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Travelling waves  
in stochastic reaction-diffusion equations

Bachelor thesis

August 19, 2018

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## Introduction

Reaction-Diffusion Equations (RDEs) are a widely described subject in mathematics and physics due to their interesting behaviour and their common occurrence in nature. A reaction-diffusion system is a system of partial differential equations that is of the form

$$\partial_t \vec{u} = D\Delta \vec{u} + \vec{f}(\vec{u}),$$

where  $\Delta$  is the diffusion operator and  $f$  is a reaction term. In chemistry  $u$  is usually higher dimensional to model the multiple compounds of a chemical reaction. For such chemical reactions the Grey-Scott model is often used, which is also a source of interesting pattern formations such as the ones that can be found in [10]. Also population dynamics, predator prey models, random motion in continuous time and other examples can be described by RDEs, see [3],[6] and [9]. A lot of higher dimensional cases have been studied with strong results on the behaviour and the stability of patterns. On the numerical analysis of RDEs there is a wide range of literature. For example [11] can be considered.

Almost all studied cases up to now are made with the assumption that we know what the parameters are and that the system is deterministic.

### The stochastic extension

In many natural setting we do not know what all the parameters are at any given moment and the parameters can vary over time in a random way. Hence it is interesting to look at these systems from a stochastic point of view. To investigate the behaviour of these systems we will introduce a stochastic partial differential equation (SPDE), that describes this behaviour. More stochastic extensions have been done by [4] and [7]. [4] deals with phenomena that can be described with additive noise, i.e. the equation is perturbed by adding small random value as noise. The second stochastic extension by [7], is done by adding some noise term  $g$ , that is multiplied by the noise, to get a SPDE. This SPDE is also the subject of this thesis and is given by

$$dU = [U_{xx} + f(U)] dt + \sigma g(U) d\beta_t.$$

In this setting it is possible to change the way the noise influences the concentration. The parameter  $\sigma$  is the noise strength that can enforce or weaken the noise in the SPDE.

The challenge is that the study on Stochastic differential equations is quite young; Kiyoshi Itô introduced the stochastic integral in the late 40's and the stochastic differential equation was born in the early 50's. For example, the stability and existence of waves in RDEs was already been proven in the late 17th century [13] and is still get a lot of interest, while a weaker version of the stability of SRDEs with multiplicative noise has just been proven by [7]. So we still have to improve the methods to analyse these stochastic differential equations.

### This thesis

In this thesis our main goal is to show numerical results for some open questions on the one dimensional SRDE with multiplicative noise. The way we do this, is by showing some main results and techniques that are commonly used to determine the speed, the travelling wave front and the stability of the wave. This includes the explanation of the travelling wave Ansatz, spectral analysis, Sturm-Liouville's theorem and the use of semigroups. Thereafter, we will extend these methods to the stochastic case. This involves an introduction to the definition of a stochastic integral and the stochastic differential equation and of course the definition of the SRDE with multiplicative noise. Then we continue with the extended methods of the ones that were introduced in the first section. These extended methods result in a weaker result than in the deterministic case on the stability. This result states that we have a bound on the norm of solutions for finite time. We will show with numerical results that this theorem is true. We will also try to define what the average stochastic speed should be, and without a proof that this quantity really exists, we analyse it numerically to show that for the examples the wave tends to have on average a constant speed. We will compute these values and show what the relation is between the noise strength and the speed of the wave.

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# 1 The deterministic model

## 1.1 Introduction

In this thesis, we will be discussing a class of PDEs, that are known as the one-dimensional reaction-diffusion equations. The model is widely used in natural sciences such as biology and physics. The equations describe the evolution of concentrations  $u$  in space over time, where these have a tendency to spread out, but are restricted by a reaction-term. For further references on derivation of such reaction-terms, consider [3] & [6].

**Definition 1.** A Reaction-Diffusion equation (RDE) for  $\mathcal{D} \subset \mathbb{R}^n$  is given by

$$u_t = \Delta u + f(u) \tag{1.1}$$

where  $u : \mathcal{D} \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$ ,  $\Delta$  is the Laplacian  $\Delta u = u_{11} + \dots + u_{nn}$ , and  $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is known as the reaction term.

In most cases  $u$  denotes the concentration of the studied object. While the Laplacian flattens out the concentration on the domain  $\mathcal{D}$ , the reaction-term counteracts this behaviour in a way that is entirely dependent on the nature of the specific problem. If the reaction-term is absent, this equation is referred to as *the heat equation* or *the diffusion equation*. However in some settings a reaction-term is needed to accurately describe the physics.

Even though experiments give a wild variety of different reaction terms, in this thesis we confine ourselves to a specific class of reaction-terms. This class is known as the class of *one dimensional bistable reaction-diffusion equations*.

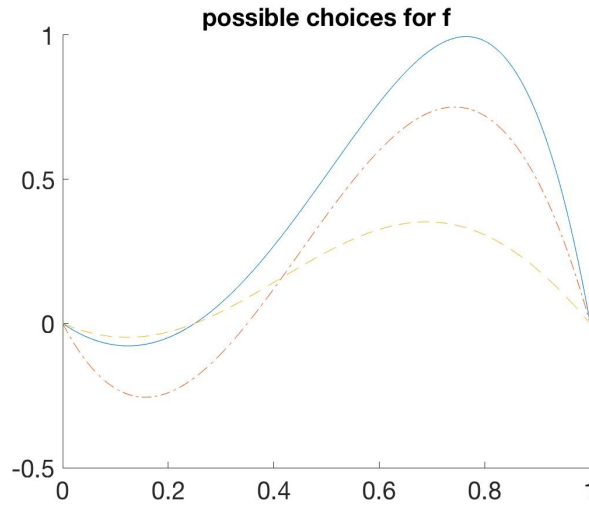


Figure 1:  $f$  for which the system is bistable

**Definition 2.** Equation (1.1) is called a one dimensional bistable RDE if  $\mathcal{D} \subset \mathbb{R}$ ,  $f : \mathcal{D} \times \mathbb{R}^+ \rightarrow \mathbb{R}$  and it has the following three properties

1.  $f : [0, 1] \mapsto \mathbb{R}$  is continuous,
2.  $f(0) = f(\alpha) = f(1)$  for some unique  $0 < \alpha < 1$ ,
3.  $f'(0) < 0$  and  $f'(1) < 0$ .

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## 1.2 Determine a wave solution

Since evaluations of certain initial conditions result in travelling waves, it is best to describe solutions relatively to those travelling waves. First, we give a formal definition of a wave, or typically in the case of a RDE, a *wave front*. Thereafter, we present a way to show the existence and a way to determine the wave explicitly.

**Definition 3.** A travelling wave front is a stationary solution  $\Phi_0 = \Phi_0(x - ct)$  that solves equation (1.1) for some parameter  $c$ , for which also holds  $\lim_{\zeta \rightarrow \pm\infty} \Phi_0(\zeta) = w_{\pm}$  where  $w_+ \neq w_-$ .

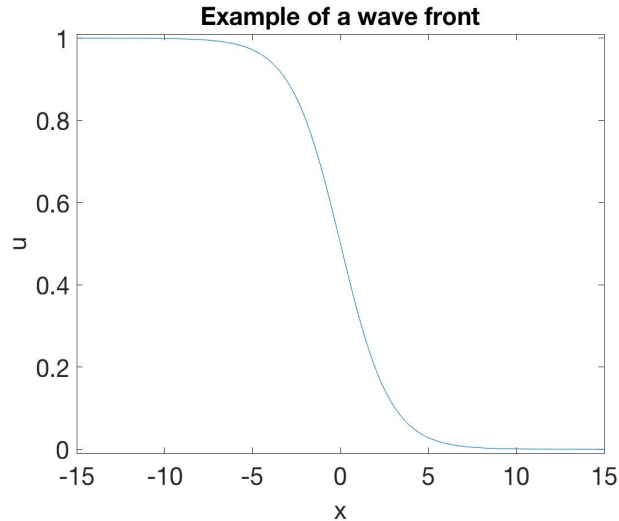


Figure 2: Example of a wave front

For an example of a wave front see Figure 2. This definition gives a way to characterise a travelling wave front. First the ‘travelling’ is defined viewing the wave in a moving reference frame.

**Definition 4.** The linear shifted frame is given by

$$\zeta = x - ct. \quad (1.2)$$

There are two important assumptions in these two definitions. The first is the speed  $c$ , for which the wave front is a stationary solution and the second is that  $w_+ \neq w_-$ . This makes sure that the wave front is not a pulse nor a constant function. These assumptions lead to the travelling wave Ansatz. This Ansatz uses both assumptions, by viewing the differential equation in the moving frame, use that the wave front is stationary in this frame and then give boundary conditions by  $w_+ = \Phi_0(-\infty) = 1$  and  $w_- = \Phi_0(\infty) = 0$ . We start viewing the PDE in the linear moving frame.

$$\frac{\partial u(x - ct, t)}{\partial t} = \frac{\partial^2 u(x - ct, t)}{\partial x^2} + f(u(x - ct, t)). \quad (1.3)$$

The chain rule on the first term gives the following

$$\frac{\partial u(x - ct, t)}{\partial t} = -c \partial_1 u(x - ct, t) + \partial_2 u(x - ct, t), \quad (1.4)$$

substituting in the previous equation we find

$$-c \partial_1 u(x - ct, t) + \partial_2 u(x - ct, t) = \frac{\partial^2 u(x - ct, t)}{\partial x^2} + f(u(x - ct, t)). \quad (1.5)$$

Note that we can also write for  $\frac{\partial^2 u(x-ct, t)}{\partial x^2} = \partial_{11} u(x-ct, t)$ . This gives

$$-c \partial_1 u(x-ct, t) + \partial_2 u(x-ct, t) = \partial_{11} u(x-ct, t) + f(u(x-ct, t)). \quad (1.6)$$

Switching to the linear moving frame  $\zeta = x - ct$  gives

$$-c \partial_1 u(\zeta, t) + \partial_2 u(\zeta, t) = \partial_{11} u(\zeta, t) + f(u(\zeta, t)). \quad (1.7)$$

Now we will change  $\partial_1$  to  $\partial_\zeta$  and  $\partial_2$  to  $\partial_t$  to get

$$u_t(\zeta, t) = u_{\zeta\zeta} u(\zeta, t) + cu_\zeta(\zeta, t) + f(u(\zeta, t)). \quad (1.8)$$

This is the same partial differential equation as the one we started with, but then in the linear moving reference frame. For the travelling wave, we knew that there was a specific speed  $c_0$  for which the wave front was a stationary solution. Hence the pair  $(\Phi_0, c_0)$  solves the ODE that is given by

$$0 = c_0 \Phi_0'(\zeta) + \Phi_0''(\zeta) + f(\Phi_0(\zeta)) \quad (1.9)$$

$$\text{with } w_+ = \Phi_0(-\infty) = 1 \text{ and } w_- = \Phi_0(\infty) = 0. \quad (1.10)$$

To this ODE the travelling wave  $\Phi_0$ , with the boundary conditions is a solution as can be found in [14]. Although  $\Phi_0(\zeta)$  is a stationary solution of (1.8), it is not clear whether it is stable or unstable.

### 1.3 Spectrum of the linearised wave equation

In order to prove that the travelling wave is in fact a stable solution, we start by linearising equation (1.8). We perturb the equation with a linear term  $v$  to show what the spectrum is. This method can only show linear stability, which will not directly be enough to prove the stability of this equation. Write  $\Phi_0 + v$  for  $u$ , to get for equation (1.8):

$$\begin{aligned} v_t(\zeta, t) &= v_{\zeta\zeta}(\zeta, t) + \Phi_{0\zeta\zeta} + c(v_\zeta + \Phi_{0\zeta}) + f(\Phi_0 + v) \\ &= \underbrace{c\Phi_0' + \Phi_0'' + f(\Phi_0)}_{=0 \text{ by (1.9)}} + f'(\Phi_0)v + R(\Phi_0, v) + cv' + v'' \\ &= v'' + cv' + f'(\Phi_0)v + R(\Phi_0, v). \end{aligned}$$

In the second equality, we take  $R(\Phi_0, v) = f(\Phi_0 + v) - f(\Phi_0) - v f'(\Phi_0)$  which is quadratic in  $v$  by the Taylor expansion of  $f(\phi_0 + v)$  around  $\Phi_0$ . From this point, we see that  $R$  is quadratic and the terms upfront are linear. Hence the first three terms determine the linear stability. We will write for (1.3)

$$v_t = \mathcal{L}v + R(\Phi_0, v), \quad (1.11)$$

where  $\mathcal{L}v$  is given by

**Definition 5.**

$$\mathcal{L}v = v'' + c_0 v' + f'(\Phi_0)v. \quad (1.12)$$

The spectrum consists of the point spectrum and the essential spectrum. First, we compute the essential spectrum, thereafter, we show an eigenvalue of the spectrum that is part of the point spectrum and show that this one is the largest eigenvalue.

#### 1.3.1 The essential spectrum

To find the essential spectrum, we have to solve the eigenvalue problem given by the ODE  $\mathcal{L}v = \lambda v$ . Rewriting the ODE as a system gives

$$\begin{aligned} v_\xi &= u \\ u_\xi &= (\lambda - f'(\Phi_0(\xi)))v - c_0 u. \end{aligned} \quad (1.13)$$

Linearisation gives the following Jacobian matrix, which determines the eigenvalues

$$J(\xi, \lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - f'(\Phi_0(\xi)) & -c_0 \end{pmatrix}. \quad (1.14)$$

Since the wave front  $\Phi_0$  has the property that  $\lim_{\xi \rightarrow -\infty} \Phi_0 = 1$  and  $\lim_{\xi \rightarrow \infty} \Phi_0 = 0$ , ([15] 1.10), we can write  $J_{\pm}(\xi, \lambda) = \lim_{\xi \rightarrow \pm\infty} J(\xi, \lambda)$ .

For  $\xi \rightarrow -\infty$  and  $\xi \rightarrow \infty$  we have  $\Phi_0 \rightarrow 1$  respectively  $\Phi_0 \rightarrow 0$ , hence we may write

$$J_- = \begin{pmatrix} 0 & 1 \\ \lambda - f'(1) & -c_0 \end{pmatrix} \text{ and } J_+ = \begin{pmatrix} 0 & 1 \\ -\lambda - f'(0) & -c_0 \end{pmatrix} \quad (1.15)$$

to get the following characteristic polynomials

$$|J_- - \zeta i I| = \begin{vmatrix} -i\zeta & 1 \\ \lambda - f'(1) & -c_0 - i\zeta \end{vmatrix} = -\zeta^2 + ic_0\zeta - \lambda + f'(1),$$

$$|J_+ - \zeta i I| = \begin{vmatrix} -i\zeta & 1 \\ \lambda - f'(0) & -c_0 - i\zeta \end{vmatrix} = -\zeta^2 + ic_0\zeta - \lambda + f'(0).$$

We find the eigenvalues by setting the characteristic polynomials equal to zero. This holds for choices of  $\lambda = -\zeta^2 + ic_0\zeta + f'(1)$  and  $\lambda = -\zeta^2 + ic_0\zeta + f'(0)$ . Theorem 3.3 of [12] implies that the full essential spectrum is the set between the two lines defined as above, which are presented in Figure 3.

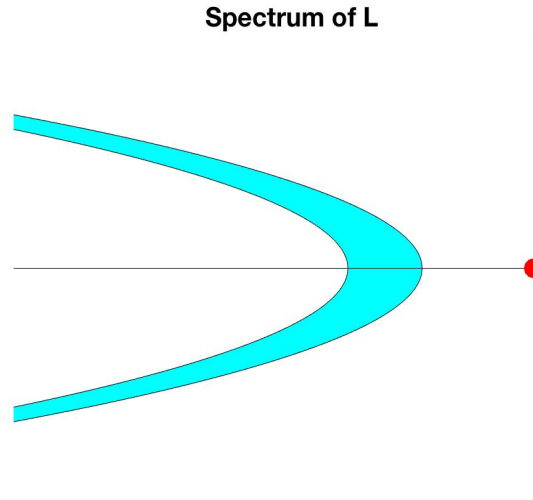


Figure 3: The spectrum of  $\mathcal{L}$

Since we impose to confine ourselves with the condition of bistable RDEs, the essential spectrum of  $\mathcal{L}$  lies entirely to the left of the imaginary axis. Because the bistability implies  $f'(0)$  and  $f'(1)$  are strictly smaller than zero see definition 2.

### 1.3.2 The point spectrum

$\Phi'_0$  is an eigenfunction for the operator  $\mathcal{L}$ , because it is a non-trivial solution to the equation  $\mathcal{L}v = 0$ . This can be seen via the following computation.

$$\begin{aligned}\mathcal{L}\Phi'_0 &= \Phi'_{0\xi\xi} + c_0\Phi'_{0\xi} + f'(\Phi_0)\Phi'_0 \\ &= \Phi'''_0 + c_0\Phi''_0 + f'(\Phi_0)\Phi'_0 \\ &= \frac{d}{d\xi}(\Phi''_0 + c_0\Phi'_0 + f(\Phi_0)) \\ &= 0.\end{aligned}\tag{1.16}$$

The last equality holds because the left hand side is the derivative of equation (1.9). The eigenvalue corresponding to this eigenfunction is 0.

Now we use Sturm-Liouville's theorem from [8] to show that 0 is the largest real eigenvalue.

**Lemma 1.** Sturm-Liouville's theorem on the real line

Let  $\mathcal{L}$  be a Sturm-Liouville operator. i.e. for  $\mathcal{L} : L^2 \rightarrow L^2$  and  $a_0, a_1 \in C^1(\mathbb{R})$

$$\begin{aligned}\mathcal{L}v &= v'' + a_1(x)v' + a_0(x)v \\ \lim_{x \rightarrow \pm\infty} e^{\nu|x|} |a_0(x) - a_0^\pm| & \lim_{x \rightarrow \pm\infty} e^{\nu|x|} |a_1(x) - a_1^\pm|\end{aligned}\tag{1.17}$$

for some  $\nu > 0$  and constants  $a_0^\pm, a_1^\pm$ , then there exists a sequence  $(\lambda_i)_{i=0}^N$  of eigenvalues with eigenfunctions  $(p_i)_{i=0}^N$  which have exactly  $i$  zeros. Furthermore, the sequence is ordered such that  $\lambda_0 > \lambda_1 > \dots > \lambda_N$  and all eigenvalues are real.

In the case of the bistable RDE, we can use theorem 3.1. of [15], which says that  $\Phi_0$  is monotonous decreasing, which implies that  $\Phi'_0$  has no roots on the real line. With the Sturm-Liouville theorem, we conclude that 0 is the largest eigenvalue by the ordering and the finitely many other eigenvalues have negative real part.

The zero eigenvalue is due to the fact that we cannot control the movement in the direction of  $\Phi'_0$ . We cannot control this behaviour because of the shift invariance of solutions, i.e. if  $\Upsilon(\cdot, t)$  is a solution to equation (1.1), then  $\Upsilon(\cdot + \epsilon, t)$  for any  $\epsilon \in \mathbb{R}$  is also a solution to the RDE. We introduce a non-linear shift, that will compensate for this lack of control.

### 1.4 The non-linear shifted frame

To compensate for the non-linear behaviour in the direction of the eigenfunction,  $\Phi'_0$ , we introduce the following, more delicate, moving reference frame.

**Definition 6.** The non-linear shifted frame is given by:

$$\xi = x - \gamma(t) \quad \text{with } \dot{\gamma} = c_0 + a(v).\tag{1.18}$$

To compute the new equation that wave solutions have to obey, we start again with equation (1.1),

$$\frac{\partial u(x - \gamma(t), t)}{\partial t} = \frac{\partial^2 u(x - \gamma(t), t)}{\partial x^2} + f(u(x - \gamma(t), t)).$$

Using the chain rule

$$\frac{\partial}{\partial t} u(x - \gamma(t), t) = -\dot{\gamma}(t)\partial_1 u(x - \gamma(t), t) + \partial_2 u(x - \gamma(t), t)$$

one finds

$$-\dot{\gamma}(t)\partial_1 u(x - \gamma(t), t) + \partial_2 u(x - \gamma(t), t) = \frac{\partial^2 u(x - \gamma(t), t)}{\partial x^2} + f(u(x - \gamma(t), t)).$$



Also notice

$$\partial_{xx}u(x - \gamma(t), t) = \partial_{11}u(x - \gamma(t), t).$$

Thus we can write:

$$-\dot{\gamma}(t)\partial_1u(x - \gamma(t), t) + \partial_2u(x - \gamma(t), t) = \partial_{11}u(x - \gamma(t), t) + f(u(x - \gamma(t), t)).$$

Now take  $\xi = x - \gamma(t)$ . This yields

$$-\dot{\gamma}u_\xi(\xi, t) + u_t(\xi, t) = u_{\xi\xi}(\xi, t) + f(u(\xi, t)).$$

As in chapter 1.3, we decompose  $u$  into  $v + \Phi_0$  to get these two equalities:

$$v_t(\xi, t) = \dot{\gamma}(v_\xi + \Phi_{0\xi}) + v_{\xi\xi} + \Phi_{0\xi\xi} + f(v + \Phi_0).$$

$$v_t(\xi, t) = (c_0 + a(v))(v_\xi + \Phi_{0\xi}) + v_{\xi\xi} + \Phi_{0\xi\xi} + f(v + \Phi_0) \quad (1.19)$$

The last observation we can make is that we can expand  $f$  around  $\Phi_0$  by  $f(\Phi_0 + v) = f(\Phi_0) + v f'(\Phi_0) + \mathcal{O}(v^2)$ . So for  $R(\Phi_0, v) = f(\Phi_0 + v) - f(\Phi_0) - v f'(\Phi_0)$ ,  $R$  is a quadratic remainder term. The result is the following equation:

$$v_t(\xi, t) = \mathcal{L}v + a(v)(\Phi'_0 + v_\xi) + R(\Phi_0, v) \quad (1.20)$$

with

$$\mathcal{L}v = v_{\xi\xi} + c_0v_\xi + f'(\Phi_0)v. \quad (1.21)$$

Notable to equation (1.20) is that  $a(v) \equiv 0$  gives exactly equation (1.11) back. This is necessary, because an alternative equation would contradict the earlier results. To study (1.20) we need another instrument.

## 1.5 Introduction to semigroups

To find an expression for the non-linear shift,  $a$ , that compensates for the non-linear behaviour, we need semigroups. The semigroup allows us to separate the non-linear shift movement from the exponential decaying part. First, we introduce the semigroup, we will use in the upcoming paragraph to split solutions.

**Definition 7.** An analytic semigroup with generator  $\mathcal{L}$  is a time dependent operator  $S(t) : L^2 \rightarrow L^2$  with the property that for any  $u \in L^2$  holds

1.  $t \rightarrow S(t)u$  is continuous for  $t \geq 0$ ,
2.  $S(t)S(\tau) = S(t + \tau)$  for  $t, \tau \geq 0$ ,
3.  $S(0) = I$ ,
4.  $\frac{d}{dt}S(t) = \mathcal{L}S(t)$  for  $t > 0$ .

Semigroups are a natural extension of the exponential function with  $\mathcal{L}$  in the exponent. Whether the semigroup is well-defined is not trivial, but it has been proven for this operator in [7]. In terms of the semigroup defined as above the solution to equation (1.20) is found to be

$$v(t) = S(t)v_0 + \int_0^t S(t-s) [a(v)\Phi'_0 + v_\xi(s) + R(\Phi_0, s)] ds. \quad (1.22)$$

Note that this equation solves (1.20), as can be seen by the following informal computation

$$\begin{aligned} \frac{d}{dt}v(t) &= \frac{d}{dt} \left( S(t)v_0 + \int_0^t S(t-s) [a(v)\Phi'_0 + v_\xi(s) + R(\Phi_0, s)] ds \right) \\ &= \mathcal{L}v(t) + [S(t-s) [a(v)\Phi'_0 + v_\xi(s) + R(\Phi_0, s)]]_0^t \\ &= \mathcal{L}v(t) + [a(v(t))(\Phi'_0 + v(t)_\xi(t)) + R(\Phi_0, t)] - [a(v(0))(\Phi'_0 + v(0)_\xi(0)) + R(\Phi_0, 0)] \\ &= \mathcal{L}v(t) + a(v)(\Phi'_0 + v_\xi(t)) + R(\Phi_0, t). \end{aligned} \quad (1.23)$$

## 1.6 Split the spectrum

Since Liouville's theorem states that there are finitely many eigenvalues in the point spectrum and the essential spectrum does not contain zero, zero cannot be an accumulation point of the spectrum. This implies, that 0 cannot be an accumulation point of the spectrum, i.e. we have a spectral gap between the point spectrum and the essential spectrum. It is possible to split solutions into a part that corresponds to the zero eigenvalue, inherently the eigenfunction  $\Phi'_0$ , and the part corresponding to the essential spectrum, which causes exponential decay. The way to do this, is to make a contour integral around the point spectrum. The resulting operator is

**Definition 8.** The projection operator on the point spectrum of  $\mathcal{L}$  is given by

$$Pf = \langle f, \psi_0 \rangle_{L^2} \Phi_0 \quad (1.24)$$

where  $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(\bar{x})dx$  and  $\psi_0$  is the normalised adjoint eigenfunction of  $\Phi'_0$  i.e.  $\mathcal{L}^*\psi_0 = 0$ , and  $\langle \Phi_0, \psi_0 \rangle = 1$ .

With use of this definition we can construct a separation of (1.22) into the part of  $P$  and  $I - P$ , where  $I$  is the identity operator:

$$\begin{aligned} v(t) = & S(t) (I - P) v(0) + \int_0^t S(t-s) (I - P) [a(v)(\Phi'_0 + v_\xi(s)) + R(\Phi_0, v)] ds \\ & + S(t) P v(0) + \int_0^t S(t-s) P [a(v)(\Phi'_0 + v_\xi(s)) + R(\Phi_0, v)] ds. \end{aligned} \quad (1.25)$$

Due to the  $I - P$ -term, the first line corresponds to the eigenvalues in the essential spectrum, that have negative real part. Hence this part decays exponentially. The second line corresponds to the point spectrum with eigenvalue 0, which does not decay. For the following choices of the free parameters, we can solve the problem. To let the second line vanish, we have to make sure that the following holds

$$\begin{aligned} 0 = & P [a(v)(\Phi'_0 + v_\xi(s)) + R(\Phi_0, v)] \\ = & \langle a(v)(\Phi'_0 + v_\xi(s)) + R(\Phi_0, v), \psi_0 \rangle \Phi_0 \\ = & [a(v) (\langle \Phi'_0, \psi_0 \rangle + \langle v_\xi, \psi_0 \rangle) + \langle R(\Phi_0, v), \psi_0 \rangle] \Phi_0 \\ = & [a(v) (1 + \langle v_\xi, \psi_0 \rangle) + \langle R(\Phi_0, v), \psi_0 \rangle] \Phi_0. \end{aligned}$$

Hence we choose

$$a(v) = -\frac{\langle R(\Phi_0, v), \psi_0 \rangle}{1 + \langle v_\xi, \psi_0 \rangle}. \quad (1.26)$$

To make sure that the equation of  $v$  is completely orthogonal to  $\psi_0$ , we want  $S(t)Pv(0) = 0$ . By definition  $\gamma$  is defined in differential form, hence we can choose the initial shift i.e.  $\gamma_0(0) := \gamma_0$ . Since we wrote  $u(\xi, t) = v(\xi, t) + \Phi_0(\xi)$ ,  $\xi = x$  for  $t = 0$ , and we can shift the initial condition  $u(\xi, 0)$  with  $\gamma_0$ , such that we can write

$$v(0) = u(x - \gamma_0, 0) - \Phi_0(x).$$

Which gives us that  $\gamma_0$  is the value for which holds:

$$\langle u(x - \gamma_0, 0) - \phi_0(x), \psi_0 \rangle = 0. \quad (1.27)$$

Proposition 2.3 in [7] tells us that there exists such a  $\gamma_0$ , for which this equality holds. For the choices of (1.26) and (1.27), the second line of (1.25) is always zero, so by plugging these constraints into (1.25), we get the following final equation

$$v(t) = S(t) [I - P] v(0) + \int_0^t S(t-s) [I - P] \left[ -\frac{\langle R(\Phi_0, v), \psi_0 \rangle}{1 + \langle v, \phi'_0 \rangle} (\Phi'_0 + v_\xi(s)) + R(\Phi_0, v(s)) \right]. \quad (1.28)$$

---

For  $v(0)$  small enough it can be proven that this equation goes to zero as can be found in [7] by choosing  $\sigma = 0$ . This reference uses a proof that is far more complicated than the equation needs, because it is a result that we need for the stochastic case in 2.7. Another way to prove that this equations goes to zero is by estimating the remainder terms from above and then show that it converges.

## 1.7 Wave solution to the Nagumo equation

The Nagumo equation is yet the first non-trivial RDE we have explicitly solved in the deterministic case as well for an extended stochastic version. This will also be elaborated later on in this thesis.

**Definition 9.** The 1D Nagumo equation *is given by the following PDE*

$$u_t = u_{xx} + u(1-u)(u-\alpha) \quad (1.29)$$

where  $\alpha \in (0, \frac{1}{2})$  is a parameter and  $u : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ .

### 1.7.1 Determine the wave solution

The following result is given by [1]. We can solve the wave equation (1.9) explicitly for the Nagumo equation by assuming that the wave solution is of the form of a hyperbolic tangent, which is also the solution to the following differential equation:

$$u' = ku(1-u), \quad k \in \mathbb{R}. \quad (1.30)$$

The derivative of (1.30) is

$$u'' = ku' - 2kuu'.$$

Combining the two gives

$$u'' = k^2u(1-u) + 2k^2u^2(1-u). \quad (1.31)$$

If we put this restriction into the wave equation of (1.9) for the Nagumo equation, it becomes

$$0 = (2k^2 - 1)u^3 + (-3k^2 - ck + 1 + \alpha)u^2 + (k^2 + ck - \alpha)u. \quad (1.32)$$

This equation must hold for any value of  $u$ , thus the following system of equations has to hold.

$$2k^2 - 1 = 0 \quad (1.33)$$

$$-3k^2 - ck + \alpha + 1 = 0 \quad (1.34)$$

$$k^2 + ck - \alpha = 0 \quad (1.35)$$

This system gives as a solution:

$$k = \pm \sqrt{\frac{1}{2}} \quad (1.36)$$

$$c = k(2\alpha - 1).$$

Next is to solve the Ansatz (1.30), a separable ODE, that is given below

$$\frac{dv}{d\zeta} = ku(1-u) \quad (1.37)$$

which is separable, integrating both sides gives:

$$\log |u| - \log |u - 1| = k\zeta + C, \text{ for some } C \in \mathbb{R}.$$

---

Since the wave lies in between 0 and 1, we write  $\log |u - 1| = \log(1 - u)$ . Rewriting this with the assumption that  $0 \leq u \leq 1$ , gives the following expression for  $u$

$$u = \frac{e^{k\zeta+C}}{1 + e^{k\zeta+C}} = 1 - \frac{1}{1 + e^{k\zeta+C}}. \quad (1.38)$$

The important part about the way the constant  $C$  appears in this equation shows us that any horizontal translation of the wave front is still a solution to (1.30). When we use the assumption that  $u(-\infty) = 1$  and  $u(\infty) = 0$ , we have to choose minus sign in equation (1.36). This results gives then an explicit wave front for the Nagumo equation that is given by

**Lemma 2.** *The travelling wave of the Nagumo equations with  $\Phi_0(-\infty) = 1$  and  $\Phi_0(\infty) = 0$  is given by*

$$\Phi_0(x - ct) = 1 - \frac{1}{1 + e^{-\frac{1}{\sqrt{2}}(x-ct)}} \quad (1.39)$$

where  $c = \sqrt{2} \left(\frac{1}{2} - \alpha\right)$ .

### 1.7.2 Stability of the wave solution

Following chapter 1.1 to 1.6 we know that it is sufficient to follow the conditions given in these chapters. The Nagumo equation satisfies definition 2, since

$f(u) = u(1 - u)(u - \alpha)$  is continuous,  $f(0) = f(\alpha) = f(1) = 0$ ,  $f(x) < 0$  for all  $x \in (0, \alpha)$  and  $f(x) > 0$  for  $x \in (\alpha, 1)$  and  $f'(0) = -\alpha$ ,  $f'(1) = \alpha - 1$ .

Also since  $f'(\Phi_0)$  exponentially decays to  $f'(0)$  and  $f'(1)$ , the conditions in Sturm-Liouville's theorem are satisfied. Thus we conclude that the one dimensional Nagumo equation has a stable wave solution, that is given by (1.39).

---

## 2 The stochastic model

### 2.1 Introduction to stochastic model

In this section, we extend the deterministic methods from the previous section to the stochastic case. We will try to prove the stochastic stability. We start with giving some intuition to what is known as the stochastic integral and explain what is meant by a Stochastic Partial Differential Equation (SPDE). Thereafter the definition of the class of Stochastic Reaction-Diffusion Equations (SRDE) is given in the setting of this thesis. We adapt the definition of a solution in this setting and then we will find such a weak solution to the stochastic differential equation that can be solved by the wave front. When the wave front is found, we will give some information on ways to show the stability of these. Although, we cannot prove the stability in full generality, we do show it for an explicit example, that is an adapted version to the Nagumo equations. For other examples, the stability is shown numerically in section 3.

### 2.2 The stochastic integral and differential equation

To understand what a stochastic differential equation is, we need to understand what Brownian motion is. Brownian motion [2], which is also often called a Wiener process, is one of the major building blocks of the stochastic integral.

**Definition 10.** *A standard Wiener process or Brownian motion is a stochastic process  $(\beta_t)_{t \geq 0}$  taking values in  $\mathbb{R}$  such that:*

1.  $\beta_0 = 0$ ,
2.  $\mathbb{P}(\{t \mapsto \beta_t\} \text{ is continuous}) = 1$ ,
3.  $\forall t, s > 0$  holds  $\beta_{t+s} - \beta_t \sim \mathcal{N}(0, s)$ ,
4. for any partition  $t_1 < t_2 < \dots < t_m$  holds that the increments  $\beta_{t_2} - \beta_{t_1}, \beta_{t_3} - \beta_{t_2}, \dots, \beta_{t_m} - \beta_{t_{m-1}}$  are independent.

Brownian motion is an example of a continuous time random process which is the object we want to study in order to find the definition of a stochastic integral. Since we want to integrate a function with respect to the Brownian motion, that is not deterministic, the integral will not be deterministic either. A way to think about the integral, without yet giving the formal definition is to see the integral as the stochastic process that has the same behaviour as the stochastic process obtained by taking limits of random evaluations of the original stochastic process.

A functional space for which the integral is well defined is  $\mathcal{H}^2[0, T]$ , as can be found in [14]. It contains all functions for which holds

$$\mathbb{E} \left[ \int_0^t f^2(\omega, t) dt \right] < \infty. \quad (2.1)$$

Furthermore there exists a space that is called  $\mathcal{H}_0^2$ , that is dense in  $\mathcal{H}^2$ . It consists of all functions  $f$  that can be written as

$$f_n(\omega, t) = \sum_{i=0}^{N-1} a_{i,n}(\omega) \mathbb{1}_{\{t_{i+1}-t_i\}} \quad (2.2)$$

**Definition 11.** *The Stochastic integral of  $f \in \mathcal{H}^2[0, T]$  with respect to the standard Brownian motion is defined as the stochastic process  $\{X_t : 0 \leq t \leq T\}$  that is a continuous martingale with respect to the standard Brownian filtration  $\mathcal{F}_t$  for which  $\{\omega : X_t(\omega) = I(m_t f)(\omega)\}$ , has probability one. In this definition  $m_t(\omega, s) = \mathbb{1}_{s \in [0, t]}$  and for functions  $\mathcal{H}^2 \ni f = \lim_{n \rightarrow \infty} f_n \in \mathcal{H}_0^2$  the integral of  $f$  is defined as*

$$I(f) = \lim_{n \rightarrow \infty} \sum_{i=0}^{N-1} a_{i,n}(\omega) \mathbb{1}_{\{B_{t_{i+1}} - B_{t_i}\}} \quad (2.3)$$

Although the notation suggests that we can interpret the stochastic integral as the usual Riemann integral, there is a substantial difference between the two. We cannot think of the integral as a *number*, since the result,  $X_t$ , is a stochastic process and the notion of the integral as the limit of rectangles is not defined. A way to think about the stochastic integral is to think of the equation to hold for all (measurable) evaluations of the Brownian motion. A simple example of a stochastic integral can be seen in Figure 4 and 5. On the left side a standard Brownian motion can be seen. On the right side, we have integrated each individual path.

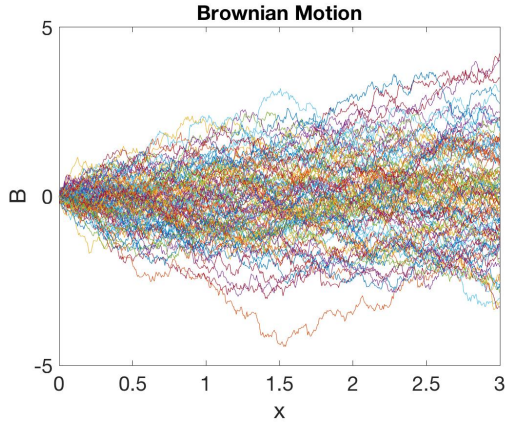


Figure 4: 100 realisations of Brownian Motion

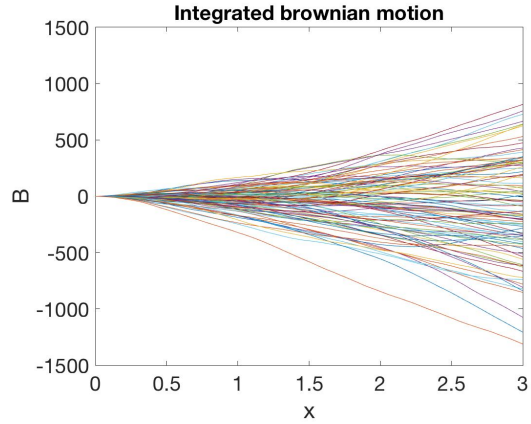


Figure 5: 100 realisations of integrated Brownian Motion

### 2.2.1 The Itô formula

In the thesis, there is a theorem that we will use extensively, due to its importance in stochastic calculus. This theorem is the stochastic equivalent to the chain rule. We will not prove it in this thesis, nor the adapted version that is used in 2.5. Proofs for the standard Itô formula, formulated below and the adapted form later on in this section, can be found in [14], respectively [5].

**Lemma 3.** For any  $f \in \mathcal{H}^2[0, T]$

$$df(t, \beta_t) = f_1(t, \beta_t)dt + \frac{1}{2}f_{22}(t, \beta_t)dt + f_2(t, \beta_t)d\beta_t. \quad (2.4)$$

This formula is known as the Itô formula.

Note the difference between the standard chain rule in Riemann calculus. The best way to see the difference, is to write the chain rule in an unconventional way, the integrated form, that is

$$f(x(t), y(t)) = f(0, 0) + \int_0^t f_1(x, t)dx + \int_0^t f_2(x, t)dt, \quad (2.5)$$

whereas, the Ito formula is in integrated form equal to:

$$f(t, \beta_t) = f(0, 0) + \int_0^t f_1(t, \beta_t)dt + \int_0^t \frac{1}{2}f_{22}(t, \beta_t)dt + \int_0^t f_2(t, \beta_t)d\beta_t. \quad (2.6)$$

The first and the third integral were expected from the usual chain rule. However, for stochastic process there is an additional deterministic term that influences the behaviour of the stochastic RDEs.

---

## 2.3 Stochastic RDEs

Since we have now introduced the stochastic framework, we can extend the class of RDEs to a new class. We call this class the one dimensional Stochastic reaction-diffusion equations (SRDE) with multiplicative noise.

**Definition 12.** For some stochastic process  $U$ , functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  and parameter  $\sigma \in \mathbb{R}^+$  A stochastic one dimensional reaction-diffusion equation with multiplicative noise is given by

$$dU = [U'' + f(U)] dt + \sigma g(U) d\beta_t. \quad (2.7)$$

This notation is an abbreviation of the integral equation, that is defined by

$$U(x, t) = U(x, 0) + \int_0^t \partial_{xx} U(x, s) + f(U(x, s)) ds + \sigma \int_0^t g(U(x, s)) d\beta_s. \quad (2.8)$$

Also note that this equation is an extension to (1.1) with an additional term that depends on  $\sigma$  to which we refer as *the noise strength*. This noise strength is multiplicative to  $g$  which means that  $\sigma$  can only amplify or weaken the stochastic values that  $b$  takes. Also, for  $\sigma = 0$  there is no stochastic part to the system. In that case we can write (2.7) in integral form as the following equation.

$$U(x, t) = U(x, 0) + \int_0^t \partial_{xx} U(x, s) + f(U(x, s)) ds$$

This is exactly the integrated form of (1.1).

Notably, if  $g(U) = 0$  then the noise does not contribute to the behaviour of the system. Formally the second derivative of  $U$  is not necessarily defined. In [7] this problem is solved using an operator  $A_* : \mathcal{H}^1 \mapsto \mathcal{H}^{-1}$ . We will omit these technicalities and refer to [7]. To continue on definition 2, we assume in this thesis that the system is bistable and we lay some restrictions on the noise term  $g$ . We assume that  $g(0) = g(1) = 0$  and  $g$  is continuous on the interval  $[0, 1]$ . Without these assumptions a travelling wave can never exist, since the noise would perturb the stable states of the system. A formal problem in the study of this equation is that  $U$  is not integrable; Especially, a wave front can never be integrable, since one of both endpoints is always non-zero, see definition 2. Therefore, we introduce a yet unknown reference function  $\Phi_{ref}$  in the following decomposition of  $U$ .

**Definition 13.** The decomposition of  $U$  is given by

$$U = X + \Phi_{ref}, \quad (2.9)$$

where  $\Phi_{ref}$  is a smooth enough stationary reference function such that  $X \in L^2$ .

This definition is just a necessity for the SPDE to be well-defined. In the introduction of this section, we defined the stochastic integral which is just defined for the space  $\mathcal{H}^2[0, T]$ . This space can be somewhat extended, but we cannot define a stochastic integral on non-integrable functions. An example of a possible reference function is an adapted hyperbolic tangent, that joins 1 and 0. Definition 13, leads to the following SPDE based on (2.7), since the reference function is independent of time.

$$dX = [X'' + \Phi_{ref}'' + f(X + \Phi_{ref})] dt + \sigma g(X + \Phi_{ref}) d\beta_t. \quad (2.10)$$

This is an abbreviation of

$$X(t) = X(0) + \int_0^t \partial_{xx} [X(s) + \Phi_{ref}] ds + \int_0^t f(X(s) + \Phi_{ref}) ds + \sigma \int_0^t g(X(s) + \Phi_{ref}) d\beta_s. \quad (2.11)$$

In this setting we look at the movement ‘on’ the reference function. The choice of the reference function is just a technical necessity. As it turns out this is the case, because equation (2.49) does not depend at all on the choice of  $\Phi_{ref}$ . We find these results by the end of section 2.6.

---

## 2.4 Stochastic shift

The behaviour of the system is heavily dependent on the value of  $\sigma$  and the shape of  $g$ .  $g$  is integrated with respect to the Brownian motion, that is not deterministic. Hence, we cannot expect a deterministic shift to be sufficient to freeze the solution. In extension of the non-linear shift of definition 6, we use this shift with an extra stochastic term that is dependent on the Brownian motion. Furthermore, we assume that the stochastic part of the shift is linear in  $\sigma$  with some unknown function  $b$  that depends on  $X$ .

Recapitulatory, we assume the shift has to be of the following form.

**Definition 14.** For  $c_\sigma \in \mathbb{R}$ ,  $a, b : \mathcal{H}^1 \rightarrow \mathbb{R}$ , the Stochastic shift is defined as  $\Gamma : \mathbb{R}^+ \rightarrow \mathbb{R}$  for which holds

$$d\Gamma(t) = [c_\sigma + a(X)] dt + \sigma b(X) d\beta_t. \quad (2.12)$$

This is an abbreviation of

$$\Gamma(t) = \Gamma_0 + \int_0^t [c_\sigma + a(X + \Phi_{ref})] ds + \sigma \int_0^t b(X + \Phi_{ref}) d\beta_s. \quad (2.13)$$

The first intuitive step that fails, is to treat  $X$  exactly as in the deterministic case. That is to write  $X(x - \Gamma(t), t)$  and derive an ODE.

This does not work because in section 1.2, we took derivatives with the chain rule. As seen in (2.4), it is possible to write out a chain rule for stochastic processes with respect to time and space. However, we cannot take derivatives of a process w.r.t. another stochastic process. Thus we cannot use this method straight away.

The solution is found in the following shift operator.

**Definition 15.**

$$T_\alpha u(x) = u(x - \alpha) \quad (2.14)$$

As before, we introduce an equivalent version to the method used in sections 1.3 and 1.4. In section 1.3, we linearise the solution around the wave front and in section 1.4 we show the behaviour of the system under some unknown shift to show the stability. In both cases, we want to know what the behaviour of  $v$ , the difference of the wave and the solution, is. In these sections, we linearised around the wave front, respectively perturbed the wave equation in the shifted frame to find stability. Again in the stochastic case, the difference between the shifted solution and the wave front is an important object. With use of the shift (2.12), the decomposition (2.9) and the shift operator (2.14), we define the difference as

**Definition 16.**

$$V(t) = T_{-\Gamma} [X(t) + \Phi_{ref}] - \Phi_\sigma. \quad (2.15)$$

## 2.5 The stochastic travelling wave equation

We want to find a differential equation that  $V$  solves, in order to find appropriate choices for  $a$  and  $b$ . This choice of  $a$  and  $b$  will make sure that if  $U$  goes to  $\Phi_\sigma$ , then it would always have the same phase. To find a differential equation, we need the notion of a weak solution.

**Definition 17.** A weak solution to a differential equation of the form  $dV = [aV'' + bV' + cV]dt$  for some test function  $\zeta \in C^\infty$  is given by

$$\langle V(t), \zeta \rangle_{L^2} = \int_0^t \langle aV, \zeta'' \rangle_{L^2} - \langle bV, \zeta' \rangle_{L^2} + \langle cV, \zeta \rangle_{L^2} dt \quad (2.16)$$

This definition means that the solution to the equation that one finds is not necessarily well defined in the appropriate space. In particular, the derivative of  $V$  does not have to exist to be a solution to the differential equation. Although, the definition is weaker, it does make sense by



equation (2.46), which explain why we can change the place of the derivative, whenever the derivative exists for  $L^2$  functions.

Using Theorem 1 as stated by [5], we examine the wave front of the SRDE. This theorem is a generalisation of the Itô formula in equation (2.4) for PDEs that contain functionals. We want to examine

$$\langle V(t), \zeta \rangle_{L^2} \quad (2.17)$$

to find the weak solution for the stochastic wave. To provide more insight in the computations, we split the functional (2.17) into two functionals. These two functionals are given by  $\phi_1$  and  $\phi_2$ .  $\phi_1$  does depend on  $X$  and  $\Gamma$  whereas  $\phi_2$  does solely depend on the stochastic shift.  $\phi_1, \phi_2$  are chosen such that they satisfy the following equation.

$$\langle V(t), \zeta \rangle_{L^2} = \phi_1(X(t), \Gamma(t)) + \phi_2(\Gamma(t)) \quad (2.18)$$

The derivative in this context is not the ordinary derivative, but is one in a functional space. It is given by the F chet derivative and is defined as

**Definition 18.** *Let  $F : X \times Y \rightarrow Z$  be a functional.  $F$  is Fr chet differentiable at a point  $(u_0, v_0)$ , if there exists a bounded linear operator  $DF(u_0) \in L(X, Z)$  such that:*

$$\lim_{h \rightarrow 0} \frac{\|F(u_0 + h, v_0) - F(u_0, v_0) - DF(u_0)h\|_Z}{\|h\|_X} = 0 \quad (2.19)$$

where  $h \rightarrow 0$  is any sequence of  $h$  such that  $\|h\|_X \rightarrow 0$ .

Now we give the definition of the first of the two functionals that construct (2.17) and then we compute the first and second order Fr chet derivative to be able to use the theorem of [5].

### 2.5.1 Derivatives of $\phi_1$

**Definition 19.**

$$\phi_1(X, \Gamma) = \langle X, T_\Gamma \zeta \rangle_{L^2} \quad (2.20)$$

For this functional  $\phi_1 : \mathcal{H}^1 \times \mathbb{R} \rightarrow \mathbb{R}$ , we compute the Fr chet derivative. To compute the Fr chet derivative of  $\phi_1$  we will expand  $\phi_1(X + v, \Gamma + \beta)$  around  $(X, \Gamma)$  to find

$$\phi_1(X + v, \Gamma + \beta) - \phi_1(X, \Gamma) = \langle v, T_\Gamma \zeta \rangle_{L^2} - \beta \langle X, T_\Gamma \zeta' \rangle_{L^2} + \mathcal{O}(\beta^2) \quad (2.21)$$

To see this, note that

$$T_{\Gamma+\beta} \zeta(x) = \zeta(x - \Gamma - \beta) = \zeta(x - \Gamma) - \beta \zeta'(x - \Gamma) + \mathcal{O}(\beta^2) = T_\Gamma \zeta(x) - \beta T_\Gamma \zeta'(x) + \mathcal{O}(\beta^2). \quad (2.22)$$

Hence we find for the Taylor expansion on  $\phi_1$

$$\begin{aligned} \langle X + v, T_{\Gamma+\beta} \zeta \rangle_{L^2} &= \langle X, T_\Gamma \zeta \rangle_{L^2} + \langle v, T_\Gamma \zeta \rangle_{L^2} + \langle X, -\beta T_\Gamma \zeta' \rangle_{L^2} + \mathcal{O}(\beta^2) \\ &= \langle X, T_\Gamma \zeta \rangle_{L^2} + \langle v, T_\Gamma \zeta \rangle_{L^2} - \beta \langle X, T_\Gamma \zeta' \rangle_{L^2} + \mathcal{O}(\beta^2). \end{aligned} \quad (2.23)$$

Now by taking the difference of  $\phi_1(X + v, \Gamma + \beta)$  and  $\phi_1(X, \Gamma)$  we find

$$\phi_1(X + v, \Gamma + \beta) - \phi_1(X, \Gamma) = \langle v, T_\Gamma \zeta \rangle_{L^2} - \beta \langle X, T_\Gamma \zeta' \rangle_{L^2} + \mathcal{O}(\beta^2). \quad (2.24)$$

With the definition of the Fr chet derivative we see that the remainder term of order  $\beta^2$  vanishes as we take limits with in the denominator of an order  $\beta$  term. Thus the Fr chet derivative is given by

$$\partial \phi_1(X, \Gamma)[v, \beta] = \langle v, T_\Gamma \zeta \rangle_{L^2} - \beta \langle X, T_\Gamma \zeta' \rangle_{L^2}. \quad (2.25)$$

For the second order Fréchet derivative, we will do a similar computation on  $\partial\phi_1(X, \Gamma)[v, \beta]$  to get

$$\begin{aligned} \partial\phi_1(X + v, \Gamma + \beta)[v, \beta] &= \langle v, T_{\Gamma+\beta}\zeta \rangle_{L^2} - \beta \langle X + v, T_{\Gamma+\beta}\zeta' \rangle_{L^2} \\ &= \langle v, T_{\Gamma}\zeta \rangle_{L^2} - \beta \langle v, T_{\Gamma}\zeta' \rangle_{L^2} - \beta \langle X, T_{\Gamma}\zeta' \rangle_{L^2} - \beta \langle v, T_{\Gamma}\zeta' \rangle_{L^2} + (-\beta)^2 \langle X, T_{\Gamma}\zeta'' \rangle_{L^2} + \mathcal{O}(\beta^3). \end{aligned} \quad (2.26)$$

Again by subtracting  $\partial\phi_1(X, \Gamma)$ , we find

$$\partial\phi_1(X + v, \Gamma + \beta)[v, \beta] - \partial\phi_1(X, \Gamma)[v, \beta] = -2\beta \langle X, T_{\Gamma}\zeta \rangle_{L^2} + \beta^2 \langle X, T_{\Gamma}\zeta'' \rangle_{L^2} + \mathcal{O}(\beta^3) \quad (2.27)$$

And so we find that the second order Fréchet derivative of  $\phi_1$  is given by

$$\partial^2\phi_1(X, \Gamma)[v, \beta][v, \beta] = -2\beta \langle X, T_{\Gamma}\zeta \rangle_{L^2} + \beta^2 \langle X, T_{\Gamma}\zeta'' \rangle_{L^2}. \quad (2.28)$$

Now theorem 1 by [5] gives the expression for the Itô formula, that is

$$\begin{aligned} \phi_1(X(t), \Gamma(t)) &= \phi_1(X(0), \Gamma(0)) \\ &+ \int_0^t \langle \partial_{xx}(X + \Phi_{ref}) + f(X(s) + \Phi_{ref}), T_{\Gamma(s)}\zeta \rangle_{L^2} ds \\ &- \int_0^t [c_\sigma + a(X(s) + \Phi_{ref})] \langle X(s) + \Phi_{ref}, T_{\Gamma(s)}\zeta' \rangle_{L^2} ds \\ &+ \sigma \int_0^t \langle g(X(s) + \Phi_{ref}), T_{\Gamma(s)}\zeta \rangle_{L^2} d\beta_s \\ &- \sigma \int_0^t b(X(s) + \Phi_{ref}) \langle X(s) + \Phi_{ref}, T_{\Gamma(s)}\zeta' \rangle_{L^2} d\beta_s \\ &- \sigma^2 \int_0^t b(X(s) + \Phi_{ref}) \langle g(X(s) + \Phi_{ref}), T_{\Gamma(s)}\zeta' \rangle_{L^2} ds \\ &+ \frac{\sigma^2}{2} \int_0^t b(X(s) + \Phi_{ref})^2 \langle X(s) + \Phi_{ref}, T_{\Gamma(s)}\zeta'' \rangle_{L^2} ds. \end{aligned} \quad (2.29)$$

### 2.5.2 Derivatives of $\phi_2$

The second functional of the decomposition of  $\langle V, \zeta \rangle$  in equation (2.18) is given by

**Definition 20.**

$$\phi_2(\Gamma) = \langle T_{-\Gamma(t)}\Phi_{ref} - \Phi_\sigma, \zeta \rangle_{L^2}. \quad (2.30)$$

In an analogous way as we found the derivatives of  $\phi_1$ , we compute the Fréchet derivatives of  $\phi_2$ . For the expansion on  $\phi_2(\Gamma + \beta)$  around  $\Gamma$  we have

$$\begin{aligned} \phi_2(\Gamma + \beta) &= \langle T_{-\Gamma-\beta}\Phi_{ref} - \Phi_\sigma, \zeta \rangle_{L^2} \\ &= \langle T_{-\Gamma-\beta}\Phi_{ref}, \zeta \rangle_{L^2} - \langle \Phi_\sigma, \zeta \rangle_{L^2} \\ &= \langle \Phi_{ref}, T_{\Gamma+\beta}\zeta \rangle_{L^2} - \langle T_{-\Gamma}\Phi_\sigma, \zeta \rangle_{L^2} \\ &= \langle \Phi_{ref}, T_{\Gamma}\zeta \rangle_{L^2} - \beta \langle \Phi_{ref}, T_{\Gamma}\zeta' \rangle_{L^2} + \mathcal{O}(\beta^2) - \langle T_{-\Gamma}\Phi_\sigma, \zeta \rangle_{L^2}. \end{aligned} \quad (2.31)$$

For equation (2.31), the second equality holds by the linearity of the inner product, the third by the shift invariance of the  $L^2$  norm and the last one is by the same Taylor expansion as in the computation of the derivative of  $\phi_1$ . By subtracting  $\phi_2(\Gamma)$  from equation (2.31), we find

$$\phi_2(\Gamma + \beta) - \phi_2(\Gamma) = -\beta \langle \Phi_{ref}, T_{\Gamma}\zeta' \rangle_{L^2} + \mathcal{O}(\beta^2). \quad (2.32)$$

Hence we find that the second order derivative of  $\Phi_2$  is given by

$$\partial\phi_2(\Gamma)[\beta] = -\beta \langle \Phi_{ref}, T_{\Gamma}\zeta' \rangle_{L^2}. \quad (2.33)$$

For the second order derivative we Taylor expand the first order partial derivative in the same manner.

$$\phi_2(\Gamma + \beta)[\beta] = -\beta \langle \Phi_{ref}, T_{\Gamma+\beta} \zeta' \rangle_{L^2} = -\beta \langle \Phi_{ref}, T_{\Gamma} \zeta' + (-\beta)^2 \langle \Phi_{ref}, T_{\Gamma} \zeta'' \rangle_{L^2} + \mathcal{O}(\beta^3). \quad (2.34)$$

Again by taking differences, we find

$$\partial^2 \phi_2(\Gamma + \beta)[\beta] - \partial^2 \phi_2(\Gamma)[\beta] = \beta^2 \langle \Phi_{ref}, T_{\Gamma} \zeta'' \rangle_{L^2} + \mathcal{O}(\beta^3). \quad (2.35)$$

So we find for the second order derivative of  $\Phi_2$ , that it is equal to

$$\partial^2 \phi_2(\Gamma)[\beta][\beta] = \beta^2 \langle \Phi_{ref}, T_{\Gamma} \zeta'' \rangle_{L^2}. \quad (2.36)$$

With theorem 1 by [5], we find an expression for  $\phi_2(\Gamma(t))$  that is given by

$$\begin{aligned} \phi_2(\Gamma(t)) &= \phi_2(\Gamma(0)) \\ &\quad - \int_0^t [c_{\sigma} + a(X(s) + \Phi_{ref})] \langle \Phi_{ref}, T_{\Gamma(s)} \zeta' \rangle_{L^2} ds \\ &\quad - \sigma \int_0^t b(X(s) + \Phi_{ref}) \langle \Phi_{ref}, T_{\Gamma(s)} \zeta' \rangle_{L^2} d\beta_s \\ &\quad + \frac{\sigma^2}{2} \int_0^t b(X(s) + \Phi_{ref})^2 \langle \Phi_{ref}, T_{\Gamma(s)} \zeta'' \rangle_{L^2} ds. \end{aligned} \quad (2.37)$$

### 2.5.3 The travelling wave equation

The travelling wave equation can be found with equality (2.18), which holds because

$$\begin{aligned} \langle V(t), \zeta \rangle_{L^2} &= \langle T_{-\Gamma} [X(t) + \Phi_{ref}] - \Phi_{\sigma}, \zeta \rangle_{L^2} \\ &= \langle T_{-\Gamma} X + T_{-\Gamma} \Phi_{ref} - \Phi_{\sigma}, \zeta \rangle_{L^2} \\ &= \langle T_{-\Gamma} X, \zeta \rangle_{L^2} + \langle T_{-\Gamma} \Phi_{ref} - \Phi_{\sigma}, \zeta \rangle_{L^2} \\ &= \langle X, T_{\Gamma} \zeta \rangle_{L^2} + \langle T_{-\Gamma} \Phi_{ref} - \Phi_{\sigma}, \zeta \rangle_{L^2} \\ &= \phi_1(X, \Gamma) + \phi_2(\Gamma). \end{aligned}$$

We can find a formula for equation (2.17) in terms of the SRDE. First we show some computational rules that give rise to the upcoming equation (2.43).

$$\langle X, T_{\Gamma} Y \rangle_{L^2} = \langle T_{-\Gamma} X, Y \rangle_{L^2} \quad (2.38)$$

$$T_{\Gamma} \partial_{xx}(X) = \partial_{xx}(T_{\Gamma} X) \quad (2.39)$$

$$T_{\Gamma} f(X) = f(T_{\Gamma} X) \quad (2.40)$$

$$T_{\Gamma} g(X) = g(T_{\Gamma} X) \quad (2.41)$$

and

$$T_{\Gamma}(X + \Phi_{ref}) = V(s) - \Phi_{\sigma}. \quad (2.42)$$

These give the following equation:

$$\begin{aligned}
\langle V(t), \zeta \rangle_{L^2} &= \langle V(0), \zeta \rangle_{L^2} \\
&+ \int_0^t \langle \partial_{xx}(V + \Phi_\sigma) + f(V(s) + \Phi_\sigma), \zeta \rangle_{L^2} \\
&- \int_0^t [c_\sigma + a(V(s) + \Phi_\sigma, \psi_0)] \langle V(s) + \Phi_\sigma, \zeta' \rangle_{L^2} ds \\
&+ \sigma \int_0^t \langle g(V(s) + \Phi_\sigma), \zeta \rangle_{L^2} d\beta_s \\
&- \sigma \int_0^t b(V(s) + \Phi_\sigma, \psi_0) \langle V(s) + \Phi_\sigma, \zeta' \rangle_{L^2} d\beta_s \\
&- \sigma^2 \int_0^t b(V(s) + \Phi_\sigma, \psi_0) \langle g(V(s) + \Phi_\sigma), \zeta' \rangle_{L^2} ds \\
&+ \frac{\sigma^2}{2} \int_0^t b(V(s) + \Phi_\sigma, \psi_0)^2 \langle V(s) + \Phi_\sigma, \zeta'' \rangle_{L^2} ds.
\end{aligned} \tag{2.43}$$

We simplify the notation by introducing the following term

$$\kappa_\sigma(V) = 1 + \frac{\sigma^2}{2} b(V, \psi_0)^2. \tag{2.44}$$

This gives us

$$\begin{aligned}
\langle V(t), \zeta \rangle_{L^2} &= \langle V(0), \zeta \rangle_{L^2} \\
&+ \int_0^t \kappa_\sigma(V + \Phi_\sigma) \langle \partial_{xx}(V(s) + \Phi_\sigma) + f(V(s) + \Phi_\sigma), \zeta \rangle_{L^2} \\
&- \int_0^t [c_\sigma + a(V(s) + \Phi_\sigma, \psi_0)] \langle V(s) + \Phi_\sigma, \zeta' \rangle_{L^2} ds \\
&+ \sigma \int_0^t \langle g(V(s) + \Phi_\sigma), \zeta \rangle_{L^2} d\beta_s \\
&- \sigma \int_0^t b(V(s) + \Phi_\sigma, \psi_0) \langle V(s) + \Phi_\sigma, \zeta' \rangle_{L^2} d\beta_s \\
&- \sigma^2 \int_0^t b(V(s) + \Phi_\sigma, \psi_0) \langle g(V(s) + \Phi_\sigma), \zeta' \rangle_{L^2} ds.
\end{aligned} \tag{2.45}$$

With the following observation we can write a differential equation that has as a solution the wave equation. To see this note that the inner product used in this context is an  $L^2$  inner product, for which we can use the following rule

$$\langle f, g' \rangle_{L^2} = -\langle f', g \rangle_{L^2}. \tag{2.46}$$

This holds since

$$\langle f, g' \rangle_{L^2} = \int_{\mathbb{R}} f g' ds = [f g']_{-\infty}^{\infty} - \int_{\mathbb{R}} f' g ds = -\langle f', g \rangle_{L^2} \tag{2.47}$$

where  $[f g']_{-\infty}^{\infty} = 0$ , since  $f$  and  $g$  are  $L^2$  functions, so both are zero at their spatial limits. Now we can write (2.45) as

$$\begin{aligned}
\langle V(t), \zeta \rangle_{L^2} &= \langle V(0), \zeta \rangle_{L^2} \\
&+ \int_0^t \kappa_\sigma(V + \Phi_\sigma) \langle \partial_{xx}(V(s) + \Phi_\sigma) + f(V(s) + \Phi_\sigma), \zeta \rangle_{L^2} ds \\
&+ \int_0^t [c_\sigma + a(V(s) + \Phi_\sigma)] \langle \partial_\xi(V(s) + \Phi_\sigma), \zeta \rangle_{L^2} ds \\
&+ \sigma \int_0^t \langle g(V(s) + \Phi_\sigma), \zeta \rangle_{L^2} d\beta_s \\
&+ \sigma \int_0^t b(V(s) + \Phi_\sigma, \psi_0) \langle \partial_\xi(V(s) + \Phi_\sigma), \zeta \rangle_{L^2} d\beta_s \\
&+ \sigma^2 \int_0^t b(V(s) + \Phi_\sigma) \langle \partial_\xi g(V(s) + \Phi_\sigma), \zeta \rangle_{L^2} ds.
\end{aligned} \tag{2.48}$$

It follows that we have a SPDE, from the definition of a weak solution.

$$\begin{aligned}
dV &= [\kappa_\sigma(V + \Phi_\sigma) \partial_{xx}(V + \Phi_\sigma) + f(V + \Phi_\sigma) + (c_\sigma + a(V + \Phi_\sigma)) \partial_\xi(V + \Phi_\sigma) \\
&+ \sigma^2 b(V + \Phi_\sigma) \partial_\xi g(V + \Phi_\sigma)] dt + [g(V + \Phi_\sigma) + b(V + \Phi_\sigma) \partial_\xi(V + \Phi_\sigma)] d\beta_t.
\end{aligned} \tag{2.49}$$

## 2.6 Fixing the free parameters

Now we will use semigroups to find the specific choices for the free parameters to show that we can choose the shifted frame in such a way that the non-decaying part of the equation vanishes. We use the same projection operator  $P$  from definition 8 and then we split equation (2.49) in the same way as equation (1.25) to get

$$\begin{aligned}
V(t) &= S(t)[I - P]V(0) + \int_0^t S(t-s)[I - P][\kappa_\sigma(V + \Phi_\sigma) \partial_{xx}(V(s) + \Phi_\sigma) + f(V(s) + \Phi_\sigma) \\
&+ (c_\sigma + a(V(s) + \Phi_\sigma)) \partial_\xi(V(s) + \Phi_\sigma) + \sigma^2 b(V(s) + \Phi_\sigma) \partial_\xi g(V(s) + \Phi_\sigma) - \mathcal{L}v] ds \\
&+ \int_0^t S(t-s)[I - P][g(V + \Phi_\sigma) + b(V + \Phi_\sigma) \partial_\xi(V + \Phi_\sigma)] d\beta_s + S(t)PV(0) \\
&+ \int_0^t S(t-s)P[\kappa_\sigma(V + \Phi_\sigma) \partial_{xx}(V(s) + \Phi_\sigma) + f(V(s) + \Phi_\sigma) + (c_\sigma + a(V(s) + \Phi_\sigma)) \partial_\xi(V(s) + \Phi_\sigma) \\
&+ \sigma^2 b(V(s) + \Phi_\sigma) \partial_\xi g(V(s) + \Phi_\sigma) - \mathcal{L}v] ds + \int_0^t S(t-s)P[g(V + \Phi_\sigma) + b(V + \Phi_\sigma) \partial_\xi(V + \Phi_\sigma)] d\beta_s.
\end{aligned} \tag{2.50}$$

In equation (2.50) the non-decaying part and the decaying part of equation 2.5 have been split. We will now choose the parameters in such a way that the non-decaying part, that corresponds to the zero eigenvalue, vanishes. First, we make sure that the fourth integral in (2.50) vanishes, by choosing  $P[g(V + \Phi_\sigma) + b(V + \Phi_\sigma) \partial_\xi(V + \Phi_\sigma)]$  equal to zero. With the definition of  $Pv = \langle v, \psi_0 \rangle \Phi_0$ , we then find

$$\begin{aligned}
0 &= P[g(V + \Phi_\sigma) + b(V + \Phi_\sigma) \partial_\xi(V + \Phi_\sigma)] \\
&= \langle g(V + \Phi_\sigma) + b(V + \Phi_\sigma) \partial_\xi(V + \Phi_\sigma), \psi_0 \rangle \Phi_0 \\
&= \langle g(V + \Phi_\sigma), \psi_0 \rangle \Phi_0 + \langle b(V + \Phi_\sigma) \partial_\xi(V + \Phi_\sigma), \psi_0 \rangle \Phi_0.
\end{aligned} \tag{2.51}$$

This can only be zero for the choice of  $b$  that is given by

$$b(V + \Phi_\sigma) = - \frac{\langle g(V + \Phi_\sigma), \psi_0 \rangle}{\langle \partial_\xi(V + \Phi_\sigma), \psi_0 \rangle}. \tag{2.52}$$

In fact this is the same case as in the deterministic one. We force the stochastic behaviour in the direction of  $\Phi'_\sigma$  to disappear. We will not prove this in this thesis, but the proof can be found in [7].

Now we determine the functional, we want to use for  $a$ .  $a$  has to make sure that all other behaviour of the non-decaying part vanishes. For the stochastic non-decaying part, we have chosen  $b$  such that it vanishes. What we are left with on non-decaying terms is

$$PV(0) + \int_0^t P[\kappa_\sigma(V + \Phi_\sigma)\partial_{xx}(V(s) + \Phi_\sigma) + f(V(s) + \Phi_\sigma) + (c_\sigma + a(V(s) + \Phi_\sigma))\partial_\xi(V(s) + \Phi_\sigma) + \sigma^2b(V(s) + \Phi_\sigma)\partial_\xi g(V(s) + \Phi_\sigma)]ds. \quad (2.53)$$

Using the definition of  $P$ , we then find an expression for  $a$  such that the integral in (2.53) vanishes. It is given by equation (2.54)

$$a(V + \Phi_\sigma) = -\frac{\langle F(V), \psi_0 \rangle_{L^2}}{\langle \partial_\xi(V + \Phi_\sigma), \psi_0 \rangle_{L^2}} \quad (2.54)$$

where  $F(V) = \kappa_\sigma(V + \Phi_\sigma)\partial_{xx}(V(s) + \Phi_\sigma) + f(V(s) + \Phi_\sigma) + c_\sigma\partial_\xi(V(s) + \Phi_\sigma) + \sigma^2b(V(s) + \Phi_\sigma)\partial_\xi g(V(s) + \Phi_\sigma)$

To make sure that  $PV(0) = 0$ , we have to choose  $\Gamma(0)$  in such a way that

$$PV(0) = \langle V(0), \psi_0 \rangle \Phi_0 = \langle T_{\Gamma(0)}U(0), \psi_0 \rangle \Phi_0 = 0$$

This value exists by proposition 2.3 of [7].

For these choices for  $a$ ,  $b$  and  $\Gamma(0)$  the only non-decaying deterministic part is given by (2.55).

$$\int_0^t S(t-s)P[\kappa_\sigma(V(s) + \Phi_\sigma)\partial_{xx}(V(s) + \Phi_\sigma) + f(V(s) + \Phi_\sigma) + (c_\sigma + a(V(s) + \Phi_\sigma))\partial_\xi(V(s) + \Phi_\sigma) + \sigma^2b(V(s) + \Phi_\sigma)\partial_\xi g(V(s) + \Phi_\sigma)]ds \quad (2.55)$$

For the travelling wave  $\Phi_\sigma$ , we see that because it is a stationary solution to the deterministic non-decaying part and orthogonal to  $\psi_0$ , the ODE that the pair  $(\Phi_\sigma, c_\sigma)$  solves is given by

$$0 = \kappa_\sigma(\Phi_\sigma)\Phi_\sigma'' + f(\Phi_\sigma) + (c_\sigma + a(\Phi_\sigma))\Phi_\sigma' + \sigma^2b(\Phi_\sigma)\Phi_\sigma'. \quad (2.56)$$

Now we note that  $a(\Phi_\sigma) = 0$  for the specific choices of  $a$  and  $b$ . This is the same as what we wanted in the deterministic case. By applying this result we find an ODE

$$0 = \kappa_\sigma(\Phi_\sigma)\Phi_\sigma'' + c_\sigma\Phi_\sigma' + f(\Phi_\sigma) + \sigma^2b(\Phi_\sigma, \psi_0)\partial_\xi g(\Phi_\sigma). \quad (2.57)$$

We choose  $(\Phi_\sigma, c_\sigma)$  in such a way that they solve this equation. The existence is guaranteed by proposition 2.2 for  $\sigma = 0$  in [7]. However, we still do not know whether the wave is stable.

## 2.7 The Stability of SRDEs

As seen in section 2.6, we have used the semigroups in the same manner as we did in the deterministic case. Unfortunately, we cannot show the exact same results as in the deterministic case for the stability of the wave. However we do have two results from [7], that involve in some sense the stability. The two theorems give criteria for which SRDEs stay bounded for a finite time. To understand these two theorems we need the notion of a special chosen norm and the escape probability. The norm is defined by

$$N_{\epsilon, \alpha, u_0} = e^{\alpha t} \|V_{u_0} t\|_{L^2}^2 + \int_0^t e^{\epsilon(s-t)} e^{\alpha s} \|V_{u_0} s\|_{\mathcal{H}^1}^2 ds \quad (2.58)$$

and the notion of the probability that this norm of  $V$  exceeds some value  $\eta$  in the time interval  $[0, T]$ , that is

$$p_{\epsilon, \alpha}(T, \eta, u_0) = \mathbb{P} \left( \sup_{0 \leq t \leq T} N_{\epsilon, \alpha, u_0} > \eta \right). \quad (2.59)$$

Theorem 2.4 and Theorem 2.5 of [7] states now that under some regularity conditions, which can be found in [7], inequality (2.60) holds.

$$P_{\epsilon, 0}(T, \eta, u_0) \leq \eta^{-1} K [ \|u_0 - \Phi_\sigma\|_{\mathcal{H}^1}^2 + \sigma^2 T ] \quad (2.60)$$

If in additions holds that  $g(\Phi_0) = \lambda \Phi_0'$  for some  $\lambda \in \mathbb{R}$ , then the inequality is even sharper. In this case, we have that there exists an  $\epsilon > 0$  and constant  $K > 0$  such that the following inequality holds.

$$P_{\epsilon, \alpha}(T, \eta, u_0) \leq \eta^{-1} K \|u_0 - \Phi_\sigma\|_{\mathcal{H}^1}^2 \quad (2.61)$$

The first inequality show that the norm of  $V$  has a small probability to grow large on a long time interval, provided that the initial condition is close to  $\Phi_\sigma$ . The second theorem states that the solution has a large probability to decay exponentially fast when the equation satisfies  $g(\Phi_0) = \lambda \Phi_0'$ . To show that these results hold and to show how sharp these estimates are, we will test these results with the numeric results in section 3.

## 2.8 The stochastic speed

Now that we have chosen our free parameters, we know what needs to be done to fix the travelling wave to stay centred. We do know from simulations that  $c_\sigma$  is the leading force and that  $c_\sigma < c_0$  for the Nagumo equations with  $g(u) = u(1 - u)$ . Hence, we cannot expect  $c_\sigma = c_0$  for general SRDEs within this setting. In the deterministic case  $c_0$  is this leading force, that is over time the only component left. So the speed of the deterministic wave tends to  $c_0$ . In the stochastic case, there is no constant speed at any later moment due to the noise. We cannot formally speak about a *speed* in the stochastic setting, because the derivative of  $\Gamma$  is not well-defined. We use an informal definition, of which we have not proven that it exists in the first place, since it is a priori not known, whether  $\mathbb{E}[\Gamma]$  is differentiable. The definition of the stochastic average speed is given by

**Definition 21.** The stochastic average ‘speed’ is given by the following equation

$$\overline{\Gamma}'_T = \frac{1}{T} \mathbb{E} [\Gamma(T)]. \quad (2.62)$$

Even though the expectation is not necessarily, well-defined, simulations suggest that the expectation converges. For  $\Gamma$  at some finite time  $T$ , we can write the following:

$$\begin{aligned} \mathbb{E} [\Gamma(T)] &= \mathbb{E} [c_\sigma T] + \mathbb{E} \left[ \int_0^T a(s) ds \right] + \mathbb{E} \left[ \int_0^T b(s) d\beta_s \right] \\ &= c_\sigma T + \mathbb{E} \left[ \int_0^T a(s) ds \right] + \mathbb{E} \left[ \int_0^T b(s) d\beta_s \right]. \end{aligned} \quad (2.63)$$

Hence the stochastic speed is given by

$$\overline{\Gamma}'_T = c_\sigma + \frac{1}{T} \mathbb{E} \left[ \int_0^T a(s) ds \right] + \frac{1}{T} \mathbb{E} \left[ \int_0^T b(s) d\beta_s \right]. \quad (2.64)$$

Our main question on the speed is what its long term behaviour is. In the deterministic case, we know that the speed converges to  $c_0$ , but for the stochastic case, this is not clear. Equation (2.64) consists of three terms. Up to now we do not know what the behaviour of the last two terms is and whether they are on average zero. If that is the case, then  $U \rightarrow \Phi_0$  with the same speed. In section 3, we will show what the contribution of  $a$  and  $b$  is by computing numerically what  $\overline{\Gamma}'_T - c_\sigma$  is for different values of  $\sigma$  and a large value of  $T$ .

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## 2.9 The stochastic Nagumo equation

In this section, we consider a very specific version of SRDEs. It is an explicit extension of section 1.7, where we chose  $f(u) = u(1-u)(u-\alpha)$ . Now we consider the stochastic equation with  $g(u) = u(1-u)$ .

First we make a remark on equation (2.57) and then we use this observation to make an Ansatz to solve the equation.

Since a travelling wave solution  $\Phi_\sigma$  moves with constant speed  $c_\sigma$ , the travelling wave can only solve equation (2.49), when it is not driven by the stochastic force. Hence, the stochastic part of the SPDE for  $V \equiv 0$ , has to be zero, i.e.

$$g(\Phi_\sigma) + b(\Phi_\sigma, \psi_0) \partial_\xi \Phi_\sigma = 0. \quad (2.65)$$

With the use of (2.52), we then find

$$g(\Phi_\sigma) - \frac{\langle g(\Phi_\sigma), \psi_0 \rangle_{L^2}}{\langle \Phi'_\sigma, \psi_0 \rangle_{L^2}} \Phi'_\sigma = 0. \quad (2.66)$$

From this, we can see  $g(\Phi_\sigma)$  has to be a scalar multiple of  $\Phi'_\sigma$ .

### 2.9.1 Ansatz to find the solution

By the previous observation on the relation between  $g(\Phi_\sigma)$  and  $\Phi'_\sigma$ , we make the Ansatz that  $g(\Phi_\sigma) = r\Phi'_\sigma$  for some parameter  $r \in \mathbb{R}$ , which is yet to be determined.

This Ansatz gives new simple expressions that are present in equation (2.57). For  $b(\Phi_\sigma, \psi_0)$  equation (2.65) gives the following equalities

$$r\Phi'_\sigma + b(\Phi_\sigma, \psi_0)\Phi'_\sigma = 0 \implies b(\Phi_\sigma, \psi_0) = -r, \quad (2.67)$$

$$\kappa_\sigma(\Phi_\sigma) = 1 + \frac{\sigma^2}{2} b(\Phi_\sigma, \psi_0)^2 = 1 + \frac{\sigma^2 r^2}{2} \quad (2.68)$$

and

$$\partial_\xi g(\Phi_\sigma) = -r\Phi''_\sigma. \quad (2.69)$$

Putting these into equation (2.57) gives the following differential equation, that is very similar to (1.9).

$$0 = \left(1 + \frac{\sigma^2 r^2}{2}\right) \Phi''_\sigma + c_\sigma \Phi'_\sigma + f(\Phi_\sigma) - \sigma^2 r^2 \Phi''_\sigma = \left(1 - \frac{\sigma^2 r^2}{2}\right) \Phi''_\sigma + c_\sigma \Phi'_\sigma + f(\Phi_\sigma) \quad (2.70)$$

This equation can be solved explicitly with the same method, introduced in section 1.7, where we used the same Ansatz. This Ansatz is given by  $u' = ku(1-u)$ . This gives a new system of equations that is obtained in the same manner as system (1.33) - (1.35). This system is given by

$$2k^2 \left(1 + \frac{\sigma^2 r^2}{2}\right) - 1 = 0 \quad (2.71)$$

$$-3k^2 \left(1 + \frac{\sigma^2 r^2}{2}\right) - ck + \alpha + 1 = 0 \quad (2.72)$$

$$k^2 \left(1 + \frac{\sigma^2 r^2}{2}\right) + ck - \alpha = 0. \quad (2.73)$$

Now note that by the Ansatz, we have:

$$\Phi'_\sigma = k(\Phi_\sigma(1-\Phi_\sigma)) = kg(\Phi_\sigma) = kr\Phi'_\sigma. \quad (2.74)$$



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So this implies that  $k = 1/r$  and putting this relation into equation (2.71), we find:

$$r = \pm \sqrt{\frac{2}{1 + \sigma^2}}. \quad (2.75)$$

We use the minus sign to obey the initial conditions, that is  $\Phi_\sigma(-\infty) = 1$  and  $\Phi_\sigma(\infty) = 0$ . By plugging this observation back into the original expressions from (2.12), we find that

$$d\Gamma = c_\sigma - \sigma \sqrt{\frac{2}{1 + \sigma^2}} d\beta_t. \quad (2.76)$$

This implies that

$$\Gamma(t) = \Gamma(0) + c_\sigma t - \sigma \sqrt{\frac{2}{1 + \sigma^2}} \beta_t. \quad (2.77)$$

It follows that the solution to the stochastic Nagumo equation with  $g(u) = u(1 - u)$  initial condition  $U(0) = \Phi_\sigma$ , with  $\Phi_\sigma(-\infty) = 1$  and  $\Phi_\sigma(\infty) = 0$  is given by

$$\begin{aligned} \Phi_\sigma(x - \Gamma(t)) &= 1 - \frac{1}{1 + e^{-\sqrt{\frac{1}{2\rho}}(x - \Gamma(t))}}, \quad \rho = \left(1 - \frac{\sigma^2}{1 + \sigma^2}\right) \\ \Gamma(t) &= \Gamma(0) + \sqrt{2\rho} \left(\frac{1}{2} - \alpha\right) t - \sigma \sqrt{\frac{2}{1 + \sigma^2}} \beta_t. \end{aligned} \quad (2.78)$$

## 2.10 Comparing the results with the deterministic case

As for equation (2.78), we see that this equation is equal to the solution for the deterministic case if  $\sigma = 0$ . For larger values of  $\sigma$ , we see that  $\rho$  decays. So apart from the stochastic noise term in  $\Gamma$ , we see that the deterministic speed, that is  $\sqrt{2\rho}(\frac{1}{2} - \alpha)$ , decays. The stochastic average speed is a factor  $\sqrt{1 - \frac{\sigma^2}{1 + \sigma^2}}$  times the deterministic speed. So for a larger noise strength, the wave is slower. For the shape of the wave front, we see that the wave gets steeper, since  $1/2\rho$  becomes larger in the exponent.

### 3 Numerical Analysis

In this section we show numerical results for the subjects on the long term behaviour, that we have not proved yet in section 2. First, we show that the numerical schemes that we use are accurate enough to draw conclusions from the produced results. We show this using sections 1.7 and 2.9. These contain exact results for SRDEs and so we can show how the numerical and exact results differ. When the differences and the accuracy are clear, we show various results of equation where we do not know exact solutions. Thereafter, we will analyse these examples and make some remarks on the open questions these examples induce.

#### 3.1 Validation of the numerical scheme

In this chapter we will use the discretised form of the  $L^2$ -norm that can be found using the Trapezoidal integration rule. It is given by

$$L_N^2[f] = \sqrt{\sum_{i=0}^N \frac{\Delta x}{2} (f_{i+1}^2 + f_i^2)}, \quad (3.1)$$

where  $N$  is the number of partitions,  $\Delta x = \frac{b-a}{N}$  and  $f_i = f(a + i\Delta x)$ . We will use this norm to show that the differences between the numerical results obtained using the program, that can be found in the appendices, and the exact solutions we found in section 2.9 are small. We will start to show the differences under the numerical norm of the numerical solutions to equation (2.57) and the exact results for  $\Phi_\sigma$  and  $c_\sigma$  from section 2.9.

##### 3.1.1 Validation of the travelling wave equation and the speed

To build the framework around the solutions that converge to a wave front, we have to find the stationary wave solution and its speed. For the stochastic Nagumo equation with  $g(u) = u(1 - u)$  and initial condition  $U(0) = \Phi_\sigma$ , we know that the exact solution is given by (2.78). We also know that for any choice of  $f$  and  $g$  the pair  $(\Phi_\sigma, c_\sigma)$  is given by the ODE (2.49).  $(\Phi_\sigma, c_\sigma)$  can be found, using the non-linear solver from *Matlab* in Appendix A.5. The use of this function requires an initial guess, which we take as the wave solution to the deterministic Nagumo equation  $\Phi_0$  and the deterministic speed  $c_0$ .  $\Phi_\sigma$  is  $\mathcal{O}(\sigma^2)$  in the  $L^2$  norm close to  $\Phi_0$  according to proposition 2.2 of [7]. Table 1 contains the norms of the differences between the results obtained in section 2.9 and the results obtained with the use of A.5. The graphs of this data can be found in figures 6, 7 and 8.

$\sigma$	$c_\sigma - c_{num}$	$L(\Phi_\sigma - \Phi_{num})$	$L(\Phi'_\sigma - \Phi'_{num})$
0.00	$2.31 \cdot 10^{-12}$	$3.71 \cdot 10^{-6}$	$1.94 \cdot 10^{-6}$
0.05	$-1.26 \cdot 10^{-8}$	$3.69 \cdot 10^{-6}$	$1.97 \cdot 10^{-6}$
0.10	$-4.97 \cdot 10^{-8}$	$3.63 \cdot 10^{-6}$	$2.03 \cdot 10^{-6}$
0.15	$-1.10 \cdot 10^{-7}$	$3.55 \cdot 10^{-6}$	$2.15 \cdot 10^{-6}$
0.20	$-1.89 \cdot 10^{-7}$	$3.44 \cdot 10^{-6}$	$2.31 \cdot 10^{-6}$
0.25	$-2.85 \cdot 10^{-7}$	$3.29 \cdot 10^{-6}$	$2.51 \cdot 10^{-6}$
0.30	$-3.93 \cdot 10^{-7}$	$3.13 \cdot 10^{-6}$	$2.77 \cdot 10^{-6}$

Table 1:  $L^2$ -norm of the Errors of the numerical waves for different values of  $\sigma$

The results show that the errors are of order  $10^{-6}$ . This is good enough for the purpose provided that the behaviour of the simulations also agrees with the theoretical results of section 2.9.

To show that the simulations do agree with the results, we will show that the explicit solution to the stochastic Nagumo equation, that is given in equation (2.78), is also good enough approached by the numerical scheme.

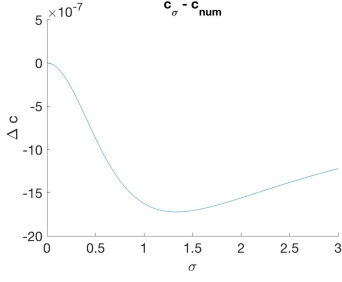


Figure 6:  $c_\sigma - c_{num}$

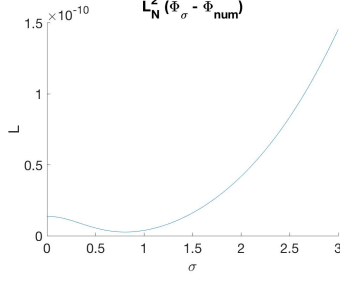


Figure 7:  $L^2(\Phi_\sigma - \Phi_{num})$

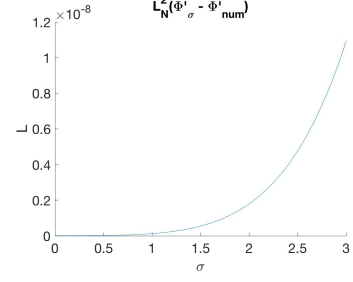


Figure 8:  $L^2(\Phi'_\sigma - \Phi'_{num})$

### 3.1.2 Validation of section 2.9

*Remark on  $\Gamma$ :*

Numerically it is inefficient to look at the entire value of  $\Gamma$ , since there is a deterministic constant part  $c_\sigma$  in it. This value is calculated when the pair  $(\Phi_\sigma, c_\sigma)$  is calculated using equation (2.57). By subtracting  $c_\sigma$  from  $\Gamma$  at every time step, we need less data in the spatial direction. This is also justified by the fact, that all the interesting behaviour occurs around the front of the wave front. Thus for large values of  $x$ ,  $U \approx 0$  and for small values of  $x$ ,  $U \approx 1$ . Also to come back at the discussion on the stochastic ‘speed’ in 2.8, the interesting behaviour of the speed is found in *a* and *b*. These will be discussed in section 3.3.

We want to show that the results from section 2.9 on the Nagumo equation with  $g(u) = u(1 - u)$  also hold numerically. We found that

$$d\Gamma = c_\sigma + \sigma \sqrt{\frac{2}{1 + \sigma^2}} d\beta_t \quad (3.2)$$

and so for  $\Gamma$  holds

$$\Gamma(t) = c_\sigma t + \sigma \sqrt{\frac{2}{1 + \sigma^2}} \beta_t. \quad (3.3)$$

This solution holds only when the initial condition is equal to  $U(0) = \Phi_\sigma$ . Figure 9 contains the graphs of the numerical  $\Gamma_{num}$  (on the left) and the exact  $\Gamma_{exact}$  (on the right), for the same evaluation of the Brownian motion in the case where  $\sigma = 0.3$ . Thereafter the difference between these two is shown in Figure 10.

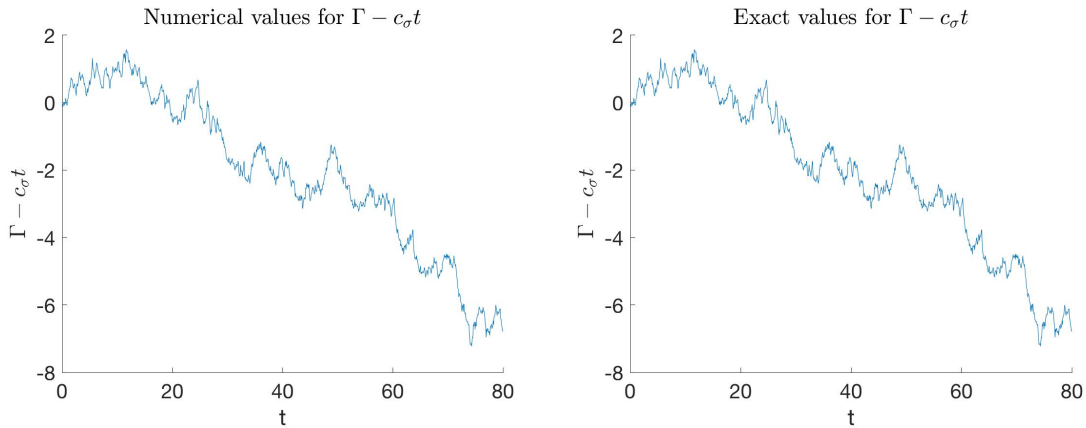


Figure 9: A single evaluation of  $\Gamma(t)$  for the Stochastic Nagumo equation for  $\alpha = 0.25, \sigma = 0.3, g(u) = u(1 - u)$  and initial condition  $U(0) = \Phi_\sigma$

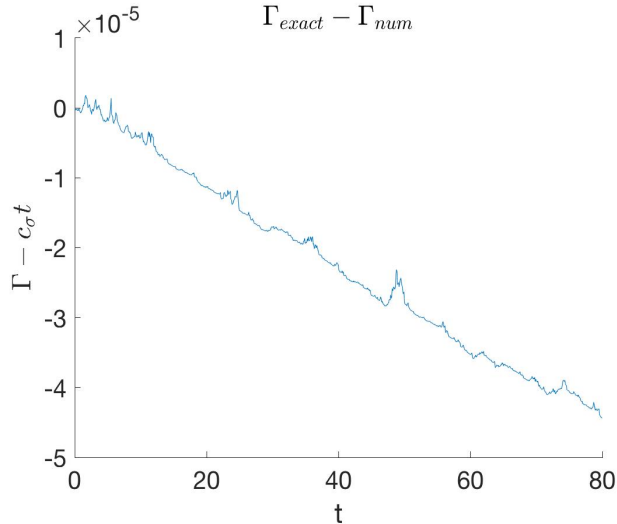


Figure 10: The difference between  $\Gamma_{exact}$  and  $\Gamma_{num}$  for the Stochastic Nagumo equation with  $g(u) = u(1 - u)$  and  $\sigma = 0.3$

To show a quantitative result on the size of the error of the numerical and the exact solution, we take 20 independent evaluations of  $\Gamma - c_\sigma t$  and compute the average speed of it. We find this value by taking the difference between the values at the endpoints of the numerical and exact solutions and divide this by the length of the time interval, that is 80. The results can be found in table 2.

$\sigma$	$\mathbb{E}[\Gamma_{exact} - \Gamma_{num}]/t$
0.00	$3.68 \cdot 10^{-12}$
0.05	$1.61 \cdot 10^{-8}$
0.10	$7.73 \cdot 10^{-8}$
0.15	$2.04 \cdot 10^{-7}$
0.20	$3.56 \cdot 10^{-7}$
0.25	$5.95 \cdot 10^{-7}$
0.30	$1.04 \cdot 10^{-6}$
0.35	$1.52 \cdot 10^{-6}$
0.40	$2.17 \cdot 10^{-6}$
0.45	$2.90 \cdot 10^{-6}$
0.50	$3.70 \cdot 10^{-6}$

Table 2: The expectation of the difference in the slope of  $\Gamma(t) - c_\sigma t$  for the Nagumo equation with  $g(u) = u(1 - u)$ , obtained by taking the average over 20 realisations.

As can be seen in table 2, the values are small, as expected for the stochastic Nagumo equation, since we know the exact solution. We also notice that the values are as good as we can expect from realisations.

To show that the stability that has been proven in section 2.9, also holds in the numerical case, we will perturbate  $U(0)$ . The perturbation we will use in this section is given by definition 22.

**Definition 22.** *The perturbation on the numerical equation is*

$$\pi(x) := A \exp(x - x_0)^2, \quad (3.4)$$

for  $0 < A \ll 1$  and  $x_0 > 0$ .

The initial condition is then  $\Phi_\sigma(x) + \pi(x)$ , which has the shape presented in Figure 11.

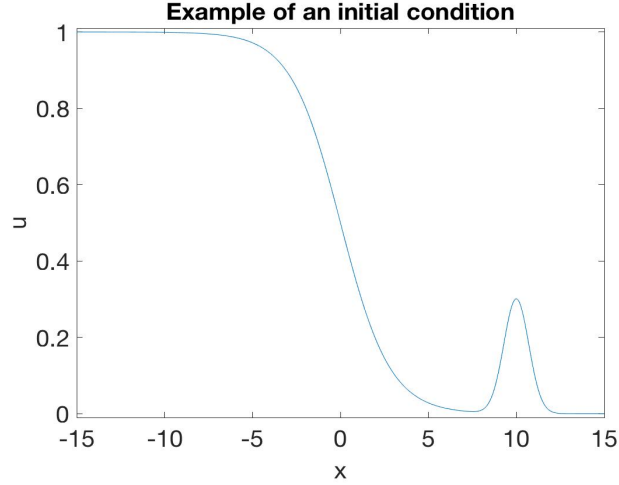


Figure 11: A perturbed initial condition

For the Nagumo equation with  $g(u) = u(1 - u)$ , we have found in section 2.7, that  $V(t)$  converges to zero over time given that the initial condition is given by  $U(0) = \Phi_\sigma + \pi$  as long as the amplitude  $A$  is small enough by theorem 2.4 in [7]. We show that for this special case also holds for the numerical solutions. Thereafter, we will show that the stability still holds for other choices of  $f$  and  $g$ , that do obey the restriction we chose in this thesis.

Realisations of the Nagumo equation with the perturbed wave as initial condition looks like Figure 12.

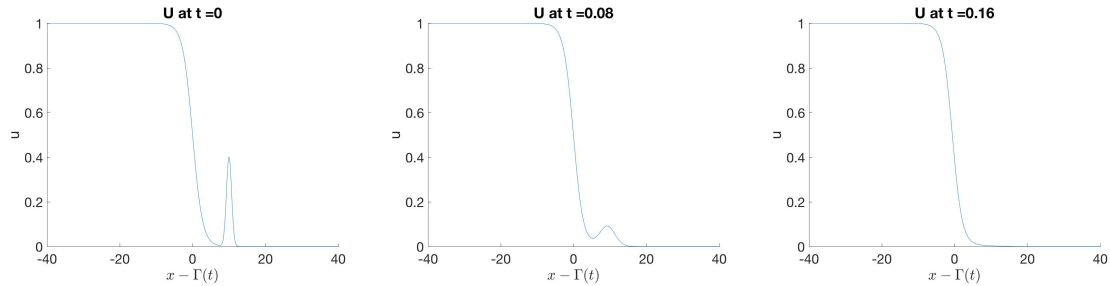


Figure 12: Realisation of  $U$  from the stochastic Nagumo equation with  $\alpha = 0.25$ ,  $\sigma = 0.3$ ,  $g(u) = u(1 - u)$  and initial condition  $\Phi_\sigma + \pi$ , where  $x_0 = 10$  and  $A = 0.4$

As expected, it seems as if the solution of  $U$  seems to converge to the travelling wave solution  $\Phi_\sigma$ . But since we cannot say that it is true by just looking at it, we will show what the difference between  $U$  and  $\Phi_\sigma$  in terms of  $V = T_{-\Gamma}U - \Phi_\sigma$  becomes small. The results can be seen in Figure 13.

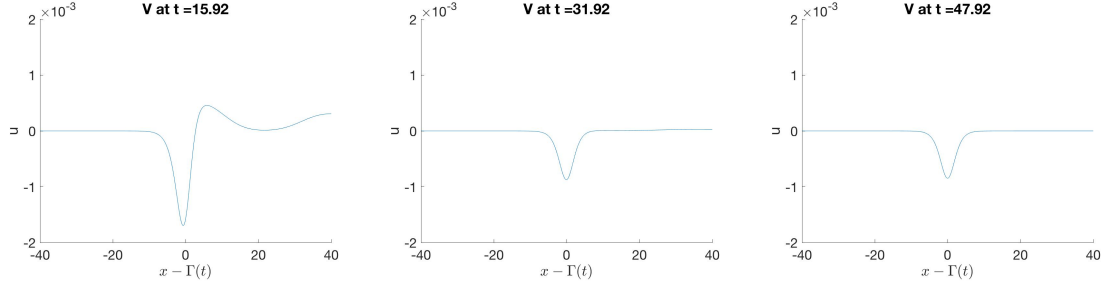


Figure 13: Realisation of  $V$  from the stochastic Nagumo equation with  $\alpha = 0.25, \sigma = 0.3, g(u) = u(1 - u)$  and initial condition  $\Phi_\sigma + \pi$ , where  $x_0 = 10$  and  $A = 0.4$

As seen in Figure 13, the maximal error of  $V$  is after some time of order  $10^{-3}$ . The shape of the bulge in Figure 13 at  $t = 47.92$  is the same as the shape of the term that remains of equation (2.50). The term that remains by the numeric error from the discretisation is found in  $PV(0)$ , because  $V(0) = U(\cdot - \Gamma(0))$ , but we can only choose  $\Gamma$  up to the precision of the discretisation. Hence we conclude that the solution is close enough to  $PV(0)$ , to say that the solution stabilises on  $\Phi_\sigma$  for this specific evaluation. However, to conclude that solutions of the Nagumo equation with  $g(u) = u(1 - u)$  converge numerically for this scheme, we have to show that all evaluations of  $U$  converge to  $\Phi_\sigma$ , which is that  $V$  converges to the zero function.

With 300 evaluations we will investigate what the general behaviour of this equation is, rather than the behaviour of a single evaluation. We do this to show that the exact results agree with the numerical results on the stability of  $U$ . The best way to show this, is to show that the  $L^2$ -norm goes to zero for  $V$  over time.

### 3.1.3 The decay of $V$ over time for the Nagumo equation

To show that the stability is preserved in the numerical case, we will compute for the Nagumo equation with  $g(u) = u(1 - u)$  and  $\sigma = 0.3$ , 300 evaluations that all start with the same initial condition, that is given by  $\Phi_\sigma + \pi$ , where  $\pi$  is the perturbation of definition 22, with  $A = 0.4$  and  $x_0 = 10$ . For Figure 14, we use a spatial discretisation of the interval  $[-40, 40]$  into  $2^{12}$  steps and the number of time steps is ten thousand from  $t = 0$  till  $t = 40$ .

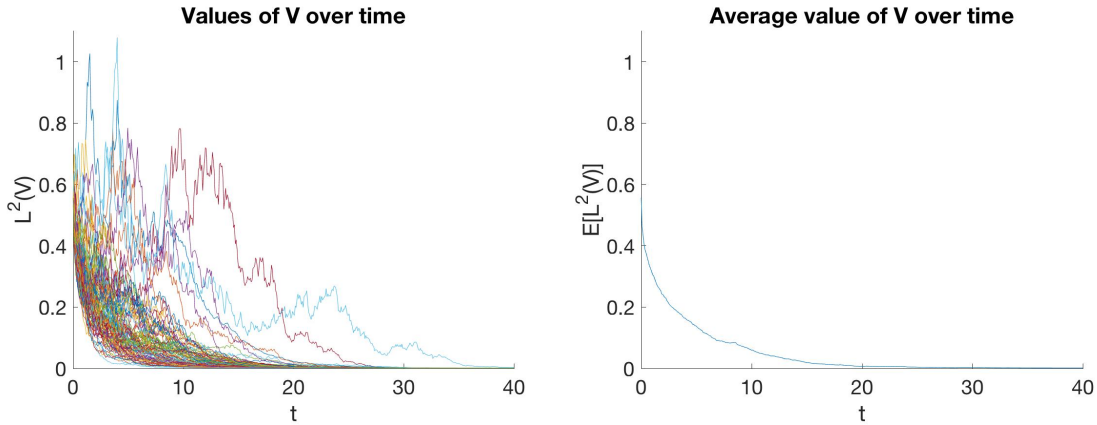


Figure 14: 300 realisation of  $V$  from the stochastic Nagumo equation with  $\alpha = 0.25, g(u) = u(1 - u), \sigma = 0.3$  and initial condition  $\Phi_\sigma + \pi$ , where  $x_0 = 10$  and  $A = 0.4$

We see that the wave front  $\Phi_\sigma$  is indeed stable, since all paths decay exponentially fast in the norm of  $V$  to zero. Also the value of  $L^2_{(2^{12})}[V]$  has at  $t = 40$  a value of order  $10^{-3}$ . Which is small

enough to conclude that  $V$  indeed goes to zero as was predicted by the result in section 2.9.

With the accuracy that we found on the wave front, the speed and the stability of the wave, we can confirm that all the expectations made on the Nagumo equations for  $g(u) = u(1 - u)$  agree with the results from the numerical scheme. Thus we can proceed to the next section to draw conclusions for the case where we do not know the explicit solutions to the SRDE.

### 3.2 Results for other choices of $f$ and $g$

As seen in the previous section, the scheme we use is accurate enough to draw conclusions from the numerical results. Hence we will now investigate two different choices for  $f$  and  $g$  that do not meet the requirements for the strong theorem by [7], but they do satisfy the theorem that was given in equation (2.60).

#### 3.2.1 The Nagumo equation with $g(u) = u^2(1 - u^2)$

The first example is an adapted version of the stochastic Nagumo that we used to find an explicit solution. The reaction term is the same as in the Nagumo equation, while  $g$  is chosen to be  $g(u) = u^2(1 - u^2)$ . For this choice, we do know that  $U$  cannot converge to  $\Phi_\sigma$ , since the method described for finding the wave solution for the stochastic Nagumo equation with  $g(u) = u(1 - u)$ , can only be done under the assumption that  $g(\Phi_\sigma) = \lambda\Phi'_\sigma$  for some  $\lambda \in \mathbb{R}$ . Hence we could never choose the stochastic term in such a way that it will completely vanish over time. On the other hand, the deterministic part of the SRDE is stable, which forces the solution to tend to the wave. This two-sided effect results in a proposition of [7] in section 2 on the orbital stability of  $L^2(V)$  to stay of order  $\sigma^2$  close to zero.

We show that these predictions hold, by using the same structure as in the validation of the numerical scheme to draw conclusions from the numerical results. We will start with an example of a specific realisation for this SRDE and explain what can be seen from the results. Thereafter, we will show what the behaviour is for the general case by taking 300 evaluations of the same process.

The initial conditions for this adapted Nagumo equation are intentionally the same as for the introductory example of the numerical analysis, that is  $\alpha = 0.25$  and  $\sigma = 0.3$ . The results are given in Figure 15. By looking at the progression with the naked eye, it seems as if the  $V$  tends quite fast to zero.

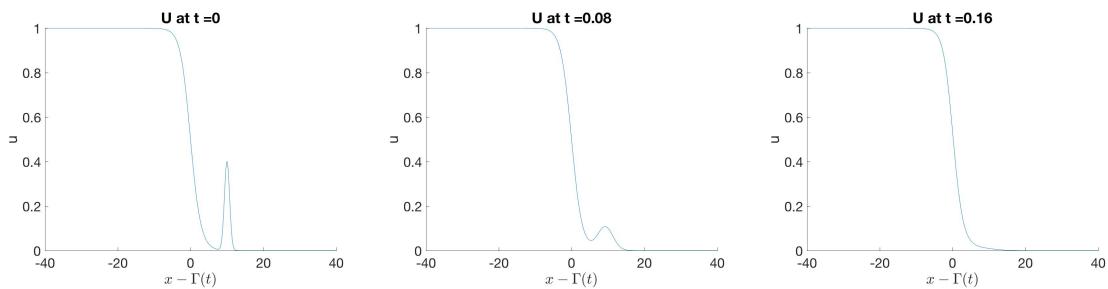


Figure 15: Realisations of  $U$  from the stochastic Nagumo equation with  $\alpha = 0.25$ ,  $\sigma = 0.3$ ,  $g(u) = u^2(1 - u^2)$  and initial condition  $\Phi_\sigma + \pi$ , where  $x_0 = 10$  and  $A = 0.4$

To illustrate that this is not convergent to  $\Phi_\sigma$ , we show figures of  $V = T_\Gamma U - \Phi_\sigma$  at later times. These can be found in Figure 16.

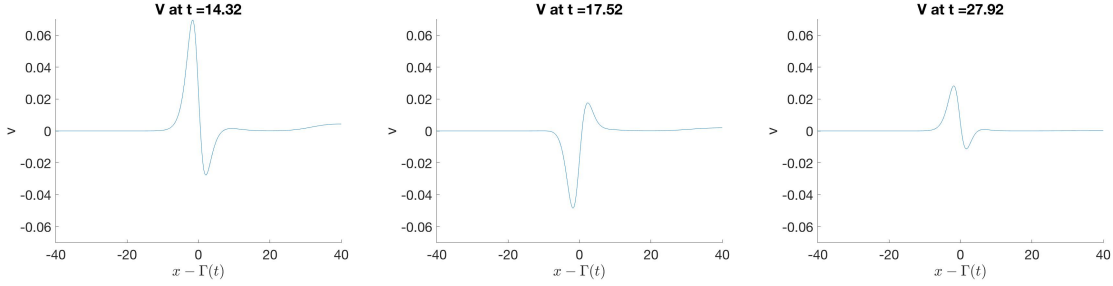


Figure 16: Realisations of  $V$  from the stochastic Nagumo equation with  $\alpha = 0.25, \sigma = 0.3, g(u) = u^2(1 - u^2)$  and initial condition  $\Phi_\sigma + \pi$ , where  $x_0 = 10$  and  $A = 0.4$

The deviation for  $V$  is indeed small, but it does not converge. In comparison to the values we get for the Nagumo equation with  $g(u) = u(1 - u)$ , we see these are a hundred times smaller. We will take 300 evaluations of this process, to see whether this is an ordinary example, i.e. the divergence is not unusual.

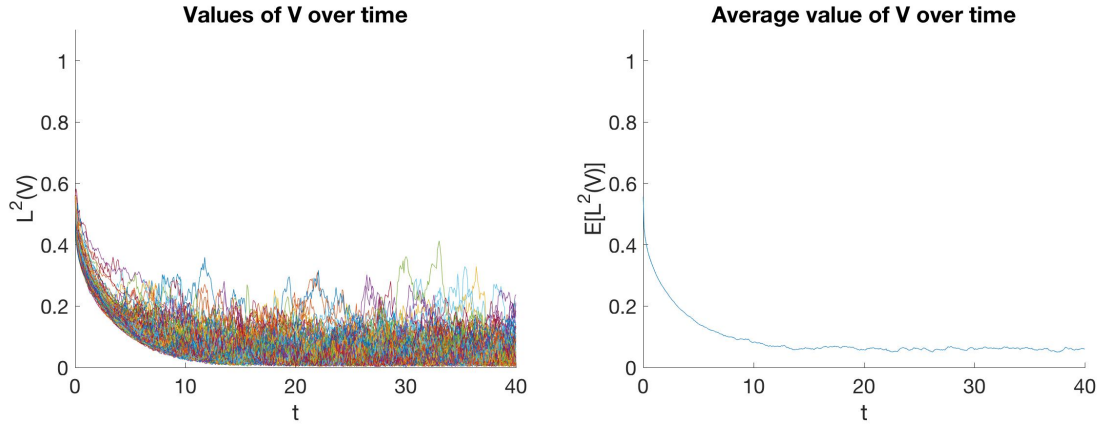


Figure 17: 300 realisation of  $V$  from the stochastic Nagumo equation with  $\alpha = 0.25, \sigma = 0.3, g(u) = u^2(1 - u^2)$  and initial condition  $\Phi_\sigma + \pi$ , where  $x_0 = 10$  and  $A = 0.4$

In Figure 17 we see that the norm of  $V$  decays fast in the beginning. However after  $t = 10$  the decay ends and the norm of  $V$  keeps fluctuating between 0 and 0.2. This is exactly the same behaviour that we found in the specific example of Figure 16, where we see that the noise is not vanishing. The theorem that was stated in section 2.6, that the norm of  $V$  should stay small with large probability is true according to these numerical results.

### 3.2.2 SRDE with $f(u) = \sin(\pi u) \sin(u - 0.25)$ and $g(u) = u(1 - u)$

For this equation we expect that  $U$  gets close to  $\Phi_\sigma$ , but it will not stabilise on the wave, since the noise term is not chosen to be the specific one, for which we can establish exponential decay with the strong theorem in 2.6, that is the one for which  $g(\Phi_\sigma) = \lambda \Phi'_\sigma$ . In this situation, there is again a two-sided effect of the reaction term that tries to bring the wave to the shape of  $\Phi_\sigma$  and on the other hand the noise term that opposes this behaviour. The result is that we expect the wave to have orbital stability of order  $\sigma^2$ . In Figure 18, we see an example of the stabilisation of the realisation for an initial condition of  $U(0) = \Phi_\sigma + \pi$ , with  $\sigma = 0.3, A = 0.4$  and  $x_0 = 10$ .



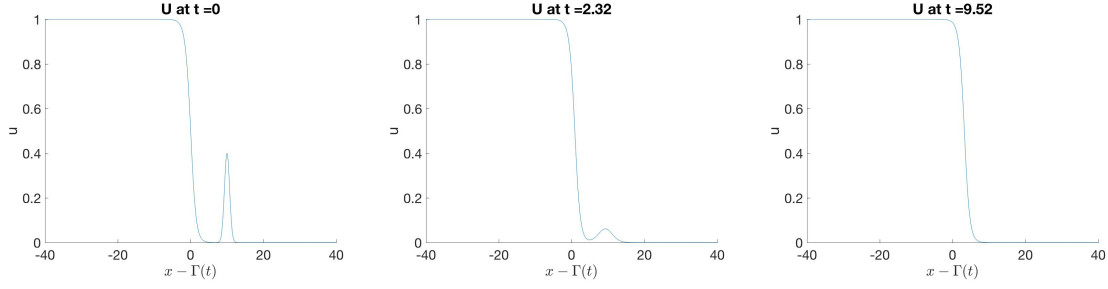


Figure 18: Realisation of  $U$  with  $f(u) = \sin(\pi u) \sin(u - 0.25)$ ,  $\sigma = 0.3$ ,  $g(u) = u(1 - u)$  and initial condition  $\Phi_\sigma + \pi$ , where  $x_0 = 10$  and  $A = 0.4$

Just like in the second example  $U$  seems to converge to  $\Phi_\sigma$ . As in the previous examples we investigate this by looking at  $V$  whether this really is the case.

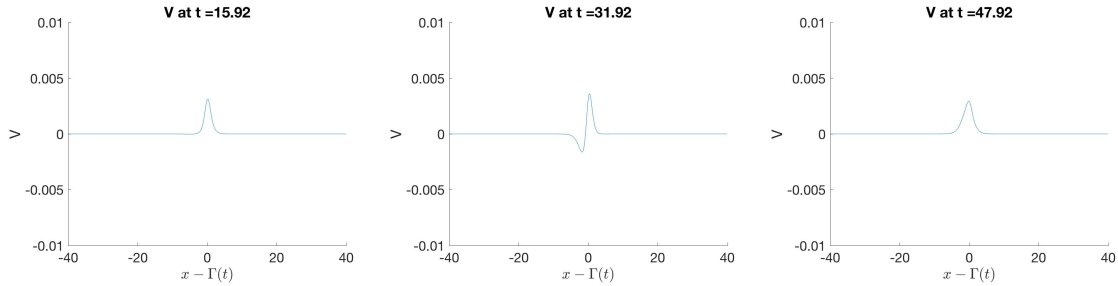


Figure 19: Realisation of  $V$  with  $f(u) = \sin(\pi u) \sin(u - 0.25)$ ,  $\sigma = 0.3$ ,  $g(u) = u(1 - u)$  and initial condition  $\Phi_\sigma + \pi$ , where  $x_0 = 10$  and  $A = 0.4$

We see that  $V$  is even smaller than the values for the Nagumo equation with  $g(u) = u(1 - u)$ , which is not to expect, since we did explicit equations on the Nagumo equation to make sure that  $V$  would decay exponentially fast. Hence we don't expect this decay to occur for an arbitrary different reaction term. To show that this behaviour is not exceptional for this equation, we will look at the behaviour of 300 evaluations. All realisations are made with the same initial conditions, that is  $U(0) = \Phi_\sigma + \pi$ , with  $x_0 = 10$  and  $A = 0.4$ , and noise strength of  $\sigma = 0.3$ . Figure 20 shows these results.

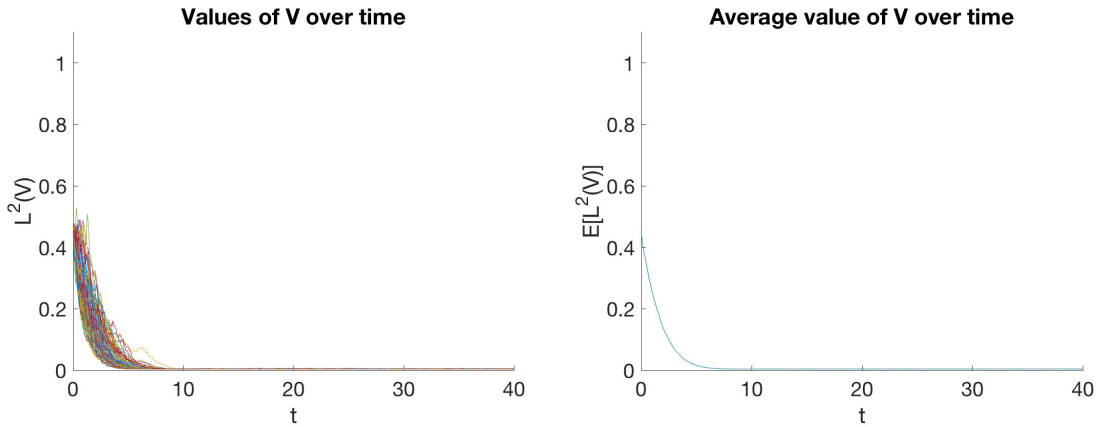


Figure 20: 300 realisation of  $V$  with  $f(u) = \sin(\pi u) \sin(u - 0.25)$ ,  $\sigma = 0.3$   $g(u) = u(1 - u)$  and initial condition  $\Phi_\sigma + \pi$ , where  $x_0 = 10$  and  $A = 0.4$

As in Figure 20 can be seen, we have exponential decay for this choice of the reaction and noise

term. Even the expected value of  $L_N^2 V$  tends to zero and the values of  $E(L_N^2(V))$  are smaller at  $t = 40$  than for the Nagumo equation. Thus we conclude that the expectation of  $U$  goes exponentially fast to  $\Phi_\sigma(\cdot - \Gamma(t))$ . This remarkable and unexpected behaviour is not what equation (2.60) predicted, but it is a lot stronger. Further research needs to be done on this subject to find a better explanation.

### 3.3 The long term behaviour

We ended section 2.8 with the discussion on the stochastic speed. Not much is known about the behaviour of  $\overline{\Gamma}_T$ . There are two main reasons for this. First of all, it is hard to prove that the definition is well defined, since we do not know whether the expectation converges theoretically. The second reason, is that our real interest is to show what the expected speed is. This speed is not well-defined for any evaluation, since  $\Gamma$  is not differentiable. Hence, to prove anything on the speed, we have to work our way around the derivative of  $\Gamma$  itself. In the numerical case, we can just take a straightforward estimator of the expected ‘speed’ of  $\Gamma$ . We will use for the expected value the average of a thousand evaluations of  $\Gamma$  from the initial condition  $U(0) = \Phi_\sigma$ .

For the Nagumo equation with  $g(u) = u(1 - u)$ , with initial condition  $\Phi_\sigma$ , we can compute the expectation of  $\Gamma$  explicitly to show that  $V$  goes to zero over time. Motivated by the remark at the beginning of 3.1.2, we compute the expectation of  $\Gamma(t) - c_\sigma t$ . The direct computation involves the fact that for any Brownian motion starting at zero the expected value is equal to zero for all  $t$ . Thus we find that

$$\mathbb{E}[\Gamma - c_\sigma t] = \mathbb{E}\left[\sigma \frac{2}{1 + \sigma^2} \beta_t\right] = 0. \quad (3.5)$$

Now we will numerically show that even for large value of the end time  $T$ , the difference between 0 and the estimated average value of  $\Gamma(t) - c_\sigma t$  is small. These results can be found in Figure 21.

#### The Nagumo equation with $g(u) = u(1 - u)$

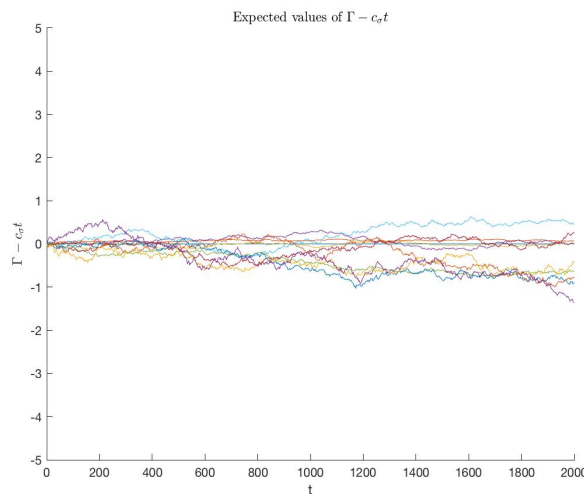


Figure 21:  $\mathbb{E}[\Gamma(t) - c_\sigma t]$  for a thousand realisations per  $\sigma$  for the stochastic Nagumo equation with  $g(u) = u(1 - u)$  and initial condition  $\Phi_\sigma$

In Figure 21, the values differ less than 1 on a time scale for 2000 time units. This means that for the most deviant average speed, the difference between the real and the estimated value of  $\overline{\Gamma}_{2000}$  is around  $\frac{1}{2000}$  which is negligible in comparison to the size of the speed, since the speed of  $\Phi_0$  is approximately 0.35. Since the expected value  $\Gamma(t) - c_\sigma t$  is zero, we expect the value of  $\overline{\Gamma}_t$  to converge to zero over time. The estimated values of  $\overline{\Gamma}_t$  per  $\sigma$  can be found in the following figure.

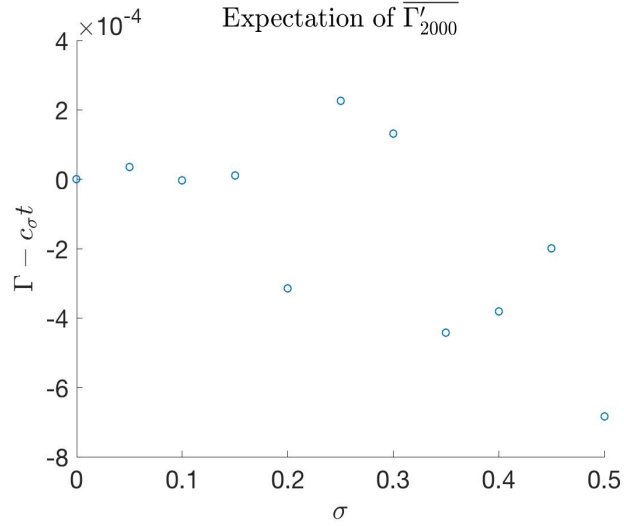


Figure 22: Expected values of  $\overline{\Gamma'_{2000}}$  per  $\sigma$  after a thousand realisations for the stochastic Nagumo equation with  $g(u) = u(1 - u)$  and initial condition  $\Phi_\sigma$ .

From Figure 22, we see that the expected values are around  $\frac{1}{100}$  of  $c_\sigma$ . Also note that we cannot see any obvious drift in this figure. If we found a clear trend line in the values of  $\overline{\Gamma'_t}$ , then we suspect that  $\sigma$  contributes to the speed apart from  $c_\sigma$ , i.e.  $a$  and  $b$  contribute to the speed in a way that is dependent on  $\sigma$ . With these two observations, we have a standard to which we can compare the results found for the other examples. We will start with the adapted stochastic Nagumo equation.

### The Nagumo equation with $g(u) = u^2(1 - u^2)$

For this stochastic Nagumo equation we cannot expect that  $V$  goes to zero and hence we also do not know if the expected values for  $\overline{\Gamma'_t}$  tend to  $c_\sigma$ . In this section we will examine the same question as we did for the Nagumo equation with  $g(u) = u(1 - u)$ . We want to know what the real contribution of  $\sigma$  is to the speed. In Figure 23, we see that the behaviour of this system is totally different from the behaviour found in Figure 21. In this example the paths are layered with the slopes decreasing as  $\sigma$  increases. The slopes ( $\overline{\Gamma'_t}$ ) of these graphs can be seen in Figure 24.

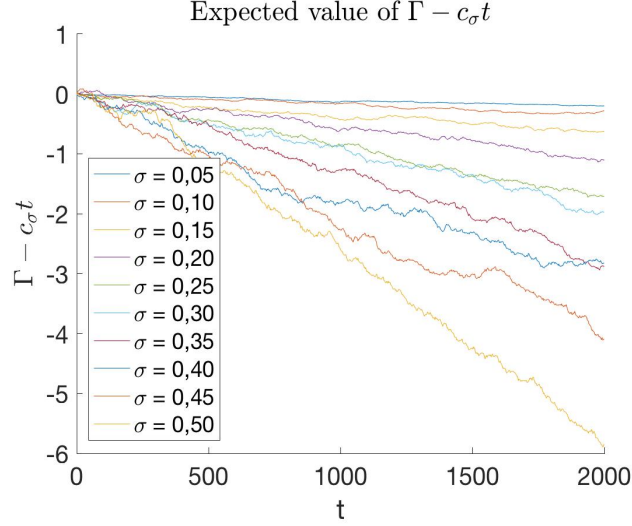


Figure 23:  $\mathbb{E}[\Gamma(t) - c_\sigma t]$  for a thousand realisations per  $\sigma$  for the stochastic Nagumo equation with  $\alpha = 0.25$ ,  $g(u) = u^2(1 - u^2)$  and initial condition  $\Phi_\sigma$

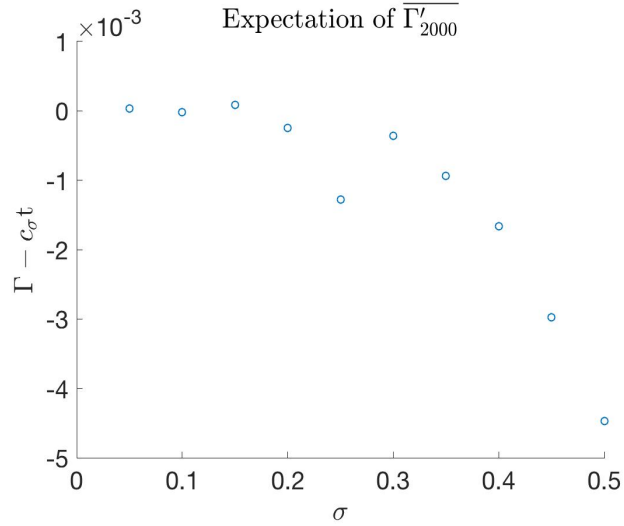


Figure 24: Expected values of  $\overline{\Gamma'_{2000}}$  per  $\sigma$  after a thousand realisations for the stochastic Nagumo equation with  $\alpha = 0.25$   $g(u) = u^2(1 - u^2)$  and initial condition  $\Phi_\sigma$

From Figure 24, we see that there is a quadratic relationship between  $\sigma$  and  $\overline{\Gamma'_t}$ . A trend line gives the following formula for the relation between the expected value for  $\overline{\Gamma'_t}$

$$\overline{\Gamma'_t} \approx c_\sigma - 0.0012\sigma^2. \quad (3.6)$$

By definition of  $\Gamma - c_\sigma t$  is equal to  $\int_0^t a(X(s))ds + \int_0^t g(X(s))d\beta_s$ . From this point of view, the observation is in particular that the functionals  $a$  and  $b$  do contribute to the speed of the wave. We can also see that the contribution of  $\int_0^t a(X(s))ds + \int_0^t g(X(s))d\beta_s$  to the speed is of order  $\sigma^2$ .

**SRDE with  $f(u) = \sin(\pi u) \sin(u - 0.25)$  and  $g(u) = u(1 - u)$**

The values we found in 20 are very small and so we can use the same argument as on the stochastic Nagumo equation with  $g(u) = u(1 - u)$ . Thus we expect the stochastic speed of  $\Gamma$  to converge

towards the speed of  $\Phi_\sigma$ , because  $T_{-\Gamma}U$  does almost converge to  $\Phi_\sigma$ . With  $\Gamma - c_\sigma t$  small, we cannot expect the influence of  $a$  and  $b$  to be big and we do not expect large deviations. The results of the average of a thousand paths of  $\Gamma - c_\sigma t$  per  $\sigma$  for each  $\sigma$  from 0 to 0.5 with steps of 0.05 can be seen in Figure 25.

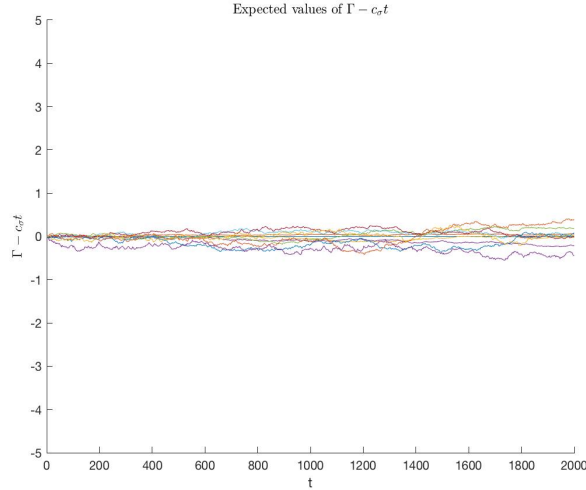


Figure 25:  $\mathbb{E}[\Gamma(t) - c_\sigma t]$  for a thousand realisations per  $\sigma$  for  $f(u) = \sin(\pi u) \sin(u - 0.25)$ ,  $g(u) = u(1 - u)$  and initial condition  $\Phi_\sigma$

From this figure we see that the deviations are of the same order as the ones we found for the Nagumo equation with  $g(u) = u(1 - u)$ . Also for the estimated values of  $\overline{\Gamma'_t}$ , we see in Figure 26, that there is no drift on the average value to one side. From this two observations we conclude that the SRDE for  $f(u) = \sin(\pi u) \sin(u - \alpha)$  and  $g(u) = u(1 - u)$  is stable and the speed of  $\Gamma$  goes to  $c_\sigma$ .

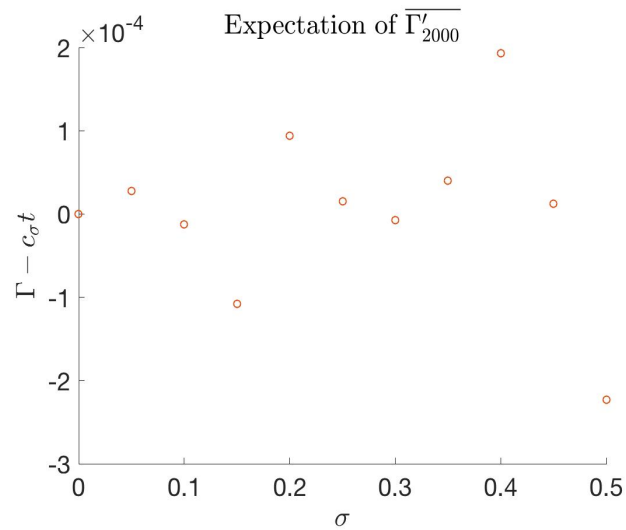


Figure 26: Expected values of  $\overline{\Gamma'_{2000}}$  per  $\sigma$  after a thousand realisations for  $f(u) = \sin(\pi u) \sin(u - 0.25)$ ,  $g(u) = u(1 - u)$  and initial condition  $\Phi_\sigma$

---

### 3.4 Conclusions

From the numerical results we find that the influence on the behaviour is far more dependent on the shape of the noise term than on the reaction term. As long as the reaction term fits definition 2, the wave will try to converge to  $\Phi_\sigma$ . The reaction term changes the shape of the wave, the time to converge and the speed of the wave. Although the theorems by [7] are true, the remarkable result for the second example, where we chose  $f(u) = \sin(\pi u) \sin(u - 0.25)$  and  $g(u) = u(1 - u)$ , is not explained by it. The theorems stated only that the norm of  $V$  would stay small with a large probability, but the numerical results showed that the value decayed exponentially fast. For further research, the functionals  $a$  and  $b$  need to be investigated to get a better understanding of the speed of the travelling wave and the stability. At this very moment we have bounds on a norm for  $V$ , that are yet too crude. Hence further research needs to be done to have sharper expectations on the behaviour of these stochastic reaction-diffusion equations.

---

# Appendices

## A The Matlab code

The Matlab code consists of 9 files. For convenience we have included a legend for the symbols used in the code with the information how it corresponds to the theoretical part of the thesis. The main document, that is A.1, uses and controls the other files. In the main file, all the parameters can be chosen. The initialisation is done with use of the two files, the boundary value problem and the stochastic boundary value problem, in the thesis section A.3 resp. A.5. Both files use the non-linear solver on two other files, which are sections A.4 and A.6. The boundary value problem solvers return the speed of the travelling wave, the wave itself and the derivative of the wave. After this initialisation, the evaluation is done in A.2. This program returns the evaluation of  $U$ ,  $T_{\Gamma}U - \Phi_{\sigma}$  and  $\Gamma$  over time. After this we can compute the values of the difference of  $T_{\Gamma}U$  with the corresponding  $\Phi_{\sigma}$ .

### A.1 The Main document

#### Contents

- Initialisation
- Specific to the Nagumo equation
- Find solution to equation for det. and stoch. travelling wave
- Perturbations and initial shifts
- Running the equations
- Shift the solution dependent on the values of Alpha
- Compute the norm of V for all time steps

`%This is the coded used to compute the travelling wave equations to SRDEs`

#### Initialisation

```
L=40;           % Max value of the spatial interval, -L minimal value
J=2^12;        % Accuracy of discretisation
h=2*L/J;       % Spatial step size
x=(-L:h:L)';  % Make discretisation of space
sigma=.3;      % Choice for noise strength
n0=1;          % Swith for choice of f: n0=1 is Nagumo,n0=2 is sine shape
N=10^3;        % Number of time steps
T=80;          % Maximal time value
Dt=T/N;        % Size of time step
t=(0:Dt:T);    % Make time array
a=0.25;        % Parameter in f
```

#### Specific to the Nagumo equation

```
d=-sqrt(2/(1+sigma^2)); % Scaling used in exact Nagumo equations
nu=1-sigma^2*d^2/2;     % Scaling used in exact Nagumo equations
c=sqrt(2*nu)*(1/2-a);   % Scaling used in exact Nagumo equations
```

#### Find solution to equation for det. and stoch. travelling wave

```
%respectively equation (1.9) and (2.46)
[u0,c0,u0dxi] = ScriptBVP(L,J,a,h,n0);
                % Generates deterministic speed and travelling wave
[us,cs,usdxi] = ScriptBVPS(L,J,a,h,n0,sigma,u0,c0,u0dxi);
```

---

```

                                % Generates stochastic speed and travelling wave
psi=exp(c0*x).*u0dxi;           % Adjoint eigenfunction
Nlz=trapz(x,psi.*u0dxi);       % Normaliser
psi=psi/Nlz                     % Normalise
l=trapz(x,usdxi.*psi);         % Necessary constant

```

## Pertubations and initial shifts

```

Amp=0.0; sp=10;                % Amplitude (A) and shift ($sp=x_0$) of perturbation
Pert=Amp*exp(-(x-sp).^2);      % Make perturbation
g0=ip(us,psi,Pert,x);          % Compute initial shift in terms of h: gamma_0
Pertg=circshift(Pert,g0);      % Make shifted perturbation
usg=circshift(us,g0);          % Shifted stochastic wave front
W=randn(1,N);                  % Create N normal dist. numbers for B. motion

```

## Running the equations

```

[t,ut,vt,Alpha]=spdeNagUVpaper(us,us+Pert,Pertg+usg-us,Dt,t,N,J,a,cs,x,sigma,usdxi,psi,g0,
                                W,u0,c0,h,l,n0);

```

```

% spdeNagUVpaper returns the time, the evolution of u (ut) over time, the
% evolution of  $V=T_{-\Gamma}U-\Phi_{\sigma}$  (vt) over time and the stochastic shift
% without the stochastic speed as described in section

```

## Shift the solution dependent on the values of Alpha

```

uta=zeros(size(ut'));
for i=1:N+1                      % Shift u by a
    Shift=-round(Alpha(i)/h);    % Number of delta x partitions to shift
    uta(i,:)=circshift(ut(:,i)',Shift); % Do the shift
    if Shift>0
        uta(i,1:Shift)=ones(1,Shift);
    end
    if Shift<0
        uta(i,end+Shift+1:end)=zeros(1,-Shift);
    end % This part creates zeros and ones where it is necessary. Since the
end % circshift takes the values on the left side and glues them on the
    % right side,  $w_{+}$ , has to be made  $w_{-}$  (see definition 3)

```

## Compute the norm of V for all time steps

```

for i=1:N+1
V(i) = sqrt(trapz(x,(uta(i,:)-us').^2)); % Take the L^2 norm
end

```

## A.2 Evaluation of the SRDE (SPDENagUVpaper.m)

### Contents

- Initialisation

```

function [t,ut,vt,Alpha]=spdeNagUVpaper(us,ICu,ICv,Dt,t,N,J,a,c,x,sigma,usdxi,psi,g0,W,u0,c0,h,l,n0)

```

```

% Solver for SRDE TW-coordinates with mult. noise

```



---

## Initialisation

```
e = ones(J+1,1); A = spdiags([-e 2*e -e], -1:1, J+1, J+1);
                                % Take Neumann boundary conditions
ind=1:J+1; A(1,2)=-2; A(end,end-1)=-2;
AdvL=spdiags([-e e ], -1:0, J+1, J+1); AdvL(1,2)=-1;
AdvR=spdiags([-e e ], 0:1, J+1, J+1); AdvR(end, end-1)=1;
A=-1/h^2*A;                      % Differentiation matrix
Adv=1/h*(AdvL+AdvR)/2;           % Advection matrix
fpu=Fpu(u0,a,n0);                % Derivative of f
EEu=speye(length(ind))-Dt*A-c*Dt*Adv; % Linear terms for u
L=A+c0*Adv+spdiags(fpu,0,J+1,J+1);
EEv=speye(length(ind))-Dt*L;     % Linear term for v
ut=zeros(J+1,length(t));         % Initialise vectors
vt=zeros(J+1,length(t));         % Initialise vectors
Alpha=zeros(length(t),1);        % Vector with shifts
Alpha(1)=g0;                     % Set initial condition such that <u,\Phi_0>=0
ut(:,1)=ICu; u_n=ICu(ind);       % Set initial condition
vt(:,1)=ICv; v_n=ICv(ind);       % Set initial condition
sqrtDt=sqrt(Dt);                 % Precompute sqrt(dt)

%%For loop to find the shifts
for k=1:N                          % Time loop
    fu=f(u_n,a,n0);               % Evaluate f
    u_new=EEu\u_n+Dt*fu+sigma*g(u_n).*sqrtDt.*W(k)); % update u orig. SRDE
    ut(ind,k+1)=u_new; u_n=u_new; % Update u
    Advv=Adv*v_n;                 % Compute v_\xi
    Noemer=1+trapz(x,Advv.*psi);   % Denominator for a and b
    usv=us+v_n;                   % Create variable $V + \Phi_{\sigma}$
    G=g(usv);                     % Evaluate g(u^{*+v})
    b=-trapz(x,psi.*G)/Noemer;     % Calculate b of Def 2.45
    kappa=1+sigma^2/2*b^2;         % Compute kappa of Def 2.39
    RHS=kappa*A*(usv)+f(usv,a,n0)+sigma^2*b*Adv*g(usv)+c*(Advv+usdxi);
    alpha=-trapz(x,psi.*RHS)/Noemer; % Calculate a of Def 2.49
    RHSa=RHS+alpha*(Advv+usdxi)-L*v_n; % Calculate RHS of eq. 2.44
    v_new=EEv\u_n+Dt*RHSa+sigma*sqrtDt*(G+(Advv+usdxi)*b)*W(k));
    vt(ind,k+1)=v_new; v_n=v_new; % Update v
    Alpha(k+1)=Alpha(k)+Dt*alpha+sigma*sqrtDt*b*W(k); %update alpha
                                % Def of $\Gamma$ 2.11
end
end
```

## A.3 Boundary Value problem (BVP.m)

### Contents

- Initial guesses
- set up differentiation matrices
- fun contains the conditions to solve
- Return the right values

```
function [u0,c0,u0dxi] = ScriptBVP(L,J,a,h,n0)
```

```
%This code computes a solution  $(\Phi_0,c_0)$  to the equation  $u''+cu'+f(u)=0$ .
x=(-L:h:L)'; % spatial discretisation
```

---

## Initial guesses

```
c0=sqrt(2)*(1/2-a); % Initial speed guess
u0=1-1./(1+exp(-x/sqrt(2))); % Initial wave guess
U0=cat(1,u0,c0); % The pair as intial guess
```

## Set up differentiation matrices

```
e = ones(J+1,1); A = spdiags([-e 2*e -e], -1:1, J+1, J+1);
A(1,2)=-2; A(end,end-1)=-2; % Take Neumann boundary conditions
AdvL=spdiags([-e e ], -1:0, J+1, J+1); AdvL(1,2)=-1;
AdvR=spdiags([-e e ], 0:1, J+1, J+1); AdvR(end, end-1)=1;
A=-1/h^2*A; % Differentiation matrix
Adv=1/h*(AdvL+AdvR)/2; % Advection Matrix
```

## Fun contains the conditions to solve

```
u = fsolve(@(u) fun(u,A,Adv,u0,a,n0),U0); % The last element of u is c
```

## Return the right values

```
u0=u(1:end-1);
c0=u(end);
u0dxi=Adv*u0;
end
```

## A.4 The non-linear solver of (1.9) (fun.m)

```
function [ Gu ] = fun( u,A,Adv,u0,a,n0 )
U=u(1:end-1); % Eliminating the speed in the array
% The non-linear solver searches zeros,hence Gu1 and Gu2 are chosen as:
Gu1=A*U+f(U,a,n0)+u(end)*Adv*U; % equation (1.9)
Gu2=sum(Adv*U.*(U-u0)); % Set phase condition  $\langle u_0, \Psi_0 \rangle = 0$ 
Gu=cat(1,Gu1,Gu2); % Put  $u_0$  and  $c_0$  together.
end
```

## A.5 The Stochastic boundary value problem (ScriptBVPS.m)

### Contents

- Set up differentiation matrices
- funS contains the conditions to solve
- Return the right values

```
function [us, cs ,usdxi] = ScriptBVPS(L,J,a,h,n0,sigma,u0,c0,u0dxi)
```

```
%This code computes a solution (u,c) to the equation  $(1+s^2/2b^2)*u''+cu'+f(u)+s^2*b*d/dx g(u)=0$ .
%f and g are specified in the files f.m and g.m
```

```
% initial guesses and initialisation
```

```
U0=cat(1,u0,c0); % Make initial guess array
x=(-L:h:L)'; % spatial discretisation
psi=exp(c0*x).*u0dxi; % Adjoint eigenfunction
psi=psi/trapz(x,u0dxi.*psi); % Normalised eigenfunction
```

---

## Set up differentiation matrices

```
e = ones(J+1,1); A = spdiags([-e 2*e -e], -1:1, J+1, J+1);
                                % Take Neumann boundary conditions
A(1,2)=-2; A(end,end-1)=-2;
AdvL=spdiags([-e e ], -1:0, J+1, J+1); AdvL(1,2)=-1;
AdvR=spdiags([-e e ], 0:1, J+1, J+1); AdvR(end, end-1)=1;
A=-1/h^2*A; % Differentiation matrix
Adv=1/h*(AdvL+AdvR)/2; % Advection matrix
```

## funS contains the conditions to solve

```
u = fsolve(@(u) funS(u,A,Adv,u0,a,n0,sigma,psi,x),U0);
```

## Return the right values

```
us=u(1:end-1);
cs=u(end);
usdxi=Adv*us;

end
```

## A.6 The non-linear solver of (2.46) (funS.m)

```
function [ Gu ] = funS( u,A,Adv,u0,a,n0,s,psi,x)
U=u(1:end-1); % Eliminating the speed in the array
b=-trapz(x,g(U).*psi)/trapz(x,(Adv*U).*psi);
kappa=1+s^2/2*b^2; % Compute values of  $b$  and  $\kappa_{\sigma}$ 
% The non-linear solver searches zeros,hence Gu1 and Gu2 are chosen as:
Gu1=kappa*A*U+f(U,a,n0)+u(end)*Adv*U+s^2*b*Adv*g(U);
Gu2=sum(Adv*U.*(U-u0)); % Gu1 is eq. 2.46 and Gu2 is  $\langle u_{\sigma}, \Psi_0 \rangle = 0$ 
Gu=cat(1,Gu1,Gu2); % Put  $u_{\sigma}$  and  $c_{\sigma}$  together
end
```

## A.7 Initial phase of U (ip.m)

```
function [ f ] = ip(us,psi,Pert,x)
% ip is an abbreviation of initial phase. This code computes the number of shifts in
% the discretisation, the perturbation and  $u_0$  have to shift to have inner
% product zero with psi for  $t=0$ .

m=10; % Large value for the inner product as initial max value.
for i=-100:1:100 % In this range of shifts, the scheme searches the smallest
                % value of the inner product.
    if abs(int(us,psi,x,Pert,i))< m
        shift =i;
        m=int(us,psi,x,Pert,i);
    end
end
f=shift;
function i = int(us,psi,x,Pert,k) %function that computes inner product
    i = -trapz(x,us.*psi)+trapz(x,circshift(us,k).*psi)+trapz(x,circshift(Pert,k).*psi);
end
end
```

---

## A.8 The reaction term (f.m)

```
function [ f ] = f(u,a,n0)
% This function returns the function $f$ that is chosen via the parameter a
% and the number of n0.
    switch n0
        case 1
            f = u.*(1-u).*(u-a);           % Standard Nagumo equation
        case 2
            f = sin(u-a).*sin(pi*u);      % Example 3 in section 3
        otherwise
            f=0;                            % Choice of a function that does not contribute to
                                            % the system.
    end
end
```

## A.9 The noise term (g.m)

```
function [ g ] = g(u)
% g is the function from the mult. noise.
g=u.*(1-u);
end
```

---

## Legend of important symbols in the Matlab code

Character in <i>Matlab</i>	Meaning in thesis
a	The parameter found in $f(\alpha)$
adv	Advection matrix
Alpha	$\Gamma - c_\sigma t$ shift
alpha	The functional $a$ from def. 14
Amp	A in definition 22
b	The functional $b$ from def. 14
c0	Speed of deterministic travelling wave ( $\Phi_0$ )
g0	Initial shift on $U$
cs	Speed of stochastic travelling wave ( $\Phi_\sigma$ )
Dt	Partition width of time ( $\Delta t$ )
h	Partition width of space ( $\Delta x$ )
J	Number of spatial partitions $standard(2^{12})$
L	Max value of spatial interval
N	Amount/number of time steps
n0	Encoding for specific choice of $f$ , n0=1 : Nagumo n0=2 : $\sin(\pi u) \sin(u - \alpha)$
Pert	Perturbation of definition 22
psi	Adjoint eigenfunction to $\Phi_0$ ( $\psi$ )
sigma	The noise strength ( $\sigma$ )
sp	$x_0$ in definition 22
T	Max value of time/end time
t	Discretised time
u0	Wave front ( $\Phi_0$ )
u0dxi	Spatial derivative of $\Phi_0$ ( $\Phi'_0$ )
us	Wave front ( $\Phi_\sigma$ ) for specific $\sigma$
usdxi	Spatial derivative of $\Phi_\sigma$ ( $\Phi'_\sigma$ )

Table 3: All important definitions of the Matlab code

---

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