$ satisfies the fixed point problem (8.35), then V satisfies (8.14) and hence $V(t) + \bar{U}(t + \tilde{\theta})$ is a solution of (2.1).*

Proof. If $V(t)$ satisfies this fixed point problem then we see that

$$\begin{aligned}
\frac{d}{dt} \left(\tilde{\mathcal{G}}(t + \tilde{\theta}, \tilde{\theta}, h)W_s \right) &= D\mathcal{F}_h \left(\bar{U}_h(t + \tilde{\theta}) \right) \tilde{\mathcal{G}}(t + \tilde{\theta}, \tilde{\theta}, h)W_s \\
&\quad + D\mathcal{F}_h \left(\bar{U}_h(t + \tilde{\theta}) \right) \mathcal{E}(t + \tilde{\theta}, \tilde{\theta}, h)\Pi_h^c(\tilde{\theta})W_s \\
&\quad - \frac{d}{dt} \left(\mathcal{E}(t + \tilde{\theta}, \tilde{\theta}, h)\Pi_h^c(\tilde{\theta})W_s \right) \\
&= D\mathcal{F}_h \left(\bar{U}_h(t + \tilde{\theta}) \right) \tilde{\mathcal{G}}(t + \tilde{\theta}, \tilde{\theta}, h)W_s + 0 - 0 \\
&= D\mathcal{F}_h \left(\bar{U}_h(t + \tilde{\theta}) \right) \tilde{\mathcal{G}}(t + \tilde{\theta}, \tilde{\theta}, h)W_s.
\end{aligned} \tag{8.38}$$

Writing

$$\mathcal{D}(t) = \frac{d}{dt} V(t) - \frac{d}{dt} \left(\tilde{\mathcal{G}}(t + \tilde{\theta}, \tilde{\theta}, h)W_s \right), \tag{8.39}$$

we can compute

$$\begin{aligned}
\mathcal{D}(t) &= \int_0^t \frac{d}{dt} \left[\tilde{\mathcal{G}}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^s(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0)) \right] dt_0 \\
&\quad + \int_\infty^t \frac{d}{dt} \left[\mathcal{E}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^c(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0)) \right] dt_0 \\
&\quad + \tilde{\mathcal{G}}(t + \tilde{\theta}, t + \tilde{\theta}, h) \Pi_h^s(t + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\tilde{\theta}, t, V(t)) \\
&\quad + \mathcal{E}(t + \tilde{\theta}, t + \tilde{\theta}, h) \Pi_h^c(t + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t, V(t)) \\
&= \int_0^t \frac{d}{dt} \left[\mathcal{G}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^s(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0)) \right] dt_0 \\
&\quad + \int_\infty^t \frac{d}{dt} \left[\mathcal{G}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^c(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0)) \right] dt_0 \\
&\quad + \mathcal{G}(t + \tilde{\theta}, t + \tilde{\theta}, h) \mathcal{M}_h^{\tilde{\theta}}(\theta, t, V(t)).
\end{aligned} \tag{8.40}$$

Exploiting $\mathcal{G}(t + \tilde{\theta}, t + \tilde{\theta}, h) = I$, this yields

$$\begin{aligned}
\mathcal{D}(t) &= \int_0^t D\mathcal{F}_h(\bar{U}_h(t + \tilde{\theta})) \mathcal{G}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^s(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0)) dt_0 \\
&\quad + \int_\infty^t D\mathcal{F}_h(\bar{U}_h(t + \tilde{\theta})) \mathcal{G}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^c(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0)) dt_0 \\
&\quad + \mathcal{M}_h^{\tilde{\theta}}(\theta, t, V(t)) \\
&= D\mathcal{F}_h(\bar{U}_h(t + \tilde{\theta})) \int_0^t \tilde{\mathcal{G}}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^s(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0)) dt_0 \\
&\quad + D\mathcal{F}_h(\bar{U}_h(t + \tilde{\theta})) \int_\infty^t \mathcal{E}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^c(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0)) dt_0 \\
&\quad + \mathcal{M}_h^{\tilde{\theta}}(\theta, t, V(t))
\end{aligned} \tag{8.41}$$

and thus

$$\begin{aligned}
\frac{d}{dt} V(t) &= D\mathcal{F}_h(\bar{U}_h(t + \tilde{\theta})) V(t) + \mathcal{M}_h^{\tilde{\theta}}(\theta, t, V(t)) \\
&= D\mathcal{F}_h(\bar{U}_h(t + \theta)) V(t) + \mathcal{N}_{h;\theta}(t, V(t)).
\end{aligned} \tag{8.42}$$

■

Lemma 8.6. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. There exists $\eta > 0$, independent of p and $\tilde{\theta}$, so that for all $W_s \in \text{Range}(\Pi_h^s(\tilde{\theta}))$ that have $\|W_s\|_{\ell^p} \leq \eta$ and all $|\theta - \tilde{\theta}| \leq \eta$, the fixed point problem (8.35) has a unique solution $\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)$ in the space $BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$.*

Proof. We first rewrite (8.35) as

$$V = T(W_s, V), \tag{8.43}$$

where

$$\begin{aligned}
T(W_s, V) &= \tilde{\mathcal{G}}(t + \tilde{\theta}, \tilde{\theta}, h) W_s + \int_0^t \tilde{\mathcal{G}}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^s(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\tilde{\theta}, t_0, V(t_0)) dt_0 \\
&\quad + \int_\infty^t \mathcal{E}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^c(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0)) dt_0.
\end{aligned} \tag{8.44}$$

Pick $V \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$ with $\|V\|_{-\tilde{\beta}/2} \leq 1$. Writing

$$\mathcal{I} = \int_0^t \tilde{\mathcal{G}}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^s(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0)) dt_0, \tag{8.45}$$

Lemma 8.3 and Lemma 8.4 imply

$$\begin{aligned}
\|\mathcal{I}\|_{\ell^p} &\leq \int_0^t \|\tilde{\mathcal{G}}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0))\|_{\ell^p} dt_0 \\
&\leq \int_0^t C_3 e^{-\tilde{\beta}(t-t_0)} \|\mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0))\|_{\ell^p} dt_0 \\
&\leq \int_0^t C_3 e^{-\tilde{\beta}(t-t_0)} C_2 \|V(t_0)\|_{\ell^p} \left[\|V(t_0)\|_{\ell^p} + |\theta - \tilde{\theta}| \right] dt_0 \\
&\leq \int_0^t C_3 e^{-\tilde{\beta}(t-t_0)} C_2 \|V\|_{-\tilde{\beta}/2} e^{-\tilde{\beta}t_0/2} \left[e^{-\tilde{\beta}t_0/2} \|V\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}| \right] dt_0 \\
&\leq C_3 C_2 \|V\|_{-\tilde{\beta}/2} \left[t e^{-\tilde{\beta}t} \|V\|_{-\tilde{\beta}/2} + \frac{2}{\tilde{\beta}} e^{-\tilde{\beta}t/2} |\theta - \tilde{\theta}| \right].
\end{aligned} \tag{8.46}$$

Observe that if we multiply this final function with $e^{\tilde{\beta}t/2}$ we still have a bounded function. Since this holds for all p we see that

$$\|\mathcal{I}\|_{-\tilde{\beta}/2} \leq C_4 \|V\|_{-\tilde{\beta}/2} \left[\|V\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}| \right] \tag{8.47}$$

for some constant C_4 , which is independent of p .

We write

$$\mathcal{J}(t) = \int_t^\infty \mathcal{E}(t + \tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^c(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, V(t_0)) dt_0. \tag{8.48}$$

Mimicking the computation above and using the explicit expression (8.3), we see that

$$\begin{aligned}
\|\mathcal{J}(t)\|_{\ell^p} &\leq \int_\infty^t C_1 C_2 \|V\|_{-\tilde{\beta}/2} e^{-\tilde{\beta}t_0/2} \left[e^{-\tilde{\beta}t_0/2} \|V\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}| \right] dt_0 \\
&= C_1 C_2 \|V\|_{-\tilde{\beta}/2} \left[\frac{1}{\tilde{\beta}} e^{-\tilde{\beta}t} \|V\|_{-\tilde{\beta}/2} + \frac{2}{\tilde{\beta}} e^{-\tilde{\beta}t/2} |\theta - \tilde{\theta}| \right].
\end{aligned} \tag{8.49}$$

Observe that if we multiply this final function with $e^{\frac{\tilde{\beta}}{2}t}$ we still have a bounded function. Since this holds for all p we see that

$$\|\mathcal{J}\|_{-\tilde{\beta}/2} \leq C_5 \|V\|_{-\tilde{\beta}/2} \left[\|V\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}| \right] \tag{8.50}$$

for some constant C_5 , which is independent of p .

Finally, Lemma 8.4 yields the bound

$$\|\tilde{\mathcal{G}}(t + \tilde{\theta}, \tilde{\theta}, h) W_s\|_{\ell^p} \leq C_3 e^{-\tilde{\beta}t} \|W_s\|_{\ell^p}, \tag{8.51}$$

which means

$$\|\tilde{\mathcal{G}}(t + \tilde{\theta}, \tilde{\theta}, h) W_s\|_{-\tilde{\beta}/2} \leq C_3 \|W_s\|_{\ell^p}. \tag{8.52}$$

This yields the bound

$$\|T(W_s, V)\|_{-\tilde{\beta}/2} \leq C_3 \|W_s\|_{\ell^p} + (C_4 + C_5) \|V\|_{-\tilde{\beta}/2} \left[\|V\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}| \right]. \tag{8.53}$$

Let $V_1 \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$ and $V_2 \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$ with $\|V_1\|_{-\tilde{\beta}/2} \leq 1$ and $\|V_2\|_{-\tilde{\beta}/2} \leq 1$. Again, we write

$$d\mathcal{M} = \mathcal{M}_h^{\tilde{\theta}}(\theta, t, V_1(t)) - \mathcal{M}_h^{\tilde{\theta}}(\theta, t, V_2(t)). \tag{8.54}$$

Using Lemma 8.3 it follows that

$$\|d\mathcal{M}\|_{\ell^p} \leq C_2 \left[\|V_1\|_{\ell^p} + \|V_2\|_{\ell^p} + |\theta - \tilde{\theta}| \right] \|V_1 - V_2\|_{\ell^p}. \tag{8.55}$$

Mimicking the above computations, this yields

$$\|T(W_s, V_1) - T(W_s, V_2)\|_{-\tilde{\beta}/2} \leq (C_4 + C_5) \left[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}| \right] \|V_1 - V_2\|_{-\tilde{\beta}/2}. \quad (8.56)$$

We now fix

$$\delta = \min\left\{1, \frac{1}{4(C_4 + C_5)}\right\} \quad (8.57)$$

and

$$\eta = \min\left\{\frac{1}{4(C_4 + C_5)}, \frac{\delta}{4C_3}\right\}. \quad (8.58)$$

For each $V \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$, $V_1 \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$ and $V_2 \in BC_{-\tilde{\beta}/2}([0, \infty), \ell^p)$ with $\|V\|_{-\tilde{\beta}/2} \leq \delta$, $\|V_1\|_{-\tilde{\beta}/2} \leq \delta$ and $\|V_2\|_{-\tilde{\beta}/2} \leq \delta$, each $\theta \in \mathbb{R}$ with $|\theta - \tilde{\theta}| < \eta$ and each $W_s \in \ell^p$ with $\|W_s\|_{\ell^p} < \eta$, we now obtain

$$\begin{aligned} \|T(W_s, V)\|_{-\tilde{\beta}/2} &\leq \frac{\delta}{4} + \delta \left[\frac{1}{4} + \frac{1}{4} \right] \\ &\leq \delta \end{aligned} \quad (8.59)$$

and

$$\|T(W_s, V_1) - T(W_s, V_2)\|_{-\tilde{\beta}/2} \leq \left[\frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right] \|V_1 - V_2\|_{-\tilde{\beta}/2}. \quad (8.60)$$

Hence we see that the equation (8.35) has a unique solution $\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)$. \blacksquare

Lemma 8.7. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. For each $W_s \in \text{Range}\left(\Pi_h^s(\tilde{\theta})\right)$ with $\|W_s\|_{\ell^p} \leq \eta$ and each $|\theta - \tilde{\theta}| \leq \eta$ we have $\Pi_h^s(\tilde{\theta})\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(0) = W_s$ and $\|\Pi_h^c(\tilde{\theta})\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(0)\|_{\ell^p} \leq C_6\|W_s\|_{\ell^p}^2$ for some constant $C_6 > 0$, which is independent of p and $\tilde{\theta}$, but may depend on h .*

Proof. It is clear that

$$\begin{aligned} \Pi_h^s(\tilde{\theta})\tilde{\mathcal{G}}(\tilde{\theta}, t_0 + \tilde{\theta}, h) &= \tilde{\mathcal{G}}(\tilde{\theta}, \tilde{\theta} + t_0, h), \\ \Pi_h^c(\tilde{\theta})\mathcal{E}(\tilde{\theta}, \tilde{\theta} + t_0, h) &= \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta}, h). \end{aligned} \quad (8.61)$$

This allows us to compute

$$\begin{aligned} \Pi_h^s(\tilde{\theta})\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(0) &= \Pi_h^s(\tilde{\theta})\tilde{\mathcal{G}}(\tilde{\theta}, \tilde{\theta}, h)W_s \\ &\quad + \Pi_h^s(\tilde{\theta}) \int_0^0 \tilde{\mathcal{G}}(\tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^s(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0)) dt_0 \\ &\quad + \Pi_h^s(\tilde{\theta}) \int_\infty^0 \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^c(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0)) dt_0 \\ &= \Pi_h^s(\tilde{\theta})W_s + \int_\infty^0 \Pi_h^s(\tilde{\theta})\mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta}, h) \Pi_h^c(t_0 + \tilde{\theta}) \mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0)) dt_0 \\ &= W_s + 0 \\ &= W_s, \end{aligned} \quad (8.62)$$

together with

$$\begin{aligned}
\Pi_h^c(\tilde{\theta})\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(0) &= \Pi_h^c(\tilde{\theta})\tilde{\mathcal{G}}(\tilde{\theta}, \tilde{\theta}, h)W_s \\
&\quad + \Pi_h^c(\tilde{\theta})\int_0^0 \tilde{\mathcal{G}}(\tilde{\theta}, t_0 + \tilde{\theta}, h)\Pi_h^s(t_0 + \tilde{\theta})\mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0))dt_0 \\
&\quad + \Pi_h^c(\tilde{\theta})\int_\infty^0 \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta}, h)\Pi_h^c(t_0 + \tilde{\theta})\mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0))dt_0 \\
&= 0 + 0 + \int_\infty^0 \Pi_h^c(\tilde{\theta})\mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta}, h)\Pi_h^c(t_0 + \tilde{\theta})\mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0))dt_0 \\
&= \int_\infty^0 \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta}, h)\Pi_h^c(t_0 + \tilde{\theta})\mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0))dt_0.
\end{aligned} \tag{8.63}$$

We assume without loss of generality that η is small enough to ensure

$$(C_4 + C_5)\|\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)\|_{-\tilde{\beta}/2}\left[\|\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}|\right] \leq \frac{1}{2}\|\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)\|_{-\tilde{\beta}/2}. \tag{8.64}$$

Using (8.53), we obtain

$$\begin{aligned}
\left\|\int_\infty^0 \mathcal{E}(\tilde{\theta}, t_0 + \tilde{\theta}, h)\Pi_h^c(t_0 + \tilde{\theta})\mathcal{M}_h^{\tilde{\theta}}(\theta, t_0, \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)(t_0))dt_0\right\|_{\ell^p} &\leq C_5\left[\|\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)\|_{-\tilde{\beta}/2} + |\theta - \tilde{\theta}|\right] \\
&\leq 4C_5C_3^2\|W_s\|_{\ell^p}^2 \\
&:= C_6\|W_s\|_{\ell^p}^2.
\end{aligned} \tag{8.65}$$

This yields the desired estimate

$$\|\Pi_h^c(\tilde{\theta})\mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s)\|_{\ell^p} \leq C_6\|W_s\|_{\ell^p}^2. \tag{8.66}$$

■

Exploiting Proposition 5.2, we pick $C_7 > 0$ in such a way that

$$|\Phi_h^\pm(\xi)| + |(\Phi_h^\pm)'(\xi)| \leq C_7e^{-\tilde{\beta}|\xi|} \tag{8.67}$$

holds for all $\xi \in \mathbb{R}$, decreasing $\tilde{\beta}$ if necessary. We have

$$(\Pi_h^c(\theta)V)_j = \frac{1}{M}\sum_{j_0 \in \mathbb{Z}} \langle \Phi_h^-(hj_0 + c_h\theta), V_{j_0} \rangle \Phi_h^+(hj + c_h\tilde{\theta}). \tag{8.68}$$

For notational compactness we write

$$\Pi_h^c(\theta)V = \lambda_h^c(\theta)(V)\Phi_h^+(h \cdot + c_h\theta). \tag{8.69}$$

Lemma 8.8. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Then there exists a constant $C_9 > 0$, independent of p , but possibly depending on h , such that for each $\theta \in \mathbb{R}$ we have the bound*

$$\|\Phi_h^+(h \cdot + \theta)\|_{\ell^p} \leq C_9. \tag{8.70}$$

In addition, for each $\theta \in \mathbb{R}$ and each sequence $\{\xi(j)\}$ with $\|\xi_j\|_\infty \leq 1$, we have the bound

$$\|(\Phi_h^-)'(h \cdot + \xi(\cdot))\|_{\ell^1} \leq C_9. \tag{8.71}$$

Proof. Note that for each $k \in \mathbb{Z}$ we have

$$\begin{aligned}
\|\Phi_h^+(h \cdot + \theta_1)\|_{\ell^p}^p &= \sum_{j \in \mathbb{Z}} |\Phi_h^+(hj + \theta_1)|^p \\
&= \sum_{j \in \mathbb{Z}} |\Phi_h^+(hj + hk + (\theta_1 - hk))|^p \\
&= \sum_{j \in \mathbb{Z}} |\Phi_h^+(hj + (\theta_1 - hk))|^p \\
&= \|\Phi_h^+(h \cdot + (\theta_1 - hk))\|_{\ell^p}^p.
\end{aligned} \tag{8.72}$$

Hence we assume without loss of generality that $|\theta_1| \leq 1$. We see with (8.67) that there is a constant $C_8 > 0$ such that

$$\begin{aligned}
\|\Phi_h^+(h \cdot + c_h \theta_1)\|_{\ell^p} &\leq C_8 \|e^{-\tilde{\beta}|h \cdot + \theta_1|}\|_{\ell^p(\mathbb{Z}, \mathbb{R})} \\
&\leq C_8 \|e^{-\tilde{\beta}|h \cdot|} e^{\tilde{\beta}|\theta_1|}\|_{\ell^p(\mathbb{Z}, \mathbb{R})} \\
&\leq C_8 e^{\tilde{\beta}|\theta_1|} \|e^{-\tilde{\beta}|h \cdot|}\|_{\ell^1(\mathbb{Z}, \mathbb{R})}^{\frac{1}{p}} \\
&\leq 2C_8 \max\{1, \|e^{-\tilde{\beta}|h \cdot|}\|_{\ell^1(\mathbb{Z}, \mathbb{R})}\} \\
&:= C_9,
\end{aligned} \tag{8.73}$$

if we assume that $\tilde{\beta}$ is small enough such that $e^{\tilde{\beta}|\theta_1|} \leq e^{\tilde{\beta}} \leq 2$. A similar calculation yields

$$\begin{aligned}
\|(\Phi_h^-)'(h \cdot + \xi(\cdot))\|_{\ell^1} &\leq C_8 \|e^{-\tilde{\beta}|h \cdot + \xi(\cdot)|}\|_{\ell^1(\mathbb{Z}, \mathbb{R})} \\
&\leq C_8 \|e^{-\tilde{\beta}|h \cdot|} e^{\tilde{\beta}}\|_{\ell^1(\mathbb{Z}, \mathbb{R})} \\
&\leq C_8 e^{\tilde{\beta}} \|e^{-\tilde{\beta}|h \cdot|}\|_{\ell^1(\mathbb{Z}, \mathbb{R})} \\
&\leq 2C_8 \max\{1, \|e^{-\tilde{\beta}|h \cdot|}\|_{\ell^1(\mathbb{Z}, \mathbb{R})}\} \\
&= C_9.
\end{aligned} \tag{8.74}$$

■

Lemma 8.9. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. For $V \in \ell^p$ and $\theta_1, \theta_2 \in \mathbb{R}$ with $|\theta_2 - \theta_1| \leq 1$ we have the bounds*

$$\|V\|_{\ell^p} \leq C_{10} |\theta_1 - \theta_2| \|V\|_{\ell^p} \tag{8.75}$$

and

$$\|\Pi_h^s(\theta_1) \Pi_h^c(\theta_2) V\|_{\ell^p} \leq C_{10} |\theta_1 - \theta_2| \|V\|_{\ell^p} \tag{8.76}$$

for some constant $C_{10} > 0$ which does not depend on p , but may depend on h .

Proof. Writing

$$\mathcal{P} = \lambda_h^c(\theta_1) \left(\Pi_h^s(\theta_2) V \right), \tag{8.77}$$

we obtain

$$\begin{aligned}
\|\mathcal{P}\|_{\ell^p} &= \left| \frac{1}{M} \sum_{j_0 \in \mathbb{Z}} \langle \Phi_h^-(hj_0 + c_h \theta_1), (\Pi_h^s(\theta_2) V)_{j_0} \rangle \right| \\
&= M^{-1} \left| \sum_{j_0 \in \mathbb{Z}} \langle \Phi_h^-(hj_0 + c_h \theta_1) - \Phi_h^-(hj_0 + c_h \theta_2), (\Pi_h^s(\theta_2) V)_{j_0} \rangle \right| \\
&\leq M^{-1} \sum_{j_0 \in \mathbb{Z}} |\Phi_h^-(hj_0 + c_h \theta_1) - \Phi_h^-(hj_0 + c_h \theta_2)| |(\Pi_h^s(\theta_2) V)_{j_0}| \\
&\leq M^{-1} \|\Phi_h^-(h \cdot + c_h \theta_1) - \Phi_h^-(h \cdot + c_h \theta_2)\|_{\ell^1} \|\Pi_h^s(\theta_2) V\|_{\ell^\infty} \\
&\leq M^{-1} |\theta_1 - \theta_2| \|c_h (\Phi_h^-)'(h \cdot + \xi(\cdot))\|_{\ell^1} \|\Pi_h^s(\theta_2) V\|_{\ell^p},
\end{aligned} \tag{8.78}$$

where each $\xi(j)$ is in between $c_h \theta_1$ and $c_h \theta_2$. Thus we obtain with Lemma 8.4 and Lemma 8.8

$$\begin{aligned}
\|\Pi_h^c(\theta_1) \Pi_h^s(\theta_2) V\|_{\ell^p} &\leq \frac{1}{|M|} C_9 |c_h| |\theta_1 - \theta_2| \|(\Pi_h^s(\theta_2) V)\|_{\ell^p} C_9 \\
&\leq \frac{1}{|M|} C_9 |c_h| |\theta_1 - \theta_2| C_3 \|V\|_{\ell^p} C_9 \\
&\leq \frac{1}{2} C_{10} |\theta_1 - \theta_2| \|V\|_{\ell^p}
\end{aligned} \tag{8.79}$$

for some constant $C_{10} > 0$ which is independent of p .

Furthermore we can compute

$$\begin{aligned}
\Pi_h^s(\theta_1)\Pi_h^c(\theta_2)V &= \left[I - \Pi_h^c(\theta_1) \right] \left[I - \Pi_h^s(\theta_2) \right] V \\
&= V - \Pi_h^c(\theta_1)V - \Pi_h^s(\theta_2)V + \Pi_h^c(\theta_1)\Pi_h^s(\theta_2)V \\
&= -\Pi_h^c(\theta_1)V + \Pi_h^c(\theta_2)V + \Pi_h^c(\theta_1)\Pi_h^s(\theta_2)V.
\end{aligned} \tag{8.80}$$

This allows us to estimate

$$\begin{aligned}
\|-\Pi_h^c(\theta_1)V + \Pi_h^c(\theta_2)V\|_{\ell^p} &\leq |\lambda_h^c(\theta_1)(V) - \lambda_h^c(\theta_2)(V)| \|\Phi_h^+(\theta_1)\|_{\ell^p} + |\lambda(\theta_2)| \|\Phi_h^+(\theta_1) - \Phi_h^+(\theta_2)\|_{\ell^p} \\
&\leq C_9|\theta_1 - \theta_2|C_9\|V\|_{\ell^p} + C_9|\theta_2|\|\theta_1 - \theta_2\| \|(\Phi_h^-)'(h \cdot + \eta(\cdot))\|_{\ell^1} \|V\|_{\ell^p} \\
&\leq C_9|\theta_1 - \theta_2|C_9\|V\|_{\ell^p} + C_9|\theta_2|\|\theta_1 - \theta_2\|C_9\|V\|_{\ell^p} \\
&\leq \frac{1}{2}C_{10}|\theta_1 - \theta_2|\|V\|_{\ell^p},
\end{aligned} \tag{8.81}$$

where each $\eta(j)$ is between $c_h\theta_1$ and $c_h\theta_2$. We thus obtain

$$\|\Pi_h^s(\theta_1)\Pi_h^c(\theta_2)V\|_{\ell^p} \leq C_{10}|\theta_1 - \theta_2|\|V\|_{\ell^p}. \tag{8.82}$$

■

Proof of Proposition 8.1. We set

$$\mathcal{U}_*^{\tilde{\theta}}(W_s) = \mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s) \tag{8.83}$$

for all $\tilde{\theta} \in \mathbb{R}$.

Fix $\tilde{\theta} \in \mathbb{R}$ and pick $\theta \in \mathbb{R}$ with $|\theta - \tilde{\theta}| \leq \eta$ and pick $W_s \in \text{Range}(\Pi_h^s(\theta))$ with $\|W_s\|_{\ell^p} \leq \eta$. By unicity of the solution to (8.35) it follows that

$$\mathcal{U}_*^\theta(W_s) = \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s^0) \tag{8.84}$$

for some $W_s^0 \in \text{Range}(\Pi_h^s(\tilde{\theta}))$. Since $\Pi_h^s(\tilde{\theta})\mathcal{W}_{*;\theta}^{\tilde{\theta}}(\cdot)(0)$ is the identity map on $\text{Range}(\Pi_h^s(\tilde{\theta}))$, it follows that

$$W_s^0 = \Pi_h^s(\tilde{\theta})\mathcal{U}_*^\theta(W_s)(0). \tag{8.85}$$

We now see

$$\begin{aligned}
W_s^0 - W_s &= \Pi_h^s(\tilde{\theta})\mathcal{U}_*^\theta(W_s)(0) - W_s \\
&= \Pi_h^s(\tilde{\theta}) \left[\mathcal{U}_*^\theta(W_s)(0) - W_s \right] + \Pi_h^s(\tilde{\theta})W_s - W_s \\
&= \Pi_h^s(\tilde{\theta})\Pi_h^c(\theta)\mathcal{U}_*^\theta(W_s)(0) + W_s - \Pi_h^c(\tilde{\theta})W_s - W_s \\
&= \Pi_h^s(\tilde{\theta})\Pi_h^c(\theta)\mathcal{U}_*^\theta(W_s)(0) - \Pi_h^c(\tilde{\theta})\Pi_h^s(\theta)W_s.
\end{aligned} \tag{8.86}$$

Lemma 8.9 hence implies

$$\begin{aligned}
\|W_s^0 - W_s\|_{\ell^p} &\leq C_{10}|\tilde{\theta} - \theta| \|\mathcal{U}_*^\theta(W_s)(0)\|_{\ell^p} + C_{10}|\tilde{\theta} - \theta| \|W_s\|_{\ell^p} \\
&\leq C_{10}|\tilde{\theta} - \theta| \left[\|W_s\|_{\ell^p} + C_6\|W_s\|_{\ell^p}^2 \right] + C_{10}|\tilde{\theta} - \theta| \|W_s\|_{\ell^p} \\
&\leq C_{11}|\tilde{\theta} - \theta| \|W_s\|_{\ell^p}.
\end{aligned} \tag{8.87}$$

Now fix $W_s^{\tilde{\theta}} \in \text{Range}\left(\Pi_h^s(\tilde{\theta})\right)$ with $\|W_s^{\tilde{\theta}}\|_{\ell^p} \leq \eta$. Then we can compute

$$\begin{aligned}
\Pi_h^c(\theta)\mathcal{U}_*^\theta(W_s)(0) - \Pi_h^c(\tilde{\theta})\mathcal{U}_*^{\tilde{\theta}}(W_s^{\tilde{\theta}})(0) &= \mathcal{U}_*^\theta(W_s)(0) - W_s - \Pi_h^c(\tilde{\theta})\mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s^{\tilde{\theta}}) \\
&= \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s^0)(0) - W_s - \mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s^{\tilde{\theta}})(0) + W_s^{\tilde{\theta}} \\
&= W_s^0 - W_s + \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s^0)(0) - W_s^0 - \mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s^{\tilde{\theta}})(0) + W_s^{\tilde{\theta}}.
\end{aligned} \tag{8.88}$$

Writing

$$\begin{aligned}
V_1 &= \mathcal{W}_{*;\theta}^{\tilde{\theta}}(W_s^0), \\
V_2 &= \mathcal{W}_{*;\tilde{\theta}}^{\tilde{\theta}}(W_s^{\tilde{\theta}}),
\end{aligned} \tag{8.89}$$

we can mimic the steps in (8.56) to obtain the estimate

$$\begin{aligned}
\|V_1(0) - W_s^0 - V_2(0) + W_s^{\tilde{\theta}}\|_{\ell^p} &\leq C_{12} \left[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\tilde{\theta} - \theta| \right] \|V_1 - V_2\|_{-\tilde{\beta}/2} \\
&\quad + C_{12} |\tilde{\theta} - \theta| \left[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} \right]
\end{aligned} \tag{8.90}$$

for some constant $C_{12} > 0$, which is independent of p and $\tilde{\theta}$. Without loss of generality we can assume that η is small enough to ensure

$$C_{12} \left[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\tilde{\theta} - \theta| \right] \leq \frac{1}{2}. \tag{8.91}$$

An estimate similar to (8.56) therefore yields

$$\begin{aligned}
\|V_1 - V_2\|_{-\tilde{\beta}/2} &\leq C_3 \|W_s^0 - W_s^{\tilde{\theta}}\|_{\ell^p} + C_{12} \left[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\tilde{\theta} - \theta| \right] \|V_1 - V_2\|_{-\tilde{\beta}/2} \\
&\quad + C_{12} |\tilde{\theta} - \theta| \left[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} \right] \\
&\leq C_3 \|W_s - W_s^{\tilde{\theta}}\|_{\ell^p} + C_3 \|W_s - W_s^0\|_{\ell^p} + \frac{1}{2} \|V_1 - V_2\|_{-\tilde{\beta}/2} \\
&\quad + C_{12} |\tilde{\theta} - \theta| \left[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} \right] \\
&\leq C_3 \|W_s - W_s^{\tilde{\theta}}\|_{\ell^p} + C_3 C_{11} |\tilde{\theta} - \theta| \|W_s\|_{\ell^p} + \frac{1}{2} \|V_1 - V_2\|_{-\tilde{\beta}/2} \\
&\quad + C_{12} |\tilde{\theta} - \theta| \left[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} \right],
\end{aligned} \tag{8.92}$$

and thus

$$\begin{aligned}
\|V_1 - V_2\|_{-\tilde{\beta}/2} &\leq 2C_3 \|W_s - W_s^{\tilde{\theta}}\|_{\ell^p} + 2C_3 C_{11} |\tilde{\theta} - \theta| \|W_s\|_{\ell^p} \\
&\quad + 2C_{12} |\tilde{\theta} - \theta| \left[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} \right].
\end{aligned} \tag{8.93}$$

Exploiting (8.65), this yields

$$\begin{aligned}
\|\Pi_h^c(\theta)\mathcal{U}_*^\theta(W_s)(0) - \Pi_h^c(\tilde{\theta})\mathcal{U}_*^{\tilde{\theta}}(W_s^{\tilde{\theta}})(0)\|_{\ell^p} &\leq C_{11} |\tilde{\theta} - \theta| \|W_s\|_{\ell^p} + \\
&\quad C_{12} \left[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} + |\tilde{\theta} - \theta| \right] \|V_1 - V_2\|_{-\tilde{\beta}/2} \\
&\quad + C_{12} |\tilde{\theta} - \theta| \left[\|V_1\|_{-\tilde{\beta}/2} + \|V_2\|_{-\tilde{\beta}/2} \right] \\
&\leq C_{13} \left[\|W_s\|_{\ell^p} + \|W_s^{\tilde{\theta}}\|_{\ell^p} + |\tilde{\theta} - \theta| \right] \|W_s - W_s^{\tilde{\theta}}\|_{\ell^p} \\
&\quad + C_{13} |\tilde{\theta} - \theta| \left[\|W_s\|_{\ell^p} + \|W_s^{\tilde{\theta}}\|_{\ell^p} \right]
\end{aligned} \tag{8.94}$$

for some constant C_{13} , which is independent of p and $\tilde{\theta}$. \blacksquare

We now expand upon the ideas developed in the second half of [20, Proposition 2.1] to foliate the state space surrounding the travelling pulses $\bar{U}_h(\cdot + \tilde{\theta})$ by the stable manifolds constructed above. We proceed by showing that these stable manifolds depend continuously on $\tilde{\theta}$. This allows us to set up an appropriate fixed point problem to establish Proposition 8.2.

We write

$$\bar{U}_h(\tilde{\theta}) = \bar{U}_h(0) - \tilde{\theta} \bar{U}'_h(0) + \mathcal{N}_1^{\tilde{\theta}}. \quad (8.95)$$

Lemma 8.10. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Then we have the bounds*

$$\|\mathcal{N}_1^{\tilde{\theta}}\|_{\ell^p} \leq C_{14} \tilde{\theta}^2 \quad (8.96)$$

and

$$\|\mathcal{N}_1^{\tilde{\theta}_2} - \mathcal{N}_1^{\tilde{\theta}_1}\|_{\ell^p} \leq C_{15} (|\tilde{\theta}_1| + |\tilde{\theta}_2|) |\tilde{\theta}_1 - \tilde{\theta}_2|, \quad (8.97)$$

for $\tilde{\theta}, \tilde{\theta}_1, \tilde{\theta}_2 \in [-\eta, \eta]$ and for some constants $C_{14} > 0$ and $C_{15} > 0$, which do not depend on p , but may depend on h .

Proof. Using Proposition 5.2 we see that there exists a sequence $\{\xi_j\}$ with $|\xi_j| \leq |\tilde{\theta}|$ such that

$$\begin{aligned} \|\mathcal{N}_1^{\tilde{\theta}}\|_{\ell^p} &= \frac{1}{2} \|\{\tilde{\theta}^2 \bar{U}''_h(\xi_j)\}\|_{\ell^p} \\ &\leq \frac{1}{2} C_7 \tilde{\theta}^2 \|\{e^{-\tilde{\beta}|hj+c_h\xi_j|}\}\|_p \\ &\leq \frac{1}{2} C_7 \tilde{\theta}^2 e^{\tilde{\beta}|c_h\tilde{\theta}|} \|\{e^{-\tilde{\beta}|hj|}\}\|_p \\ &\leq C_7 \tilde{\theta}^2 \|\{e^{-\tilde{\beta}|hj|}\}\|_p \\ &:= C_{14} \tilde{\theta}^2, \end{aligned} \quad (8.98)$$

where C_{14} does not depend on p as before. We can hence write

$$\bar{U}_h(\tilde{\theta}) = \bar{U}_h(0) - \tilde{\theta} \bar{U}'_h(0) + \mathcal{N}_1^{\tilde{\theta}} \quad (8.99)$$

with

$$\|\mathcal{N}_1^{\tilde{\theta}}\|_{\ell^p} \leq C_{14} \tilde{\theta}^2. \quad (8.100)$$

Furthermore, using Proposition 5.2, we see that we can find sequences $\{\xi_j\}$ and $\{\eta_j\}$ with ξ_j between $hj + c_h\tilde{\theta}_1$ and $hj + c_h\tilde{\theta}_2$ and η_j between hj and $hj + c_h\tilde{\theta}_1$ so that

$$\begin{aligned} \|\mathcal{N}_1^{\tilde{\theta}_2} - \mathcal{N}_1^{\tilde{\theta}_1}\|_{\ell^p} &= \|\bar{U}_h(\tilde{\theta}_1) + \tilde{\theta}_1 \bar{U}'_h(0) - \bar{U}_h(\tilde{\theta}_2) - \tilde{\theta}_2 \bar{U}'_h(0)\|_{\ell^p} \\ &\leq |\tilde{\theta}_1 - \tilde{\theta}_2|^2 \|\{\bar{U}''_h(\xi_j)\}\|_{\ell^p} + |\tilde{\theta}_1 - \tilde{\theta}_2| \|\bar{U}'_h(0) - \bar{U}'_h(\tilde{\theta}_1)\|_{\ell^p} \\ &\leq |\tilde{\theta}_1 - \tilde{\theta}_2|^2 \|\{\bar{U}''_h(\xi_j)\}\|_{\ell^p} + |\tilde{\theta}_1 - \tilde{\theta}_2| |\tilde{\theta}_1| \|\{\bar{U}''_h(\eta_j)\}\|_{\ell^p} \\ &\leq C_{15} (|\tilde{\theta}_1| + |\tilde{\theta}_2|) |\tilde{\theta}_1 - \tilde{\theta}_2|, \end{aligned} \quad (8.101)$$

similarly to the calculations from Lemma 8.3, where $|\xi_j| \leq |\tilde{\theta}_2 - \tilde{\theta}_1|$ and $|\eta_j| \leq |\tilde{\theta}_1|$. \blacksquare

We write

$$\mathcal{N}_2^{\tilde{\theta}}(W) = \mathcal{U}_*^{\tilde{\theta}}(\Pi_h^s(\tilde{\theta})W)(0) - \Pi_h^s(\tilde{\theta})W \quad (8.102)$$

for $W \in \text{Range}(\Pi_h^s(0))$ with $\|\Pi_h^s(\tilde{\theta})W\|_{\ell^p} < \eta$. We note that Lemma 8.4 implies that this inequality holds if $\|W\|_{\ell^p} < \frac{\eta}{C_3}$.

Lemma 8.11. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. Recall the constants C_6 and C_{13} appearing in Proposition 8.1 and Lemma 8.4. Then for any $\tilde{\theta} \in [-\eta, \eta]$ and $W \in \text{Range}(\Pi_h^s(0))$ with $\|W\|_{\ell^p} < \frac{\eta}{C_3}$ we have the bound*

$$\|\mathcal{N}_2^{\tilde{\theta}}(W)\|_{\ell^p} \leq C_6 \|V\|_{\ell^p}^2. \quad (8.103)$$

In addition, for any $\tilde{\theta}_1, \tilde{\theta}_2 \in [-\eta, \eta]$ and $W_1, W_2 \in \text{Range}(\Pi_h^s(0))$ with $\|W_1\|_{\ell^p} < \frac{\eta}{C_3}$ and $\|W_2\|_{\ell^p} < \frac{\eta}{C_3}$, we have

$$\begin{aligned} \|\mathcal{N}_2^{\tilde{\theta}_2}(W_2) - \mathcal{N}_2^{\tilde{\theta}_1}(W_1)\|_{\ell^p} &\leq C_{13} \left[\|W_1\|_{\ell^p} + \|W_2\|_{\ell^p} + |\theta_1 - \theta_2| \right] \|W_1 - W_2\|_{\ell^p} \\ &\quad + C_{13} |\theta_1 - \theta_2| \left[\|W_1\|_{\ell^p} + \|W_2\|_{\ell^p} \right]. \end{aligned} \quad (8.104)$$

Proof. Note that

$$\Pi_h^s(\tilde{\theta}) \mathcal{N}_2^{\tilde{\theta}}(W) = 0, \quad (8.105)$$

so that

$$\begin{aligned} \mathcal{N}_2^{\tilde{\theta}}(W) &= \Pi_h^c(\tilde{\theta}) U_*^{\tilde{\theta}}(W)(0) - \Pi_h^c(\tilde{\theta}) \Pi_h^s(\tilde{\theta}) W \\ &= \Pi_h^c(\tilde{\theta}) U_*^{\tilde{\theta}}(W)(0). \end{aligned} \quad (8.106)$$

Therefore both bounds follow from Proposition 8.1. \blacksquare

Let $\delta > 0$ be a small constant, which we will determine later. Pick U^0 in such a way that $\|U^0 - \bar{U}_h(0)\|_{\ell^p} < \delta$. We write $U^0 = \bar{U}_h(0) + V_0$.

Our goal is to find a small $W \in \text{Range}(\Pi_h^s(0))$ and a small $\tilde{\theta}$ in such a way that

$$V^0 + \bar{U}_h(0) = \bar{U}_h(\tilde{\theta}) + U_*^{\tilde{\theta}}(\Pi_h^s(\tilde{\theta})W)(0). \quad (8.107)$$

Using our notation from above we see that

$$\begin{aligned} \bar{U}_h(\tilde{\theta}) + U_*^{\tilde{\theta}}(\Pi_h^s(\tilde{\theta})W)(0) &= \bar{U}_h(\tilde{\theta}) + \Pi_h^s(\tilde{\theta})W + \mathcal{N}_2^{\tilde{\theta}}(W) \\ &= \bar{U}_h(0) + \tilde{\theta} \bar{U}'_h(0) + \mathcal{N}_1^{\tilde{\theta}} + \Pi_h^s(\tilde{\theta})W + \mathcal{N}_2^{\tilde{\theta}}(W) \\ &= \bar{U}_h(0) + \tilde{\theta} \bar{U}'_h(0) + \mathcal{N}_1^{\tilde{\theta}} + W - \Pi_h^c(\tilde{\theta})W + \mathcal{N}_2^{\tilde{\theta}}(W), \end{aligned} \quad (8.108)$$

which means that (8.107) can be written as

$$V^0 = \tilde{\theta} \bar{U}'_h(0) + \mathcal{N}_1^{\tilde{\theta}} + W - \Pi_h^c(\tilde{\theta})W + \mathcal{N}_2^{\tilde{\theta}}(W) \quad (8.109)$$

We write $\lambda_h^c : \text{Range}(\Pi_h^c(0)) \rightarrow \mathbb{R}$ for the map $\mu \bar{U}'_h \mapsto \mu$. This allow us to rephrase (8.109) as the fixed point problem

$$\begin{cases} \Pi_h^s(0)V^0 &= \Pi_h^s(0)\mathcal{N}_1^{\tilde{\theta}} + W - \Pi_h^c(0)\Pi_h^c(\tilde{\theta})(W) + \Pi_h^s(0)\left(\mathcal{N}_2^{\tilde{\theta}}(W)\right) \\ \lambda_h^c \left[\Pi_h^c(0)V^0 \right] &= \tilde{\theta} + \lambda_h^c \left[\Pi_h^c(0)\mathcal{N}_1^{\tilde{\theta}} \right] - \lambda_h^c \left[\Pi_h^c(0)\Pi_h^c(\tilde{\theta})(W) \right] + \lambda_h^c \left[\Pi_h^c(0)\left(\mathcal{N}_2^{\tilde{\theta}}(W)\right) \right]. \end{cases} \quad (8.110)$$

We show that equation (8.110) has a solution in the space

$$X_{\kappa, \varepsilon_\theta} := \{V \in \text{Range}(\Pi_h^s(0)) : \|V\|_{\ell^p} \leq \kappa\} \times [-\varepsilon_\theta, \varepsilon_\theta] \quad (8.111)$$

for some $\kappa, \varepsilon_\theta$ which we will determine later. Without loss of generality we assume that $\kappa, \varepsilon_\theta$ are small enough such that all previous inequalities hold.

Lemma 8.12. *Assume that (HP1), (HP2), (HS), (H α 1) and (H α 2) are satisfied. There are small constants $\delta > 0$, $\kappa > 0$ and $\varepsilon_\theta > 0$, independent of p , but not necessarily of h , such that for each $V^0 \in \ell^p$ with $\|V^0\|_{\ell^p} < \delta$ (8.110) has a unique solution $(W, \tilde{\theta}) \in X_{\kappa, \varepsilon_\theta}$. Moreover there is a constant $C_{19} > 0$ such that we have the bound*

$$\|W\|_{\ell^p} \leq C_{19} \|V^0\|_{\ell^p}. \quad (8.112)$$

Proof. We show that the map

$$T : (W, \theta) \mapsto \left(\begin{array}{l} \Pi_h^s(0)V^0 - \Pi_h^s(0)\mathcal{N}_1^{\tilde{\theta}} + \Pi_h^s(0)\Pi_h^c(\tilde{\theta})(W) - \Pi_h^s(0)(\mathcal{N}_2^{\tilde{\theta}}(W)) \\ \lambda_h^c \left[\Pi_h^c(0)V^0 \right] - \lambda_h^c \left[\Pi_h^c(0)\mathcal{N}_1^{\tilde{\theta}} \right] + \lambda_h^c \left[\Pi_h^c(0)\Pi_h^c(\tilde{\theta})(W) \right] - \lambda_h^c \left[\Pi_h^c(0)(\mathcal{N}_2^{\tilde{\theta}}(W)) \right] \end{array} \right) \quad (8.113)$$

maps $X_{\kappa, \varepsilon_\theta}$ into $X_{\kappa, \varepsilon_\theta}$ and is a contraction. Recall the constant C_1 from (8.3). Note that

$$\begin{aligned} \Pi_h^c(0)\Pi_h^c(\tilde{\theta})(W) &= \Pi_h^c(0)(W) - \Pi_h^c(0)\Pi_h^s(\tilde{\theta})(W) \\ &= -\Pi_h^c(0)\Pi_h^s(\tilde{\theta})(W) \end{aligned} \quad (8.114)$$

We see, using Lemma 8.9, Lemma 8.10 and Lemma 8.11, and setting

$$C_{16} = C_1 \sum_{j \in \mathbb{Z}} e^{-\tilde{\beta}|hj|} \quad (8.115)$$

that

$$\begin{aligned} \|T(W, \tilde{\theta})^{(1)}\|_{\ell^p} &\leq (1 + C_{16})\|V^0 - \mathcal{N}_1^{\tilde{\theta}} - \mathcal{N}_2^{\tilde{\theta}}(W)\|_{\ell^p} + C_{10}|\tilde{\theta}|\|W\|_{\ell^p} \\ &\leq (1 + C_{16})\left(\|V^0\|_{\ell^p} + C_{14}\tilde{\theta}^2 + C_6\|W\|_{\ell^p}^2\right) + C_{10}|\tilde{\theta}|\|W\|_{\ell^p} \\ &\leq C_{17}\left[\|V^0\|_{\ell^p} + \tilde{\theta}^2 + \|W\|_{\ell^p}^2 + |\tilde{\theta}|\|W\|_{\ell^p}\right] \end{aligned} \quad (8.116)$$

and

$$\begin{aligned} |T(W, \tilde{\theta})^{(2)}| &\leq \frac{C_{16}(\|V^0\|_{\ell^p} + C_{14}\tilde{\theta}^2 + C_6\|W\|_{\ell^p}^2) + (1 + C_{16})C_{10}|\tilde{\theta}|\|W\|_{\ell^p}}{|M|} \\ &\leq C_{17}\left[\|V^0\|_{\ell^p} + \tilde{\theta}^2 + \|W\|_{\ell^p}^2 + |\tilde{\theta}|\|W\|_{\ell^p}\right] \end{aligned} \quad (8.117)$$

for some constant $C_{17} > 0$, which is independent of p . Note that

$$\begin{aligned} \Pi_h^c(\tilde{\theta}_2)(W_2) - \Pi_h^c(\tilde{\theta}_1)(W_1) &= \Pi_h^c(\tilde{\theta}_2)W_2 - \Pi_h^c(\tilde{\theta}_1)W_2 + \Pi_h^c(\tilde{\theta}_1)(W_1 - W_2) \\ &= \Pi_h^c(\tilde{\theta}_2)W_2 - \Pi_h^c(\tilde{\theta}_1)W_2 + \Pi_h^c(\tilde{\theta}_1)(W_1 - W_2) - \Pi_h^c(0)(W_1 - W_2). \end{aligned} \quad (8.118)$$

Using Lemma 8.9, (8.81), Lemma 8.10 and Lemma 8.11 we obtain

$$\begin{aligned} \|T(W_1, \tilde{\theta}_1)^{(1)} - T(W_2, \tilde{\theta}_2)^{(1)}\|_{\ell^p} &\leq (1 + C_{16})\left(\|\mathcal{N}_1^{\tilde{\theta}_2} - \mathcal{N}_1^{\tilde{\theta}_1}\|_{\ell^p} + \|\Pi_h^c(\tilde{\theta}_2)(W_2) - \Pi_h^c(\tilde{\theta}_1)(W_1)\|_{\ell^p} \right. \\ &\quad \left. + \|\mathcal{N}_2^{\tilde{\theta}_2}(W_2) - \mathcal{N}_2^{\tilde{\theta}_1}(W_1)\|_{\ell^p}\right) \\ &\leq (1 + C_{16})\left(C_{15}(|\tilde{\theta}_1| + |\tilde{\theta}_2|)|\tilde{\theta}_2 - \tilde{\theta}_1| \right. \\ &\quad \left. + C_{10}|\tilde{\theta}_1 - \tilde{\theta}_2|\|W_2\| + C_{10}|\tilde{\theta}_1|\|W_1 - W_2\| \right. \\ &\quad \left. + C_{13}\left[\|W_1\|_{\ell^p} + \|W_2\|_{\ell^p} + |\tilde{\theta}_1 - \tilde{\theta}_2|\right]\|W_1 - W_2\|_{\ell^p} \right. \\ &\quad \left. + C_{13}|\tilde{\theta}_2 - \tilde{\theta}_1|\left[\|W_1\|_{\ell^p} + \|W_2\|_{\ell^p}\right]\right) \\ &\leq C_{18}\left[|\tilde{\theta}_1| + |\tilde{\theta}_2| + \|W_1\|_{\ell^p} + \|W_2\|_{\ell^p}\right]\left[|\tilde{\theta}_1 - \tilde{\theta}_2| + \|W_1 - W_2\|_{\ell^p}\right], \end{aligned} \quad (8.119)$$

together with

$$\begin{aligned}
|T(W_1, \tilde{\theta}_1)^{(2)} - T(W_2, \tilde{\theta}_2)^{(2)}| &\leq \frac{C_{16} \|\mathcal{N}_1^{\tilde{\theta}_2} - \mathcal{N}_1^{\tilde{\theta}_1} + \Pi_h^c(\tilde{\theta}_2)(W_2) - \Pi_h^c(\tilde{\theta}_1)(W_1) + \mathcal{N}_2^{\tilde{\theta}_2}(W_2) - \mathcal{N}_2^{\tilde{\theta}_1}(W_1)\|_{\ell^p}}{|M|} \\
&\leq C_{18} \left[|\tilde{\theta}_1| + |\tilde{\theta}_2| + \|W_1\|_{\ell^p} + \|W_2\|_{\ell^p} \right] \left[|\tilde{\theta}_1 - \tilde{\theta}_2| + \|W_1 - W_2\|_{\ell^p} \right]
\end{aligned} \tag{8.120}$$

for some constant $C_{18} > 0$, which is independent of p . First we let $0 < \kappa_{\max} < 1$ and $0 < \theta_{\max} < 1$ be constants such that all inequalities above hold for all $|\kappa| \leq \kappa_{\max}$ and all $|\theta| \leq \theta_{\max}$. In particular, we demand that $\kappa_{\max} < \eta$, $\kappa_{\max} < \frac{\eta}{C_3}$ and $\theta_{\max} < \eta$. Finally we write

$$\delta = \kappa = \varepsilon_\theta = \frac{1}{20} \min\{\kappa_{\max}, \theta_{\max}, \frac{1}{C_{17}}, \frac{1}{C_{18}}\} > 0. \tag{8.121}$$

With these choices we obtain the estimate

$$\begin{aligned}
\|T(W, \tilde{\theta})^{(1)}\|_{\ell^p} &\leq C_{17} \left[\|V^0\|_{\ell^p} + \tilde{\theta}^2 + \|W\|_{\ell^p}^2 + |\tilde{\theta}| \|W\|_{\ell^p} \right] \\
&\leq \frac{1}{20} \kappa + \frac{1}{20} \kappa + \frac{1}{20} \kappa + \frac{1}{20} \kappa \\
&\leq \frac{1}{2} \kappa.
\end{aligned} \tag{8.122}$$

Furthermore we see that

$$|T(W, \tilde{\theta})^{(2)}| \leq \frac{1}{2} \varepsilon_\theta. \tag{8.123}$$

Hence we see that the map T indeed maps $X_{\kappa, \varepsilon_\theta}$ into $X_{\kappa, \varepsilon_\theta}$.

In addition, (8.119) implies

$$\begin{aligned}
\|T(W_1, \tilde{\theta}_1)^{(1)} - T(W_2, \tilde{\theta}_2)^{(1)}\|_{\ell^p} &\leq C_{18} \left[|\tilde{\theta}_1| + |\tilde{\theta}_2| + \|W_1\|_{\ell^p} + \|W_2\|_{\ell^p} \right] \left[|\tilde{\theta}_1 - \tilde{\theta}_2| + \|W_1 - W_2\|_{\ell^p} \right] \\
&\leq \frac{4}{20} |\tilde{\theta}_1 - \tilde{\theta}_2| + \frac{4}{20} \|W_1 - W_2\|_{\ell^p},
\end{aligned} \tag{8.124}$$

while (8.120) yields

$$|T(W_1, \tilde{\theta}_1)^{(2)} - T(W_2, \tilde{\theta}_2)^{(2)}| \leq \frac{4}{20} |\tilde{\theta}_1 - \tilde{\theta}_2| + \frac{4}{20} \|W_1 - W_2\|_{\ell^p}. \tag{8.125}$$

Therefore the map T is a contraction and thus the fixed point problem (8.110) has a unique solution $(W, \tilde{\theta})$. Moreover we see that

$$\begin{aligned}
\|(W, \tilde{\theta})\|_{\ell^p \times \mathbb{R}} &\leq \|T(W, \tilde{\theta}) - T(0, 0)\|_{\ell^p \times \mathbb{R}} + \|T(0, 0)\|_{\ell^p \times \mathbb{R}} \\
&\leq \frac{1}{2} \|(W, \tilde{\theta})\|_{\ell^p \times \mathbb{R}} + 2C_{17} \|V^0\|_{\ell^p},
\end{aligned} \tag{8.126}$$

which yields

$$\begin{aligned}
\|(W, \tilde{\theta})\|_{\ell^p \times \mathbb{R}} &\leq 4C_{17} \|V^0\|_{\ell^p} \\
&= C_{19} \|V^0\|_{\ell^p}
\end{aligned} \tag{8.127}$$

as desired. ■

Proof of Proposition 8.2. If $(W, \tilde{\theta})$ satisfies (8.110) then we see from (8.108) that

$$\begin{aligned}
U^0 &= \bar{U}_h(0) + V^0 \\
&= \bar{U}_h(\tilde{\theta}) + \mathcal{U}_*^{\tilde{\theta}}(W)(0),
\end{aligned} \tag{8.128}$$

as desired. ■

Proof of Theorem 2.3. Let U be the solution of (2.1) with an initial condition $U(0) = U^0$ for which $\|U^0 - \bar{U}_h(0)\|_{\ell^p} < \delta$. By Proposition 8.2 and by uniqueness of the solution we see that

$$U = \bar{U}_h(\tilde{\theta}) + \mathcal{U}_*^{\tilde{\theta}}(W) \quad (8.129)$$

for some small $\tilde{\theta} \in \mathbb{R}$ and $W \in \ell^p$ with

$$\|W\|_{\ell^p} \leq C_{19}\|U^0 - \bar{U}_h(0)\|_{\ell^p}. \quad (8.130)$$

Hence we obtain

$$\begin{aligned} \|U(t) - \bar{U}_h(t + \tilde{\theta})\|_{\ell^p} &\leq e^{-\frac{\tilde{\beta}}{2}t} \|\mathcal{U}_*^{\tilde{\theta}}(\Pi_h^c(\tilde{\theta})(W))\|_{-\tilde{\beta}/2} \\ &\leq 2C_3 e^{-\frac{\tilde{\beta}}{2}t} \|\Pi_h^c(\tilde{\theta})(W)\|_{\ell^p} \\ &\leq 2C_3 e^{-\frac{\tilde{\beta}}{2}t} C_{16} \|W\|_{\ell^p} \\ &\leq 2C_3 C_{16} C_{19} \|U^0 - \bar{U}_h(0)\|_{\ell^p} e^{-\frac{\tilde{\beta}}{2}t}, \end{aligned} \quad (8.131)$$

as desired. ■

9 Discussion

In this paper, we considered the discrete FitzHugh-Nagumo equation

$$\begin{aligned}\dot{u}_j &= \frac{1}{h^2} \sum_{k>0} \alpha_k [u_{j+k} + u_{j-k} - 2u_j] + g(u_j) - w \\ \dot{w}_j &= \rho[u_j - \gamma w_j]\end{aligned}\tag{9.1}$$

posed on the one-dimensional lattice with $j \in \mathbb{Z}$ and where the cubic nonlinearity g is given by $g(u) = u(1-u)(u-r_0)$ and where we allowed infinitely many of the coefficients $\{\alpha_k\}_{k>0}$ to be non-zero. We were interested in finding travelling pulse solutions of the form $U_j(t) = V(hj + ct)$ for certain wavespeed c which satisfy the conditions that $\lim_{|\xi| \rightarrow \infty} V(\xi) = 0$ for the variable $\xi = hj + ct$.

In §4 we showed the existence of these travelling pulse solutions. In §6 we analyzed the spectrum of the linearisation of our problem around the travelling pulse solution and proved that in a suitable halfplane it consisted solely of the points $k2\pi i \frac{c}{h}$ for $k \in \mathbb{Z}$, which are all simple eigenvalues. Finally, in §8 we used Green's functions to establish the nonlinear stability of our travelling pulse solutions.

When lifting the method from [1] to the system case in §3, it turned out that when controlling the various cross terms, we were aided by the relative simplicity of the equation for w , since it has constant coefficients. When generalizing this method to an even larger class of systems where this is not the case, it might turn out that additional techniques will be required to control these cross terms.

Although most of our results in §6 and earlier are uniform in h , this is not the case for our main nonlinear stability result. Indeed, the constants appearing in the statement of Theorem 2.3 could in principle blow up or shrink to zero as $h \downarrow 0$. We do not believe that this is the case, but chose to not fully pursue the matter in this paper. We are confident that Lemma 5.7 can be stated in an h -independent fashion, but this requires delicate and involved estimates. In principle, this would allow the exponents appearing in §7 to be chosen independently of h . However, the constants in [3, Prop. 3.3] and [3, Lemma 4.3], results we apply multiple times, cannot be chosen independently of h . In particular, using the notation from [3, Prop. 3.3], there is no constant K for which $|\tilde{\mathcal{L}}(s) + II| \leq K$ holds for all small $h > 0$. This is caused by the factor $\frac{1}{h^2}$ appearing in our differential equation. In addition, the integration path in (7.26) becomes unbounded as $h \downarrow 0$.

The methods used to prove our results circumvent the need to use a statespace as in [20], where $C(I; \mathbb{R}^2)$ was used with $I = [-1, 1]$. In the infinite-range case considered here, one would be forced to take $I = \mathbb{R}$, which leads to many technical difficulties. For example, the Arzelà-Ascoli theorem no longer holds. Nevertheless, it would be convenient to have exponential dichotomies and tools like Lin's method for the general setting in this paper.

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