Center Manifold Theory for Functional Differential Equations of Mixed Type

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Mixed Type Functional Differential Equations (MFDE)

We are interested in nonlinear differential equations of the form

$$\dot{x}(t) = G(x_t).$$

- x is a continuous function with $x(t) \in \mathbb{R}$.
- $x_t \in C([-1, 1])$ is the **state** of x at t, i.e.,

$$x_t(\theta) = x(t + \theta), \qquad \theta \in [-1, 1] \ .$$

- $G : C([-1, 1]) \to \mathbb{R}$ is sufficiently smooth.

Note that

- $\dot{x}(t)$ depends on both past and future values of x.

- We will look for $x(t)$ near equilibria $\overline{x}$, $G(\overline{x}) = 0$.

Results

Starting point is the MFDE

$$\dot{x}(t) = Lx_t + R(x_t).$$

- $x_t \in X = C([-1, 1])$ is the state of x at t.
- $L : X \to \mathbb{R}$ is (for example) the linear operator

$$\phi \mapsto A_0\phi(0) + A_-\phi(-1) + A_+\phi(+1).$$

- $R : X \to \mathbb{R}$ is a nonlinear smooth operator with $R(0) = 0$ and $DR(0) = 0$.

Characteristic equation given by $\Delta(z) = 0$, with

$$\Delta(z) = z - A_0 - A_-e^{-z} - A_+e^{z}.$$

We are specially interested in cases where there are eigenvalues on the imaginary axis, i.e., $\Delta(i\omega) = 0$ for some $\omega \in \mathbb{R}$.

Results II

Recall

$$\dot{x}(t) = Lx_t + R(x_t). \tag{1}$$

As for delay equations, can define spectral projection $Q_0 : X \to X_0 \subset X$ onto finite dimensional subspace X_0 spanned by elements of the form

$$\phi : t \mapsto t^l e^{i\omega t} \text{ with } \Delta(i\omega) = 0 \text{ and } \dot{\phi}(t) = L\phi_t.$$

Main result gives "smooth" $u^* : X_0 \to \bigcap_{\eta>0} BC^1_\eta$ such that

(i) Sufficiently small solutions x to (1) are captured via $x = u^*(Q_0 x_0)$.

(ii) Any $\phi \in X_0$ such that $u^*\phi$ is sufficiently small, yields a solution $x = u^*\phi$ to (1).

(iii) Dynamics on X_0 is captured by ODE (with $A = L_{|X_0}$)

$$\begin{aligned} \dot{\Phi}(t) &= A\Phi(t) + f(\Phi(t)), \text{ where} \\ f(\psi) &= Q_0[L(u^*\psi - \psi)_\theta + R((u^*\psi)_\theta)]. \end{aligned}$$

MFDE

Weird "interaction from future" often raises doubts about usefullness of MFDE in modelling applications.

- Lattice differential equations.

  One studies travelling wave solutions to infinite dimensional differential systems on discrete lattices.

  - Material science (crystals)
  - Image processing (recognizing edges / outlines in pictures)
  - Biology (signal propagation through nerves with discrete gaps)

- Solving optimal control problems with delays.

- Direct models that, indeed, contain past + future terms.

Modelling

Ferdinand Banks in 'Energy Economics: a modern introduction'

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The difference between science and economics is
that science aims at the understanding of the
behaviour of nature, while economics is involved
with an understanding of models- and many of
these models have no relation to any state
of nature that has ever existed on this planet [...]
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Our examples will come from economics.

- Optimal control capital dynamics with delay
- Direct life cycle model

Optimal Control Capital Market Dynamics

Consider an economy that starts at time $t = 0$. Total amount of capital in economy given by $k(t) \geq 0$. Investments given by $u(t)$.

Production takes time! (Rustichini, 1989)

Consumption $c(t)$ that is technologically feasible depends on investments and available capital, i.e.,
$c(t) = C(u(t-\tau), k(t-\tau))$.

Total welfare is given by $\ln c(t) > 0$ to ensure "spreading out" of consumption.

Optimal control problem: maximize

$$\int_0^\infty e^{-\rho t} \ln C\big(u(t-\tau), k(t-\tau)\big) dt,$$

subject to $\dot{k}(t) = u(t-\tau) - gk(t-\tau)$.

Here g is a form of capital decay rate and ρ is the discount rate. This is a factor to correct for the fact that future welfare is rated to be less important than present welfare.

Euler Lagrange with Delays

Consider the problem to maximize the functional

$$J(y) = \int_0^\infty f(t, y(t-\tau), y(t), \dot{y}(t-\tau), \dot{y}(t))dt.$$

Introduce notation $x(t) = y(t-\tau)$ and $z(t) = y(t+\tau)$.

Theorem 1 (Hughes 1968). *If y maximizes J, then the following MFDE is satisfied*

$$\begin{aligned} \underbrace{D_3}_{y(t)} f(t, x, y, \dot{x}, \dot{y}) \quad &+ \quad \underbrace{D_2}_{y(t-\tau)} f(t+\tau, y, z, \dot{y}, \dot{z}) \\ &= \quad (d/dt)[\underbrace{D_5}_{\dot{y}(t)} f(t, x, y, \dot{x}, \dot{y}) \\ &\qquad + \underbrace{D_4}_{\dot{y}(t-\tau)} f(t+\tau, y, z, \dot{y}, \dot{z})]. \end{aligned}$$

- Solving an optimal control problem with delays thus leads to a MFDE!

Market Dynamics with Delays

Application of Hughes' result to maximize

$$\int_0^\infty e^{-\rho t} \ln C\big(u(t-\tau), k(t-\tau)\big) dt,$$

leads to MFDE

$$\begin{aligned}&e^{-\rho(t+\tau)}[gD_1C/C + D_2C/C]\big(\dot{k}(t+\tau) + gk(t), k(t)\big)\\ &= \tfrac{d}{dt}\Big(e^{-\rho t}D_1C/C(\dot{k}(t) + gk(t-\tau), k(t-\tau))\Big).\end{aligned} \tag{2}$$

Our example: $C(u,k) = \sqrt{u} - k$. Steady state solution

$$\overline{k} = \frac{e^{-2\rho\tau}}{4(\rho + ge^{-\rho\tau})^2}.$$

Linearizing around steady state and trying exponential solutions e^{zt} yields characteristic equation

$$\begin{aligned}\Delta(z) \quad &= \quad (z - \rho e^{-(z-\rho)\tau})(z - \rho + \rho e^{z\tau})\\ &\qquad -\tfrac{1}{2}(\rho + ge^{-\rho\tau})(2\rho e^{\rho\tau} + g) = 0.\end{aligned}$$

Market Dynamics with Delay

- Benhabib & Nishimura (1979) analyzed model without delays, but with at least three different economic goods.

- They found a pair of distinct eigenvalues that cross the imaginary axis when varying the parameters (ρ, g), leading to Hopf bifurcation.

- Using CM reduction we are able to do the same for MFDE.

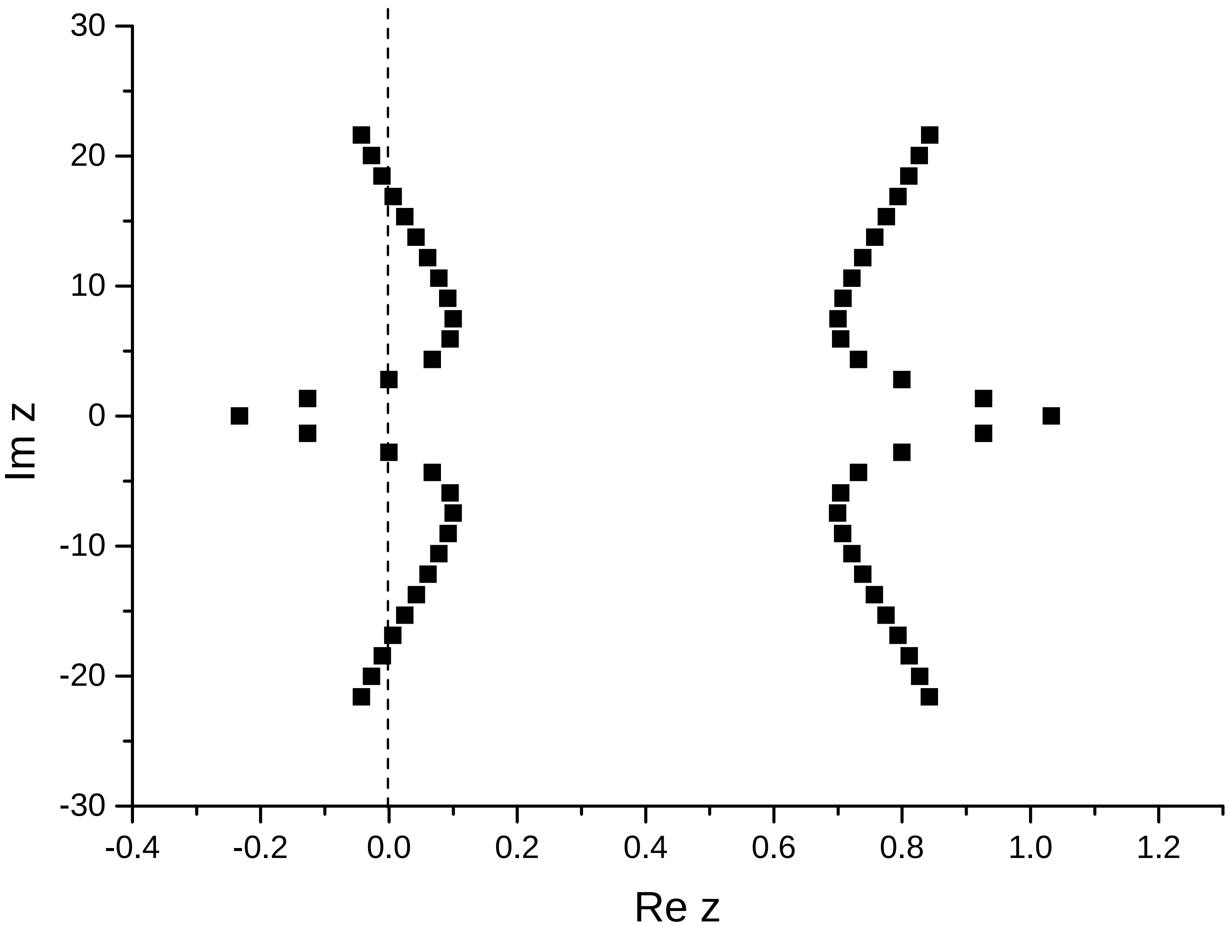
- $\Delta(z)$ is transcendental function with infinitely many zeroes to the right and left of imaginary axis.

- Numerically study zeroes in a neighbourhood of origin.

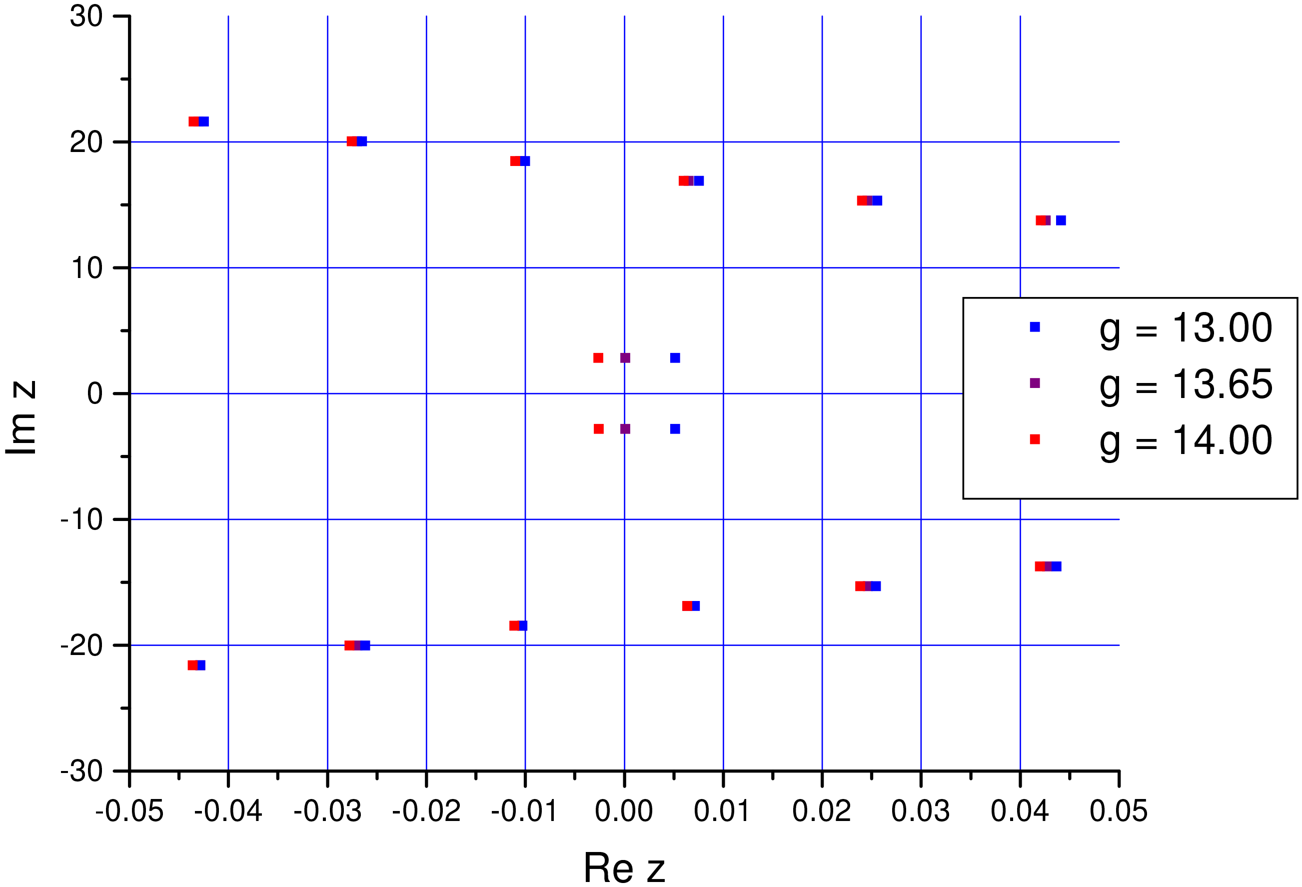
- Economically interesting periodic solutions for models with only one production and one consumption good.

Zeroes

Roots of $\Delta(z)$ in rectangle $[-5, 5] \times [-23, 23]$, calculated using complex bisection.



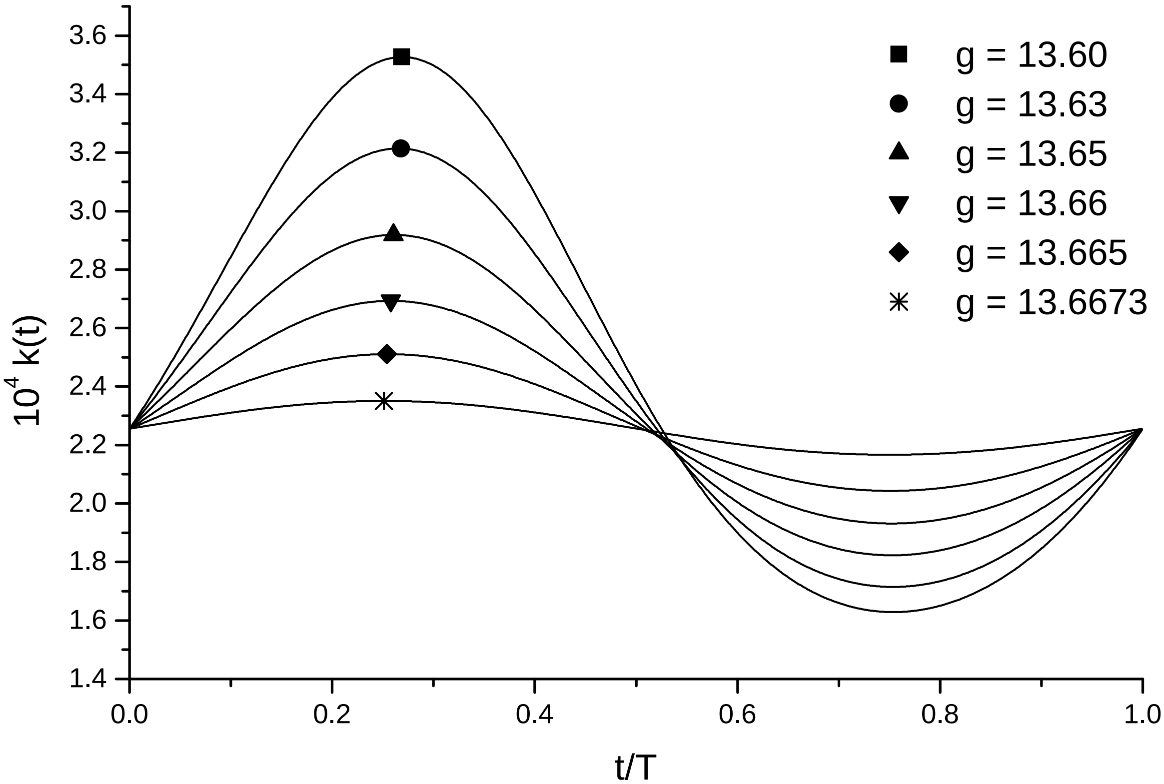
Zeroes - Detail

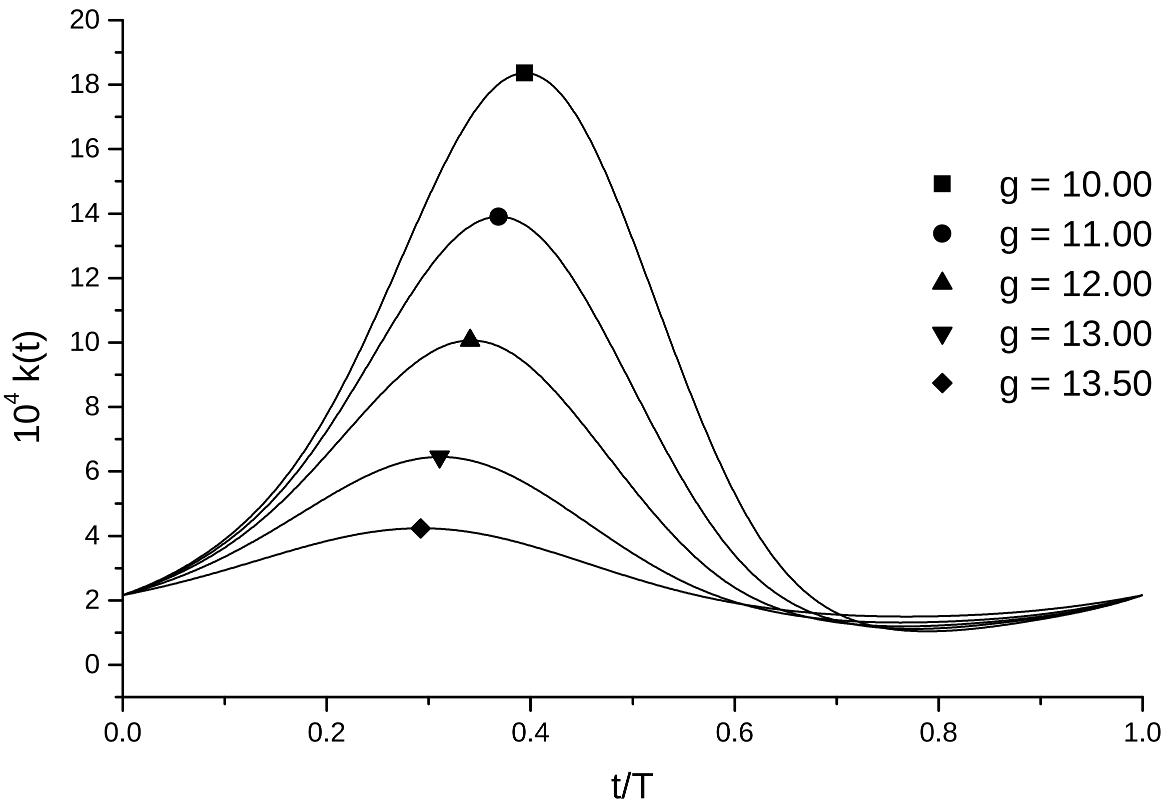


Observe crossing of imaginary axis at $g \approx 13.667698$, $\rho = 0.80$, $\tau = 4$.

Hopf bifurcation

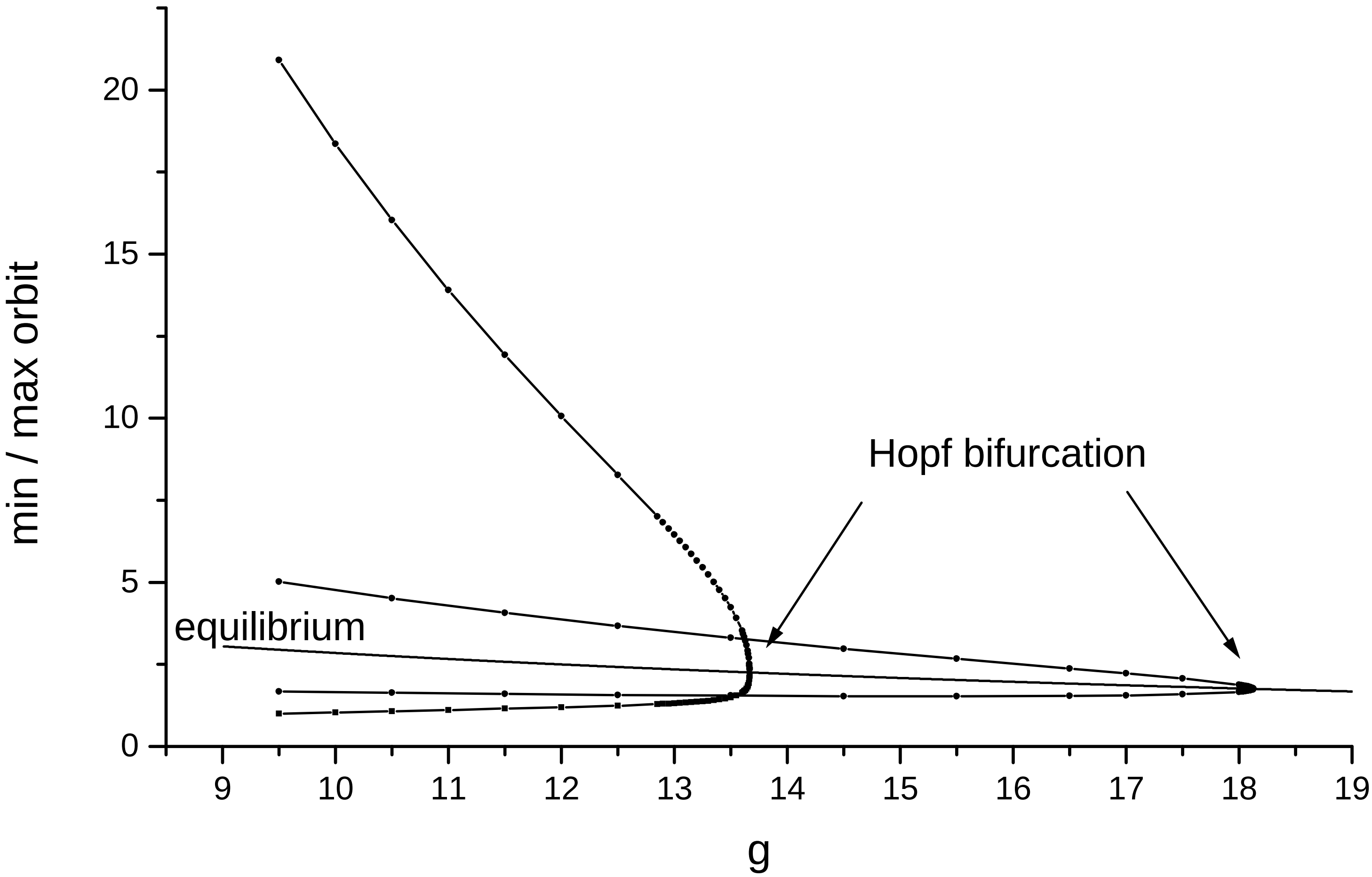
Periodic solutions were computed for capital $k(t)$.





Hopf bifurcation

Using computed periodic solutions, can construct local bifurcation diagram.



Explicit formula available for computing direction of bifurcation, derived by "lifting" finite dimensional Hopf bifurcation on CM.

Direct modelling: Life cycle model

Albis et al. (2004) consider a population model consisting of overlapping generations, that leads directly to MFDE.

- Fixed size population normalized to one. Each individual lives for time $T = 1$.

- Individuals born at time s have assets $a(s,t)$ at time t.

- At birth, assets are zero, i.e., $a(s,s) = 0$.

- One does not die in debt, i.e., $a(s,s+1) \geq 0$.

- Age-independent wages $w(t)$ are received.

- Interest rate $r(t)$.

- Individuals born at time s consume $c(s,t)$ at time t.

Individual budget constraint:

$$\frac{\partial a(s,t)}{\partial t} = r(t)a(s,t) + w(t) - c(s,t).$$

Life cycle model II

Goal of every individual born at time s is to maximize his lifetime welfare, given by

$$\int_s^{s+1} \ln c(s,\tau)d\tau.$$

Solving this optimization problem shows that the optimal asset distribution $a^*(s,t)$ depends on the interest rates and wages during the lifetime of an individual, i.e.,

$$a^*(s,t) = F(r_{s+}, w_{s+}, t-s),$$

for some F. Here $r_{s+} \in C([0,1])$ is defined by $r_{s+}(\theta) = r(s+\theta)$.

The total amount of capital at any time t is given by

$$k(t) = \int_{t-1}^{t} a^*(\sigma,t)d\sigma,$$

namely the total amount of assets owned by living individuals.

Life cycle model III

The economy has a single market good, that can be used for both production and consumption. It is produced at the rate Q given by

$$Q(k(t), e(t), l(t)) = Ak(t)^{\alpha}(e(t)l(t))^{\beta}.$$

- $l(t)$ is the labour supply, in our case $l(t) = 1$.
- $e(t)$ accounts for the increase in labour efficiency over time.
- $A, \alpha, \beta > 0$ are parameters.
- Q above is known as a Cobb-Douglas production function.

Note that interest rate $r(t)$ is, (by definition), the price of capital. Similarly, the wages $w(t)$ are the price of labour. They can be found by partial differentiation of Q.

$$\begin{aligned} r(t) &= \frac{\partial Q}{\partial k} = \alpha A k(t)^{\alpha-1}(e(t)l(t))^{\beta}, \\ w(t) &= \frac{\partial Q}{\partial l} = \beta A k(t)^{\alpha} e(t)^{\beta} l(t)^{\beta-1}. \end{aligned}$$

Life cycle model IV

We can now put everything back together.

$$
\begin{aligned}
k(t) &= \int_{t-1}^{t} a^*(\sigma, t)d\sigma, \\
a^*(s,t) &= F(r_{s+}, w_{s+}, t-s), \\
r(t) &= \tfrac{\partial Q}{\partial k} = \alpha A k(t)^{\alpha-1}(e(t)l(t))^{\beta}, \\
w(t) &= \tfrac{\partial Q}{\partial l} = \beta A k(t)^{\alpha} e(t)^{\beta} l(t)^{\beta-1}.
\end{aligned}
$$

Substituting everything into the first equation, we arrive at

$$k(t) = G(k_t, \alpha, \beta),$$

in which $k_t \in C([-1,1])$ is given by $k_t(\theta) = k(t+\theta)$.

Life cycle model V

Threefold differentiation of

$$k(t) = G(k_t, \alpha, \beta). \tag{3}$$

using the explicit form of G yields an MFDE

$$\begin{aligned} k'''(t) = f(k(t), k'(t), \quad & k''(t), k(t-1), k(t+1), \\ & \int_t^{t+1} k(\tau)^{\alpha+\beta-1} d\tau, \int_t^{t-1} k(\tau)^{\alpha+\beta-1} d\tau). \end{aligned} \tag{4}$$

Albis et al. choose $\alpha + \beta = 1$, in which case the MFDE becomes linear.

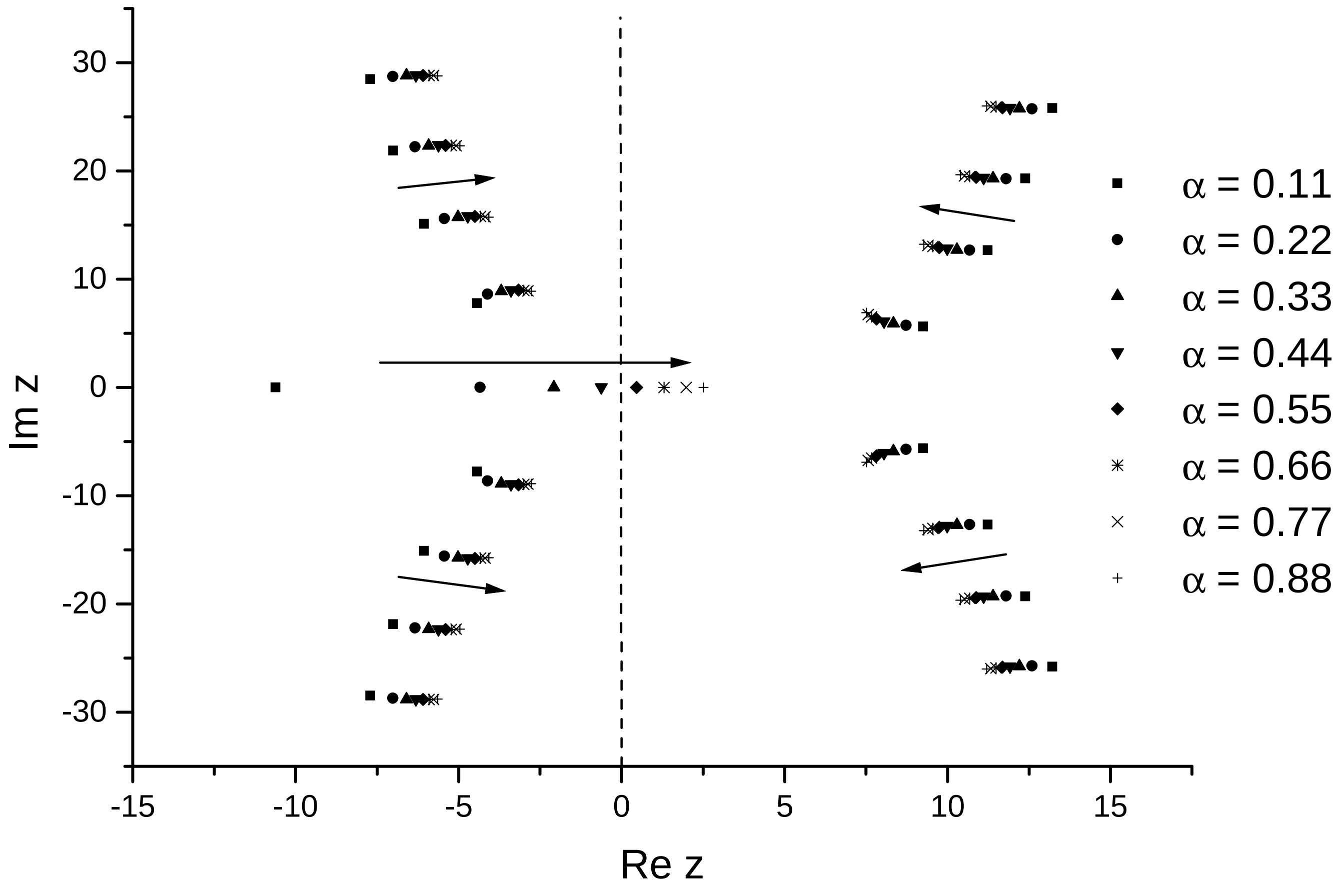
We are interested in $\alpha + \beta \neq 1$, and we find that for every $\gamma = \alpha + \beta \neq 1$, (4) has a unique strictly positive equilibrium solution $\overline{k}$.

Linearization around $\overline{k}$ yields the characteristic equation with $w = \alpha A \overline{k}^{\gamma-1}$,

$$\begin{aligned} \Delta(z+w) &= \alpha z^3 + w(\alpha - \gamma^2) z^2 - \gamma w^2 (1-\gamma) z \\ &- w\gamma((\gamma-1)w^2 + 2\beta) \\ &+ (z+w)^{-1}[2\beta w(\gamma z + w)\cosh z + \\ & \quad w^2(\gamma-1)(\gamma w^2 + 2\beta)]. \end{aligned}$$

Life cycle model VI

Roots of $\Delta(z)$ with $\alpha = \beta$.



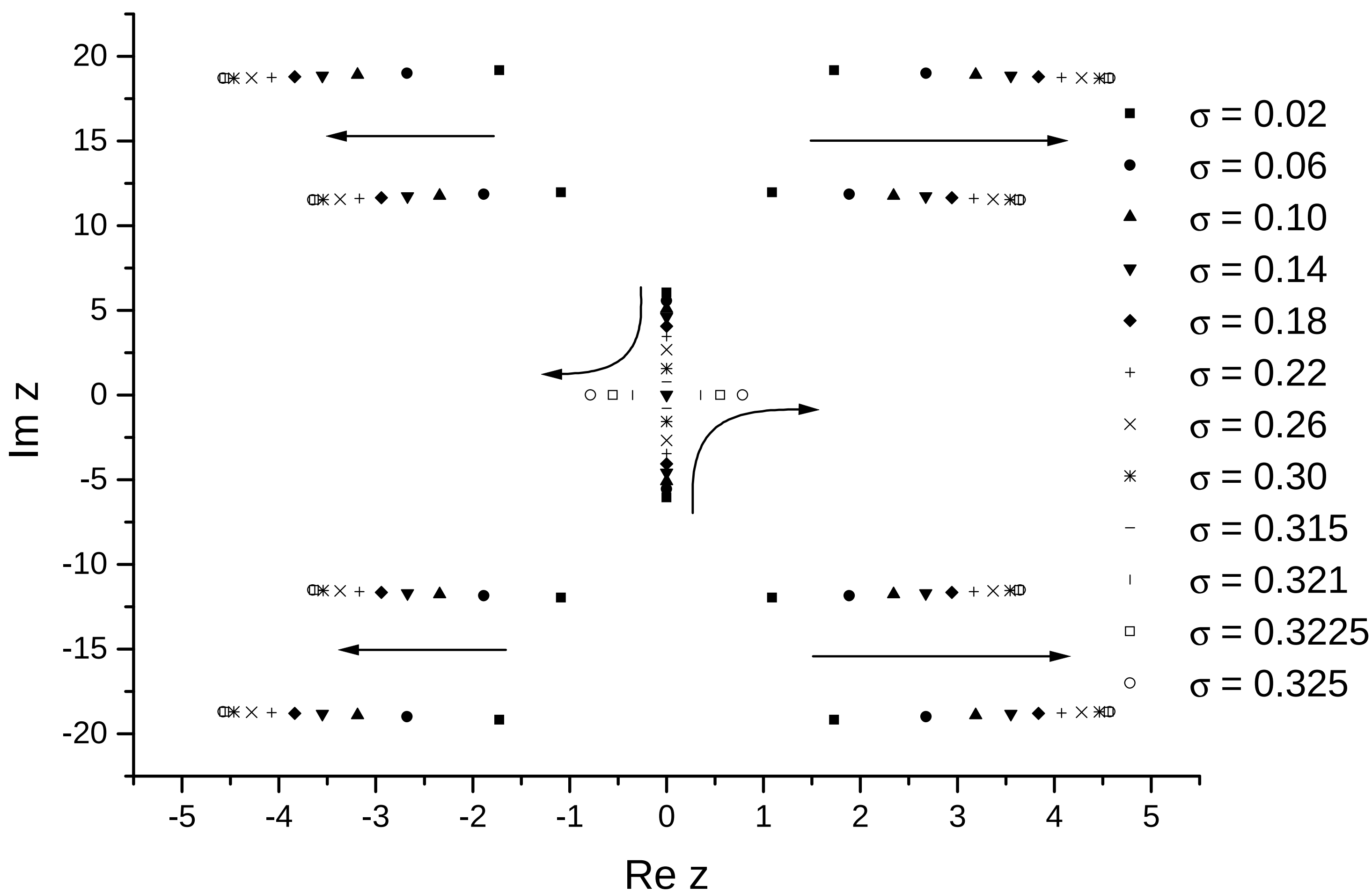
- Real root crosses the imaginary axis at $\alpha + \beta = 1$
- Apparently no imaginary pair of roots that cross imaginary axis.
- Since we demand $k > 0$, eigenmodes with $\mathrm{Re}\, z > 0$ and $\mathrm{Im}\, z \neq 0$ do not interest us.
- Loss of stability when $\alpha + \beta > 1$?

Recent Model (Albis et al. 2005)

Similar model, now with fixed wages only in the time periode $[\alpha, 1-\alpha] \subset [0,1]$ of an individual's life.

Albis et al. attempt to find Hopf bifurcations for characteristic equation

$$\Delta(z) = -\frac{\int_\alpha^{1-\alpha} e^{z\tau} d\tau}{1-2\alpha} + (1-\sigma)\int_0^1 e^{z\tau} d\tau + \frac{\sigma}{\int_0^1 e^{-z\tau} d\tau}.$$



Fixed $\alpha = 0.2$. Conditions for Hopf bifurcation not satisfied!

The program

- Analyze the linear homogeneous equation

$$\dot{x}(\xi) = Lx_\xi$$

  and determine set $\mathcal{N}_0$ of solutions that grow at most polynomially.
- Analyze the linear inhomogeneous equation

$$\dot{x}(\xi) = Lx_\xi + f(\xi) \tag{5}$$

  and find an "inverse" $\mathcal{K}$ such that $x = \mathcal{K}f$ solves (5) and $\mathcal{K}$ projects out $\mathcal{N}_0$ in some sense.
- Analyze functional

$$\mathcal{G}(u, y) = y + \mathcal{K}(R(u)),$$

  for $y \in \mathcal{N}_0$. Use fixpoint arguments to find fixpoint $\overline{u}(y)$ for $\mathcal{G}(\cdot, y)$ for sufficiently small y. Note that $\overline{u}(y)$ solves

$$\dot{x}(\xi) = Lx_\xi + R(x_\xi).$$

Comparison with delay equations

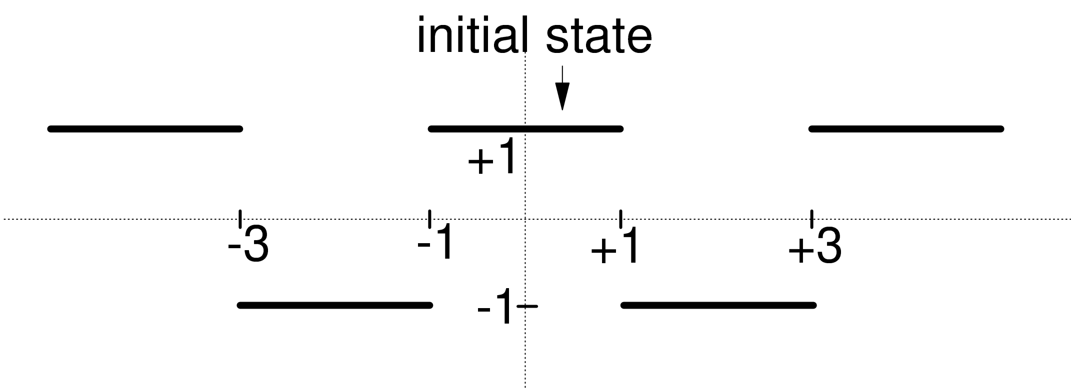
The homogeneous linear equation

$$\dot{x}(\xi) = Lx_\xi,$$

is for general MFDE not a well-posed initial value problem, if we demand x continuous.

Example: $x|_{[-1,1]} = 1$ and

$$x'(\xi) = x(\xi - 1) + x(\xi + 1).$$



Comparison with delay equations II

Linear delay equations (RFDE) however do allow unique forward continuation of solutions, i.e., the problem

$$\dot{x}(\xi) = L_d x_{\xi -}, \qquad x_{0-} = \phi,$$

in which $x_{\xi -} \in C([-1,0])$ is defined by $x_{\xi -}(\theta) = x(\xi + \theta)$, has solution

$$x_{t-} = S(t)\phi,$$

in which $S(t)$ is an eventually compact semigroup, with generator

$$\begin{array}{l} A : \mathcal{D}(A) = \{\phi \in C^1([-1,0]) \mid \dot{\phi}(0) = L_d \phi\} \to C([-1,0]) \\ A\phi = \dot{\phi} \end{array}$$

One can thus employ all the strong results from semigroup theory, in particular variation of constants formula!

As in study of elliptic PDEs (Mielke, Kirchgässner), need to construct pseudo inverse $\mathcal{K}$ by hand for MFDE.

Linear inhomogeneous equations

The important step is to analyze the linear inhomogeneous equation

$$\dot{x}(\xi) = Lx_\xi + f(\xi), \tag{6}$$

for functions $f : \mathbb{R} \to \mathbb{R}$.

Mallet-Paret established result for hyperbolic versions of (6), using Laplace transform techniques.

Theorem 2. *Suppose that $\Delta(z) = 0$ has no roots on the imaginary axis. Then for every $f \in L^\infty$, (6) has a unique solution in $W^{1,\infty}$, given by*

$$x(\xi) = \int_{-\infty}^{\infty} G(\xi - s) f(s) ds,$$

where G has Fourier transform $\hat{G}(\eta) = \Delta(i\eta)^{-1}$.

Nonhyperbolic inhomogeneous equations

However, when we have spectrum on the imaginary axis,

$$\dot{x}(\xi) = Lx_\xi$$

has set of solutions $\mathcal{N}_0$ that are bounded or grow at most at a polynomial rate. Need to find a "pseudo-inverse" that projects out these solutions in some way.

Want to apply Mallet-Paret theorem, but need to "shift" the eigenvalues off the imaginary axis first.

This can be done by multiplying the equations with exponentials $e^{\eta\xi}$.

Laplace transform enables us to link the "projecting out" of solutions in $\mathcal{N}_0$, to the projection Q_0 onto the spectral subspace X_0.

Dynamics on the Center Manifold

For any $\phi \in X_0$, define the function $\Phi : \mathbb{R} \to X_0$ by

$$\Phi(t) = Q_0[(u^*\phi)_t].$$

Then Φ satisfies an ODE on the center manifold

$$\begin{aligned} \dot{\Phi}(\xi) &= A\Phi(\xi) + f(\Phi(\xi)), \text{ where} \\ f(\psi) &= Q_0[L(u^*\psi - \psi)_\theta + \widetilde{R}((u^*\psi)_\theta)]. \end{aligned} \tag{7}$$

Here $\widetilde{R}$ is "cut-off" version of R, i.e., globally bounded. Conversely, if Ψ satisfies (7), then $x = u^*\Psi(0)$ satisfies

$$\dot{x}(t) = Lx_t + \widetilde{R}(x_t)$$

and in addition, $x_t = (u^*\Psi(t))_0$.

For delay eqs. (RFDE), variation of constants approach yields

$$f_d(\psi) \quad = \quad Q_0[\widetilde{R}((u^*\psi)_0)\delta(\theta)].$$

Although this appears to differ from (7), examples indicate that our f restricted to RFDE yields same Taylor expansions as f_d.

Further issues

- In the context of RFDE, one can also study invariant stable and unstable manifolds that capture sufficiently small solutions on the halflines $\mathbb{R}_{\pm}$.

- This feature is absent from our analysis here, due to the ill-posedness of the initial value problem. We cannot define a suitable solution operator for linear systems on half lines.

- Mallet - Paret and Verduyn Lunel (2001) have some results in this direction

- Need to develop Floquet theory to analyze periodic solutions of MFDE.

- Need to do some work on stability analysis. Spectrum is unbounded to the left and right of imaginary axis. But perhaps results restricted to positive solutions? Useful for modelling applications.

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