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Fredholm index of nonlocal differential operators via spectral flow and exponential dichotomy

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1 Introduction

1.1 Practical aspect

One of the most well-known equations that is used to model the spread of genetic traits and the propagation of nerve pulses in a nerve axon, is the Nagumo-equation[14, 15], which is given by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), \tag{1.1}$$

with $D > 0$ the diffusion constant and $f(u)$ a non-linear term that typically looks like

$$f(u) = u(1-u)(u-a), \tag{1.2}$$

where $0 < a < 1$. A discretisation of (1.1) yields the equation

$$\dot{u}_n = D(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \quad n \in \mathbb{Z} \tag{1.3}$$

where n can be interpreted as a certain space variable, for instance on a lattice, and $t \in \mathbb{R}$ the usual time variable. To find a traveling wave solution, we substitute the Ansatz $u_n(t) = U(n+ct)$ into (1.3) and obtain the equation [19],

$$cU'(z) = D(U(z-1) - 2U(z) + U(z+1)) + f(U(z)), \quad z = n + ct. \tag{1.4}$$

Such equations are called functional differential equations of mixed type (MFDEs), which typically have a shift in their arguments.
Note that in (1.3), when interpreting n as some space variable, the derivative $\dot{u}_n$ depends only on the function value of its two nearest neighbours (u_{n-1} and u_{n+1}) and itself (u_n). Equations like (1.4), which has only a finite range of interactions with its neighbours, have been studied by John Mallet-Paret in [1].

However, there are also models where one has to look at equations with a infinite range of interactions [17, 20, 21, 22]. For example, in the paper of Schouten and Hupkes [16], they study the discrete infinite-range FitzHugh-Nagumo equation

$$\begin{aligned} \dot{u}_j &= \frac{\kappa}{h^2}\sum_{k=1}^{\infty} e^{-k^2}[u_{j+k} + u_{j-k} - 2u_j] + f(u_j) - w_j, \\ \dot{w}_j &= [\rho u_j - \gamma w_j], \end{aligned} \tag{1.5}$$

where f is given by (1.2) and $h, \kappa, \rho, \gamma > 0$ are constants. Plugging in the traveling wave Ansatz $u_j(t) = U_h(hj + c_h t)$ and $w_j(t) = W_h(hj + ct)$ and writing $\xi = hj + ct$ gives the MFDE

$$\begin{aligned} c_h\dot{U}_h(\xi) &= \frac{\kappa}{h^2}\sum_{k=1}^{\infty} e^{-k^2}[U_h(\xi + kh) + U_h(\xi - kh) - 2U_h(\xi)] + f(U_n(\xi)) - W_h(\xi), \\ c_h\dot{W}_h(\xi) &= [\rho U_h(\xi) - \gamma W_h(\xi)]. \end{aligned} \tag{1.6}$$

Such equations are used to model the dynamics of large neuron networks [23, Equation (3.31)].

1.2 Theoretical aspect

1.2.1 Fredholm operator

Several of the main results of this thesis concern Fredholm operators and Fredholm index. Therefore, we will give their definitions.

Definition 1.1. *Let X, Y be two Banach spaces. A bounded linear operator $T : X \to Y$ is a* ***Fredholm Operator*** *if the following holds:*

- *The kernel $\mathcal{K}(T) \subset X$ is finite dimensional;*
- *The range $\mathcal{R}(T) \subset Y$ is closed;*
- *$\mathcal{R}(T)$ has finite codimension in Y.*

Recall that the codimension of a subspace $Z \subset Y$ is defined as $\dim(Y/Z)$, namely the dimension of the quotient group Y/Z. In case that Y is finite dimensional, we have that $\operatorname{codim}(Z) = \dim(Y) - \dim(Z)$.

The **Fredholm index** of a Fredholm operator T is defined as

$$\operatorname{ind}(T) = \dim \mathcal{K}(T) - \operatorname{codim}(\mathcal{R}(T)). \tag{1.7}$$

The most simple examples of Fredholm operators are the matrices. Recall the following formula for matrices.

Dimension formula for matrices: For $A : \mathbb{R}^n \to \mathbb{R}^m$ a matrix, we have

$$\dim \ker(A) + \operatorname{rk}(A) = n. \tag{1.8}$$

This gives us a very easy way to calculate the Fredholm index of A, namely

$$\begin{aligned} \operatorname{ind}(A) &= \dim \ker(A) - \operatorname{codim}(A) \\ &= \dim \ker(A) - (m - \operatorname{rk} A) \\ &= \dim \ker(A) - (m - n + \dim \ker(A)) \\ &= n - m. \end{aligned} \tag{1.9}$$

We see that if A is invertible, then we must have $\operatorname{ind}(A) = 0$, which means $m = n$ and that is a well-known result in Linear Algebra.
For matrices, we also have the Fredholm alternative theorem, which relates the range of a matrix to the kernel of its adjoint.

Theorem 1.2 (Fredholm Alternative Theorem)**.** *Let $A \in \operatorname{Mat}(m \times n, \mathbb{R})$ be a $m \times n$ real valued matrix and $b \in \mathbb{R}^m$. Then we have*

$$Ax = b \text{ has a solution } \Leftrightarrow \forall y \in \ker(A^\top) : \ \langle y, b \rangle = 0$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^m$.

One way to interpret this theorem is if we have a inhomogeneous equation $Ax = b$, we can check the existence of a solution by checking the orthogonality of the inhomogeneous term b with the solutions of the homogeneous adjoint equation. This allows us to avoid solving an inhomogeneous equation.
The same idea can also be applied to differential equations. For example, consider the linear first order differential equation

$$\dot{x} = ax + f, \ a > 0 \ \ f \text{ bounded and continuous on } \mathbb{R}. \tag{1.10}$$

Suppose we are interested in the existence of an unique solution that is bounded on $\mathbb{R}$, for each bounded and continuous f. One way to investigate this is to solve the equation explicitly. Therefore, we first examine the homogeneous equation

$$\dot{x} = ax \tag{1.11}$$

which has the general solution

$$x(t) = Ce^{at}. \tag{1.12}$$

To solve the inhomogeneous equation, we use the method of variation of parameters and consider C as a function of t

$$x(t) = C(t)e^{at}. \tag{1.13}$$

Substituting this into the equation yields

$$C(t) = \int_{t_0}^{t} f(s)e^{-as}ds + K. \tag{1.14}$$

Hence, we obtain the solution to the inhomogeneous equation

$$x(t) = e^{at}\left(\int_{t_0}^{t} f(s)e^{-as}ds + K\right). \tag{1.15}$$

Note that $\int_{t_0}^{\infty} f(s)e^{-as}ds < \infty$ due to the assumption that f is bounded on $\mathbb{R}$. By choosing $K = -\int_{t_0}^{\infty} f(s)e^{-as}ds$, it can be shown that $x(t)$ is the unique solution that is bounded on $\mathbb{R}$, for each bounded and continuous f.

We can also analyze this problem from an algebraic point of view by defining the differential operator

$$\Lambda_L x(t) := \left(\frac{d}{dt} - a\right)x := \frac{dx}{dt}(t) - ax(t). \tag{1.16}$$

Then, equation (1.10) can be written as

$$\Lambda_L x = f, \tag{1.17}$$

which has the form $Ax = b$ as in Theorem 1.2. In this way, questions about the existence and uniqueness of the solutions of (1.10) are equivalent to asking whether Λ_L is an isomorphism.
If Λ_L satisfies similar Fredholm properties like in (Theorem 1.2) (which is true according to Theorem A in Section 3), then we can solve the equation by considering the kernel of the adjoint operator, which is defined as

$$(\Lambda_L)^* := \left(-\frac{d}{dt} - a\right) \tag{1.18}$$

that has the adjoint property

$$\langle \Lambda_L x, y\rangle = \langle x, (\Lambda_L)^* y\rangle, \qquad x \in L^p, \quad y \in L^q \tag{1.19}$$

for $1 \leq p, q \leq \infty$ and $\langle f, g\rangle = \int_{-\infty}^{\infty} f(s)g(s)ds$ is the dot product. Note that the minus sign in (1.18) comes from verifying (1.19) using integration by parts.
We observe that the kernel of the adjoint consists of the solutions of the homogeneous equation

$$x' = -at, \tag{1.20}$$

which are

$$x(t) = Ce^{-at} \tag{1.21}$$

for $C \in \mathbb{R}$. Since we are looking for bounded solution on $\mathbb{R}$, we must choose $C = 0$. Hence, according to the Fredholm alternative theorem, this implies that the range of Λ_L is equal to its codomain, which means that Λ_L is surjective. Similarly, we can show that the kernel of Λ_L is also trivial. Hence, we see that Λ_L is an isomorphism, which agrees with our former conclusions.

It might seem very unnecessary to analyze (1.10) with this algebraic approach since we know how to solve (1.10) explicitly. But when dealing with equations like (1.4) which are not standard ODE, the operator Λ_L can give us more insight about the solutions of such equations.

1.2.2 Fredholm index and roots of characteristic equation

Let us now consider the next example of one-dimensional first order linear equation

$$\dot{x} = a(t)x + f, f \text{ bounded and continuous on } \mathbb{R}, \tag{1.22}$$

where

$$a(t) = \begin{cases} -1 & \text{if } t \geq 0 \\ 1 & \text{if } t < 0. \end{cases} \tag{1.23}$$

Suppose we would like to calculate the Fredholm index of the Λ_L operator for this system, which is

$$\Lambda_L = \frac{d}{dt} - a(t). \tag{1.24}$$

By definition, we have $\operatorname{ind}(\Lambda_L) = \ker(\Lambda_L) - \operatorname{codim}(\mathcal{R}(\Lambda_L))$. We will first examine the kernel

$$\dot{x}(t) = \begin{cases} -x(t) & \text{if } t \geq 0 \\ x(t) & \text{if } t < 0. \end{cases} \tag{1.25}$$

It is well-known that the general solution is given by

$$x(t) = \begin{cases} Ce^{-t} & \text{if } t \geq 0 \\ De^{t} & \text{if } t < 0. \end{cases} \tag{1.26}$$

Since we are looking for bounded and continuous solution, we require that $C = D$. This shows that the kernel has dimension one.
For $\operatorname{codim}(\mathcal{R}(\Lambda_L))$, we can show that Λ_L is surjective with the method of variation of parameters. Indeed, for all $K \in \mathbb{R}$, we have the bounded continuous solution

$$x(t) = \begin{cases} e^{-t}\left(\int_0^t f(s)e^{s}ds + K\right) & t \geq 0 \\ e^{t}\left(-\int_t^0 f(s)e^{-s}ds + K\right) & t \leq 0 \end{cases} \tag{1.27}$$

for each f. Hence, we conclude that $\operatorname{codim}(\mathcal{R}(\Lambda_L)) = 0$ and thus $\operatorname{ind}(\Lambda_L) = 1$.

We can also calculate the index through another approach. Since Λ_L is a Fredholm operator and has similar property as in (Theorem 1.2), we have that $\operatorname{codim}(\mathcal{R}(\Lambda_L)) = \dim\ker((\Lambda_L)^*)$. This means that to calculate the index, we only need to examine the kernels, which are solutions of a linear homogeneous system.

For a linear homogeneous equation $x'(t) = Ax(t)$, we have $x(t) = e^{\lambda t}v$ is a solution if and only if λ and v are respectively the eigenvalue and eigenvector of A. Hence, finding a solution is equivalent to solving the characteristic equation $\det(sI - A) = 0$. For the kernel of Λ_L, we have for $t \geq 0$, the "positive" linear system $x'(t) = -x(t)$, which means we need to look at the characteristic equation

$$\Delta_+(s) := s + 1. \tag{1.28}$$

While for $t < 0$, we have the "negative" linear system $x'(t) = x(t)$ and

$$\Delta_-(s) := s - 1. \tag{1.29}$$

We see that we have one stable eigenvalue for $t \geq 0$ and one unstable eigenvalue for $t < 0$. This means that both systems have a one dimensional solution spaces(which we require to only contain bounded and continuous solutions). Intuitively, we would like to "paste" these half-line solutions to create a solution on $\mathbb{R}$. But since we also require the solution to be continuous, this means that although we have two degree of freedoms from the two system, we lose one degree of freedom by solving a system of equation for the

parameter at $t = 0$. This gives us $\dim\ker(\Lambda_L) = 1$.
For the kernel of the adjoint, we have

$$\dot{x}(t) = \begin{cases} x(t) & \text{if } t \geq 0 \\ -x(t) & \text{if } t < 0, \end{cases} \tag{1.30}$$

which clearly does not have non-trivial bounded solution. Hence, we again conclude that $\mathrm{ind}(\Lambda_L) = 1$.

Consider now the d-dimensional linear equation

$$\dot{x}(t) = A(t)x(t) \tag{1.31}$$

where

$$A(t) = \begin{cases} A_+ & \text{if } t \geq 0 \\ A_- & \text{if } t < 0, \end{cases} \tag{1.32}$$

with A_+ and A_- constant matrices that only have real eigenvalues. Suppose A_+ has m stable eigenvalues and A_- has n unstable eigenvalues. Then, for $\ker(\Lambda_L)$ we have the following bounded solution

$$x(t) = \begin{cases} C_1 e^{\lambda_1 t} v_1 + \cdots C_m e^{\lambda_m t} v_m & t \geq 0 \\ D_1 e^{\lambda_1 t} w_1 + \cdots D_n e^{\lambda_n t} w_n & t \leq 0. \end{cases} \tag{1.33}$$

and just as above, we would like to create a solution on $\mathbb{R}$ by "pasting" them. But in order to assure the continuity, we need to solve the d-dimensional system of equation

$$C_1 v_1 + \cdots + C_m v_m = D_1 w_1 + \cdots + D_n w_n. \tag{1.34}$$

If $v_1, \cdots, v_m, w_1, \cdots, w_s$ is a basis for $\mathbb{R}^d$ for certain $s \leq n$ and $s + m = d$, then we can rewrite (1.34) to

$$C_1 v_1 + \cdots + C_m v_m + D_1 w_1 + \cdots + D_s w_s = D_{s+1} w_{s+1} + \cdots + D_n w_n. \tag{1.35}$$

We see that a solution to this equation requires d fixed parameters, which leaves us only $m + n - d$ free parameters to choose. Hence, we conclude

$$\dim(\ker(\Lambda_L)) = m + n - d. \tag{1.36}$$

In case $\dim\ker((\Lambda_L)^*) = 0$, we would have calculated the index by only considering the eigenvalues.

This intuition turns out to hold in a more general case. It is shown in [24, Theorem 2.1] that for linear ODE $x'(\xi) = A(\xi)x(\xi)$ such that $\lim_{\xi\to\pm\infty} A(\xi)$ both exist, the index can be calculated by the formula

$$\text{index} = \dim W^u(A_-) - \dim W^u(A_+), \tag{1.37}$$

where $A_\pm := \lim_{\xi\to\pm\infty} A(\xi)$ and $\dim W^u(A)$ is the number of unstable eigenvalues (counting with multiplicity) of a matrix A. Note that formula (1.37) is exactly (1.36) with $\dim W^u(A_-) = n$ and $\dim W^u(A_+) = d - m$. Formula (1.37) also shows that the calculation of the index is only a matter of finding the roots of the characteristic equation, which allows us to avoid solving the differential equation itself. Furthermore, it shows that the index depends only on the limiting matrices, which is not very surprising since only the asymptotic behavior of a continuous solution determines its boundedness on $\mathbb{R}$.

However, in case we consider an equation which has a shift in its argument, for example

$$\dot{x}(t) = x(t+1), \tag{1.38}$$

then, the formula in (1.37) does not hold. This is typically because the characteristic equation, which is defined as $\Delta(s) = s - e^s$, has infinitely many zeros in the complex plane. As a consequence, one of the dimension in (1.37) can be infinite.
Hence, we need a generalized version of (1.37). This is done by counting the number of eigenvalues that crossed the imaginary axis during a certain continuous path (See Theorem B and C, Section 3).

1.2.3 Exponential dichotomy

In ODE, we know that the initial value problem is well-posed. In other words, given a specific initial value condition, one can find a unique solution that satisfy such condition. For MFDEs, we have a similar question. Suppose we have a continuous function $\phi \in C([-1,1];\mathbb{R})$ as a initial profile, can we extend such a ϕ to a solution of (1.4) on the whole real lines or at least on the half-lines $[0,\infty)$ or $(-\infty,0]$. Unfortunately, this is typically not the case. However, one can use exponential dichotomy to study the state space $C([-1,1];\mathbb{R})$, which is a powerful tool to analyze such ill-posed initial value problems and to construct traveling pulse solution to the discrete FitzHugh-Nagumo equations (see [18]). The idea is to split the initial profile ϕ into two components ϕ_1 and ϕ_2 that each can be extended on the half-lines, instead of trying to find a solution that extend ϕ itself. In other words, one has the splitting

$$C([-1,1];\mathbb{R}) = P \oplus Q \tag{1.39}$$

where Q contains all initial conditions $\psi \in C([-1,1];\mathbb{R})$ for which a bounded continuous function $x[\psi] : [-1,\infty) \to \mathbb{R}$ exists such that $x(\xi)$ is a solution for $\xi \geq 0$ and $x(\xi) = \psi(\xi)$ for $-1 \leq \xi \leq 1$. Similarly, P contains all initial conditions that can be extended on the negative half-lines $[-\infty, 0)$.

We say that the initial state space $C([-1,1];\mathbb{R})$ has a dichotomy if (1.39) holds. If in addition, we have the estimates

$$\|x[\psi](\xi)\| \leq Ce^{-\epsilon\xi}\|\psi\|_{C([-1,1];\mathbb{R})}, \tag{1.40}$$

for $\psi \in Q$ and similar estimations for the space P, then we say that $C([-1,1];\mathbb{R})$ has an exponential dichotomy. We shall see (Proposition 7.5) that in case the shifts for (2.1) are bounded, we have an exponential dichotomy for a certain initial state space.

2 Settings, definitions and main results

2.1 Settings

The system that we are going to study is given by

$$x'(\xi) = \sum_{j=1}^{\infty} A_j(\xi)x(\xi + r_j) + h(\xi) \tag{2.1}$$

for which $r_j \in \mathbb{R}$ are called the **shifts**. As a matter of notation. we require that

$$\begin{aligned} &r_1 = 0 \\ &r_j \neq r_k \text{ if } j \neq k. \end{aligned} \tag{2.2}$$

For the matrices $A_j(\xi) : \mathbb{R} \to \mathbb{C}^{d\times d}$, we require that they are uniformly bounded and we define

$$\|A_j\| := \sup_{\xi\in\mathbb{R}} |A_j(\xi)|. \tag{2.3}$$

Moreover, we require

$$\sum_{j=1}^{\infty} \|A_j\| e^{l|r_j|} < \infty \tag{2.4}$$

for a certain $l > 0$. Note that this is the same condition as proposed in [3].

For $\xi \in \mathbb{R}$, we define the linear functional $L(\xi) : L^\infty(\mathbb{R}, \mathbb{C}^d) \to \mathbb{C}^d$ with

$$L(\xi)(\phi) = \sum_{j=1}^{\infty} A_j(\xi)\phi(r_j),\ \ \phi \in L^\infty(\mathbb{R}, \mathbb{C}^d). \tag{2.5}$$

Then we can write (2.1) as

$$x'(\xi) = L(\xi)x_\xi \tag{2.6}$$

with $x_\xi(\theta) = x(\xi + \theta)$.
Note that L is well-defined because from (2.4) we know that $\sum_{j=1}^{\infty} \|A_j\| \leq \sum_{j=1}^{\infty} \|A_j\| e^{l|r_j|} < \infty$. Hence, we have that

$$\sum_{j=1}^{\infty} \|A_j(\xi)\phi(\xi + r_j)\| \leq \sum_{j=1}^{\infty} \|A_j\| \|\phi\|_{L^\infty} < \infty. \tag{2.7}$$

So indeed $\sum_{j=1}^{\infty} A_j(\xi)\phi(\xi + r_j)$ converges.

In case the matrices don't depend on ξ, we have the constant coefficient operator

$$L_0\phi = \sum_{j=1}^{\infty} A_{j,0}\phi(r_j). \tag{2.8}$$

Then (2.1) becomes

$$x'(\xi) = L_0 x_\xi + h(\xi) = \sum_{j=1}^{\infty} A_{j,0}x(\xi + r_j) + h(\xi) \tag{2.9}$$

or in the homogenous case

$$x'(\xi) = L_0 x_\xi. \tag{2.10}$$

Associated to equation (2.6) we define the linear operator $\Lambda_L : W^{1,p}(\mathbb{R}, \mathbb{C}^d) \to L^p(\mathbb{R}, \mathbb{C}^d)$ by

$$(\Lambda_L \phi)(\xi) = \phi'(\xi) - L(\xi)(\phi(\xi)) = \phi'(\xi) - \sum_{j=1}^{\infty} A_j(\xi)\phi(\xi + r_j). \tag{2.11}$$

The adjoint equation of (2.5) is the equation defined as

$$y'(\xi) = L^*(\xi)y_\xi, \tag{2.12}$$

where

$$L^*(\xi)\phi = -\sum_{j=1}^{\infty} A_j(\xi - r_j)^* \psi(\xi - r_j) \tag{2.13}$$

and $A_j(\xi - r_j)^*$ denotes the adjoint of the matrix $A_j(\xi - r_j)$.
For the adjoint equation we define the adjoint operator Λ_L^* of Λ_L to be

$$(\Lambda_L^* y)(\xi) = -y'(\xi) + L^*(\xi)y_\xi = -y'(\xi) - \sum_{j=1}^{\infty} A_j(\xi - r_j)^* y(\xi - r_j). \tag{2.14}$$

That is, $\Lambda_L^* = -\Lambda_{L^*}$. It is verified in [2, Proposition B.2] that the identity

$$\int_{-\infty}^{\infty} \overline{y(\xi)}(\Lambda_L x)(\xi)d\xi = \int_{-\infty}^{\infty} \overline{(\Lambda_L^* y)(\xi)}x(\xi)d\xi \tag{2.15}$$

holds for all $x \in W^{1,p}(\mathbb{R}, \mathbb{C}^d),\ y \in W^{1,q}(\mathbb{R}, \mathbb{C}^d)$, where $p^{-1} + q^{-1} = 1$.

2.2 Sobolev spaces

Recall the spaces of $\mathcal{L}^p$ functions

$$\mathcal{L}^p(\mathbb{R}, \mathbb{C}^d) = \{f : f \text{ is measurable and } \|f\|_p < \infty\}, \tag{2.16}$$

where

$$\|f\|_p = \left(\int_{\mathbb{R}} |f|^p d\lambda\right)^{\frac{1}{p}} \tag{2.17}$$

for $1 \leq p < \infty$ and λ is the Lebesgue measure. For $p = \infty$, we have

$$\|f\|_\infty = \text{esssup}|f|, \tag{2.18}$$

where

$$\text{esssup}|f| = \inf\{a \in \mathbb{R} : \lambda|f|^{-1}((a, \infty)) = 0\}. \tag{2.19}$$

Recall the equivalence relation $f \sim g \Leftrightarrow f = g,\ a.e.$. This gives us the quotient space
$L^p(\mathbb{R}, \mathbb{C}^d) := \mathcal{L}^p(\mathbb{R}, \mathbb{C}^d)/\sim$, for $1 \leq p \leq \infty$. For simplicity, we shall write L^p for the space $L^p(\mathbb{R}, \mathbb{C}^d)$.

For $1 \leq p \leq \infty$, we denote the Sobolev space $W^{1,p}$ as the set

$$W^{1,p} := \{f \in L^p : f \text{ is absolutely continuous and } f' \in L^p\}. \tag{2.20}$$

with the norm $\|.\|_{W^{1,p}}$ defined as

$$\|f\|_{W^{1,p}} := \|f\|_{L^p} + \|f'\|_{L^p}. \tag{2.21}$$

Furthermore, [4,Thm 8.8] yields that if $f \in W^{1,p}$ then also $f \in L^\infty$, and we have that

$$\|f\|_{L^\infty} \leq C\|f\|_{W^{1,p}} \tag{2.22}$$

for some $C > 0$.

2.3 Asymptotically hyperbolic system

For L_0 as in (2.8), we define the characteristic equation $\Delta_{L_0} : D \subset \mathbb{C} \to \mathbb{C}^{d\times d}$

$$\Delta_{L_0}(s) = sI - \sum_{j=1}^{\infty} A_{j,0}e^{sr_j}. \tag{2.23}$$

We have the following remarks regarding the characteristic equation (2.23).

Remark 2.1.

(1) In case the shifts are zero, we obtain the familiar characteristic equation $\Delta_{L_0}(s) = sI - A$ which is defined in Linear Algebra.

(2) Condition (2.4) ensures that the sum $\sum_{j=1}^{\infty} A_{j,0}e^{sr_j}$ converges absolutely for $|\mathrm{Re}(s)| < l$. Hence, $\Delta_{L_0}(s)$ is well-defined inside this strip.

(3) It is proven in [2, Proposition 2.17] that $s \mapsto \Delta_{L_0}(s)$ is holomorphic in the region $|Re(s)| < l$.

(4) By (3), it follows that $\det(\Delta_{L_0}(s))$ is holomorphic in the region $|\mathrm{Re}(s)| < l$, since it is a polynomial expression of holomorphic functions.

(5) We have that λ is a root of the equation $\det(\Delta_{L_0}(s)) = 0$ if and only if $x(\xi) = e^{\lambda\xi}v$ is a solution of (2.10) from some vector $v \neq 0$. Hence, we also call λ an eigenvalue.

(6) The characteristic equation satisfies the asymptotic formula

$$\Delta_{L_0}(s) = sI + O(1), \ |\mathrm{Im}(s)| \to \infty \tag{2.24}$$

uniformly in each vertical strip $|\mathrm{Re}(s)| \leq k < l$.

Definition 2.2. *We call a system with constant coefficients hyperbolic if for all $y \in \mathbb{R}$, we have*

$$\det(\Delta_{L_0}(iy)) \neq 0. \tag{2.25}$$

Lemma 2.3. *Recall the constant coefficient operator L_0 as in (2.8). Then, $\det(\Delta_{L_0}(s))$ has only finitely many zeros in the region $|\mathrm{Re}(s)| < l$.*

Proof. For $s \neq 0$ we have that $\Delta_{L_0}(s) = I - \frac{1}{s}\sum_{j=1}^{\infty} A_{j,0}e^{sr_j}$. From the asymptotic formula above it follows that $\|\frac{1}{s}\sum_{j=1}^{\infty} A_{j,0}e^{sr_j}\| < 1$ when $|\mathrm{Im}\ s| \geq M$ for certain $M > 0$. Then $\Delta_{L_0}(s)$ is invertible. So $\det(\Delta_{L_0}(s))$ has no zeros for $|\mathrm{Im}(s)|$ sufficiently large. Suppose now that in some compact region $|\mathrm{Re}(s)| \leq l$, $|\mathrm{Im}(s)| \leq M$, that the holomorphic function $\det(\Delta_{L_0}(s))$ has infinitely many zeros, which means there is a sequence of distinct zeros $\{\lambda_j\}_{j=1}^{\infty}$. By the Bolzano-Weierstrass theorem this sequence has a convergent subsequence. So the set $\{z \in \mathbb{C} : \det(\Delta_{L_0}(s)) = 0\}$ has an accumulation point. So by the identity theorem for analytic functions [5, Theorem III.3.2], we have that $\det(\Delta_{L_0}(s))$ is the zero function. By the asymptotic formula above, this is clearly a contradiction. □

Definition 2.4. *A system L is called asymptotic hyperbolic to $\pm\infty$ if there is a hyperbolic constant coefficient system $L_\pm$ with*

$$L_\pm(\phi(\xi)) = \sum_{j=1}^{\infty} A_{j,\pm}\phi(\xi + r_j) \tag{2.26}$$

and an $M_\pm(\xi)$ such that

$$L(\xi)(\phi(\xi)) = L_\pm(\phi(\xi)) + M_\pm(\xi)(\phi(\xi)) \tag{2.27}$$

with

$$\lim_{\xi\to\pm\infty} \|M_\pm(\xi)\| = 0. \tag{2.28}$$

2.4 Main Results

Before we establish our main results, we will first state the following theorem that is proved by J.M.Bos [2, Theorem 1.4], which is a generalization of [1, Theorem A].

Theorem A. (The Fredholm Alternative) *Assume L as in (2.5) is asymptotically hyperbolic. Then for each p with $1 \leq p \leq \infty$, the operator Λ_L from $W^{1,p}$ to L^p is a Fredholm operator. The kernel $\mathcal{K}_L^p \subset W^{1,p}$ of Λ_L is independent of p, so we denote $\mathcal{K}_L^p = \mathcal{K}_L$, and similarly $\mathcal{K}_{L^*}^p = \mathcal{K}_{L^*}^p = \mathcal{K}_{L^*}$ for the kernel of the operator Λ_L^* associated to the adjoint L^*. The range $\mathcal{R}_L^p \subset L^p$ of Λ_L in L^p is given by*

$$\mathcal{R}_L^p = \{h \in L^p | \int_{-\infty}^{\infty} \overline{y(\xi)} h(\xi) d\xi = 0, \text{ for all } y \in \mathcal{K}_{L^*}\}. \tag{2.29}$$

In particular

$$\dim \mathcal{K}_{L^*} = codim \mathcal{R}_L^p, \ \dim \mathcal{K}_L = codim \mathcal{R}_{L^*}^p, \ ind(\Lambda_L) = -ind(\Lambda_L^*), \tag{2.30}$$

where ind denotes the Fredholm index.
Finally, when $L = L_0$ is a hyperbolic constant coefficient operator, we have

$$codim \mathcal{R}_{L_0}^p = 0, \ \dim \mathcal{K}_{L_0} = 0, \ ind(\Lambda_{L_0}) = 0. \tag{2.31}$$

In particular, Λ_{L_0} is an isomorphism.

Note that in the statement of Theorem A we have that $\mathcal{K}_L^p \subset W^{1,p} \subset L^p$, for each p, and similarly for K_{L^*}. In particular, $K_{L^*} \subset L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$. So by Hölder's inequality, the integral in (2.29) exists.

With theorem A as our tool, we are able to establish the following two theorems which are generalizations of [1, Theorem B] and [1, Theorem C].

Theorem B. (The Cocycle Property) *Assume L is asymptotically hyperbolic. Then the Fredholm index of Λ_L depends only on the limiting operators $L_\pm$, namely the limits of $L(\xi)$ as $\xi \to \pm\infty$. Denoting*

$$ind(\Lambda_L) = \iota(L_-, L_+), \tag{2.32}$$

we have that

$$\iota(L_1, L_2) + \iota(L_2, L_3) = \iota(L_1, L_3) \tag{2.33}$$

for any triplet (L_1, L_2, L_3) of hyperbolic constant coefficient operators.

As mentioned earlier, in the case of an ODE $x'(\xi) = A(\xi)x(\xi)$, the index can be calculated by the formula [24, Theorem 2.1]

$$\iota(L_-, L_+) = \dim W^u(L_-) - \dim W^u(L_+) \tag{2.34}$$

which immediately implies the cocycle property stated above. However, in the general case with shifts, formula (2.34) does not hold. This is typically because $\dim W^u = \infty$ for either L_+ or L_-. Nevertheless, the index can be calculated by the spectral flow formula, which generalizes (2.34).

Theorem C. (The Spectral Flow Property) *Let $L_{0,\eta}^\rho$, for $\eta > 0$, $-1 \leq \rho \leq 1$, be a continuously varying one parameter family of constant coefficient operators such that the corresponding matrices satiesfy $\sum_{j=0}^{\infty} |A_{j,0}^\rho| e^{\eta |r_j|} < \infty$ for all $-1 \leq \rho \leq 1$, and suppose the operators $L_\pm = L_{0,\eta}^{\pm 1}$ are hyperbolic. Suppose further there are only finitely many values*

$$\{\rho_1, \rho_2, \cdots, \rho_J\} \subset (-1, 1) \tag{2.35}$$

of ρ for which $L_{0,\eta}^\rho$ is not hyperbolic. Then

$$\iota(L_-, L_+) = -cross(L_{0,\eta}^\rho) \tag{2.36}$$

is the net number of eigenvalues of (2.10) which cross the imaginary axis from left to right as ρ increases from -1 to $+1$.

Here, we use the same definition of $\mathrm{cross}(L^\rho)$ that is defined in [1] as follow: Let $\{\lambda_{j,k}\}_{k=1}^{K_j}$ denote those eigenvalues of equation (2.10), with $L_0 = L_0^{\rho_j}$, on the imaginary axis, $\mathrm{Re}(\lambda_{j,k}) = 0$. We list these eigenvalues with repetitions, according to their multiplicity as roots of the characteristic equation. Let M_j be the sum of their multiplicities. For ρ near ρ_j, with $\pm(\rho - \rho_j) > 0$, this equation has exactly M_j eigenvalues (counting multiplicity) near the imaginary axis due to Rouché's Theorem (see Proposition 6.1 for more justification), $M_j^{L\pm}$ with $\mathrm{Re}(\lambda) < 0$, and $M_j^{R\pm}$ with $\mathrm{Re}(\lambda) > 0$, where $M_j^{L\pm} + M_j^{R\pm} = M_j$. The net crossing number of eigenvalues at $\rho = \rho_j$ is therefore given by $M_j^{R+} - M_j^{R-}$. As such, we define

$$\mathrm{cross}(L^\rho) = \sum_{j=1}^{J} (M_j^{R+} - M_j^{R-}). \tag{2.37}$$

Note that in the case of ODE, the characteristic equation $\Delta(s, \rho) = s - A(\rho)$ has only finitely many zeros during the entire continuous path that is parametrized with $-1 \le \rho \le 1$. Hence, we can calculate the cross by only considering the difference in the number of unstable eigenvalues between the limit operators, which are the endpoints of such continuous path. This gives exactly (2.34).

These two results are also given in the paper of Scheel and Faye [3]. But instead of using abstract compact operator theorems as in [3], we will follow the outline in [1] by examining the Laplace transform of (2.1) and obtain the above two results through explicit calculations. Furthermore, we will work out some of the claims that were made in [1]. We shall also see that by considering the Laplace transform of (2.1), we are able to prove similar exponential dichotomy result as in [10], which is stated below.

Proposition A. *Let L be as defined in (2.8) with coefficients that satisfies condition (2.4) for a certain $\tilde{\eta} > 0$ and suppose that the shifts as in (2.2) are bounded and the closure of the set of shifts are countable. Pick $\eta \in \mathbb{R}$ with $|\eta| < \tilde{\eta}$ such that the characteristic equation $\Delta_L(s)$ as in (2.23) is invertible for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \eta$. Then we have the splitting*

$$C([r_{\min}, r_{\max}]; \mathbb{C}^d) = P_L(\eta) \oplus Q_L(\eta) \tag{2.38}$$

where $Q_L(\eta)$ is the set of right half-line solutions $x : [r_{\min}, \infty) \to \mathbb{C}^d$ such that $x(\xi)$ satisfies equation (2.10) and the bound $|x(\xi)| \le Ke^{\eta\xi}$, for certain $K > 0$ and for all $\xi \ge 0$. Similarly, $P_L(\eta)$ is the set of left half-line solutions $y : (-\infty, r_{\max}] \to \mathbb{C}^d$ such that $y(\xi)$ satisfies (2.10) and has the bound $|x(\xi)| \le Ke^{\eta\xi}$ for all $\xi \le 0$.

The proof of this proposition also follow the same idea as in [10, Proposition 5.6].

3 Preliminaries

3.1 Roots of analytic functions

Theorem 3.1 (Rouché's theorem)**.** *Suppose $f, g : U \to \mathbb{C}$ are analytic functions on an open set $U \subset \mathbb{C}$. If $\overline{D_0} \subset U$ is bounded and has continuous boundary and if for each $z \in \partial D_0$,*

$$|f(z) - g(z)| < |f(z)| + |g(z)|, \tag{3.1}$$

holds, then the number of zeros of f in D_0 equals the number of zeros of g in D_0, counting the multiplicities.

Proof. See [6, Lemma 2.3.3]. □

3.2 Interchanging limits

We recall the following theorems that allow us to exchange integrals, sums and limits under certain conditions, which will be used in many of the proofs in this thesis.

Theorem 3.2 (Tonelli-Fubini)**.** *Let $(X, A_1, \mu_1), (Y, A_2, \mu_2)$ be sigma-finite measure spaces and $f : X \times Y \to [-\infty, \infty]$ be $\mathcal{A}_1 \otimes \mathcal{A}_2$-measurable. If one of the three conditions*

$$\int_X \int_Y |f(x, y)| d(\mu_1 \otimes \mu_2) < \infty, \quad \int_X \int_Y |f(x, y)| d\mu_2 d\mu_1 < \infty, \quad \int_X \int_Y |f(x, y)| d\mu_1 d\mu_2 < \infty \tag{3.2}$$

holds, then we have

$$\int_X \int_Y f(x, y) d(\mu_1 \otimes \mu_2) = \int_X \int_Y f(x, y) d\mu_2 d\mu_1 = \int_Y f(x, y) d\mu_1 d\mu_2. \tag{3.3}$$

Proof. See [12, Tonelli and Fubini Theorem]. □

Theorem 3.3 (Interchanging limits and sums)**.** *Let $F(x) := \sum_{n=1}^{\infty} f_n(x)$ with $f_n : [a, b] \to \mathbb{R}$ be continuous functions. If the sum converges uniformly on $[a, b]$, then F is also continuous and*

$$\lim_{x \to c} F(x) = \sum_{n=1}^{\infty} \lim_{x \to c} f_n(x). \tag{3.4}$$

Proof. Pick an $N \in \mathbb{N}$ such that $\left|\sum_{n \geq N+1}^{\infty} f_n(x)\right| < \epsilon/3$ for all $x \in [a, b]$. Choose a sufficiently small $\delta > 0$ such that for $|x - c| < \delta$, we have

$$\left|\sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{\infty} f_n(c)\right| \leq \left|\sum_{n=1}^{N} f_n(x) - f_n(c) + \sum_{n=N+1}^{\infty} f_n(x) - f_n(c)\right| \leq 3 \cdot \epsilon/3 = \epsilon. \tag{3.5}$$

□

Remark 3.4. *For limits like $x \to -\infty$ and $x \to \infty$, we may also bring it inside the summation if the sum converges uniformly on respectively $(-\infty, c]$ and $[d, \infty)$ for certain $c, d \in \mathbb{R}$.*

Theorem 3.5 (Termwise Differentiation)**.** *Suppose that $f_n : [a, b] \to \mathbb{R}$ is C^1 on $[a, b]$ (at the endpoints of a and b this means one-sided derivative) for $n \geq 1$ and suppose furthermore that*

(1) $\sum_{n=1}^{\infty} f_n(x_0)$ converges pointwise for all $x_0 \in [a, b]$.

(2) $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[a, b]$.

Then, for the function $F(x) := \sum_{n=1}^{\infty} f_n(x)$, we have that

(3) $F(x)$ is differentiable for $x \in [a, b]$ with $F'(x) = \sum_{n=1}^{\infty} f_n'(x)$.

(4) $F(x)$ converges uniformly on $[a,b]$.

Proof. See [13, Theorem 2]. $\square$

Theorem 3.6 (Termwise differentiation for analytic functions)**.** *Let $f_1, f_2, \cdots : D \to \mathbb{C}$, $D \subset \mathbb{C}$ open and non-empty, be a sequence of analytic functions which converges locally uniformly. Then the limit function f is analytic and the sequence of the derivatives (f_n') converges locally uniformly to f'.*

Proof. See [5, Theorem III.1.3]. $\square$

Lemma 3.7. *Let*

$$f_0, f_1, f_2, \cdots : D \to \mathbb{C},\ D \subset \mathbb{C} \tag{3.6}$$

be a sequence of continuous functions which converges locally uniformly to f. Then for any piecewise smooth curve $\alpha : [a,b] \to D$, we have

$$\lim_{n\to\infty} \int_\alpha f_n(\zeta) d\zeta = \int_\alpha f(\zeta) d\zeta. \tag{3.7}$$

Proof. See [5, Remark III.1.2]. $\square$

Lemma 3.8 (Differentiation lemma)**.** *Let I be a non-degenerate (meaning, containing more than one point) interval in $\mathbb{R}$, and $f : I \times \Omega \to \mathbb{R}$ be a function with the properties*

(a) $\omega \mapsto f(x,\omega)$ is μ-integrable for each $x \in I$;

(b) $x \mapsto f(x,\omega)$ is differentiable on I for each $\omega \in \Omega$, the derivative at x being denoted by $f'(x,\omega)$;

(c) there is a μ-integrable function $h \geq 0$ on Ω such that

$$|f'(x,\omega)| \leq h(\omega) \quad \text{for all } (x,\omega) \in I \times \Omega. \tag{3.8}$$

Then, the function defined on I by

$$\varphi(x) := \int f(x,\omega)\mu(d\omega) \tag{3.9}$$

is differentiable, for each $x \in I$ the function $\omega \mapsto f'(x,\omega)$ is μ-integrable, and

$$\varphi'(x) = \int f'(x,\omega)\mu(d\omega) \quad \text{for every } x \in I. \tag{3.10}$$

Proof. See [12, Lemma 16.2]. $\square$

3.3 Fourier Transform

We recall the formula for the Fourier Transform $\hat{f} \in L^2$ (or sometimes denoted as $\mathcal{F}(f)$) of a function $f \in L^2$, and the formula for the inverse transform $\check{g}$ of any $g \in L^2$ (or sometimes denoted as $\mathcal{F}^{-1}(f)$), which is given by

$$\hat{f}(\eta) = \int_{-\infty}^{\infty} e^{-i\xi\eta} f(\xi) d\xi, \quad \check{g}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi\eta} g(\eta) d\eta. \tag{3.11}$$

Following [1,page 8], if f or $\hat{f}$ is not integrable, then we define the integrals in (3.11) differently. For $f \in L^2$, we define (3.11) as

$$\hat{f}(\eta) = \lim_{\Omega\to\infty} \int_{-\Omega}^{\Omega} e^{-i\xi\eta} f(\xi) d\xi, \tag{3.12}$$

which converges according to [1,page 8].

The Fourier transform can be also defined in the language of distributions. For f a tempered distribution, the Fourier Transformation $\mathcal{F}(f)$ of f is given by

$$(\mathcal{F}(f), \zeta) = (f, \mathcal{F}(\zeta)) \tag{3.13}$$

[2, Definition 2.12]. The Fourier Transform is an isometric isomorphism on L^2 [7, Chapter 5.64].

3.4 Laplace transform and shifting the path of integration

For a function $f : [0, \infty) \to \mathbb{C}^d$ that satisfies $f(\xi) = O(e^{-a\xi})$ as $\xi \to \infty$ (we used the big oh notation, which means that $e^{a\xi} f(\xi)$ is bounded as $\xi \to \infty$), we define the Laplace transform $\tilde{f}$ of f (sometimes also denoted as $\mathcal{L}(f)$) to be

$$\tilde{f}(s) = \int_0^\infty e^{-s\xi} f(\xi) d\xi, \tag{3.14}$$

for complex s satisfying $\mathrm{Re}(s) > -a$. In fact, $\tilde{f}$ is holomorphic for such s. The inverse transform is given by

$$f(\xi) = \frac{1}{2\pi i} \lim_{\Omega \to \infty} \int_{k-i\Omega}^{k+i\Omega} e^{s\xi} \tilde{f}(s) ds, \tag{3.15}$$

for any $k > -a$, with $\xi \geq 0$. The integral (3.15) is taken in the Fourier sense as explained above. This Laplace transform is related to the Fourier transform as follows: let $g \in L^2$ be given by $g(\xi) = e^{-k\xi} f(\xi)$ for $\xi \geq 0$, and $g(\xi) = 0$ for $\xi < 0$. Then $\hat{g}(\eta) = \tilde{f}(k + i\eta)$. With this last relation, one can easily derive (3.15) from the inverse Fourier transform.

For a meromorphic function g in a region of the complex plane, and $\lambda \in \mathbb{C}$ a pole of g in that region, we recall the residue theorem [5, Theorem III.6.3.]

$$\mathrm{res}(g, \lambda) = \frac{1}{2\pi i} \int_{|s-\lambda|=\epsilon} g(s) ds$$

with ϵ sufficiently small.

Following [1,page 8], for $f(\xi) = O(e^{-a\xi})$ as $\xi \to \infty$ such that for some $b \geq a$ the Laplace transform $\tilde{f}$ is meremorphic in the closed half plane $\mathrm{Re}(s) \geq -b$ and is holomorphic on the vertical line $\mathrm{Re}(s) = -b$, then we may shift the path of integration in (3.15) to the line $\mathrm{Re}(s) = -b$ and obtain

$$f(\xi) = \sum \mathrm{res}(e_\xi \tilde{f}, \lambda) + \frac{1}{2\pi i} \lim_{\Omega \to \infty} \int_{-b-i\Omega}^{-b+i\Omega} e^{s\xi} \tilde{f}(s) ds, \tag{3.16}$$

where for $\xi \in \mathbb{C}$, $e_\xi : \mathbb{C} \to \mathbb{C}$ is defined as

$$e_\xi(s) = e^{\xi s}. \tag{3.17}$$

The sum in (3.16) is taken over all poles λ of $\tilde{f}$ in the strip $-b < \mathrm{Im}(s) \leq -a$. Furthermore, we need the next three extra conditions in order to shift the path of integration

- $\tilde{f}$ has only finitely many poles in $-b < \mathrm{Re}(s) \leq -a$;
- $\tilde{f}(s) \to 0$ uniformly in the strip $-b \leq \mathrm{Re}(s) \leq k$, as $|\mathrm{Im}(s)| \to \infty$;
- the function $\eta \mapsto \tilde{f}(-b + i\eta)$ belongs to L^2.

4 Proof of Theorem B

To prove theorem B, we shall use a classical result about Fredholm operators, which states that the Fredholm index is a locally constant function. This means that on a connected domain, the Fredholm index is a constant function. As a consequence, we shall see that the Fredholm index stays constant if we vary the Fredholm operators continuously.

Lemma 4.1. *The set* $\text{Fred}(X, Y)$ *of Fredholm operators between Banach spaces* X *and* Y *is a open subset of* $B(X, Y)$*, the set of bounded linear operators from* X *to* Y*. Furthermore, the function*

$$\text{ind} : \text{Fred}(X, Y) \to \mathbb{Z}$$

is locally constant.

The proof of this can be found in [9]. We will also use the following two results.

Lemma 4.2. *Let* $f : X \to Y$ *be a locally constant function between two topological spaces* X *and* Y*. If* X *is connected, then* f *is constant on* X*.*

Proof. Let $x_0 \in X$ and $c = f(x_0)$. Consider the set $A := \{x \in X | f(x) = c\}$. By the definition of locally constant, for each $x \in X$, there exists an open $U_x \subset X$ such that $x \in U_x$ and $f(U_x) = f(x)$. Hence, we have

$$A \subset \bigcup_{x \in A} U_x \subset A. \tag{4.1}$$

Thus, A is open. We now consider $B := X \setminus A$. Then we also have

$$B \subset \bigcup_{x \in B} U_x \subset B. \tag{4.2}$$

Hence A is also closed. By the definition of connectivity, we have that $A = X$ or $A = \emptyset$. Since $x_0 \in A$, we have $A = X$. Thus f is constant on X. □

Lemma 4.3. *Let* X, Y, Z *be topological spaces and* $f : X \to Y$ *and* $g : Y \to Z$ *be such that* g *is locally constant and* f *is continuous. Then* $g \circ f$ *is also locally constant.*

Proof. Let $x_0 \in X$, then $f(x_0) \in Y$. Because g is locally constant, there exists a $U_{f(x_0)} \subset Y$ open that contains $f(x_0)$ and such that $g(U_{f(x_0)}) = g(f(x_0))$. Since f is continuous, we have that $V_{x_0} := f^{-1}(U_{f(x_0)}) \subset X$ is open and contain x_0. By definition we have $g(f(V_{x_0}) = g(f(x_0))$. Hence $g \circ f$ is indeed also locally constant. □

Sketch of the proof of Theorem B. The proof for Theorem B is identically to [1, Theorem B]. Therefore, we will only give a sketch of this proof and work out some of the claims in the proof. To prove that the Fredholm index of Λ_L depends only on the limiting operators $L_\pm$. we construct the following (continuous) homotopy

$$L^\rho(\xi) = (1 - \rho)L^0(\xi) + \rho L^1(\xi) \tag{4.3}$$

between two operators L^0, L^1 that have the same limit operators, Then L^ρ is asymptotically hyperbolic, since we have the limit

$$\lim_{\xi \to \infty} L^\rho(\xi) = (1 - \rho) \lim_{\xi \to \infty} L^0(\xi) + \rho \lim_{\xi \to \infty} L^1(\xi) = (1 - \rho)L_+ + \rho L_+ = L_+ \tag{4.4}$$

and the same holds for $\xi \to -\infty$.
Therefore, by theorem A, and the above Lemma's, we have that $\text{ind}(\Lambda_{L^0}) = \text{ind}(\Lambda_{L^1})$. So indeed the index depends only on the limiting operators.

To prove the cocycle property (2.33), we will again construct a homotopy. Let L_1, L_2 and L_3 as given in the statement of the theorem. For $0 \leq \rho \leq 1$, consider the system $z'(\xi) = L^\rho(\xi) z_\xi$, given by

$$z'(\xi) = (1 - H(\xi)) L_{1,2} z_\xi + H(\xi) R(\rho) L_{2,3} R(-\rho) z_\xi \tag{4.5}$$

in twice the number 2d of variables, where

$$z(\xi) = \begin{pmatrix} x(\xi) \\ y(\xi) \end{pmatrix}, \; L_{j,k} z_\xi = \begin{pmatrix} L_j x_\xi \\ L_k y_\xi \end{pmatrix}, \; R(\rho) = \begin{pmatrix} \cos(\frac{\pi\rho}{2}) I_d & \sin(\frac{\pi\rho}{2}) I_d \\ -\sin(\frac{\pi\rho}{2}) I_d & \cos(\frac{\pi\rho}{2}) I_d \end{pmatrix}, \tag{4.6}$$

with H is the Heaviside function, thus $H(\xi) = 0$ if $\xi < 0$ en $H(\xi) = 1$ if $\xi \geq 0$, and I_d is the $d \times d$ identity matrix.
We will check that (4.5) is asymptotically hyperbolic for each ρ, which is only claimed in the proof of [1, Theorem B].
For $\xi \to -\infty$, we have

$$\lim_{\xi \to -\infty} (1 - \alpha(\xi)) L_{1,2} + \alpha(\xi) R(\rho) L_{2,3} R(-\rho) = L_{1,2}. \tag{4.7}$$

We will show that $L_{1,2}$ is hyperbolic, which by definition means that $\det(sI_{2d} - \sum_{j=1}^\infty B_j e^{s r_j}) \neq 0$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \neq 0$ and B_j the matrices corresponds to the operator $L_{1,2}$. Note that we have the following expression for B_j.

$$B_j = \begin{pmatrix} A_j^1 & 0 \\ 0 & A_j^2 \end{pmatrix}, \tag{4.8}$$

where A_j^i is the j-th matrix of system L^i. Hence, we have

$$\begin{aligned} \det(sI_{2d} - \sum_{j=1}^\infty B_j e^{s r_j}) &= \det \begin{pmatrix} sI_d - \sum_{j=1}^\infty A_j^1 e^{s r_j} & 0 \\ 0 & sI_d - \sum_{j=1}^\infty A_j^2 e^{s r_j} \end{pmatrix} \\ &= \det(\Delta_{L^1}(s)) \cdot \det(\Delta_{L^2}(s)), \end{aligned} \tag{4.9}$$

since L_1 and L_2 are hyperbolic, this gives that $L_{1,2}$ is indeed hyperbolic as well.
For the $+\infty$ limit operator,

$$\lim_{\xi \to \infty} (1 - \alpha(\xi)) L_{1,2} + \alpha(\xi) R(\rho) L_{2,3} R(-\rho) = R(\rho) L_{2,3} R(-\rho). \tag{4.10}$$

We want to show that $K := R(\rho) L_{2,3} R(-\rho)$ is hyperbolic for each ρ. Denote C_j for the matrices corresponding to the system K. Then, we have the following expression for C_j

$$C_j = \begin{pmatrix} R(\rho) A_j^2 R(-\rho) & 0 \\ 0 & R(\rho) A_j^3 R(-\rho) \end{pmatrix}. \tag{4.11}$$

Hence we have

$$\begin{aligned} \det(sI_{2d} - \sum_{j=1}^\infty C_j e^{s r_j}) &= \det \begin{pmatrix} sI_d - \sum_{j=1}^\infty R(\rho) A_j^2 R(-\rho) e^{s r_j} & 0 \\ 0 & sI_d - \sum_{j=1}^\infty R(\rho) A_j^3 R(-\rho) e^{s r_j} \end{pmatrix} \\ &= \det \begin{pmatrix} sI_d - R(\rho) \left(\sum_{j=1}^\infty A_j^2 e^{s r_j} \right) R(-\rho) & 0 \\ 0 & sI_d - R(\rho) \left(\sum_{j=1}^\infty A_j^3 e^{s r_j} \right) R(-\rho) \end{pmatrix} \\ &= \det \left(sI_d - R(\rho) \left(\sum_{j=1}^\infty A_j^2 e^{s r_j} \right) R(-\rho) \right) \cdot \det \left(sI_d - R(\rho) \left(\sum_{j=1}^\infty A_j^3 e^{s r_j} \right) R(-\rho) \right) \\ &= \det(\Delta_{L_2}(s)) \cdot \det(\Delta_{L_3}(s)), \end{aligned} \tag{4.12}$$

where the last equality follows from the fact that $R(-\rho) = (R(\rho))^{-1}$ for each $0 \leq \rho \leq 1$ (note that it has the form of a rotation matrix) and that similar matrices have the same eigenvalues. Hence, we see that K is indeed hyperbolic and thus (4.5) gives a homotopy of Fredholm operators.
The proof now continues identically as in [1, Theorem B].

$\square$

5 Asymptotic behavior of solutions

Before we proceed to the proof of Theorem C, we will first give a result about the asymptotic behavior of solutions of the inhomogeneous constant coefficient system (2.9). This result will be useful for proving Theorem C. It will also give us important tools to establish the results in upcoming sections regarding exponential dichotomies.

We will first show that if λ is an eigenvalue of the homogeneous system (2.10) and if f is any $\mathbb{C}^d$ valued function which is holomorphic in a neighbourhood of λ in $\mathbb{C}$, then for $\epsilon > 0$ sufficiently small, the function

$$x(\xi) = \operatorname{res}(e_\xi \Delta_{L_0}^{-1} f, \lambda) = \frac{1}{2\pi i} \int_{|s-\lambda|=\epsilon} e^{s\xi} \Delta_{L_0}(s)^{-1} f(s) ds \tag{5.1}$$

is an eigensolution of (2.10) corresponding to λ. Here, e_ξ is as in (3.17) and with eigensolution we mean a solution of the form $e^{\lambda\xi} p(\xi)$ for $p(\xi)$ a polynomial. We will first show that x defined as in (5.1) is a solution for (2.10).

We have that

$$x'(\xi) = \frac{1}{2\pi i} \int_{|s-\lambda|=\epsilon} e^{s\xi} s \Delta_{L_0}(s)^{-1} f(s) ds, \quad x(\xi + r) = \frac{1}{2\pi i} \int_{|s-\lambda|=\epsilon} e^{s\xi} e^{sr} \Delta_{L_0}(s)^{-1} f(s) ds. \tag{5.2}$$

Note we can bring $\frac{d}{d\xi}$ into the integral due to the Leibniz rule. By direct substitution into (2.10), we have on the right-hand side

$$\sum_{j=1}^{\infty} A_{j,0} x(\xi + r_j) = \frac{1}{2\pi i} \sum_{j=1}^{\infty} A_{j,0} \int_{|s-\lambda|=\epsilon} e^{s\xi} e^{sr_j} \Delta_{L_0}(s)^{-1} f(s) ds. \tag{5.3}$$

Recall that $\Delta_{L_0}(s)$ is an analytic function in a certain region (which contains λ). So the function $h(s,\xi) := e^{s\xi} e^{sr_j} \Delta_{L_0}(s)^{-1} f(s)$ is an analytic function of s in a neighbourhood of λ except at the point $s = \lambda$. But since we are integrating over the curve $|s - \lambda| = \epsilon$, we may switch the summation and the integral above due to Lemma 3.7. This yields

$$\begin{aligned} \sum_{j=1}^{\infty} A_{j,0} x(\xi + r_j) &= \int_{|s-\lambda|=\epsilon} \sum_{j=1}^{\infty} A_{j,0} e^{s\xi} e^{sr_j} \Delta_{L_0}(s)^{-1} f(s) ds \\ &= \int_{|s-\lambda|=\epsilon} (sI - \Delta_{L_0}(s)) e^{s\xi} \Delta_{L_0}(s)^{-1} f(s) ds \\ &= \int_{|s-\lambda|=\epsilon} s e^{s\xi} \Delta_{L_0}(s)^{-1} f(s) ds - \int_{|s-\lambda|=\epsilon} e^{s\xi} f(s) ds. \end{aligned} \tag{5.4}$$

Note that the latter integral is zero because it is a integral of an analytic function over a closed curve. This yields

$$\sum_{j=1}^{\infty} A_{j,0} x(\xi + r_j) = \int_{|s-\lambda|=\epsilon} s e^{s\xi} \Delta_{L_0}(s)^{-1} f(s) ds = x'(\xi). \tag{5.5}$$

Hence, $x(\xi)$ indeed satisfies the homogeneous equation. Now we want to show that $x(\xi)$ is an eigensolution, i.e. that is of the form $x(\xi) = e^{\lambda\xi} p(\xi)$ for some polynomial p. This is done by substituting the Taylor and Laurent series of the functions

$$e^{s\xi} = e^{\lambda\xi} \sum_{j=1}^{\infty} \frac{(s-\lambda)^j \xi^j}{j!}, \quad \Delta_{L_0}(s)^{-1} f(s) = \sum_{j=-M}^{\infty} C_j (s-\lambda)^j. \tag{5.6}$$

Substituting these power series into $x(\xi)$ gives

$$\begin{aligned} x(\xi) &= \frac{1}{2\pi i}\int_{|s-\lambda|=\epsilon}\left(e^{\lambda\xi}\sum_{j=0}^{\infty}\frac{(s-\lambda)^j\xi^j}{j!}\right)\cdot\left(\sum_{j=-M}^{\infty}C_j(s-\lambda)^j\right)ds \\ &= e^{\lambda\xi}\frac{1}{2\pi i}\int_{|s-\lambda|=\epsilon}\left(\sum_{j=0}^{\infty}\frac{(s-\lambda)^j\xi^j}{j!}\right)\cdot\left(\sum_{j=-M}^{\infty}C_j(s-\lambda)^j\right)ds. \end{aligned} \tag{5.7}$$

We have that the function $s\mapsto\sum_{j=0}^{\infty}\frac{(s-\lambda)^j\xi^j}{j!}\cdot\sum_{j=-M}^{\infty}C_j(s-\lambda)^j$ is an analytic function in a neighbourhood of λ with in λ a pole. So by the residue theorem, we obtain

$$x(\xi) = e^{\lambda\xi}\operatorname{res}(h,\lambda). \tag{5.8}$$

Note that the residue is equal to the coefficient of the $(s-\lambda)^{-1}$ term, which is a polynomial in ξ.

We are now able to establish the following result, which is similar to [1, Proposition 6.1].

Proposition 5.1. *Let $x:\mathbb{R}\to\mathbb{C}^d$ be a solution of equation (2.9) for almost all $\xi\in\mathbb{R}$, with some $h:\mathbb{R}\to\mathbb{C}^d$. Let $\eta>0$ be such that the system of (2.9) satisfies $\sum_{j=1}^{\infty}|A_{j,0}|e^{\eta|r_j|}<\infty$. Assume for some real numbers $-\eta<a<b<\eta$ that*

$$|x(\xi)|\le Ke^{-a|\xi|},\ \forall\ \xi\in\mathbb{R},\ h(\xi)=O(e^{-b\xi}),\ \xi\to\infty. \tag{5.9}$$

Then for every $\epsilon>0$, we have that

$$x(\xi)=z(\xi)+O(e^{-(b-\epsilon)\xi}),\ \xi\to\infty, \tag{5.10}$$

where z is an eigensolution corresponding to the set of eigenvalues

$$\Lambda=\{\lambda\in\mathbb{C}|-b<\ \operatorname{Re}\lambda\le -a, \det(\Delta_{L_0}(\lambda))=0\}. \tag{5.11}$$

The analogous result for $\xi\to-\infty$ also holds, for which the assumption is replaced by $a>b$ and the statement is replaced by $x(\xi)=z(\xi)+O(e^{-(b+\epsilon)\xi})$, $\xi\to-\infty$.

The proof for this proposition follows the same idea as in [1, Proposition 7.1], which can be summarized into the following steps,

(1) We apply the Laplace transform to equation (2.1) and rewrite the transformed equation into the form

$$\Delta_{L_0}(s)\tilde{x}(s)=\psi(s)+\tilde{h}(s), \tag{5.12}$$

where

$$\psi(s)=x(0)-\sum_{j=1}^{\infty}A_{j,0}\int_{-r_j}^{0}e^{-s\xi}x(\xi+r_j)d\xi. \tag{5.13}$$

We shall see that equation (5.12) will give us an extension of $\tilde{x}(s)$ to a meromorphic function.

(2) We will then apply the inverse Laplace transform to the function $\tilde{x}(s)$, where we will shift the path of integration to pick up the residues, which are eigensolutions according to (5.1). In other words, we shall obtain $x(\xi)=z(\xi)+w(\xi)$, where $z(\xi)$ is the residue and $w(\xi)$ the remaining term.

(3) Finally, we shall show that $w(\xi)$ has the desired bound.

We shall go through these steps individually in the following sections.

5.1 Step (1) of the proof

Let x be a solution that satisfies the conditions in the statement. We perform the Laplace transform of (2.1). For $|a| < \mathrm{Re}(s) \leq \eta$, the left hand side yields:

$$\begin{aligned}\mathcal{L}(x') = \int_0^\infty e^{-s\xi} x'(\xi) d\xi &= [e^{-s\xi} x(\xi)]_0^\infty + \int_0^\infty s e^{-s\xi} x(\xi) d\xi \\ &= -x(0) + s\tilde{x}(s).\end{aligned} \tag{5.14}$$

The right-hand side yields

$$\mathcal{L}\left(\sum_{j=1}^\infty A_{j,0} x(\xi + r_j) + h(\xi)\right) = \int_0^\infty e^{-s\xi} \sum_{j=1}^\infty A_{j,0} x(\xi + r_j) d\xi + \tilde{h}(s). \tag{5.15}$$

Proposition 5.2. *The Laplace transform of equation(2.1) can be written in the form*

$$\Delta_{L_0}(s)\tilde{x}(s) = \psi(s) + \tilde{h}(s) \tag{5.16}$$

with

$$\psi(s) = x(0) - \sum_{j=1}^\infty A_{j,0} \int_{-r_j}^0 e^{-s\xi} x(\xi + r_j) d\xi. \tag{5.17}$$

Proof. We will first show that the sum

$$\mathcal{I} := \sum_{j=1}^\infty A_{j,0} \int_{-r_j}^0 e^{-s\xi} x(\xi + r_j) d\xi \tag{5.18}$$

converges absolutely and uniformly for $|\mathrm{Re}(s)| \leq \eta$. This implies that $\psi(s)$ is an analytic function for $|\mathrm{Re}(s)| < \eta$, since it is an infinite sum of analytic functions that converges uniformly (recall Theorem 3.6). A first estimate yields

$$|\mathcal{I}| \leq \sum_{j=1}^\infty |A_{j,0}| \left| \int_{-r_j}^0 e^{-s\xi} x(\xi + r_j) d\xi \right| =: \sum_{j=1}^\infty |A_{j,0}| \mathcal{I}_j. \tag{5.19}$$

Note that if $|\mathrm{Re}(s)| \leq \eta$, we have $\mathrm{Re}(s) + a \leq \eta + a$ and $-\mathrm{Re}(s) + a \leq \eta + a$. Furthermore, for $r_j \geq 0$, we have that $0 \leq \xi + r_j \leq r_j$. Hence, we can estimate

$$\begin{aligned}\mathcal{I}_j \leq \int_{-r_j}^0 e^{-Re(s)\xi} |x(\xi + r_j)| d\xi &\leq \int_{-r_j}^0 e^{-Re(s)\xi} K e^{-a|\xi + r_j|} d\xi \\ &\leq \int_{-r_j}^0 e^{-Re(s)\xi} K e^{-a(\xi + r_j)} d\xi \\ &\leq K e^{-ar_j} \int_{-r_j}^0 e^{-(Re(s)+a)\xi} d\xi \\ &\leq K e^{-ar_j} \int_0^{r_j} e^{(Re(s)+a)\xi} d\xi \\ &\leq K e^{-ar_j} \int_0^{r_j} e^{(\eta+a)\xi} d\xi \\ &\leq \frac{K}{\eta + a} e^{\eta|r_j|}.\end{aligned} \tag{5.20}$$

In contrast, for $r_j < 0$, we have $r_j \leq \xi + r_j \leq 0$. Hence, we can estimate

$$
\begin{aligned}
\mathcal{I}_j \leq \left| \int_0^{|r_j|} e^{-s\xi} x(\xi + r_j) d\xi \right| &\leq \int_0^{|r_j|} e^{-Re(s)\xi} |x(\xi + r_j)| d\xi \\
&\leq \int_0^{|r_j|} e^{-Re(s)\xi} K e^{-a|\xi + r_j|} d\xi \\
&\leq \int_0^{|r_j|} e^{-Re(s)\xi} K e^{a(\xi + r_j)} d\xi \\
&\leq K e^{a r_j} \int_0^{|r_j|} e^{(-Re(s)+a)\xi} d\xi \\
&\leq K e^{-a|r_j|} \int_0^{|r_j|} e^{(\eta+a)\xi} d\xi \\
&\leq \frac{K}{\eta + a} e^{\eta |r_j|}.
\end{aligned} \tag{5.21}
$$

Note that $|a| \neq \eta$ so we are not dividing by zero. We now make the following estimation

$$
\sum_{j=1}^{\infty} |A_{j,0}| \mathcal{I}_j \leq \sum_{j=1}^{\infty} |A_{j,0}| \frac{K}{\eta + a} e^{\eta |r_j|} \leq \frac{K}{\eta + a} \sum_{j=1}^{\infty} |A_{j,0}| e^{\eta |r_j|} < \infty. \tag{5.22}
$$

Thus, we see that this sum indeed converges absolutely and uniformly, if $|\mathrm{Re}(s)| \leq \eta$.
We return to our Laplace-transformed equation:

$$
\begin{aligned}
s\tilde{x}(s) = x(0) + \int_0^{\infty} e^{-s\xi} \sum_{j=1}^{\infty} A_{j,0} x(\xi + r_j) d\xi - \sum_{j=1}^{\infty} A_{j,0} \int_{-r_j}^{0} e^{-s\xi} x(\xi + r_j) d\xi \\
+ \sum_{j=1}^{\infty} A_{j,0} \int_{-r_j}^{0} e^{-s\xi} x(\xi + r_j) d\xi + \tilde{h}(s),
\end{aligned} \tag{5.23}
$$

which we can rewrite to

$$
s\tilde{x}(s) - \sum_{j=1}^{\infty} A_{j,0} \int_{-r_j}^{0} e^{-s\xi} x(\xi + r_j) d\xi - \int_0^{\infty} e^{-s\xi} \sum_{j=1}^{\infty} A_{j,0} x(\xi + r_j) d\xi = \psi(s) + \tilde{h}(s). \tag{5.24}
$$

In order to write it in the form of (5.16), we would like to apply Fubini's theorem to the expression $\int_0^{\infty} e^{-s\xi} \sum_{j=1}^{\infty} A_{j,0} x(\xi + r_j) d\xi$. To do this we must show that this integral is bounded.

$$
\begin{aligned}
\int_0^{\infty} \sum_{j=1}^{\infty} \left| e^{-s\xi} A_{j,0} x(\xi + r_j) \right| d\xi &\leq \int_0^{\infty} e^{-Re(s)\xi} \sum_{j=1}^{\infty} |A_{j,0}| |x(\xi + r_j)| d\xi \\
&\leq \int_0^{\infty} e^{-Re(s)\xi} \sum_{j=1}^{\infty} |A_{j,0}| K e^{-a|\xi + r_j|} d\xi \\
&\leq \int_0^{\infty} e^{-Re(s)\xi} \sum_{j=1}^{\infty} |A_{j,0}| K e^{|a|(\xi + |r_j|)} d\xi \\
&\leq \int_0^{\infty} e^{(-Re(s)+|a|)\xi} \sum_{j=1}^{\infty} |A_{j,0}| K e^{|a||r_j|} d\xi \\
&\leq \sum_{j=1}^{\infty} |A_{j,0}| K e^{\eta |r_j|} \int_0^{\infty} e^{(-Re(s)+|a|)\xi} d\xi \\
&< \infty.
\end{aligned} \tag{5.25}
$$

Thus, we may indeed apply Fubini and the left-hand side of the above expression becomes

$$\begin{aligned}\psi(s)+\tilde{h}(s) &= s\tilde{x}(s)-\sum_{j=1}^{\infty}A_{j,0}\int_{-r_j}^{0}e^{-s\xi}x(\xi+r_j)d\xi-\int_{0}^{\infty}e^{-s\xi}\sum_{j=1}^{\infty}A_{j,0}x(\xi+r_j)d\xi\\ &= s\tilde{x}(s)-\sum_{j=1}^{\infty}A_{j,0}\int_{-r_j}^{0}e^{-s\xi}x(\xi+r_j)d\xi-\sum_{j=1}^{\infty}A_{j,0}\int_{0}^{\infty}e^{-s\xi}x(\xi+r_j)d\xi\\ &= s\tilde{x}(s)-\sum_{j=1}^{\infty}A_{j,0}\int_{0}^{r_j}e^{-s(\xi-r_j)}x(\xi)d\xi-\sum_{j=1}^{\infty}A_{j,0}\int_{r_j}^{\infty}e^{-s(\xi-r_j)}x(\xi)d\xi\\ &= \Delta_{L_0}(s)\tilde{x}(s).\end{aligned}\tag{5.26}$$

This yields equation (5.16). □

We note that (5.9) gives that $\tilde{x}$ and $\tilde{h}$ are holomorphic in the half planes $\mathrm{Re}(s)>-a$ and $\mathrm{Re}(s)>-b$ respectively. Moreover, we have that $\psi(s)$ is holomorphic if $\mathrm{Re}(s)\leq\eta$ (see Proposition 5.2). Because $-b<\eta$ and $-a<\eta$, we can extend the analytic function $\tilde{x}$ uniquely to a meromorphic function in the region $-b<\mathrm{Re}(s)\leq -a$ with relation (5.12). In other words, we have the meromorphic function

$$\tilde{y}(\xi)=\begin{cases}\Delta_{L_0}^{-1}(s)(\psi(s)+\tilde{h}(s)) & \text{if } -b\leq \mathrm{Re}(s)\leq -a\\ \tilde{x}(s) & \text{if } \mathrm{Re}(s)>-a.\end{cases}\tag{5.27}$$

5.2 Step (2) of the proof

Since x satisfies the growth condition $|x(\xi)|\leq Ke^{-\alpha|\xi|}$ for all $\xi\in\mathbb{R}$, we have for any $k>-a$, the inverse Laplace transform

$$x(\xi)=\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{k-i\Omega}^{k+i\Omega}e^{s\xi}\tilde{x}(s)ds\tag{5.28}$$

for $\xi\geq 0$ (recall Section 3.4).

We would like to shift the path of this integral to the line $\mathrm{Re}(s)=-b$ and obtain the residues. In order to perform this shift, we need to show the following (recall Section 3.4).

(1) $\tilde{y}$ has only finitely many poles in the strip $-b<\mathrm{Re}(s)\leq -a$;

(2) $\tilde{y}(s)\to 0$ uniformly in the strip $-b\leq\mathrm{Re}(s)\leq k$, as $|\mathrm{Im}(s)|\to\infty$;

(3) the function $k\mapsto\tilde{y}(-b+ik)$ belongs to L^2.

Since the only singularities of $\tilde{y}(\xi)$ in the strip $-b<\mathrm{Re}(s)\leq -a$ are the roots of $\Delta_{L_0}(s)$, for which there are only finitely many. Hence, (1) is indeed true.
We have that ψ and $\tilde{h}$ are uniformly bounded on the vertical strip $-b+\epsilon\leq\mathrm{Re}(s)\leq k$ for each $k<\eta$ (note that $-\eta<-b+\epsilon$ since $b<\eta$). This is obviously true for $\tilde{h}$. For ψ, this is also true by the estimation in (5.22). Together with the fact $\Delta_{L_0}^{-1}(s)=O(|s|^{-1})$, we have $\tilde{y}(s)=O(|s|^{-1})$, uniformly as $|\mathrm{Im}\ s|\to\infty$, on each such strip. This verifies (2) and (3). Hence, we may shift the path of integration.

Assume that $\epsilon>0$ is small enough such that the strip $-b<\mathrm{Re}(s)\leq -b+\epsilon/2$ does not contain any eigenvalues and also that $-b+\epsilon/2<k$. Then, together with the above observations, we shift the path of integration in (5.28) to the line $\mathrm{Re}(s)=-b+\epsilon/2$ and obtain

$$x(\xi)=z(\xi)+w(\xi),\quad z(\xi)=\sum_{\lambda\in\Lambda}\mathrm{res}(e_\xi\tilde{x},\lambda),\quad w(\xi)=\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{-b+\epsilon/2-i\Omega}^{-b+\epsilon/2+i\Omega}e^{s\xi}\tilde{y}(s)ds.\tag{5.29}$$

Recall that in this way z is an eigensolution to (2.10).

5.3 Step (3) of the proof

It remains to show that

$$w(\xi) = O(e^{-(b-\epsilon)\xi}),\ \xi \to \infty. \tag{5.30}$$

We know that the function $t \mapsto \tilde{y}(-b+\epsilon/2+it)$ is in L^2. Hence, we can take its Inverse Fourier Transformation.

$$\begin{aligned} L^2 \ni \mathcal{F}^{-1}(\tilde{y}(-b+\epsilon/2+i\eta))(\xi) &= \frac{1}{2\pi}\lim_{\Omega\to\infty}\int_{-\Omega}^{\Omega} e^{i\xi\eta}\tilde{y}(-b+\epsilon/2+i\eta)d\eta \\ &= \frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{-b+\epsilon/2-i\Omega}^{-b+\epsilon/2+i\Omega} e^{(s+b-\epsilon/2)\xi}\tilde{y}(s)ds \\ &= e^{(b-\epsilon/2)\xi}\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{-b+\epsilon/2-i\Omega}^{-b+\epsilon/2+i\Omega} e^{s\xi}\tilde{y}(s)ds \\ &= e^{(b-\epsilon/2)\xi}w(\xi). \end{aligned} \tag{5.31}$$

Let $u(\xi) = e^{(b-\epsilon/2)\xi}w(\xi)$ and $v(\xi) = \begin{cases} e^{-\epsilon\xi}u(\xi) = e^{(b-\epsilon)\xi}w(\xi) & \text{if } \xi \geq 0 \\ e^{\epsilon\xi}u(\xi) = e^{b\xi}w(\xi) & \text{if } \xi < 0. \end{cases}$ By the above calculation, it follows that $u \in L^2(\mathbb{R}, \mathbb{C}^d)$. Hence it follows that $v \in L^1(\mathbb{R}, \mathbb{C}^d)$. Because z satisfies the homogeneous equation, we have that w, like x, satisfies the inhomogeneous equation. For $\xi \geq 0$, we conclude that v satisfies the equation

$$v'(\xi) = (b-\epsilon)v(\xi) + \sum_{j:r_j\geq 0} e^{-(b-\epsilon)r_j}A_{j,0}v(\xi+r_j) + \sum_{j:r_j<0} e^{-br_j}A_{j,0}v(\xi+r_j) + e^{(b-\epsilon)\xi}h(\xi). \tag{5.32}$$

We want to show that the right-hand side is integrable on $[0,\infty)$, which means we only need to show that $\sum_{j:r_j\geq 0} e^{-(b-\epsilon)r_j}A_{j,0}v(\xi+r_j) + \sum_{j:r_j<0} e^{-br_j}A_{j,0}v(\xi+r_j)$ is integrable. Using Fatou's lemma, we can estimate

$$\begin{aligned} &\int_0^\infty \left| \sum_{j:r_j\geq 0} e^{-(b-\epsilon)r_j}A_{j,0}v(\xi+r_j) + \sum_{j:r_j<0} e^{-br_j}A_{j,0}v(\xi+r_j)\right| d\xi \\ &\leq \int_0^\infty \left|\sum_{j:r_j\geq 0} e^{-(b-\epsilon)r_j}A_{j,0}v(\xi+r_j)\right| d\xi + \int_0^\infty \left|\sum_{j:r_j<0} e^{-br_j}A_{j,0}v(\xi+r_j)\right| d\xi \\ &\leq \sum_{j:r_j\geq 0}|A_{j,0}|e^{|-(b-\epsilon)||r_j|}\int_0^\infty |v(\xi+r_j)|d\xi + \sum_{j:r_j<0}|A_{j,0}|e^{|b||r_j|}\int_0^\infty |v(\xi+r_j)|d\xi \\ &\leq \|v\|_1 \sum_{j:r_j\geq 0}|A_{j,0}|e^{\eta|r_j|} + \|v\|_1\sum_{j:r_j<0}|A_{j,0}|e^{\eta|r_j|} \\ &\leq \|v\|_1\sum_{j=1}^{\infty}|A_{j,0}|e^{\eta|r_j|} \\ &< \infty. \end{aligned} \tag{5.33}$$

This shows that $v'(\xi)$ is integrable on $[0,\infty)$. Therefore, we can write $v(\xi) = v(0) + \int_0^\xi v'(s)ds$, for $\xi \geq 0$. Hence, we conclude that $v(\xi)$ is bounded as $\xi \to \infty$. This means that indeed $w(\xi) = O(e^{-(b-\epsilon)\xi})$.

For the analogous result for $\xi \to -\infty$, the proof is similar. The Laplace Transform is then carried out from $-\infty$ to 0, and instead of shifting the path of integral to the left, we shift it to the right for the $\xi \to -\infty$ case.

6 Proof of Theorem C

The goal is to calculate the index $\iota(L_-, L_+)$ given two hyperbolic limit operators L_- and L_+. Suppose as in the hypothesis of Theorem C, we have a continuous varying path of constant matrix coefficients $\sigma(\rho)$, with $-1 \leq \rho \leq 1$ such that (with a little abuse of notation) $\sigma(-1) = L_-$ and $\sigma(1) = L_+$. Then, we can look at the index of

$$\iota(L_-, L_\rho), \tag{6.1}$$

where L_ρ is the system with $\sigma(\rho)$ as coefficients. Since $\sigma(\rho)$ is continuous and the Fredholm index stays constant under continuous path, this means that $\iota(L_, L_\rho) = \iota(L_-, L_-) = 0$ as long as L_ρ is hyperbolic, which is the case when the corresponding $\Delta_{L_\rho}(s)$ has no eigenvalues on the imaginary axis, according to Theorem A. This shows that during the path $\sigma(\rho)$, a change of index happens only when the corresponding eigenvalues cross the imaginary axis. Theorem C tells us exactly how the index changes during such a cross.

The idea of proving Theorem C is to approximate the family L_0^ρ of operators in the statement with a generic family. We will then show that the crossing number for the approximation is unchanged.

To formalize the ideas above, we start by introducing the following notations and definitions.
Throughout this whole section, we will keep the shifts r_j fixed. For $\eta > 0$, we define the following vector space over $\mathbb{C}$

$$\ell^1_\eta(\mathbb{C}^{d\times d}) := \{\mathbf{A} = (A_{1,0}, A_{2,0}, \cdots) : \sum_{j=1}^{\infty} |A_{j,0}|e^{\eta|r_j|} < \infty, A_{j,0} \in \mathbb{C}^{d\times d}\}. \tag{6.2}$$

It is easy to verify that this is a vector space. We choose the norm $\|\mathbf{A}\|_\eta = \sum_{j=1}^{\infty} |A_{j,0}|e^{\eta|r_j|}$. Note that if $\eta_1 < \eta_2$, then $\ell^1_{\eta_2} \subset \ell^1_{\eta_1}$. We denote paths in this set by

$$\sigma : [-1, 1] \to \ell^1_\eta(\mathbb{C}^{d\times d})$$

between coefficients $\sigma(\pm 1) = \mathbf{A}_\pm$. We say that such a path is continuous if it is continuous with respect to the $\|\mathbf{A}\|_\eta$ norm. Note that if $\eta_1 < \eta_2$, then a continuous path in $\ell^1_{\eta_2}$ is also continuous in $\ell^1_{\eta_1}$.

We will now show that $\mathrm{cross}(L^\rho)$ as in (2.37) is well-defined.

Proposition 6.1. *Let $\sigma : [-1, 1] \to \ell^1_\eta(\mathbb{C}^{d\times d})$ be a continuous path of coefficients. Denote $g(s, \rho) := sI - \sum_{j=1}^{\infty} A_{j,0}(\rho)e^{sr_j}$ and $h(s, \rho) := \det(sI - \sum_{j=1}^{\infty} A_{j,0}(\rho)e^{sr_j})$, where $\{A_{j,0}(\rho)\}_{j\geq 1} = \sigma(\rho)$ and $|\mathrm{Re}(s)| < \eta$. Then for each $\rho_0 \in [-1, 1]$, there exists a $\delta > 0$ and an $\epsilon > 0$ such that for all $\rho \in [-1, 1]$ with $|\rho - \rho_0| < \delta$, we have that $g(s, \rho)$ and $g(s, \rho_0)$ has the same number of zeros, counting with multiplicities, in the region $R_\epsilon := \{s \in \mathbb{C} : \mathrm{Re}(s) < \epsilon\}$.*

To prove this proposition, we need the following lemmas as support.

Lemma 6.2. *Let $\sigma : [-1, 1] \to \ell^1_\eta(\mathbb{C}^{d\times d})$ be a continuous path of coefficients. Denote $g(s, \rho) := sI - \sum_{j=1}^{\infty} A_{j,0}(\rho)e^{sr_j}$ where $\{A_{j,0}(\rho)\}_{j\geq 1} = \sigma(\rho)$ and $|\mathrm{Re}(s)| < \eta$. Then there exists an $M > 0$ such that if $|\mathrm{Im}(s)| \geq M$, then $g(s, \rho)$ is invertible for all $\rho \in [-1, 1]$. In other words, there are no eigenvalues for all $\rho \in [-1, 1]$, if $|\mathrm{Im}(s)|$ is large enough.*

Proof. We have $|\sum_{j=1}^{\infty} A_{j,0}(\rho)e^{sr_j}| \leq \|\mathbf{A}(\rho)\|_\eta$. Since $[-1, 1]$ is compact and $\sigma(\rho)$ together with the norm function are continuous, we have that $\|\mathbf{A}(\rho)\|_\eta \leq \max_{\rho\in[-1,1]} \|\mathbf{A}(\rho)\|_\eta =: N$. Now choose M such that $N/M < 1$. $\square$

Lemma 6.3. *Let $\sigma : [-1, 1] \to \ell^1_\eta(\mathbb{C}^{d\times d})$ be a continuous path of coefficients. Let $D = \{s \in \mathbb{C} : |\mathrm{Re}(s)| < \eta\}$. Denote $g(s, \rho) := sI - \sum_{j=1}^{\infty} A_{j,0}(\rho)e^{sr_j}$ and $h(s, \rho) := \det(sI - \sum_{j=1}^{\infty} A_{j,0}(\rho)e^{sr_j})$, where $\{A_{j,0}(\rho)\}_{j\geq 1} = \sigma(\rho)$ and $s \in D$. Then $h(s, \rho)$ is analytic in s for all $\rho \in [-1, 1]$, and h is uniformly continuous in ρ, which means for all $\rho_0 \in [-1, 1]$, there exists a $\delta > 0$ such that for all ρ with $|\rho - \rho_0| < \delta$, we have $|h(s, \rho) - h(s, \rho_0)| < \epsilon$, for all $s \in D$.*

Proof. Since $h(s,\rho) = \det(g(s,\rho))$, we have that $h(s,\rho)$ satisfies the above statements if $g(s,\rho)$ does, since the determinant is polynomial.
The analyticity follows from Remark 2.1 and the uniform continuity follows from the assumption that σ is continuous in ρ with respect to the $\|\mathbf{A}(\rho)\|_\eta$ norm. $\square$

Proof of Proposition 6.1. Since $h(s,\rho_0)$ has only finitely many zeros, we take $\epsilon_1 > 0$ such that there are no roots on the vertical line $|\mathrm{Re}(s)| = \epsilon_1$. Furthermore, from Lemma 6.2, there is an $M > 0$ such that $h(s,\rho)$ has no roots for all $\rho \in [-1,1]$, if $|\mathrm{Im}(s)| \geq M$. Let $\partial R := \{s \in \mathbb{C} : |\mathrm{Re}(s)| = \epsilon_1, |\mathrm{Im}(s)| = M\}$. Then $\tilde{\epsilon} := \min_{s\in\partial R} |h(s,\rho)| > 0$ since there are no roots on ∂R. Lemma 6.3 now gives us a $\delta > 0$ such that for all $\rho \in [-1,1]$ such that $|\rho - \rho_0| < \delta$, we have $|h(s,\rho) - h(s,\rho_0)| < \tilde{\epsilon} \leq |h(s,\rho_0)|$ on $\tilde{R}$. Hence, we may apply Rouché's theorem and conclude that $h(s,\rho)$ and $h(s,\rho_0)$ have the same number of zeros inside the region bounded by ∂R. The last statement follows by choosing $R_\epsilon = R_{\epsilon_1}$. $\square$

For any continuous path σ, we define

$$\mathrm{NH}(\sigma) = \{\rho \in [-1,1]\ |\text{equation (2.10), with coefficients at } \mathbf{A} = \sigma(\rho), \text{ is not hyperbolic}\}. \tag{6.3}$$

Thus σ satisfies the conditions of Theorem C if and only if $\mathrm{NH}(\sigma) \subset (-1,1)$ is a finite set.

For $|\gamma| < \eta$, it will be useful to introduce the shift operator

$$S_\gamma : \ell^1_\eta(\mathbb{C}^{d\times d}) \to \ell^1_{\eta-|\gamma|}(\mathbb{C}^{d\times d}), \tag{6.4}$$
$$(A_{1,0}, A_{2,0}, \cdots) \mapsto (A_{1,0} + \gamma I, e^{-\gamma r_2} A_{2,0}, e^{-\gamma r_3} A_{3,0}, \cdots). \tag{6.5}$$

This transformation S_γ arises from the change of variables $y(\xi) = e^{\gamma\xi} x(\xi)$ in equation (2.10). It is clear that this is well-defined and one can easily check that

$$\Delta_{S_\gamma L_0}(s) = \Delta_{L_0}(s - \gamma). \tag{6.6}$$

That is, the operator S_γ shifts all eigenvalues to the right by an amount γ.
For convenience, we define the function

$$\mathrm{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases} \tag{6.7}$$

In order to follow the proof of [1, Proof of Theorem C], we will introduce the following definitions of classes of constant coefficients.

Definition 6.4. *We say the constant coefficient equation (2.10), with (2.8) (or simply the coefficients themselves) satisfies **Property** $\mathcal{G}_0$ if there exists at most one $\tilde{\eta} \in \mathbb{R}$ such that $\lambda = i\tilde{\eta}$ is an eigenvalue, and if moreover, this is a simple root of the characteristic equation* $\det \Delta_{L_0}(s) = 0$.

Remark 6.5. *Note that the coefficients $A_{j,0}$ can be complex matrices, which means the in general eigenvalues does not have to occur in complex conjugate pairs.*

Definition 6.6. *Let $\sigma \in C^1([-1,1], \ell^1_\eta(\mathbb{C}^{d\times d})$ be a smooth one parameter family of coefficients for (2.8). We say the corresponding family of equations (2.10) (or simply the coefficients themselves) satisfies **Property** $\mathcal{G}_1$ if*

(i) for each $\rho \in [-1,1]$, equation (2.10) with coefficients at $\boldsymbol{A} = \sigma(\rho)$ satisfies Property $\mathcal{G}_0$,

(ii) at $\rho = \pm 1$ equation (2.10) is hyperbolic,

(iii) all eigenvalues $\lambda = \lambda(\rho)$ of equation (2.10) on the imaginary axis $\mathrm{Re}(\lambda(\rho_0)) = 0$, *for some $\rho = \rho_0 \in (-1,1)$, cross the axis transversely with ρ, that is* $\mathrm{Re}\ \lambda'(\rho_0) \neq 0$.

Remark 6.7. *Note that for the above definition we implicitly assumed that* $\mathrm{Re}(\lambda)$ *is a differentiable function.*

We also note that a family of constant coefficient equations satisfying Property $\mathcal{G}_1$ also satisfies the hypotheses of Theorem C. This is mainly due to the following result.

Lemma 6.8. *Let $f : [a, b] \to \mathbb{R}$ be differentiable. Suppose there is no $x \in [a, b]$ such that $f(x) = f'(x) = 0$. Then f has only finitely many zeros.*

Proof. Suppose f has infinitely many zeros. Then we have a distinct sequence of zeros of f, namely $\{x_n\}_{n=1}^{\infty} \subset [a, b]$. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence. Hence, we may assume without loss that the sequence $\{x_n\}_{n=1}^{\infty}$ converges. Let $x := \lim_{n\to\infty} x_n$, then $x \in [a, b]$ since $[a, b]$ is closed. Because f is continuous, we have $f(x) = 0$. Then we have

$$f'(x) = \lim_{n\to\infty} \frac{f(x_n) - f(x)}{|x_n - x|} = \lim_{n\to\infty} \frac{0}{|x_n - x|} = 0, \tag{6.8}$$

which is a contradiction. □

Let $\lambda_j(\rho)$ denote the eigenvalue as in (iii) above for which Re $\lambda_j(\rho_j) = 0$, with $\rho_j \in \mathrm{NH}(\sigma)$. We shall also denote

$$\text{Re } \lambda_j'(\rho_j) = \mu_j \neq 0, \tag{6.9}$$

and we note that

$$\text{cross}(L^\rho) = \sum_{j=1}^{J} \text{sign}(\text{Re } \lambda_j'(\rho_j)). \tag{6.10}$$

The following result shows that paths satisfying Property $\mathcal{G}_1$, and joining given coefficients $\mathbf{A}_\pm$, are dense among all continuous paths joining these endpoints.

Proposition 6.9. *Let $\sigma \in C([-1, 1], \ell^1_\eta(\mathbb{C}^{d\times d}))$ be such that the corresponding parameter family of differential equations (2.10), with (2.8), satisfies the hypotheses of Theorem C. Then given $\epsilon > 0$, there exists $\tilde{\sigma}([-1, 1], \ell^1_l(\mathbb{C}^{d\times d}))$ for certain $l > 0$, such that*

(i) $\tilde{\sigma}(\pm 1) = \sigma(\pm 1)$

(ii) $|\sigma(\rho) - \tilde{\sigma}(\rho)| < \epsilon$ *for all* $\rho \in [-1, 1]$

(iii) the family of differential equations (2.10), with (2.8), satisfies Property $\mathcal{G}_1$.

In order to prove this, we will first refer to a result which shows that continuous path $\sigma : [-1, 1] \to \ell^1_\eta(\mathbb{C}^{d\times d})$ can be approximated by a path $\hat{\sigma}$ that satisfies property $\mathcal{G}_0$. The proof of this can be found in [3, Chapter 4].

Proposition 6.10. *Fix any two points $\mathbf{A}_\pm \in \ell^1_\eta(\mathbb{C}^{d\times d})$, and assume for some $1 < j < k$ that r_j/r_k is irrational. Then there is a residual (and thus dense) subset $Y \subset X$ of the space of curves*

$$X = \{\sigma \in C^1([-1, 1], \ell^1_\eta(\mathbb{C}^{d\times d})) | \sigma(\pm 1) = \mathbf{A}_\pm\}$$

joining these points, such that for any $\sigma \in Y$, and for any $\rho \in (-1, 1)$, $\sigma(\rho)$ satisfies property $\mathcal{G}_0$.

Remark 6.11. *The condition that r_j/r_k has to be irrational is not a restriction, since we can always introduce an additional shift r with a zero coefficient matrix $A_r = 0$.*

Proof of Proposition 6.9. By Proposition 6.10, we may assume without loss that σ in the statement of Proposition 6.9 satisfies Property $\mathcal{G}_0$. Note that σ may not have a finite $\mathrm{NH}(\sigma)$ as defined in (6.3) after applying Proposition 6.10. It is enough to perturb σ to a nearby $\tilde{\sigma}$ with the same endpoints, such that all eigenvalues cross the imaginary axis transversely with ρ.

Claim: There exists an $\epsilon_1 > 0$ such that for all eigenvalues λ of equation (2.10) satisfying $|\text{Re } \lambda| < \epsilon_1$, for any $\mathbf{A} = \sigma(\rho)$, are simple. Moreover, the eigenvalues in this region are locally parametrized by a smooth function (at least C^1) on some maximal open interval $I \subset (-1,1)$ for which $|\mathrm{Re}\ \lambda(\rho)| < \epsilon_1$. Finally, there are only finitely many of such parametrizations.

Proof of the claim: Let $\mathrm{NH}(\sigma)$ be as defined in (6.3). Then $\mathrm{NH}(\sigma)$ is closed, because for each $\rho \in \mathrm{NH}(\sigma)^c$, we have that $\sigma(\rho)$ is hyperbolic. Let $g(s, \tilde{\rho}) = \det(\Delta_{L_0^\rho}(s))$. By Proposition 6.1, there is an open neighbourhood of ρ such that $\sigma(\rho)$ is still hyperbolic. Clearly we also have $\mathrm{NH}(\sigma) \subset (-1,1)$ is bounded, hence it is compact. Note that the set $\mathrm{NH}(\sigma)$ is a union of closed intervals and/or singletons, since the eigenvalues varies continuously and some eigenvalues might remain on the imaginary axis for certain time (see Figure 1).

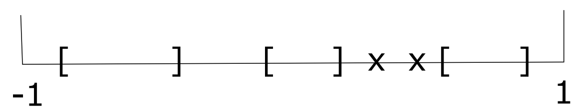

Figure 1: An example of possible configuration for the set $\mathrm{NH}(\sigma)$. It consists of closed intervals and singletons, depending on whether the eigenvalue remains on the imaginary axis.

For each $\rho^* \in \mathrm{NH}(\sigma)$, we have that due to property $\mathcal{G}_0$, there is only one λ_{ρ^*} that lies on the imaginary axis and is simple. Applying the implicit function theorem to the function $\det(\Delta_{L_0^\rho}(s))$ yields an open neighbourhood of λ_{ρ^*}, such that λ_{ρ^*} can be parametrized uniquely and smoothly (at least C^1) on an open interval around ρ^* and λ_{ρ^*} remains simple. In case ρ^* lies in an interval J of $\mathrm{NH}(\sigma)$, we may extend this parametrization to be on an open interval such that the boundaries of J are also covered, since we have a unique smooth parametrization for each $\rho \in \mathrm{NH}(\sigma)$ and an interval is connected (see Figure 2).

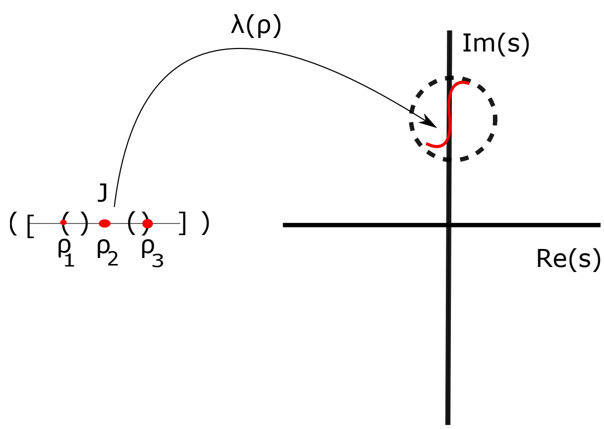

Figure 2: For each ρ in such an interval J of the set $\mathrm{NH}(\sigma)$, we can find a open neighbourhood of ρ to locally parametrize the simple eigenvalue on the imaginary axis. By "gluing" these parametrizations together (which we may do since the conditions of the implicit function theorem guarantee the overlap between the parametrizations), we obtain a parametrization with a domain that covers J.

Moreover, J is compact, thus we have due to Proposition 6.1 a minimal ϵ_3 such that the only eigenvalue inside the strip $\{s \in \mathbb{C} || \mathrm{Re}(s)| < \epsilon_3\}$ is the eigenvalue that lies on the imaginary axis, for all $\rho \in J$. This shows that for each such interval J there exists an ϵ_4 such that inside the strip $\{s \in \mathbb{C} || \mathrm{Re}(s)| < \epsilon_4\}$ the eigenvalue can be smoothly parametrized on a maximal open interval U (that covers J) for which $|\mathrm{Re}\ \lambda(\rho)| < \epsilon_4$ (meaning that outside this maximal interval, the eigenvalue has real part greater or equal to ϵ_4). Since we can find such an U for each intervals and singletons of $\mathrm{NH}(\sigma)$. We have due to the compactness of $\mathrm{NH}(\sigma)$ finitely many of such coverings which we denote as $\bigcup_{j=1}^{n} U_j$. This yields a minimal $\tilde{\epsilon}_1$ for all $\rho \in \mathrm{NH}(\sigma)$, such that all eigenvalues inside the strip $\{s \in \mathbb{C} || \mathrm{Re}(s)| < \tilde{\epsilon}_1\}$ can be parametrized on a maximal open interval for which $|\mathrm{Re}\ \lambda(\rho)| < \tilde{\epsilon}_1$ (see Figure 3).

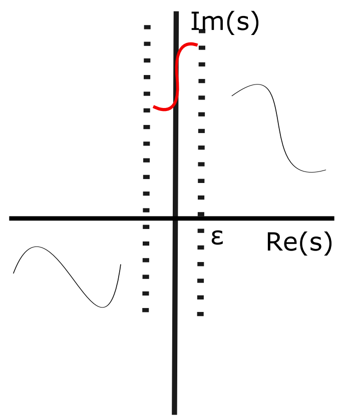

Figure 3: An example of the trajectories of the eigenvalues. As explained above, we can find an $\epsilon > 0$ such that the strip $\{s \in \mathbb{C} : |\mathrm{Re}(s)| < \epsilon\}$ "isolates" all other eigenvalues from the one simple eigenvalue inside the strip that we can parametrize with the implicit function theorem. This parametrization is defined on a maximal open interval for which the eigenvalue remains inside the strip.

Let $W = (-1, 1) \setminus \bigcup_{j=1}^{n} U_j$. Let $\tilde{\epsilon}_2 = \inf_{\rho \in W}\{|\mathrm{Re}(\lambda)| : \lambda \text{ is an eigenvalue of } \sigma(\rho)\}$. Suppose $\tilde{\epsilon}_2 = 0$, then there exists a sequence $\{\rho_n\}_{n=1}^{\infty}$ such that there is a corresponding $\{\lambda_n\}_{n=1}^{\infty}$ such that $|\mathrm{Re}(\lambda_n)| \to 0$ as $n \to \infty$. With Bolzano-Weierstrass theorem we may assume without loss of generality that ρ_n converges to a certain $\tilde{\rho} \in [-1, 1]$. Note that $\tilde{\rho} \notin \mathrm{NH}(\sigma)$. Therefore, it follows from Proposition 6.1 that there exists a $\delta > 0$ such that for all ρ with $|\rho - \tilde{\rho}| < \delta$, that $g(s, \rho)$ has no roots near the imaginary axis, which is a contradiction to the assumption that $\mathrm{Re}(\lambda_n) \to 0$ as $n \to \infty$. Hence $\tilde{\epsilon}_2 > 0$. We now take $\epsilon_1 = \min\{\tilde{\epsilon}_1, \tilde{\epsilon}_2\}$, and the claim now follows. Note that due to the hyperbolicity at ± 1, there exists an ϵ_2 such that are no eigenvalues in the strip $|\mathrm{Re}(z)| < \epsilon_1$, for $\rho \in [-1, -1 + \epsilon_2] \cup [1 - \epsilon_2, 1]$.

The proof now proceeds the same as in [1, Proof of Proposition 8.1]. □

Remark 6.12. *If ϵ is small enough then one has*

$$\mathrm{cross}(L_0^{\rho}) = \mathrm{cross}(\tilde{L}_0^{\rho}). \tag{6.11}$$

This is simply a consequence of the fact that as the eigenvalues are roots of a holomorphic function, in any bounded region they vary continuously as a set in the Hausdorff topology (this follows from the Rouché's theorem).

We will need the following lemma, which is stated in [1, Lemma 8.2].

Lemma 6.13. *Let $f(s, \rho)$ and $\hat{f}(s, \rho)$, for $(s, \rho) \in \mathbb{C} \times \mathbb{R}$, be two $d \times d$ matrix valued functions which are holomorphic in s in a neighbourhood of $\lambda_0 \in \mathbb{C}$, and which also are C^1 in ρ in a neighbourhood of $\rho_0 \in \mathbb{R}$. Denote $g(s, \rho) = \det f(s, \rho)$ and $\hat{g}(s, \rho) = \det \hat{f}(s, \rho)$. Assume that*

$$f(s, \rho_0) = \hat{f}(s, \rho_0) \tag{6.12}$$

identically in a neighbourhood of $s = \lambda_0$, and that

$$g(\lambda_0, \rho_0) = 0, \ \ D_1 g(s, \rho_0)|_{s=\lambda_0} \neq 0, \tag{6.13}$$

where D_k denotes the derivative of a function with respect to its k-th argument.
Denote by $s = \lambda(\rho)$ and $s = \hat{\lambda}(\rho)$ the unique solutions of $g(s,\rho) = 0$ and $\hat{g}(s,\rho) = 0$ near $s = \lambda_0$, for ρ near ρ_0 (note the uniqueness follows from the implicit function theorem). Assume that

$$\text{Re } \lambda'(\rho_0) = \text{Re}\hat{\lambda}' = \mu \neq 0, \tag{6.14}$$

that is, the two roots $\lambda(\rho)$ and $\hat{\lambda}(\rho)$ cross the imaginary axis transversely with the same horizontal speed and direction. For $0 \leq \beta \leq 1$ set

$$f(s,\rho,\beta) = (1-\beta)f(s,\rho) + \beta\hat{f}(s,\rho), \;\; g(s,\rho,\beta) = \det f(s,\rho,\beta) \tag{6.15}$$

and let $s = \lambda(\rho,\beta)$ denote the unique solution of $g(s,\rho,\beta) = 0$ near $s = \lambda_0$, for ρ near ρ_0 and all $\beta \in [0,1]$. Then for (ρ,β) near $\{\rho_0\} \times [0,1]$, we have that

$$\text{Re } \lambda(\rho,\beta) = \text{Re } \lambda_0 \textit{ if and only if } \rho = \rho_0. \tag{6.16}$$

The following result [1, Proposition 8.3] shows that, without loss, we may assume the eigenvalues cross the imaginary axis by means of a rigid shift of the spectrum.

Proposition 6.14. *Let $\sigma \in C^1([-1,1],(\ell^1_\eta(\mathbb{C}^{d\times d}))$ be a one parameter family of coefficients for (2.10) satisfying Property $\mathcal{G}_1$. Then there exists another such $\tilde{\sigma} \in C^1([-1,1],(\ell^1_l(\mathbb{C}^{d\times d}))$ for certain $l > 0$, joining the same endpoints $\tilde{\sigma}(\pm 1) = \sigma(\pm 1)$, and also satisfying Property $\mathcal{G}_1$, such that $\text{NH}(\tilde{\sigma}) = \text{NH}(\sigma)$, with $\text{NH}(\sigma)$ defined in (6.3). Moreover, at each $\rho_j \in \text{NH}(\sigma)$, we have $\mu_j = \tilde{\mu}_j$, with $\tilde{\mu}_j$ corresponding to the family $\tilde{\sigma}$. Thus the eigenvalues of the two families σ and $\tilde{\sigma}$ cross the imaginary axis at the same values of ρ, and moving in the same direction, left or right.*
In addition, the family $\tilde{\sigma}$ has the form

$$\tilde{\sigma}(\rho) = S_{\mu_j(\rho-\rho_j)}\tilde{\sigma}(\rho_j) \tag{6.17}$$

for ρ in a neighbourhood of each ρ_j. That is, for the family $\tilde{\sigma}$ the eigenvalues cross the imaginary axis by a shift of the entire spectrum to the left ($\mu_j < 0$) or to the right ($\mu_j > 0$) for ρ near ρ_j.

Proof. The proof is identical to [1, Proposition 8.3]. However, we will add some details to the proof. Let $\epsilon > 0$ be small enough that $[\rho_j - \epsilon, \rho_j + \epsilon] \subset (-1,1)$ are disjoint intervals for $\rho_j \in \text{NH}(\sigma)$, and such that also for all j, $|\mu_j(\rho - \rho_j)| < \eta$ for $|\rho - \rho_j| < \epsilon$. Let $l = \max_{j\in J}\{|\mu_j|\epsilon\}$. Define $\hat{\sigma} : [-1,1] \to \ell^1_l(\mathbb{C}^{d\times d})$ by

$$\hat{\sigma}(\rho) = S_{\mu_j(\rho-\rho_j)}\sigma(\rho_j), \;\; |\rho - \rho_j| \leq \epsilon \tag{6.18}$$

with $\hat{\sigma}(\rho)$ arbitrary for other values of ρ. Letting L^ρ_0 and $\hat{L}^\rho_0$ denote the operators associated to σ and $\hat{\sigma}$ respectively, we set $L^{\rho,\beta}_0 = (1-\beta)L^\rho_0 + \beta\hat{L}^\rho_0$ and consider

$$\Delta_{L^{\rho,\beta}_0}(s) = (1-\beta)\Delta_{L^\rho_0}(s) + \beta\Delta_{\hat{L}^\rho_0}(s) \tag{6.19}$$

for $0 \leq \beta \leq 1$.

Claim: The function $f(s,\rho) = \Delta_{L^\rho_0}(s)$ and $\hat{f}(s,\rho) = \Delta_{\hat{L_0}^\rho}(s)$ are C^1 in a neighbourhood of ρ_j, for all j.

Proof of the claim. We know that $\hat{\sigma}(\rho) = S_{\mu_j(\rho-\rho_j)}\sigma(\rho)$, we have that $\Delta_{\hat{L}^\rho_0}(s) = \Delta_{L^\rho_0}(s - \mu_j(\rho-\rho_j))$ and $\mu_j(\rho - \rho_j)$ is clearly C^1 near ρ_j. So we only need to show that $\Delta_{L^\rho_0}(s)$ is C^1 in a neighbourhood of ρ_j.
From the assumption that σ is continuous with respect to the $\|\mathbf{A}\|_\eta$ norm, it is immediate that $f(s,\rho)$ is continuous near ρ_j . In fact, $f(s,\rho)$ is uniformly continuous in ρ for all $s \in \mathbb{C}$ with $|Re(s)| < \eta$, which means that there exists a $\delta > 0$ such that for all ρ with $|\rho - \rho_0| < \delta$, we have $|f(s,\rho) - f(s,\rho_0)| < \epsilon$, for all s with $|Re(s)| < \eta$.
We also assumed that σ is differentiable in ρ with respect to the $\|\mathbf{A}\|_\eta$ norm. Denote $\sigma'(\rho) = (A'_1(\rho), A'_2(\rho), \cdots)$. Then we have that $\frac{\partial}{\partial\rho}f(s,\rho) = \frac{\partial}{\partial\rho}\left(sI - \sum_{j=1}^\infty A_j(\rho)e^{sr_j}\right)$. However, since σ is C^1 with respect to the $\|\mathbf{A}\|_\eta$

norm, it easily follows that $\frac{\partial}{\partial\rho}\left(sI - \sum_{j=1}^{\infty} A_j(\rho)e^{sr_j}\right) = -\sum_{j=1}^{\infty} A_j'(\rho)e^{sr_j}$. The only thing we need to verify is that $\sigma'(\rho) \in \ell_\eta^1$. This follows from the triangle inequality:

$$\|\sigma'(\rho)\|_\eta \leq \|\frac{\sigma(\rho+h)-\sigma(\rho)}{h} - \sigma'(\rho)\|_\eta + \|\frac{\sigma(\rho+h)-\sigma(\rho)}{h}\|_\eta < \infty$$

for h close to 0. ☐

The proof now proceeds the same as in [1, Proposition 8.3] ☐

Remark 6.15. *Clearly,* $\mathrm{cross}(L_0^\rho) = \mathrm{cross}(\tilde{L}_0^\rho)$ *for the operators* L_0^ρ *and* $\tilde{L}_0^\rho$ *associated to* σ *and* $\tilde{\sigma}$.

Before we proceed to the next proposition, we will first need the following lemma regarding solutions of certain type of non-hyperbolic system.

Lemma 6.16. *Let* L_0 *be given as in (2.8). If* L_0 *has only one simple eigenvalue* $i\hat{\eta}$ *on the imaginary axis (i.e.* $\det(\Delta_{L_0})(i\hat{\eta}) = 0$*). Then we have that* $\mathcal{K}_L \subset W^{1,p}$*, for* $1 \leq p \leq \infty$*, where* $\mathcal{K}_L$ *is the kernel of* Λ_L*. Furthermore, for* $x \in \mathcal{K}_L$ *we have* $x(\xi) = e^{i\hat{\eta}\xi}v$*, for a kernel element* $v \in \mathbb{C}^d$ *associated to* $\Delta_{L_0}(i\hat{\eta})$*.*

Proof. The proof for $\mathcal{K}_L \subset W^{1,p}$, for $1 \leq p \leq \infty$ is already done in [2, Proposition B.1]. Also, according to the proof in [2, Theorem 3.7], for $x \in W^{1,p}$ a solution of (2.10), we have that the Fourier transform of (2.10) is given by

$$i\eta\hat{x}(\eta) = \left(\sum_{j=1}^{\infty} A_{j,0}e^{i\eta r_j}\right)\hat{x}(\eta). \tag{6.20}$$

Hence $\Delta_{L_0}(i\eta)\hat{x}(\eta) = 0$. Because we know that only for $\eta = \hat{\eta}$, we have that $\det(\Delta_{L_0}(i\hat{\eta})) = 0$, we have that $\hat{x}(\eta) = 0$ if $\eta \neq \tilde{\eta}$. And for $\eta = \hat{\eta}$, we have that $\hat{x}(\hat{\eta}) \in \ker(\Delta_{L_0}(i\hat{\eta}))$. But since we know that $i\hat{\eta}$ is a simple eigenvalue and thus has algebraic multiplicity equal to 1, we know that $\dim\ker(\Delta_{L_0}(i\hat{\eta})) = 1$. Hence we know that $\hat{x}(\hat{\eta}) = \lambda v$ for some eigenvector $v \in \mathbb{C}^d$. Hence we have that $\hat{x}(\eta) = \lambda v\delta(\eta - \hat{\eta})$. Now taking the inverse-Fourier transform of $\hat{x}$:

$$\begin{aligned}(\mathcal{F}^{-1}\hat{x}, \zeta) = (\hat{x}, \mathcal{F}^{-1}(\zeta)) &= \int_{-\infty}^{\infty} \lambda v\delta(\eta - \tilde{\eta})\mathcal{F}^{-1}(\zeta)(\eta)d\eta \\ &= \lambda v\mathcal{F}^{-1}(\tilde{\eta}) \\ &= \frac{\lambda v}{2\pi}\int_{-\infty}^{\infty} e^{ix\tilde{\eta}}\zeta(x)dx \\ &= (\frac{\lambda v}{2\pi}e^{ix\tilde{\eta}}, \zeta(x))\end{aligned} \tag{6.21}$$

So indeed we see that $x(\xi) = e^{i\tilde{\eta}\xi}v$ with a kernel vector $v \in \mathbb{C}^d$. ☐

Proposition 6.17. *Suppose that* $s = ir$*, with* $r \in \mathbb{R}$*, is a simple eigenvalue of equation (2.10), and suppose there are no other eigenvalues with* $\mathrm{Re}(\lambda) = 0$*. Then for* $\gamma \in \mathbb{R}$ *with* $|\gamma| > 0$ *sufficiently small, we have that*

$$\iota(S_{-\gamma}L_0, S_\gamma L_0) = -\mathrm{sign}\ \gamma. \tag{6.22}$$

Proof. The proof of this proposition is identical to [1, Proposition 8.4], where a change of variable is used to calculate the index. However, instead of using [1, Proposition 7.1] in that proof, we use Proposition 5.1, which is a similar result but stated in our setting with infinite shifts. During this proof, Lemma 6.16 is also used. ☐

Proof of Theorem C. With all the results stated above as tools, the proof for theorem C is identical to [1, Proof of Theorem C]. ☐

7 Exponential dichotomy

Recall from Proposition 5.1, we obtained the following equivalent expression for the Laplace transform of equation (2.1)

$$\Delta_{L_0}(s)\tilde{x}(s) = x(0) - \sum_{j=1}^{\infty} A_{j,0} \int_{-r_j}^{0} e^{-s\xi} x(\xi + r_j) d\xi, \tag{7.1}$$

if $h = 0$ and the solution $x(\xi)$ has certain exponential behavior.
By considering a strip in the complex plane where $\Delta_{L_0}(s)$ is invertible (let us say for the strip $\mathrm{Re}(s) = b$ for certain $b \in \mathbb{R}$), we were able to perform the inverse Laplace transform and obtain a solution

$$w(\xi) = \frac{1}{2\pi i} \lim_{\Omega \to \infty} \int_{b-i\Omega}^{b+i\Omega} e^{\xi s} \Delta_{L_0}^{-1}(s) \left(x(0) - \sum_{j=1}^{\infty} A_{j,0} \int_{-r_j}^{0} e^{-s\xi'} x(\xi + r_j) d\xi' \right) \tag{7.2}$$

on the half-line $\xi \in [0, \infty)$, that also has certain exponential behavior.
This does not seem very surprising, since we started with a function that already is a solution and has certain exponential property. However, it does raise a question of whether $w(\xi)$ can still be a solution on the half line $[0, \infty)$, if we were to replace x by some arbitrary continuous function ϕ.

Assume that the shifts are bounded, i.e. there exist $r_{\min} \leq 0 \leq r_{\max}$ with $r_{\min} \leq r_j \leq r_{\max}$ for all $j \geq 1$. Inspired by the above observation, for a continuous function $\phi \in C([r_{\min}, r_{\max}]; \mathbb{C}^d)$ and $v \in \mathbb{C}^d$, we introduce the following expression

$$\mathcal{T}_{L;\eta}[\phi, v](\xi) = \frac{1}{2\pi i} \lim_{\Omega \to \infty} \int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s} \Delta_L(s)^{-1} \left(v - \sum_{j=1}^{\infty} A_j \int_{0}^{r_j} e^{-s(\sigma - r_j)} \phi(\sigma) d\sigma \right) ds, \tag{7.3}$$

where $\eta \in \mathbb{R}$ is such that $\Delta_L(s)$ is invertible. We shall see that this is well-defined (also if ϕ is not continuous at $\xi = 0$) and it provides a powerful tool to split each function ϕ into components that can be extended to half-line solutions for the homogeneous differential equation. In other words, we will show that the initial state space $C([r_{\min}, r_{\max}]; \mathbb{C}^d)$ has an exponential dichotomy.

In this section, we closely follow the outline of [10, §5], which describes a similar result in the setting where the variable x in (2.1, with $h = 0$) takes values in a Banach space with finitely many shifts. For convenience, we introduce the set

$$\mathcal{R} = \{0\} \cup \{-r_j\}_{j=1}^{\infty}. \tag{7.4}$$

Throughout this section, we shall always assume that our system is autonomous. Hence, we simply write L instead of L_0. We will also fix the shifts and use $\tilde{\eta}$ to denote the positive number such that (2.4) is satisfied. Furthermore, we shall assume that the closure $\overline{\mathcal{R}}$ is countable. We also assume without loss of generality that $\mathcal{R} = \overline{\mathcal{R}}$, since we can always add extra shifts by adding zero matrices.

In order to formulate results regarding the $\mathcal{T}_{L;\eta}$-function, we introduce the following same notations as in [10, §5].
For any $\eta \in \mathbb{R}$ and any interval $I \subset \mathbb{R}$, we introduce the function space

$$BC_\eta(I, \mathbb{C}^d) = \{x \in C(I, \mathbb{C}^d) : \|x\|_\eta := \sup_{\xi \in I} e^{-\eta\xi} |x(\xi)| < \infty\}. \tag{7.5}$$

In other words, for $x \in BC_\eta(I, \mathbb{C}^d)$, we have that x is continuous on I and satisfies $|x(\xi)| \leq K e^{\eta\xi}$ for some constant $K > 0$.

We will also define the following two families

$$\begin{aligned} BC_\eta^{\ominus}(\mathbb{C}^d) &= BC_\eta((-\infty, r_{\max}); \mathbb{C}^d), \\ BC_\eta^{\oplus}(\mathbb{C}^d) &= BC_\eta(r_{\min}, \infty); \mathbb{C}^d), \end{aligned} \tag{7.6}$$

together with the solutions spaces

$$
\begin{aligned}
\mathfrak{P}_L(\eta) &= \{x \in BC_\eta^{\ominus}(\mathbb{C}^d) : x'(\xi) = L\mathrm{ev}_\xi x, \text{ for all } \xi \leq 0\}, \\
\mathfrak{Q}_L(\eta) &= \{x \in BC_\eta^{\oplus}(\mathbb{C}^d) : y'(\xi) = L\mathrm{ev}_\xi y, \text{ for all } \xi \geq 0\},
\end{aligned}
\tag{7.7}
$$

where we have defined the function

$$
[\mathrm{ev}_\xi x](\sigma) = x(\xi + \sigma) \text{ for all } r_{\min} \leq \sigma \leq r_{\max}. \tag{7.8}
$$

The initial segments of these solutions are contained in the spaces

$$
\begin{aligned}
P_L(\eta) &= \{\phi \in C([r_{\min}, r_{\max}]; \mathbb{C}^d) : \phi = \mathrm{ev}_0 x \text{ for some } x \in \mathfrak{P}_L(\eta)\}, \\
Q_L(\eta) &= \{\phi \in C([r_{\min}, r_{\max}]; \mathbb{C}^d) : \phi = \mathrm{ev}_0 y \text{ for some } y \in \mathfrak{Q}_L(\eta)\}.
\end{aligned}
\tag{7.9}
$$

In order to allow solutions of (2.10) to have a jump discontinuity at $\xi = 0$, we introduce the notations

$$
\begin{aligned}
BC_\eta^-(\mathbb{C}^d) &= BC_\eta((-\infty, 0]; \mathbb{C}^d), \\
BC_\eta^+(\mathbb{C}^d) &= BC_\eta([0, \infty); \mathbb{C}^d),
\end{aligned}
\tag{7.10}
$$

together with the families of functions spaces

$$
\begin{aligned}
\widehat{BC}_\eta^{\oplus}(\mathbb{C}^d) &= C([r_{\min}, 0]; \mathbb{C}^d) \times BC_\eta^+(\mathbb{C}^d), \\
\widehat{BC}_\eta^{\ominus}(\mathbb{C}^d) &= BC_\eta^-(\mathbb{C}^d) \times C([0, r_{\max}]; \mathbb{C}^d).
\end{aligned}
\tag{7.11}
$$

For $\hat{y} = (y_1, y_2) \in \widehat{BC}^{\oplus}$, we write for $\xi \neq 0$ that $\hat{y}(\xi) = y_1 \mathbb{1}_{[r_{\min}, 0)} + y_2 \mathbb{1}_{(0, \infty)}$. We write $\hat{y}(0^+) := \lim_{\xi \downarrow 0} \hat{y}(\xi)$ and $\hat{y}(0^-) := \lim_{\xi \uparrow 0} \hat{y}(\xi)$ to resolve the ambiguity at $\xi = 0$. For any $0 \leq \xi \leq -r_{\min}$ and $\hat{y} \in \widehat{BC}_\eta^{\oplus}$, we introduce the notation

$$
\hat{\mathrm{ev}}_\xi \hat{y} = (\phi^l, \phi^r) \in C([r_{\min}, -\xi]; \mathbb{C}^d) \times C([-\xi, r_{\max}]; \mathbb{C}^d) \tag{7.12}
$$

where

$$
\begin{aligned}
\phi^l((\sigma)) &= \begin{cases} \hat{y}(\xi + \sigma) & r_{\min} \leq \sigma < -\xi \\ \hat{y}(0^-) & \sigma = -\xi, \end{cases} \\
\phi^r(\sigma) &= \begin{cases} \hat{y}(\xi + \sigma) & -\xi < \sigma \leq r_{\max} \\ \hat{y}(0^-) & \sigma = -\xi. \end{cases}
\end{aligned}
\tag{7.13}
$$

We then write

$$
\widehat{L}_+ \widehat{\mathrm{ev}}_\xi \hat{y} = \sum_{r_j = -\xi} A_j \phi^r(r_j) + \sum_{r_j > -\xi} A_j \phi^r(r_j) + \sum_{r_j < -\xi} A_j \phi^l(r_j). \tag{7.14}
$$

The plus sign hence stands for the fact that every reference to $\hat{y}(0)$ is interpreted as $\hat{y}(0^+)$.
For $\hat{x} \in \widehat{BC}_\eta^{\ominus}$ and $-r_{\max} \leq \xi \leq 0$, we again write $\widehat{ev}_\xi \hat{x} = (\phi^l, \phi^r)$ with (ϕ^l, ϕ^r) defined as in (7.13), with $\hat{y}$ replaced by $\hat{x}$. We then write

$$
\widehat{L}_- \widehat{\mathrm{ev}}_\xi \hat{x} = \sum_{r_j = -\xi} A_j \phi^l(r_j) + \sum_{r_j > -\xi} A_j \phi^r(r_j) + \sum_{r_j < -\xi} A_j \phi^l(r_j). \tag{7.15}
$$

Note that any discontinuities in the functions $\xi \mapsto \widehat{L}_+ \widehat{\mathrm{ev}}_\xi \hat{y}$ and $\xi \mapsto \widehat{L}_- \widehat{\mathrm{ev}}_\xi \hat{x}$ only occur when $\xi \in \mathcal{R}$. Hence for $\xi \notin \mathcal{R}$, we simply use the notation $\hat{L}$ since there is no cause for confusion.

We are now ready to introduce the solution spaces

$$
\begin{aligned}
\widehat{\mathfrak{P}}_L(\eta) &= \{\hat{x} \in \widehat{BC}_\eta^{\ominus}(\mathbb{C}^d) : \hat{x}'(\xi) = \hat{L}\hat{\mathrm{ev}}_\xi \hat{x}, \text{ for all } \xi \in (-\infty, 0) \setminus \mathcal{R}\}, \\
\widehat{\mathfrak{Q}}_L(\eta) &= \{\hat{y} \in \widehat{BC}_\eta^{\oplus}(\mathbb{C}^d) : \hat{y}'(\xi) = \hat{L}\hat{\mathrm{ev}}_\xi \hat{y}, \text{ for all } \xi \in (0, \infty) \setminus \mathcal{R}\},
\end{aligned}
\tag{7.16}
$$

together with the associated initial segment spaces

$$\begin{aligned}\widehat{P}_L(\eta) &= \{\hat{\phi} \in C([r_{\min}, 0]; \mathbb{C}^d) \times C([0, r_{\max}]; \mathbb{C}^d) : \hat{\phi} = \widehat{\mathrm{ev}}_0 \hat{x} \text{ for some } \hat{x} \in \widehat{\mathfrak{P}}_L(\eta)\}, \\ \widehat{Q}_L(\eta) &= \{\hat{\phi} \in C([r_{\min}, 0]; \mathbb{C}^d) \times C([0, r_{\max}]; \mathbb{C}^d) : \hat{\phi} = \widehat{\mathrm{ev}}_0 \hat{y} \text{ for some } \hat{y} \in \widehat{\mathfrak{Q}}_L(\eta)\}.\end{aligned} \tag{7.17}$$

We are now ready to state the following main results regarding the expression $\mathcal{T}_{L;\eta}$, which are all similar to the results as in [10, §5].

Proposition 7.1. *Let L be as defined in (2.8) with coefficients that satisfy condition (2.4) for a certain $\tilde{\eta} > 0$. Pick $\eta \in \mathbb{R}$ with $|\eta| < \tilde{\eta}$ such that the characteristic function $\Delta_L(s)$ is invertible for all $s \in \mathbb{C}$ with* $\mathrm{Re}(s) = \eta$.
Then for any $\hat{\phi} \in C([r_{\min}, 0]; \mathbb{C}^d) \times C([0, r_{\max}]; \mathbb{C}^d)$ and any $v \in \mathbb{C}^d$, the symbol $\mathcal{T}_{L;\eta}[\hat{\phi}, v](\xi)$ introduced in (7.3) is well-defined for $\xi \neq 0$. In addition, we have

$$T_{L;\eta}[\hat{\phi}, v] \in BC^-_\eta(\mathbb{C}^d), \text{ for } \xi < 0, \tag{7.18}$$

$$T_{L;\eta}[\hat{\phi}, v] \in BC^+_\eta(\mathbb{C}^d), \text{ for } \xi > 0. \tag{7.19}$$

The limit at zero exists both side and the jump of discontinuity is given by

$$T_{L;\eta}[\hat{\phi}, v](0^+) - T_{L;\eta}[\hat{\phi}, v](0^-) = v. \tag{7.20}$$

Moreover, there exists a constant $K > 0$ that does not depend on $\hat{\phi}$ and v such that

$$\|\mathcal{T}_{L;\eta}[\hat{\phi}, v]\|_{BC^-_\eta(\mathbb{C}^d)} + \|\mathcal{T}_{L;\eta}[\hat{\phi}, v]\|_{BC^+_\eta(\mathbb{C}^d)} \le K[\|\hat{\phi}\|_\infty + |v|], \tag{7.21}$$

where $\|\hat{\phi}\|_\infty$ denotes the supremum norm.

The map $\mathcal{T}_{L;\eta}$ can be used to relate functions in $\widehat{\mathfrak{P}}_L(\eta)$ and $\widehat{\mathfrak{Q}}_L(\eta)$ back to their initial segments. The following result is a consequence of Proposition 5.1.

Proposition 7.2. *Let L be as defined in (2.8) with coefficients that satisfy condition (2.4) for a certain $\tilde{\eta} > 0$. Pick $\eta \in \mathbb{R}$ with $|\eta| < \tilde{\eta}$ such that $\Delta_L(s)$ is invertible for all $s \in \mathbb{C}$ with* $\mathrm{Re}(s) = \eta$.
Then for any $\hat{q} \in \widehat{\mathfrak{Q}}_L(\eta)$ we have

$$\hat{q}(\xi) = \mathcal{T}_{L;\eta}[\widehat{ev}_0 \hat{q}, \hat{q}(0^+)](\xi) \tag{7.22}$$

for all $\xi > 0$, while for any $\hat{p} \in \widehat{\mathfrak{P}}_L(\eta)$ we have

$$\hat{p}(\xi) = -\mathcal{T}_{L;\eta}[\widehat{ev}_0 \hat{p}, \hat{p}(0^-)](\xi) \tag{7.23}$$

for all $\xi < 0$.

Proof. The proof is identical to [10, proof of Proposition 5.3], where almost-everywhere pointwise convergence of the inverse Laplace transform is used. A shift of integration of path is also used in that proof, which is justified similarly as in the proof for Proposition 5.1. □

Let η be such that $\Delta_L(s)$ is invertible for $\mathrm{Re}(s) = \eta$. As in [10, Equation (5.11)], we introduce the two operators

$$E_{\widehat{Q}_L(\eta)} : C([r_{\min}, 0]; \mathbb{C}^d) \times C([0, r_{\max}]; \mathbb{C}^d) \to \widehat{BC}^{\oplus}_\eta(\mathbb{C}^d), \tag{7.24}$$

$$E_{\widehat{P}_L(\eta)} : C([r_{\min}, 0]; \mathbb{C}^d) \times C([0, r_{\max}]; \mathbb{C}^d) \to \widehat{BC}^{\ominus}_\eta(\mathbb{C}^d) \tag{7.25}$$

with

$$[E_{\widehat{Q}_L(\eta)} \hat{\phi}](\xi) = \begin{cases} \mathcal{T}_{L;\eta}[\hat{\phi}, \hat{\phi}(0^+)](\xi) & \xi > 0 \text{ and } \xi = 0^+ \\ \hat{\phi}(\xi) + \mathcal{T}_{L;\eta}[\hat{\phi}, \hat{\phi}(0^+)](\xi) & r_{\min} \le \xi < 0 \text{ and } \xi = 0^-, \end{cases} \tag{7.26}$$

together with

$$[E_{\widehat{P}_L(\eta)}\hat{\phi}](\xi) = \begin{cases} -\mathcal{T}_{L;\eta}[\hat{\phi}, \hat{\phi}(0^+)](\xi) & \xi < 0 \text{ and } \xi = 0^- \\ \hat{\phi}(\xi) - \mathcal{T}_{L;\eta}[\hat{\phi}, \hat{\phi}(0^+)](\xi) & 0 < \xi \le r_{\max} \text{ and } \xi = 0^+, \end{cases} \tag{7.27}$$

which leads us to the following direct consequence of Proposition 7.2.

Corollary 7.3. *Consider the setting of Proposition 7.2. Then for any $\hat{\phi} \in \hat{P}_L(\eta)$ we have*

$$\hat{\phi} = \widehat{\mathrm{ev}}_0 E_{\hat{P}_L(\eta)}\hat{\phi}, \tag{7.28}$$

while for any $\hat{\phi} \in \widehat{Q}_L(\eta)$ we have

$$\hat{\phi} = \widehat{\mathrm{ev}}_0 E_{\hat{Q}_L(\eta)}\hat{\phi}. \tag{7.29}$$

The next result shows that for a general $\hat{\phi}$, its image under the operators $E_{\widehat{Q}_L(\eta)}$ and $E_{\widehat{P}_L(\eta)}$ are solutions on the half-lines. Furthermore, it shows that these solutions are exponentially bounded.

Proposition 7.4. *Let L be as defined in (2.8) with coefficients that satisfy condition (2.4) for a certain $\tilde{\eta} > 0$. Pick $\eta \in \mathbb{R}$ with $|\eta| < \tilde{\eta}$ such that $\Delta_L(s)$ is invertible for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \eta$.*
Then for any $\hat{\phi} \in C([r_{\min}, 0]; \mathbb{C}^d) \times C([0, r_{\max}]; \mathbb{C}^d)$, we have the inclusions

$$\begin{aligned} E_{\widehat{Q}_L(\eta)}\hat{\phi} &\in \widehat{\mathfrak{Q}}_L(\eta), \\ E_{\widehat{P}_L(\eta)}\hat{\phi} &\in \widehat{\mathfrak{P}}_L(\eta). \end{aligned} \tag{7.30}$$

In addition, there exists constants $K > 0$ and $\epsilon > 0$ that do not depend on $\hat{\phi}$ such that the estimates

$$\left|[E_{\widehat{Q}_L(\eta)}\hat{\phi}](\xi)\right| \le K e^{(\eta-\epsilon)\xi}[|\hat{\phi}(0^+)| + \|\hat{\phi}\|_\infty] \tag{7.31}$$

hold for all $\xi > 0$, while the estimates

$$\left|[E_{\widehat{P}_L(\eta)}\hat{\phi}](\xi)\right| \le K e^{(\eta+\epsilon)\xi}[|\hat{\phi}(0^+)| + \|\hat{\phi}\|_\infty] \tag{7.32}$$

hold for all $\xi < 0$.

Note that if $\phi \in C([r_{\min}, r_{\max}]; \mathbb{C}^d)$, we will then obtain continuity at $\xi = 0$ for the operators $[E_{\widehat{P}_L(\eta)}\hat{\phi}](\xi)$ and $[E_{\widehat{Q}_L(\eta)}\hat{\phi}](\xi)$ due to identity (7.20). Hence we have the restrictions

$$E_{\widehat{P}_L(\eta)}(C([r_{\min}, r_{\max}]; \mathbb{C}^d)) \subset BC_\eta^\ominus(\mathbb{C}^d), \quad E_{\widehat{Q}_L(\eta)}(C([r_{\min}, r_{\max}]; \mathbb{C}^d)) \subset BC_\eta^\oplus(\mathbb{C}^d). \tag{7.33}$$

This shows that the operators $E_{\widehat{P}_L(\eta)}$ and $E_{\widehat{Q}_L(\eta)}$ can be interpreted as the projections operators that give us the desired exponential splitting of the state space $C([r_{\min}, r_{\max}]; \mathbb{C}^d)$.

Proposition 7.5. *Let L be as defined in (2.8) with coefficients that satisfy condition (2.4) for a certain $\tilde{\eta} > 0$. Pick $\eta \in \mathbb{R}$ with $|\eta| < \tilde{\eta}$ such that $\Delta_L(s)$ is invertible for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \eta$. Then the spaces $P_L(\eta)$ and $Q_L(\eta)$ are both closed and we have the splitting*

$$C([r_{\min}, r_{\max}]; \mathbb{C}^d) = P_L(\eta) \oplus Q_L(\eta) \tag{7.34}$$

which can be made explicit by writing

$$\phi = \mathrm{ev}_0 E_{\widehat{P}_L(\eta)}\phi + \mathrm{ev}_0 E_{\widehat{Q}_L(\eta)}\phi \tag{7.35}$$

for any $\phi \in C([r_{\min}, r_{\max}]; \mathbb{C}^d)$.

7.1 Decomposition of $\mathcal{T}_{L;\eta}$

To obtain the above results, we need a more detailed understanding of the $\mathcal{T}_{L;\eta}$ function. This is done by splitting the function $\mathcal{T}_{L;\eta}$ into different components by isolating the slowest decaying portion of $\Delta_L^{-1}(s)$ and study them individually. In other words, we introduce the expression

$$R_{L;\alpha}(s) = \Delta_L(s)^{-1} - \frac{1}{s-\alpha}I - \frac{Le^{s\cdot} - \alpha I}{(s-\alpha)^2} \tag{7.36}$$

with $\alpha \in \mathbb{R}$ and $\alpha \neq \mathrm{Re}(s)$. We recall that $Le^{s\cdot} = \sum_{j=1}^{\infty} A_j e^{sr_j}$

Lemma 7.6. *Let L be as defined in (2.8) with coefficients that satisfy condition (2.4) for a certain $\tilde{\eta} > 0$. Pick $\eta \in \mathbb{R}$ with $|\eta| < \tilde{\eta}$ such that $\Delta_L(s)$ is invertible for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \eta$.*
Then for $\alpha \neq \eta$ and $|\alpha| < \tilde{\eta}$, there exists constants $\epsilon > 0$ and $K > 0$ so that

$$|R_{L;\alpha}(s)| \leq \frac{K}{|s|^3} \tag{7.37}$$

for all $s \in \mathbb{C}$ with $|\mathrm{Re}(s) - \eta| < \epsilon$.

Proof. The proof uses geometric series and is identical to [10, Lemma 5.7]. □

To exploit the decompositions of (7.36), we define the following expressions:

$$\begin{aligned} \mathcal{M}^1_\alpha[v] &= \frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s}\left[\frac{1}{s-\alpha} + \frac{Le^{s\cdot}-\alpha}{(s-\alpha)^2}\right] v, \\ \mathcal{R}^1_\alpha[v](\xi) &= \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s} R_{L;\alpha}(s) v ds \end{aligned} \tag{7.38}$$

together with

$$\begin{aligned} \mathcal{M}^2_\alpha[\hat{\phi}](\xi) &= \frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s}\left[\frac{1}{s-\alpha} + \frac{Le^{s\cdot}-\alpha}{(s-\alpha)^2}\right]\left(\sum_{j=1}^{\infty} A_j \int_{r_j}^{0} e^{-s(\sigma - r_j)}\hat{\phi}(\sigma)d\sigma\right) ds, \\ \mathcal{R}^2_\alpha[\hat{\phi}](\xi) &= \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s} R_{L;\alpha}(s)\left(\sum_{j=1}^{\infty} A_j \int_{r_j}^{0} e^{-s(\sigma - r_j)}\hat{\phi}(\sigma)d\sigma\right) ds. \end{aligned} \tag{7.39}$$

Note that we have dropped the limit with respect to Ω in the expressions for $\mathcal{R}^1_\alpha$ and $\mathcal{R}^2_\alpha$, because Lemma 7.6 shows that the integrands are integrable.

The expressions (7.38) and (7.39) give us the following decompositions of $\mathcal{T}_{L;\eta}$

$$\mathcal{T}_{L;\eta}[\hat{\phi}, v](\xi) = \mathcal{M}^1_\alpha[v](\xi) + \mathcal{M}^2_\alpha[\hat{\phi}](\xi) + \mathcal{R}^1_\alpha[v](\xi) + \mathcal{R}^2_\alpha[\hat{\phi}](\xi). \tag{7.40}$$

For convenience, we recall the Heaviside function

$$H(\xi) = \begin{cases} 1 & \xi > 0 \\ \frac{1}{2} & \xi = 0 \\ 0 & \xi < 0. \end{cases} \tag{7.41}$$

To shorten the notations, we will sometimes denote $\psi[\hat{\phi}](s) := \sum_{j=1}^{\infty} A_j \int_0^{r_j} e^{-s(\sigma - r_j)}\hat{\phi}(\sigma)d\sigma$, inspired by (5.13).

Our next goal is to gain a more explicit expressions for the terms in (7.40). This is mainly done by computing the following integrals.

Lemma 7.7. *For any $\alpha > \eta$ and $\xi \in \mathbb{R}$, we have*

$$\frac{1}{2\pi i} \lim_{\Omega\to\infty} \int_{\eta - i\Omega}^{\eta + i\Omega} e^{\xi s} \frac{1}{s-\alpha} ds = -e^{\alpha s} H(-\xi) \tag{7.42}$$

and

$$\frac{1}{2\pi i} \lim_{\Omega\to\infty} \int_{\eta - i\Omega}^{\eta + i\Omega} e^{\xi s} \frac{1}{(s-\alpha)^2} ds = -\xi e^{\alpha s} H(-\xi). \tag{7.43}$$

Proof. see [10, Lemma 5.8]. □

For $\xi \neq 0$, the above integrals can be calculated using the Jordan's lemma and the residue theorem. For $\xi > 0$, this explicit calculation can be found in [11]. During the calculation in [11], it shows that for $\xi > 0$ and $C_\Omega := \eta + \Omega e^{it}$ for $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$, we can estimate the error term

$$\left| \int_{C_\Omega} e^{\xi s} \frac{1}{s-\alpha} ds \right| \leq \frac{\epsilon(\Omega)\pi e^{\eta\xi}}{\xi} [1 - e^{-\xi\Omega}] \tag{7.44}$$

where $\epsilon(\Omega) \to 0$ if $\Omega \to \infty$. This comes from the fact that $\frac{1}{|s|} \to 0$, for $|s| \to \infty$. For $\xi < 0$, we take $D_\Omega := \eta + \Omega e^{it}$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ and follow the same estimation as in [11]. We then obtain

$$\begin{aligned}
\left| \int_{D_\Omega} e^{\xi s} \frac{1}{s-\alpha} ds \right| &\leq \epsilon(\Omega) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |e^{\xi(\eta + \Omega\cos(\theta) + i\Omega\sin(\theta))}| \cdot |\Omega i e^{i\theta}| d\theta \\
&\leq \epsilon(\Omega)\Omega e^{\eta\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\xi\Omega\cos(\theta)} d\theta \\
&\leq 2\epsilon(\Omega)\Omega e^{\eta\xi} \int_0^{\frac{\pi}{2}} e^{\xi\Omega\sin(\theta)} d\theta \\
&\leq 2\epsilon(\Omega)\Omega e^{\eta\xi} \int_0^{\frac{\pi}{2}} e^{\xi\Omega\frac{2}{\pi}\theta} d\theta \\
&\leq \frac{\epsilon(\Omega)\pi e^{\eta\xi}}{|\xi|} [1 - e^{-|\xi|\Omega}].
\end{aligned} \tag{7.45}$$

Hence, for $\xi \neq 0$, we see that the error is bounded by the function $E(\xi, \Omega) := \frac{\epsilon(\Omega)\pi e^{\eta\xi}}{|\xi|}(1 - e^{-\Omega|\xi|})$, where the function $\xi \mapsto \frac{1}{|\xi|}(1 - e^{-\Omega|\xi|})$ is a bounded function for all $\xi \in \mathbb{R}$ and all $\Omega \geq 0$. This means we can make the estimation $|E(\xi, \Omega)| = E(\xi, \Omega) \leq \epsilon(\Omega)\pi e^{\eta\xi} M$, where M is the constant that bounds the function $\xi \mapsto \frac{1}{|\xi|}(1 - e^{-\Omega|\xi|})$. We now make the following conclusions.

Remark 7.8.

(1). For any fixed $\eta \in \mathbb{R}$, the error term satisfies the limit $E(\xi, \Omega) \to 0$, as $\Omega \to \infty$, uniformly for every compact sets of ξ.

(2). If $\eta = 0$, then $E(\xi, \Omega) \to 0$, as $\Omega \to \infty$, uniformly for all $\xi \in \mathbb{R}$.

(3). If $\eta > 0$, then, for any $c \in \mathbb{R}$, $E(\xi, \Omega) \to 0$, as $\Omega \to \infty$, uniformly for all $\xi \in (-\infty, c)$.

(4). If $\eta < 0$, then, for any $a \in \mathbb{R}$, $E(\xi, \Omega) \to 0$, as $\Omega \to \infty$, uniformly for all $\xi \in (a, \infty)$.

(5). For $\eta \neq 0$, it does not hold that $E(\xi, \Omega) \to 0$, as $\Omega \to \infty$, uniformly for all $\xi \in \mathbb{R}$.

We shall see that (5) is the reason that we make the boundedness assumption for the shifts.

7.1.1 Characterization of the term $\mathcal{M}^1_\alpha$.

With Lemma 7.7 in hand, we can determine the following explicit expression for $\mathcal{M}^1_\alpha[v]$ and obtain characterizations similar to [10, Lemma 5.9].

Proposition 7.9. *For $\mathcal{M}^1_\alpha$ as defined in (7.39) with $\alpha > \eta$ such that $|\alpha| < \tilde{\eta}$, we have the explicit expression*

$$\mathcal{M}^1_\alpha[v] = \left[-e^{\alpha\xi}H(-\xi) - \sum_{j=1}^{\infty} A_j e^{\alpha(\xi+r_j)}(\xi+r_j)H(-(\xi+r_j)) + \alpha\xi e^{\alpha\xi}H(-\xi) \right] v. \tag{7.46}$$

Proof. In order to obtain the above expression using Lemma 7.7, it only remains to show that we may apply Fubini to the following expression

$$\frac{1}{2\pi i} \lim_{\Omega\to\infty} \int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s} \frac{1}{(s-\alpha)^2} \sum_{j=1}^{\infty} A_j e^{sr_j} v ds. \tag{7.47}$$

Observe that

$$\begin{aligned} \int_{\eta-i\infty}^{\eta+i\infty} \left| e^{\xi s} \frac{1}{(s-\alpha)^2} \sum_{j=1}^{\infty} A_j e^{sr_j} v \right| ds &\leq \int_{\eta-i\infty}^{\eta+i\infty} |e^{s\xi}| \frac{1}{|s-\alpha|^2} \left| \sum_{j=1}^{\infty} A_j e^{sr_j} v \right| ds \\ &\leq \int_{\eta-i\infty}^{\eta+i\infty} e^{\eta\xi} \frac{1}{|s-\alpha|^2} \sum_{j=1}^{\infty} |A_j| e^{\tilde{\eta}|r_j|} |v| ds \\ &< \infty. \end{aligned} \tag{7.48}$$

Hence, we have that the integrand in (7.47) is L^1 and thus we may drop the limit with respect to Ω sign. Furthermore, we have shown that we may apply Fubini. □

Lemma 7.10. *For any $\alpha > \eta$ such that $|\alpha| < \tilde{\eta}$ and sufficiently small $\epsilon > 0$, we have the function*

$$\mathbb{C}^d \ni v \mapsto \mathcal{M}^1_\alpha[v](\xi) \in BC^-_{\eta+\epsilon}(\mathbb{C}^d) \cap BC^+_{\eta-\epsilon}(\mathbb{C}^d) \tag{7.49}$$

is well-defined and bounded. For each fixed v, we have the jump discontinuity

$$\mathcal{M}^1_\alpha[v](0^+) - \mathcal{M}^1_\alpha[v](0^-) = v. \tag{7.50}$$

Proof. Identity (7.49) is is clear for terms like $e^{\alpha\xi}H(-\xi)$ and $\alpha\xi e^{\alpha\xi}H(-\xi)$. We will mainly examine the summation term, for $\xi \leq 0$

$$\begin{aligned} e^{-(\eta+\epsilon)\xi} \left| \sum_{j=1}^{\infty} A_j(\xi+r_j)e^{\alpha(\xi+r_j)}H(-(\xi+r_j)) \right| &\leq \sum_{j=1}^{\infty} |A_j| e^{(\alpha-(\eta+\epsilon))\xi} |\xi| e^{\alpha r_j} |H(\xi+r_j)| \\ &\quad + \sum_{j=1}^{\infty} |A_j| e^{(\alpha-(\eta+\epsilon))\xi} |r_j| e^{\alpha r_j} |H(\xi+r_j)|. \end{aligned} \tag{7.51}$$

This is bounded for all $\xi \leq 0$, for $\epsilon > 0$ sufficiently small. For $\xi \geq 0$, we have that $H(-(\xi+r_j)) = 0$ for $\xi \geq M$ with certain $M > 0$, since the shifts are bounded. Thus, the bounds in (7.49) indeed hold.

Note that the series $\sum_{j=1}^{\infty} A_j e^{\alpha(\xi+r_j)}(\xi+r_j)H(-(\xi+r_j))$ is (locally) normal convergent in ξ. Hence, we have

$$\begin{aligned} \lim_{\xi\uparrow 0} \mathcal{M}^1_\alpha[v](\xi) &= \lim_{\xi\uparrow 0} \left[-e^{\alpha\xi}H(-\xi) - \sum_{j=1}^{\infty} A_j e^{\alpha(\xi+r_j)}(\xi+r_j)H(-(\xi+r_j)) + \alpha\xi e^{\alpha\xi}H(-\xi) \right] v \\ &= \lim_{\xi\uparrow 0} -e^{\alpha\xi}H(-\xi)v - \sum_{j=1}^{\infty} \lim_{\xi\uparrow 0} A_j e^{\alpha(\xi+r_j)}(\xi+r_j)H(-(\xi+r_j))v + \lim_{\xi\uparrow 0} \alpha\xi e^{\alpha\xi}H(-\xi)v \\ &= -v - \sum_{j:r_j\neq 0}^{\infty} A_j e^{\alpha r_j}(r_j)H(-(r_j))v, \end{aligned} \tag{7.52}$$

while

$$\lim_{\xi\downarrow 0}\mathcal{M}^1_\alpha[v](\xi) = \lim_{\xi\downarrow 0}\left[-e^{\alpha\xi}H(-\xi) - \sum_{j=1}^{\infty} A_j e^{\alpha(\xi+r_j)}(\xi+r_j)H(-(\xi+r_j)) + \alpha\xi e^{\alpha\xi}H(-\xi)\right] v$$

$$= \lim_{\xi\downarrow 0} -e^{\alpha\xi}H(-\xi)v - \sum_{j=1}^{\infty}\lim_{\xi\downarrow 0} A_j e^{\alpha(\xi+r_j)}(\xi+r_j)H(-(\xi+r_j))v + \lim_{\xi\downarrow 0}\alpha\xi e^{\alpha\xi}H(-\xi)v \tag{7.53}$$

$$= -\sum_{j:r_j\neq 0}^{\infty} A_j e^{\alpha r_j}(r_j)H(-(r_j))v.$$

Thus, we obtain identity (7.50). □

We will show that $\mathcal{M}^1_\alpha$ also has a continuous derivative for $\xi\in\mathbb{R}\setminus\mathcal{R}$.

Proposition 7.11. *$\mathcal{M}^1_\alpha[v]$ is continuously differentiable on $\mathbb{R}\setminus\mathcal{R}$ and we may apply term-wise differentiation and obtain*

$$\mathcal{M}^1_\alpha[v]'(\xi) = \alpha\mathcal{M}^1_\alpha[v](\xi) - \sum_{j=1}^{\infty} A_j e^{\alpha(\xi+r_j)}H(-(\xi+r_j))v + \alpha e^{\alpha\xi}H(-\xi)v. \tag{7.54}$$

Proof. We make the following computations. For $\xi\in\mathbb{R}\setminus\{0\}$, we have

$$\frac{d}{d\xi}[-e^{\alpha\xi}H(-\xi)] = -\alpha e^{\alpha\xi}H(-\xi) \tag{7.55}$$

and

$$\frac{d}{d\xi}[\alpha\xi e^{\alpha\xi}H(-\xi)] = \alpha e^{\alpha\xi}H(-\xi) + \alpha^2\xi e^{\alpha\xi}H(-\xi). \tag{7.56}$$

We would like to compute $\mathcal{M}'_\alpha[v](\xi)$ by term wise differentiation, i.e., we want to show that

$$\begin{aligned}
&\frac{d}{d\xi}\left[\sum_{j=1}^{\infty} A_j e^{\alpha(\xi+r_j)}(\xi+r_j)H(-(\xi+r_j))\right]\\
&= \sum_{j=1}^{\infty}\left[\frac{d}{d\xi}A_j e^{\alpha(\xi+r_j)}(\xi+r_j)H(-(\xi+r_j))\right]\\
&= \sum_{j=1}^{\infty} A_j\left[\alpha e^{\alpha(\xi+r_j)}(\xi+r_j)H(-(\xi+r_j)) - e^{\alpha(\xi+r_j)}H(-(\xi+r_j))\right].
\end{aligned} \tag{7.57}$$

In order to do this, we need that the derivative terms $\sum_n f'_n(\xi)$ (thus the last sum of (7.57)) are locally C^1 in $\xi\in\mathbb{R}\setminus\mathcal{R}$ and converge locally uniformly in $\xi\in\mathbb{R}\setminus\mathcal{R}$. The C^1-condition is clearly true, for the locally uniform convergence, we can estimate

$$\begin{aligned}
&\sum_{j=1}^{\infty}|A_j|\left|\alpha e^{\alpha(\xi+r_j)}(\xi+r_j)H(-(\xi+r_j)) - e^{\alpha(\xi+r_j)}H(-(\xi+r_j))\right|\\
&\leq \sum_{j=1}^{\infty}|A_j|\left|\alpha e^{\alpha(\xi+r_j)}(\xi+r_j)\right| + |A_j|\left|e^{\alpha(\xi+r_j)}\right|\\
&\leq \sum_{j=1}^{\infty}|A_j||\alpha|e^{\alpha(\xi+r_j)}|\xi| + \sum_{j=1}^{\infty}|A_j||\alpha|e^{\alpha(\xi+r_j)}|r_j| + \sum_{j=1}^{\infty}|A_j|e^{\alpha(\xi+r_j)}\\
&\leq |\alpha||\xi|e^{\alpha\xi}\sum_{j=1}^{\infty}|A_j|e^{\alpha r_j} + |\alpha|e^{\alpha\xi}\max\{|r_{\min}|, r_{\max}\}\sum_{j=1}^{\infty}|A_j|e^{\alpha r_j} + e^{\alpha\xi}\sum_{j=1}^{\infty}|A_j|e^{\alpha r_j}\\
&\leq \left(|\alpha||\xi|e^{\alpha\xi} + |\alpha|e^{\alpha\xi}\max\{|r_{\min}|, r_{\max}\} + e^{\alpha\xi}\right)\sum_{j=1}^{\infty}|A_j|e^{\alpha r_j}.
\end{aligned} \tag{7.58}$$

From the above expression we see that the sum indeed converges locally uniformly for every compact sets of ξ. Hence, we may perform term wise differentiation. With some algebraic manipulation, identity (7.54) follows. $\square$

Lemma 7.12. *For $\mathcal{M}^1_\alpha[v]'$ as defined in (7.54) , we have for all $j \geq 1$ the discontinuity*

$$\mathcal{M}^1_\alpha[v]'(-r_j^+) - \mathcal{M}^1_\alpha[v]'(-r_j^-) = A_j v. \tag{7.59}$$

Proof. We note that for $r_k \neq 0$, the jump only comes from the summation term of (7.46). Hence, we exploit

$$\begin{aligned}
&\lim_{\xi \downarrow -r_k} -\sum_{j=1}^{\infty} A_j e^{\alpha(\xi + r_j)} H(-(\xi + r_j))v \\
&= \sum_{j=1}^{\infty} \lim_{\xi \downarrow -r_k} -A_j e^{\alpha(\xi + r_j)} H(-(\xi + r_j))v \\
&= \sum_{j: r_j < r_k} -A_j e^{\alpha(-r_k + r_j)} v - \sum_{j: r_j > r_k} \lim_{\xi \downarrow -r_k} A_k [e^{\alpha(\xi + r_k)} H(-(\xi + r_k))] v \\
&= -\sum_{j: r_j < r_k} A_j e^{\alpha(-r_k + r_j)} v,
\end{aligned} \tag{7.60}$$

while

$$\begin{aligned}
&\lim_{\xi \uparrow -r_k} -\sum_{j=1}^{\infty} A_j e^{\alpha(\xi + r_j)} H(-(\xi + r_j))v \\
&= \sum_{j=1}^{\infty} \lim_{\xi \uparrow -r_k} -A_j e^{\alpha(\xi + r_j)} H(-(\xi + r_j))v \\
&= \sum_{j: r_j < r_k} -A_j e^{\alpha(-r_k + r_j)} v - \sum_{j: r_j > r_k} \lim_{\xi \uparrow -r_k} A_k [e^{\alpha(\xi + r_k)} H(-(\xi + r_k))] v \\
&= -\sum_{j: r_j < r_k} A_j e^{\alpha(-r_k + r_j)} v - A_k v,
\end{aligned} \tag{7.61}$$

which gives us identity (7.59). $\square$

At last, we will substitute the expression $\mathcal{M}^1_\alpha[v]$ into the differential equation (2.10). To shorten the notation, we introduce

$$\mathcal{H}_{\mathcal{M}^1_\alpha}[v](\xi) = \mathcal{M}^1_\alpha[v]'(\xi) - \hat{L}\widehat{\text{ev}}_\xi \mathcal{M}^1_\alpha[v]. \tag{7.62}$$

By direct computation, we then have

$$\begin{aligned}
\mathcal{H}_{\mathcal{M}^1_\alpha}[v](\xi) = &\sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j e^{\alpha(\xi + r_k + r_j)} (\xi + r_k + r_j) H(-(\xi + r_k + r_j))v \\
&- 2\alpha \sum_{j=1}^{\infty} A_j (\xi + r_j) e^{\alpha(\xi + r_j)} H(-(\xi + r_j))v + \alpha^2 \xi e^{\alpha\xi} H(-\xi)
\end{aligned} \tag{7.63}$$

for all $\xi \in \mathbb{R} \setminus \mathcal{R}$. Note that all sums in (7.63) converges due to condition (2.4), which is assumed to hold for certain $\tilde{\eta}$ and by assumption we also have $|\eta| < \tilde{\eta}$.

7.1.2 Characterization of the term $\mathcal{M}_\alpha^2$.

We will now study the term $\mathcal{M}_\alpha^2$, for which we will first state a result regarding the following two terms. For any $\alpha \in \mathbb{R} \setminus \{\eta\}$, any $\hat{\phi} \in L^2([r_{\min}, r_{\max}]; \mathbb{C}^d)$ and any $\theta \in [r_{\min}, r_{\max}]$, we define the two expressions

$$\begin{aligned}
\mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}](\xi) &= \frac{1}{2\pi i} \lim_{\Omega\to\infty} \int_{\eta - i\Omega}^{\eta + i\Omega} e^{\xi s} \frac{1}{s-\alpha} e^{s\theta} \int_\theta^0 e^{-s\sigma} \hat{\phi}(\sigma) d\sigma ds, \\
\mathcal{J}^{(2)}_{\alpha,\theta}[\hat{\phi}](\xi) &= \frac{1}{2\pi i} \lim_{\Omega\to\infty} \int_{\eta - i\Omega}^{\eta + i\Omega} e^{\xi s} \frac{1}{(s-\alpha)^2} e^{s\theta} \int_\theta^0 e^{-s\sigma} \hat{\phi}(\sigma) d\sigma ds.
\end{aligned} \tag{7.64}$$

Note that both expressions are zero if $\theta = 0$. Then, we have the following characterizations of the expressions in (7.64), which is [10, Lemma 5.10].

Lemma 7.13. *Suppose that $r_{\min} \le 0 \le r_{\max}$ and pick $\alpha > \eta$ and $\theta \in [r_{\min}, r_{\max}] \setminus \{0\}$. Then for all sufficiently small $\epsilon > 0$, the maps*

$$\begin{aligned}
L^2([r_{\min}, r_{\max}]; \mathbb{C}^d) \ni \hat{\phi} &\mapsto \mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}] \in BC_\eta(\mathbb{R}; \mathbb{C}^d) \cap BC^-_{\eta+\epsilon}(\mathbb{C}^d) \cap BC^+_{\eta-\epsilon}(\mathbb{C}^d), \\
L^2([r_{\min}, r_{\max}]; \mathbb{C}^d) \ni \hat{\phi} &\mapsto \mathcal{J}^{(2)}_{\alpha,\theta}[\hat{\phi}] \in BC_\eta(\mathbb{R}; \mathbb{C}^d) \cap BC^-_{\eta+\epsilon}(\mathbb{C}^d) \cap BC^+_{\eta-\epsilon}(\mathbb{C}^d), \\
L^2([r_{\min}, r_{\max}]; \mathbb{C}^d) \ni \hat{\phi} &\mapsto \mathcal{J}^{(2)}_{\alpha,\theta}[\hat{\phi}]' \in BC_\eta(\mathbb{R}; \mathbb{C}^d) \cap BC^-_{\eta+\epsilon}(\mathbb{C}^d) \cap BC^+_{\eta-\epsilon}(\mathbb{C}^d)
\end{aligned} \tag{7.65}$$

are well-defined and bounded. Upon fixing $\hat{\phi} \in C([r_{\min}, 0]; \mathbb{C}^d) \times C([0, r_{\max}]; \mathbb{C}^d)$, we have the explicit identities

$$\begin{aligned}
\mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}](\xi) &= -e^{\alpha(\xi+\theta)} \int_\theta^0 H(\sigma - \xi - \theta) e^{-\alpha\sigma} \hat{\phi}(\sigma) d\sigma \\
&= -e^{\alpha(\xi+\theta)} \int_{\max\{\theta, \xi+\theta\}}^{\max\{0, \xi+\theta\}} e^{-\alpha\sigma} \hat{\phi}(\sigma) d\sigma
\end{aligned} \tag{7.66}$$

together with

$$\begin{aligned}
\mathcal{J}^{(2)}_{\alpha,\theta}[\hat{\phi}](\xi) &= -e^{\alpha(\xi+\theta)} \int_\theta^0 (\xi + \theta - \sigma) H(\sigma - \xi - \theta) e^{-\alpha\sigma} \hat{\phi}(\sigma) d\sigma \\
&= -e^{\alpha(\xi+\theta)} \int_{\max\{\theta, \xi+\theta\}}^{\max\{0, \xi+\theta\}} (\xi + \theta - \sigma) e^{-\alpha\sigma} \hat{\phi}(\sigma) d\sigma,
\end{aligned} \tag{7.67}$$

which both hold for any $\xi \in \mathbb{R}$.
In addition, for any $\xi \notin \{0, -\theta\}$, we have

$$\mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}]'(\xi) = \alpha \mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}](\xi) - \mathrm{sign}(\theta) \hat{\phi}(\xi + \theta) \mathbb{1}_{\min\{-\theta, 0\} < \xi < \max\{-\theta, 0\}}, \tag{7.68}$$

while for any $\xi \in \mathbb{R}$, we have

$$\mathcal{J}^{(2)}_{\alpha,\theta}[\hat{\phi}]'(\xi) = \alpha \mathcal{J}^{(2)}_{\alpha,\theta}[\hat{\phi}](\xi) + \mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}](\xi). \tag{7.69}$$

In particular, if $\theta < 0$, then we have

$$\mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}]' \in BC_\eta((-\infty, 0]; \mathbb{C}^d) \cap C([0, -\theta]; \mathbb{C}^d) \cap BC_\eta([-\theta, \infty); \mathbb{C}^d) \tag{7.70}$$

with jumps

$$\begin{aligned}
\mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}]'(0^+) - \mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}]'(0^-) &= \hat{\phi}(\theta), \\
\mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}]'(-\theta^+) - \mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}]'(-\theta^-) &= -\hat{\phi}(0^-).
\end{aligned} \tag{7.71}$$

On the other hand, if $\theta > 0$, then we have

$$\mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}]' \in BC_\eta((-\infty, -\theta]; \mathbb{C}^d) \cap C([-\theta, 0]; \mathbb{C}^d) \cap BC_\eta([0, \infty); \mathbb{C}^d) \tag{7.72}$$

with jumps

$$\begin{aligned}\mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}]'(0^+) - \mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}]'(0^-) &= \hat{\phi}(\theta),\\ \mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}]'(-\theta^+) - \mathcal{J}^{(1)}_{\alpha,\theta}[\hat{\phi}]'(-\theta^-) &= -\hat{\phi}(0^+).\end{aligned} \tag{7.73}$$

With Lemma 7.13 as tools, we will now determine an explicit expression for $\mathcal{M}^2_\alpha$. Recall that by definition, we have

$$\mathcal{M}^2_\alpha[\hat{\phi}](\xi) = \frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s}\left[\frac{1}{s-\alpha} + \frac{Le^{s\cdot}-\alpha}{(s-\alpha)^2}\right]\left(\sum_{j=1}^{\infty} A_j \int_{r_j}^{0} e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma\right) ds. \tag{7.74}$$

Lemma 7.14. *For any $\alpha > \eta$ with $|\alpha| < \tilde{\eta}$. We have the explicit expression*

$$\begin{aligned}&\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s}\frac{1}{s-\alpha}\left(\sum_{j=1}^{\infty} A_j \int_{r_j}^{0} e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma\right) ds\\ &= \frac{1}{2\pi i}\sum_{j=1}^{\infty} A_j \lim_{\Omega\to\infty}\int_{r_j}^{0}\hat{\phi}(\sigma)\int_{\eta-i\Omega}^{\eta+i\Omega} e^{s(\xi-\sigma+r_j)}\frac{1}{s-\alpha}dsd\sigma\\ &= \sum_{j=1}^{\infty} A_j \mathcal{J}^{(1)}_{\alpha,r_j}[\hat{\phi}](\xi)\end{aligned} \tag{7.75}$$

for all $\xi \in \mathbb{R}$.

Proof. Fix $\xi \in \mathbb{R}$. We observe that

$$\sum_{j=1}^{\infty}\int_{\eta-i\Omega}^{\eta+i\Omega}\left|e^{\xi s}\frac{1}{s-\alpha}A_j\int_{r_j}^{0} e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma\right| ds = \sum_{j=1}^{\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\eta\xi}\frac{1}{|s-\alpha|}|A_j|\left|\int_{r_j}^{0} e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma\right| ds. \tag{7.76}$$

Note that $\hat{\phi} \in C([r_{\min},0],\mathbb{C}^d)\times C([0,r_{\max}],\mathbb{C}^d)$, so it is bounded. If $\eta = 0$, this gives

$$\begin{aligned}\sum_{j=1}^{\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\eta\xi}\frac{1}{|s-\alpha|}|A_j|\left|\int_{r_j}^{0} e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma\right| ds &\leq \sum_{j=1}^{\infty}\int_{-i\Omega}^{i\Omega}\frac{1}{|s-\alpha|}|A_j|\|\hat{\phi}\|_\infty |r_j|\\ &\leq \sum_{j=1}^{\infty}|A_j|\|\hat{\phi}\|_\infty e^{\tilde{\eta}|r_j|}\int_{-i\Omega}^{i\Omega}\frac{1}{|s-\alpha|}ds < \infty.\end{aligned} \tag{7.77}$$

If $\eta \neq 0$, we have

$$\begin{aligned}&\sum_{j=1}^{\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\eta\xi}\frac{1}{|s-\alpha|}|A_j|\left|\int_{r_j}^{0} e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma\right| ds\\ &\leq \sum_{j=1}^{\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\eta\xi}\frac{1}{|s-\alpha|}|A_j|\|\hat{\phi}\|_\infty \mathrm{sign}(r_j)\left[\frac{1}{\eta}e^{\eta r_j} - \frac{1}{\eta}\right] ds\\ &\leq \sum_{j=1}^{\infty}|A_j|\|\hat{\phi}\|_\infty \mathrm{sign}(r_j)\left[\frac{1}{\eta}e^{\eta r_j} - \frac{1}{\eta}\right]\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\eta\xi}\frac{1}{|s-\alpha|}ds < \infty.\end{aligned} \tag{7.78}$$

Hence, we may apply Fubini and obtain

$$\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s}\frac{1}{s-\alpha}\left(\sum_{j=1}^{\infty} A_j\int_{r_j}^{0} e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma\right) = \sum_{j=1}^{\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s}\frac{1}{s-\alpha}A_j\int_{r_j}^{0} e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma. \tag{7.79}$$

Hence we have

$$\begin{aligned}
&\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s}\frac{1}{s-\alpha}\left(\sum_{j=1}^{\infty}A_j\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma\right)ds \\
&=\frac{1}{2\pi i}\lim_{\Omega\to\infty}\sum_{j=1}^{\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s}\frac{1}{s-\alpha}A_j\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma ds.
\end{aligned}\tag{7.80}$$

By applying Fubini to the summands, we have that

$$\begin{aligned}
&\frac{1}{2\pi i}\lim_{\Omega\to\infty}\sum_{j=1}^{\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s}\frac{1}{s-\alpha}A_j\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma ds \\
&=\frac{1}{2\pi i}\lim_{\Omega\to\infty}\sum_{j=1}^{\infty}A_j\int_{r_j}^{0}\hat{\phi}(\sigma)\int_{\eta-i\Omega}^{\eta+i\Omega}e^{s(\xi-\sigma+r_j)}\frac{1}{s-\alpha}dsd\sigma.
\end{aligned}\tag{7.81}$$

We will now show that

$$\begin{aligned}
&\frac{1}{2\pi i}\lim_{\Omega\to\infty}\sum_{j=1}^{\infty}A_j\int_{r_j}^{0}\hat{\phi}(\sigma)\int_{\eta-i\Omega}^{\eta+i\Omega}e^{s(\xi-\sigma+r_j)}\frac{1}{s-\alpha}dsd\sigma \\
&=\frac{1}{2\pi i}\sum_{j=1}^{\infty}A_j\lim_{\Omega\to\infty}\int_{r_j}^{0}\hat{\phi}(\sigma)\int_{\eta-i\Omega}^{\eta+i\Omega}e^{s(\xi-\sigma+r_j)}\frac{1}{s-\alpha}dsd\sigma.
\end{aligned}\tag{7.82}$$

As mentioned in Remark 7.8, we have that the limit in (7.42) holds uniformly for every compact sets of ξ. Because our shifts are bounded, we have that $|\xi-\sigma+r_j|\le M$ for all j and all $\sigma\in[r_{\min},r_{\max}]$. Hence, we can find an $N\in\mathbb{R}$ such that for all $\Omega\ge N$,

$$\begin{aligned}
&\left|\sum_{j=1}^{\infty}A_j\int_{r_j}^{0}\hat{\phi}(\sigma)\int_{\eta-i\Omega}^{\eta+i\Omega}e^{s(\xi-\sigma+r_j)}\frac{1}{s-\alpha}dsd\sigma-\sum_{j=1}^{\infty}A_j\int_{r_j}^{0}\hat{\phi}(\sigma)\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega}e^{s(\xi-\sigma+r_j)}\frac{1}{s-\alpha}dsd\sigma\right| \\
&\le\sum_{j=1}^{\infty}|A_j|\left|\int_{r_j}^{0}\hat{\phi}(\sigma)\left(\int_{\eta-i\Omega}^{\eta+i\Omega}e^{s(\xi-\sigma+r_j)}\frac{1}{s-\alpha}ds-\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega}e^{s(\xi-\sigma+r_j)}\frac{1}{s-\alpha}ds\right)d\sigma\right| \\
&\le\sum_{j=1}^{\infty}|A_j|\int_{r_j}^{0}|\hat{\phi}(\sigma)|\left|\int_{\eta-i\Omega}^{\eta+i\Omega}e^{s(\xi-\sigma+r_j)}\frac{1}{s-\alpha}ds-\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega}e^{s(\xi-\sigma+r_j)}\frac{1}{s-\alpha}ds\right|d\sigma \\
&\le\sum_{j=1}^{\infty}|A_j|\int_{r_{\min}}^{r_{\max}}\|\hat{\phi}\|_\infty\left|\int_{\eta-i\Omega}^{\eta+i\Omega}e^{s(\xi-\sigma+r_j)}\frac{1}{s-\alpha}ds-\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega}e^{s(\xi-\sigma+r_j)}\frac{1}{s-\alpha}ds\right|d\sigma \\
&\le\epsilon\sum_{j=1}^{\infty}|A_j|\|\hat{\phi}\|_\infty(r_{\max}-r_{\min}).
\end{aligned}\tag{7.83}$$

Thus, we see that (7.82) is justified. Applying Fubini again to the summands yields (7.75). □

Lemma 7.15. *For any $\alpha>\eta$ with $|\alpha|<\tilde{\eta}$. We have the explicit expression*

$$\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega}e^{\xi s}\frac{Le^{s\cdot}}{(s-\alpha)^2}\sum_{j=1}^{\infty}A_j\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma=\sum_{k=1}^{\infty}A_k\sum_{j=1}^{\infty}A_j\mathcal{J}^{(2)}_{\alpha,r_j}[\hat{\phi}](\xi+r_k)\tag{7.84}$$

for all $\xi\in\mathbb{R}$.

Proof. Fix $\xi \in \mathbb{R}$. By definition, we have

$$\begin{aligned}
&\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s}\frac{Le^{s\cdot}}{(s-\alpha)^2}\sum_{j=1}^{\infty}A_j\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma \\
&=\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s}\frac{1}{(s-\alpha)^2}\sum_{k=1}^{\infty}A_k e^{sr_k}\sum_{j=1}^{\infty}A_j\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma.
\end{aligned} \tag{7.85}$$

We estimate

$$\begin{aligned}
&\int_{\eta-i\infty}^{\eta+i\infty}|e^{\xi s}|\frac{1}{|s-\alpha|^2}\left|\sum_{k=1}^{\infty}A_k e^{sr_k}\right|\left|\sum_{j=1}^{\infty}A_j\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma\right|ds \\
&\le e^{\eta\xi}\int_{\eta-i\infty}^{\eta+i\infty}\frac{1}{|s-\alpha|^2}\sum_{k=1}^{\infty}|A_k|e^{\eta r_k}\sum_{j=1}^{\infty}|A_j|\left|\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma\right|ds.
\end{aligned} \tag{7.86}$$

Since the sum $\sum_{k=1}^{\infty}|A_j|e^{\eta r_k}$ converges, we can bring it outside the integral to obtain

$$\begin{aligned}
&e^{\eta\xi}\int_{\eta-i\infty}^{\eta+i\infty}\frac{1}{|s-\alpha|^2}\sum_{k=1}^{\infty}|A_k|e^{\eta r_k}\sum_{j=1}^{\infty}|A_j|\left|\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma\right|ds \\
&\le\sum_{k=1}^{\infty}|A_k|e^{\eta(\xi+r_k)}\int_{\eta-i\infty}^{\eta+i\infty}\frac{1}{|s-\alpha|^2}\sum_{j=1}^{\infty}|A_j|\left|\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma\right|ds \\
&\le\sum_{k=1}^{\infty}|A_k|e^{\eta(\xi+r_k)}\int_{\eta-i\infty}^{\eta+i\infty}\frac{1}{|s-\alpha|^2}\sum_{j=1}^{\infty}|A_j|\mathrm{sign}(r_j)\left[\frac{1}{\eta}e^{\eta r_j}-\frac{1}{\eta}\right]ds \\
&\le\sum_{k=1}^{\infty}|A_k|e^{\eta(\xi+r_k)}\sum_{j=1}^{\infty}|A_j|\mathrm{sign}(r_j)\left[\frac{1}{\eta}e^{\eta r_j}-\frac{1}{\eta}\right]\int_{\eta-i\infty}^{\eta+i\infty}\frac{1}{|s-\alpha|^2}ds \\
&<\infty.
\end{aligned} \tag{7.87}$$

Hence, we see that the integrand of (7.85) is in L^1, which allows us to drop the limit sign with respect to Ω. This yields

$$\begin{aligned}
&\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\eta-i\Omega}^{\eta+i\Omega} e^{\xi s}\frac{1}{(s-\alpha)^2}\sum_{k=1}^{\infty}A_k e^{sr_k}\sum_{j=1}^{\infty}A_j\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma ds \\
&=\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s}\frac{1}{(s-\alpha)^2}\sum_{k=1}^{\infty}A_k e^{sr_k}\sum_{j=1}^{\infty}A_j\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma ds.
\end{aligned} \tag{7.88}$$

Furthermore, it follows from the estimates in (7.86) and (7.87), that we may apply Fubini to switch the sums and the integral. Hence, we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s}\frac{1}{(s-\alpha)^2}\sum_{k=1}^{\infty}A_k e^{sr_k}\sum_{j=1}^{\infty}A_j\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma ds \\
&=\sum_{k=1}^{\infty}A_k\sum_{j=1}^{\infty}A_j\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}e^{s(\xi+r_k)}\frac{1}{(s-\alpha)^2}\int_{r_j}^{0}e^{-s(\sigma-r_j)}\hat{\phi}(\sigma)d\sigma ds \\
&=\sum_{k=1}^{\infty}A_k\sum_{j=1}^{\infty}A_j\mathcal{J}^{(2)}_{\alpha,r_j}[\hat{\phi}](\xi+r_k).
\end{aligned} \tag{7.89}$$

□

Proposition 7.16. *For any $\alpha > \eta$ with $|\alpha| < \tilde{\eta}$. We have the explicit expression*

$$M_\alpha^2[\hat{\phi}](\xi) = \sum_{j=1}^{\infty} A_j \mathcal{J}_{\alpha,r_j}^{(1)}[\hat{\phi}](\xi) + \sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j \mathcal{J}_{\alpha,r_j}^{(2)}[\hat{\phi}](\xi + r_k) - \alpha \sum_{j=1}^{\infty} A_j \mathcal{J}_{\alpha,r_j}^{(2)}[\hat{\phi}](\xi). \tag{7.90}$$

for all $\xi \in \mathbb{R}$.

Proof. Fix $\xi \in \mathbb{R}$. This follows from Lemma 7.14 and Lemma 7.15. □

Remark 7.17. *For the proof of Proposition 7.16, we were able to bring the limit inside at (7.82) because our shifts are bounded (see Remark 7.8). However, for unbounded shifts we may not. Hence, more work has to be done if we wish to generalize our results to unbounded shifts.*

We also obtain the next similar result regarding $\mathcal{M}_\alpha^2$ as in [10, Lemma 5.11].

Lemma 7.18. *For $\alpha > \eta$ with $|\alpha| < \tilde{\eta}$ and for any sufficiently small $\epsilon > 0$, we have*

$$M_\alpha^2[\hat{\phi}] \in BC_\eta(\mathbb{R}, \mathbb{C}^d) \cap BC_{\eta+\epsilon}^{-}(\mathbb{C}^d) \cap BC_{\eta-\epsilon}^{+}(\mathbb{C}^d) \tag{7.91}$$

for any $\hat{\phi} \in L^2([r_{\min}, r_{\max}]; \mathbb{C}^d)$.

Proof. We first estimate the terms

$$e^{-(\eta\pm\delta)\xi} \left| \sum_{j=1}^{\infty} A_j \mathcal{J}_{\alpha,r_j}^{(1)}[\hat{\phi}](\xi) \right| \leq \sum_{j=1}^{\infty} |A_j| e^{(\alpha-(\eta\pm\delta))\xi} e^{\alpha r_j} \left| \int_{\max\{r_j,\xi+r_j\}}^{\max\{0,\xi+r_j\}} e^{-\alpha\sigma} \hat{\phi}(\sigma) d\sigma \right|. \tag{7.92}$$

It is clear that (7.92) is bounded on a compact interval of ξ. Moreover, due to the boundedness of the shifts, we have that the integral on the right-hand side becomes zero for ξ sufficiently large. Also, for ξ sufficiently close to $-\infty$, we have that the integral becomes $|\int_{r_j}^{0} e^{-\alpha\sigma} \hat{\phi}(\sigma) d\sigma|$, which is a bounded term that does not depends on ξ. Because $\alpha > \eta$ by assumption, we have that $\alpha - (\eta \pm \delta) > 0$ for $\delta \geq 0$ sufficiently small. Hence we conclude that the first term of $M_\alpha^{(2)}$ satisfies the bounds in (7.91).

We proceed to estimate the term

$$\begin{aligned} &e^{-(\eta\pm\delta)\xi} \left| \sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j \mathcal{J}_{\alpha,r_j}^{(2)}[\hat{\phi}](\xi + r_k) \right| \\ &\leq \sum_{k=1}^{\infty} |A_k| \sum_{j=1}^{\infty} |A_j| e^{(\alpha-(\eta\pm\delta))\xi} e^{\alpha(r_k+r_j)} \left| \int_{\max\{r_j,\xi+r_k+r_j\}}^{\max\{0,\xi+r_k+r_j\}} e^{-\alpha\sigma} (\xi + r_k + r_j - \sigma) \hat{\phi}(\sigma) d\sigma \right|. \end{aligned} \tag{7.93}$$

Again, we see that the integral on the right-hand side is zero for ξ sufficiently large, and for ξ sufficiently close to $-\infty$, the integral becomes

$$\begin{aligned} &\left| \int_{r_j}^{0} e^{-\alpha\sigma} (\xi + r_k + r_j - \sigma) \hat{\phi}(\sigma) d\sigma \right| \\ &\leq \|\hat{\phi}\|_\infty \left[\frac{|\xi + r_k + r_j|}{|\alpha|} + \frac{1}{\alpha^2} + \frac{|\xi + r_k| e^{-\alpha r_j}}{|\alpha|} + \frac{e^{-\alpha r_j}}{\alpha^2} \right]. \end{aligned} \tag{7.94}$$

We see that the exponential term does not depends on ξ, which means that the term $e^{(\alpha-(\eta\pm\delta))\xi}$ is conserved. Because $\alpha - (\eta \pm \delta) > 0$, we have that $e^{(\alpha-(\eta\pm\delta))\xi}|\xi|$ is bounded for $\xi \leq 0$. Terms that are independent of ξ also stay bounded together with the sum. Hence, we conclude that the second term of $M_\alpha^{(2)}$ satisfies the bounds in (7.91). Since the third term of $M_\alpha^{(2)}$ can be estimated similarly as the second term, we conclude that the bounds in (7.91) are justified. □

We will now determine the derivative of $\mathcal{M}_\alpha^2$. We need the following lemmas.

Lemma 7.19. *The sum*

$$\sum_{j=1}^{\infty} A_j \mathcal{J}^{(1)}_{\alpha,r_j}[\hat{\phi}]'(\xi) \tag{7.95}$$

with $\mathcal{J}^{(1)}_{\alpha,r_j}[\hat{\phi}]'(\xi)$ *as defined in (7.68), converges locally uniformly in* ξ*, for all* $\xi \in \mathbb{R} \setminus \mathcal{R}$.

Proof. We first estimate

$$\begin{aligned}
\left|\sum_{j=1}^{\infty} A_j \mathcal{J}^{(1)}_{\alpha,r_j}[\hat{\phi}]'(\xi)\right| &= \left|\sum_{j=1}^{\infty} A_j \alpha \mathcal{J}^{(1)}_{\alpha,r_j}[\hat{\phi}](\xi) - A_j \operatorname{sign}(r_j)\hat{\phi}(\xi+r_j)\mathbb{1}_{\min\{-r_j,0\}<\xi<\max\{-r_j,0\}}\right| \\
&\le \sum_{j=1}^{\infty} |A_j||\alpha \mathcal{J}^{(1)}_{\alpha,r_j}[\hat{\phi}](\xi)| + \sum_{j=1}^{\infty} |A_j||\hat{\phi}(\xi+r_j)|\mathbb{1}_{\min\{-r_j,0\}<\xi<\max\{-r_j,0\}} \\
&\le \sum_{j=1}^{\infty} |A_j||\alpha| e^{\alpha(\xi+r_j)} \left|\int_{\max\{r_j,\xi+r_j\}}^{\max\{0,\xi+r_j\}} e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma\right| + \sum_{j=1}^{\infty} |A_j|\|\hat{\phi}\|_\infty.
\end{aligned} \tag{7.96}$$

Since $\sum_{j=1}^{\infty} |A_j|\|\hat{\phi}\|_\infty$ is just a constant, we only need to look at

$$\sum_{j=1}^{\infty} |A_j||\alpha| e^{\alpha(\xi+r_j)} \left|\int_{\max\{r_j,\xi+r_j\}}^{\max\{0,\xi+r_j\}} e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma\right|. \tag{7.97}$$

For $\xi \ge 0$, we get

$$\begin{aligned}
&\sum_{j=1}^{\infty} |A_j||\alpha| e^{\alpha(\xi+r_j)} \left|\int_{\max\{r_j,\xi+r_j\}}^{\max\{0,\xi+r_j\}} e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma\right| \\
&= \sum_{j:r_j\ge-\xi} |A_j||\alpha| e^{\alpha(\xi+r_j)} \left|\int_{\xi+r_j}^{\xi+r_j} e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma\right| + \sum_{j:r_j<-\xi}^{\infty} |A_j||\alpha| e^{\alpha(\xi+r_j)} \left|\int_{\xi+r_j}^{0} e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma\right| \\
&\le \sum_{j:r_j<-\xi} |A_j||\alpha|\|\hat{\phi}\|_\infty e^{\alpha(\xi+r_j)} \left[-\frac{1}{\alpha}e^{-\alpha\sigma}\right]_{\xi+r_j}^{0} \\
&\le \sum_{j:r_j<-\xi}^{\infty} |A_j||\alpha|\|\hat{\phi}\|_\infty e^{\alpha(\xi+r_j)} \left[-\frac{1}{\alpha} + \frac{1}{\alpha}e^{-\alpha(\xi+r_j)}\right] \\
&\le \sum_{j:r_j<-\xi} |A_j|\|\hat{\phi}\|_\infty e^{\alpha(\xi+r_j)} \left[1 + e^{-\alpha(\xi+r_j)}\right] \\
&\le \sum_{j=1}^{\infty} |A_j|\|\hat{\phi}\|_\infty e^{\alpha(\xi+r_j)} \left[1 + e^{-\alpha(\xi+r_j)}\right] \\
&\le \sum_{j=1}^{\infty} |A_j|\|\hat{\phi}\|_\infty + e^{\alpha\xi} \sum_{j=1}^{\infty} |A_j|\|\hat{\phi}\|_\infty e^{\tilde{\eta}|r_j|} < \infty.
\end{aligned} \tag{7.98}$$

For $\xi < 0$, we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} |A_j||\alpha| e^{\alpha(\xi+r_j)} \left| \int_{\max\{r_j,\xi+r_j\}}^{\max\{0,\xi+r_j\}} e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma \right| \\
&= \sum_{j:r_j<-\xi} |A_j||\alpha| e^{\alpha(\xi+r_j)} \left| \int_{r_j}^{0} e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma \right| + \sum_{j:r_j\geq-\xi} |A_j||\alpha| e^{\alpha(\xi+r_j)} \left| \int_{r_j}^{\xi+r_j} e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma \right| \\
&= \sum_{j:r_j<-\xi} |A_j||\alpha| e^{\alpha(\xi+r_j)} \left| \int_{r_j}^{0} e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma \right| + \sum_{j:r_j\geq-\xi} |A_j||\alpha| e^{\alpha(\xi+r_j)} \left| \int_{\xi+r_j}^{r_j} e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma \right| \\
&\leq \sum_{j:r_j<-\xi} |A_j||\alpha|\|\hat{\phi}\|_\infty e^{\alpha(\xi+r_j)} \cdot \left(-\mathrm{sign}(r_j) \left[-\frac{1}{\alpha} e^{-\alpha\sigma} \right]_{r_j}^{0} \right) + \sum_{j:r_j\geq-\xi} |A_j||\alpha| e^{\alpha(\xi+r_j)} \|\hat{\phi}\|_\infty \left[-\frac{1}{\alpha} e^{-\alpha\sigma} \right]_{\xi+r_j}^{r_j} \\
&\leq \sum_{j:r_j<-\xi}^{\infty} |A_j||\alpha|\|\hat{\phi}\|_\infty e^{\alpha(\xi+r_j)} \left[\frac{1}{|\alpha|} + \frac{1}{|\alpha|} e^{-\alpha r_j} \right] + \sum_{j:r_j\geq-\xi} |A_j||\alpha| e^{\alpha(\xi+r_j)} \|\hat{\phi}\|_\infty \left[\frac{1}{|\alpha|} e^{-\alpha r_j} + \frac{1}{|\alpha|} e^{-\alpha(\xi+r_j)} \right] \\
&\leq \sum_{j:r_j<-\xi} |A_j|\|\hat{\phi}\|_\infty e^{\alpha(\xi+r_j)} \left[1 + e^{-\alpha r_j} \right] + \sum_{j:r_j\geq-\xi} |A_j| e^{\alpha(\xi+r_j)} \|\hat{\phi}\|_\infty \left[e^{-\alpha r_j} + e^{-\alpha(\xi+r_j)} \right] \\
&\leq \sum_{j=1}^{\infty} |A_j|\|\hat{\phi}\|_\infty e^{\alpha(\xi+r_j)} \left[1 + e^{-\alpha r_j} \right] + \sum_{j=1}^{\infty} |A_j| e^{\alpha(\xi+r_j)} \|\hat{\phi}\|_\infty \left[e^{-\alpha r_j} + e^{-\alpha(\xi+r_j)} \right] \\
&\leq \|\hat{\phi}\|_\infty e^{\alpha\xi} \sum_{j=1}^{\infty} |A_j| e^{\tilde{\eta}|r_j|} + 2\|\hat{\phi}\|_\infty e^{\alpha\xi} \sum_{j=1}^{\infty} |A_j| + \|\hat{\phi}\|_\infty \sum_{j=1}^{\infty} |A_j|.
\end{aligned} \tag{7.99}$$

Hence, we conclude that $\sum_{j=1}^{\infty} A_j \mathcal{J}^{(1)}_{\alpha,r_j}[\hat{\phi}]'(\xi)$ converges locally uniformly in ξ. □

Lemma 7.20. *The sum*

$$\sum_{k=1}^{\infty} |A_k| \sum_{j=1}^{\infty} |A_j||\mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}]'(\xi + r_k)| \tag{7.100}$$

and the sum

$$\sum_{k=1}^{\infty} |A_k| \sum_{j=1}^{\infty} |A_j||\mathcal{J}^{(1)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k)| \tag{7.101}$$

with $\mathcal{J}^{(1)}_{\alpha,r_j}[\hat{\phi}]'(\xi)$ as defined in (7.68) and $\mathcal{J}^{(2)}_{\alpha,r_j}[\hat{\phi}]'(\xi)$ as defined in (7.69), converge both locally uniformly in ξ, for all $\xi \in \mathbb{R} \setminus \mathcal{R}$.

Proof. We have

$$\begin{aligned}
&\sum_{k=1}^{\infty} |A_k| \sum_{j=1}^{\infty} |A_j||\mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}]'(\xi + r_k)| \\
&\leq \sum_{k=1}^{\infty} |A_k| \sum_{j=1}^{\infty} |A_j||\alpha\mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k) + \mathcal{J}^{(1)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k)| \\
&\leq \sum_{k=1}^{\infty} |A_k| \sum_{j=1}^{\infty} |A_j||\alpha\mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k)| + |A_j||\mathcal{J}^{(1)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k)|.
\end{aligned} \tag{7.102}$$

We examine the term

$$\begin{aligned}
&\sum_{k=1}^{\infty}|A_k|\sum_{j=1}^{\infty}|A_j||\alpha\mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi+r_k)| \\
&=\sum_{k=1}^{\infty}|A_k|\sum_{j=1}^{\infty}|A_j||\alpha|e^{\alpha(\xi+r_k+r_j)}\left|\int_{\max\{r_j,\xi+r_k+r_j\}}^{\max\{0,\xi+r_k+r_j\}}(\xi+r_k+r_j-\sigma)e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma\right| \\
&=: \sum_{k=1}^{\infty}|A_k|\sum_{j=1}^{\infty}|A_j|e^{\alpha(\xi+r_k+r_j)}\mathcal{I}_{j,k}(\xi).
\end{aligned} \tag{7.103}$$

We examine the four different outcomes of the above integral, if $\xi+r_k+r_j\geq 0$ and $\xi+r_k+r_j\geq r_j$, then

$$\mathcal{I}_{j,k}(\xi)=\left|\int_{\xi+r_k+r_j}^{\xi+r_k+r_j}(\xi+r_k+r_j-\sigma)e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma\right|=0. \tag{7.104}$$

if $\xi+r_k+r_j\leq 0$ and $\xi+r_k+r_j\geq r_j$, then

$$\begin{aligned}
\mathcal{I}_{j,k}(\xi)&=\left|\int_{\xi+r_k+r_j}^{0}(\xi+r_k+r_j-\sigma)e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma\right| \\
&\leq\|\hat{\phi}\|_\infty\int_{\xi+r_k+r_j}^{0}-(\xi+r_k+r_j-\sigma)e^{-\alpha\sigma}d\sigma \\
&\leq\|\hat{\phi}\|_\infty\left[-\frac{e^{-\alpha\sigma}(\alpha(\sigma-(\xi+r_k+r_j))+1)}{\alpha^2}\right]_{\xi+r_k+r_j}^{0} \\
&\leq\|\hat{\phi}\|_\infty\left[\frac{|\xi+r_k+r_j|}{|\alpha|}+\frac{1}{\alpha^2}+\frac{e^{-\alpha(\xi+r_k+r_j)}}{\alpha^2}\right].
\end{aligned} \tag{7.105}$$

Note that

$$\begin{aligned}
&\sum_{k=1}^{\infty}|A_k|\sum_{j=1}^{\infty}|A_j|\|\hat{\phi}\|_\infty e^{\alpha(\xi+r_k+r_j)}\left[|\xi+r_k+r_j|+\frac{1}{|\alpha|}+\frac{e^{-\alpha(\xi+r_k+r_j)}}{|\alpha|}\right] \\
&\leq\sum_{k=1}^{\infty}|A_k|\sum_{j=1}^{\infty}|A_j|\|\hat{\phi}\|_\infty e^{\alpha(\xi+r_k+r_j)}|\xi+r_k+r_j|+\sum_{k=1}^{\infty}|A_k|\sum_{j=1}^{\infty}|A_j|\|\hat{\phi}\|_\infty e^{\alpha(\xi+r_k+r_j)}\frac{1}{|\alpha|} \\
&\qquad+\sum_{k=1}^{\infty}|A_k|\sum_{j=1}^{\infty}|A_j|\|\hat{\phi}\|_\infty\frac{1}{|\alpha|} \\
&\leq\|\hat{\phi}\|_\infty\left((|\xi|e^{\alpha\xi}+\frac{1}{|\alpha|})\sum_{k=1}^{\infty}|A_k|\sum_{j=1}^{\infty}|A_j|+M(e^{\alpha\xi}+\frac{1}{|\alpha|})\left(\sum_{k=1}^{\infty}|A_k|e^{\tilde{\eta}|r_k|}\right)\left(\sum_{j=1}^{\infty}|A_j|e^{\tilde{\eta}|r_j|}\right)\right),
\end{aligned} \tag{7.106}$$

which converges locally uniformly in ξ. Note that we used the estimation that for all $x\geq 0$, we have $x\leq M(\epsilon)e^{\epsilon x}$ for some constant $M(\epsilon)>0$ and $\epsilon>0$ such that $|\alpha|+\epsilon\leq\tilde{\eta}$.

If $\xi + r_k + r_j \geq 0$ and $\xi + r_k + r_j \leq r_j$, then

$$
\begin{aligned}
\mathcal{I}_{j,k}(\xi) &= \left| \int_{r_j}^{\xi+r_k+r_j} (\xi + r_k + r_j - \sigma) e^{-\alpha\sigma} \hat{\phi}(\sigma) d\sigma \right| \\
&\leq \|\hat{\phi}\|_\infty \int_{r_j}^{\xi+r_k+r_j} |\xi + r_k + r_j - \sigma| e^{-\alpha\sigma} d\sigma \\
&\leq \|\hat{\phi}\|_\infty \int_{r_j}^{\xi+r_k+r_j} (\xi + r_k + r_j - \sigma) e^{-\alpha\sigma} d\sigma \\
&\leq \|\hat{\phi}\|_\infty \left[\frac{e^{-\alpha\sigma}(\alpha(\sigma - (\xi + r_k + r_j)) + 1)}{\alpha^2} \right]_{r_j}^{\xi+r_k+r_j} \\
&\leq \|\hat{\phi}\|_\infty \left[\frac{e^{-\alpha(\xi+r_k+r_j)}}{\alpha^2} + \frac{|\xi + r_k| e^{-\alpha r_j}}{|\alpha|} + \frac{e^{-\alpha r_j}}{\alpha^2} \right].
\end{aligned}
\tag{7.107}
$$

It can be shown, with the same estimation technique as in (7.106), that

$$
\sum_{k=1}^{\infty} |A_k| \sum_{j=1}^{\infty} |A_j| \|\hat{\phi}\|_\infty e^{\alpha(\xi+r_k+r_j)} \left[\frac{e^{-\alpha(\xi+r_k+r_j)}}{\alpha^2} + \frac{|\xi + r_k| e^{-\alpha r_j}}{|\alpha|} + \frac{e^{-\alpha r_j}}{\alpha^2} \right] \tag{7.108}
$$

converges locally uniformly in ξ.
At last, if $\xi + r_k + r_j \leq 0$ and $\xi + r_k + r_j \leq r_j$, we have

$$
\mathcal{I}_{j,k}(\xi) = \left| \int_{r_j}^{0} (\xi + r_k + r_j - \sigma) e^{-\alpha\sigma} \hat{\phi}(\sigma) d\sigma \right|. \tag{7.109}
$$

For $r_j \leq 0$, we have $\xi + r_k \leq 0$. Thus,

$$
\begin{aligned}
&\left| \int_{r_j}^{0} (\xi + r_k + r_j - \sigma) e^{-\alpha\sigma} \hat{\phi}(\sigma) d\sigma \right| \\
&\leq \|\hat{\phi}\|_\infty \int_{r_j}^{0} |\xi + r_k + r_j - \sigma| e^{-\alpha\sigma} d\sigma \\
&\leq \|\hat{\phi}\|_\infty \int_{r_j}^{0} -(\xi + r_k + r_j - \sigma) e^{-\alpha\sigma} d\sigma \\
&\leq \|\hat{\phi}\|_\infty \left[\frac{e^{-\alpha\sigma}(\alpha(\sigma - (\xi + r_k + r_j)) + 1)}{\alpha^2} \right]_{r_j}^{0} \\
&\leq \|\hat{\phi}\|_\infty \left[\frac{(\xi + r_k + r_j)}{\alpha} - \frac{1}{\alpha^2} - \frac{(\xi + r_k) e^{-\alpha r_j}}{\alpha} + \frac{e^{-\alpha r_j}}{\alpha^2} \right] \\
&\leq \|\hat{\phi}\|_\infty \left[\frac{|\xi + r_k + r_j|}{|\alpha|} + \frac{1}{\alpha^2} + \frac{|\xi + r_k| e^{-\alpha r_j}}{|\alpha|} + \frac{e^{-\alpha r_j}}{\alpha^2} \right].
\end{aligned}
\tag{7.110}
$$

For $r_j \geq 0$, we have

$$\begin{aligned}
&\left|\int_{r_j}^{0} (\xi + r_k + r_j - \sigma)e^{-\alpha\sigma}\hat{\phi}(\sigma)d\sigma\right| \\
&\leq \|\hat{\phi}\|_\infty \int_0^{r_j} |\xi + r_k + r_j - \sigma|e^{-\alpha\sigma}d\sigma \\
&\leq \|\hat{\phi}\|_\infty \int_0^{r_j} -(\xi + r_k + r_j - \sigma)e^{-\alpha\sigma}d\sigma \\
&\leq \|\hat{\phi}\|_\infty \int_{r_j}^{0} (\xi + r_k + r_j - \sigma)e^{-\alpha\sigma}d\sigma \\
&\leq \|\hat{\phi}\|_\infty \left[\frac{|\xi + r_k + r_j|}{|\alpha|} + \frac{1}{\alpha^2} + \frac{|\xi + r_k|e^{-\alpha r_j}}{|\alpha|} + \frac{e^{-\alpha r_j}}{\alpha^2}\right].
\end{aligned} \tag{7.111}$$

It can be shown, with the same estimation technique as in (7.106), that

$$\sum_{k=1}^{\infty}|A_k|\sum_{j=1}^{\infty}|A_j|\|\hat{\phi}\|_\infty e^{\alpha(\xi+r_k+r_j)}\left[\frac{|\xi + r_k + r_j|}{|\alpha|} + \frac{1}{\alpha^2} + \frac{|\xi + r_k|e^{-\alpha r_j}}{|\alpha|} + \frac{e^{-\alpha r_j}}{\alpha^2}\right] \tag{7.112}$$

converges locally uniformly in ξ. Hence, with identities (7.106)-(7.112), we conclude that (7.103) converges locally uniformly in ξ.

With a similar estimation technique as for (7.97), it can be shown that the term

$$\sum_{k=1}^{\infty}|A_k|\sum_{j=1}^{\infty}|A_j||\mathcal{J}^{(1)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k)| \tag{7.113}$$

converges locally uniformly in ξ. □

Proposition 7.21. *$\mathcal{M}^2_\alpha[\hat{\phi}]$ is continuously differentiable on $\mathbb{R}\backslash\mathcal{R}$ and we may apply term-wise differentiation and obtain*

$$\begin{aligned}
\mathcal{M}^2_\alpha[\hat{\phi}]'(\xi) = &\sum_{j=1}^{\infty} A_j\left[\alpha\mathcal{J}^{(1)}_{\alpha,r_j}[\hat{\phi}](\xi) - \text{sign}(r_j)\hat{\phi}(\xi + r_j)\mathbb{1}_{\min\{-r_j,0\}<\xi<\max\{-r_j,0\}}\right] \\
&+ \sum_{k=1}^{\infty}A_k\sum_{j=1}^{\infty}A_j(\alpha\mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k) + \mathcal{J}^{(1)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k)) \\
&- \alpha\sum_{j=1}^{\infty}A_j(\alpha\mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi) + \mathcal{J}^{(1)}_{\alpha;r_j}[\hat{\phi}](\xi)) \\
= &\,\alpha\mathcal{M}^2_\alpha[\hat{\phi}](\xi) - \sum_{j=1}^{\infty}A_j\text{sign}(r_j)\hat{\phi}(\xi + r_j)\mathbb{1}_{\min\{-r_j,0\}<\xi<\max\{-r_j,0\}} \\
&+ \sum_{k=1}^{\infty}A_k\sum_{j=1}^{\infty}A_j\mathcal{J}^{(1)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k) - \alpha\sum_{j=1}^{\infty}A_j\mathcal{J}^{(1)}_{\alpha;r_j}[\hat{\phi}](\xi).
\end{aligned} \tag{7.114}$$

In particular, for $\xi \in (-\infty, 0) \setminus \mathcal{R}$, we have

$$\begin{aligned}
\mathcal{M}^2_\alpha[\hat{\phi}]'(\xi) = &\,\alpha\mathcal{M}^2_\alpha[\hat{\phi}](\xi) - \sum_{r_j>0}A_j\hat{\phi}(\xi + r_j)\mathbb{1}_{-r_j<\xi<0} \\
&+ \sum_{k=1}^{\infty}A_k\sum_{j=1}^{\infty}A_j\mathcal{J}^{(1)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k) - \alpha\sum_{j=1}^{\infty}A_j\mathcal{J}^{(1)}_{\alpha;r_j}[\hat{\phi}](\xi)
\end{aligned} \tag{7.115}$$

and for $\xi \in (0,\infty) \setminus \mathcal{R}$, we obtain

$$\begin{aligned} \mathcal{M}^2_\alpha[\hat{\phi}]'(\xi) &= \alpha \mathcal{M}^2_\alpha[\hat{\phi}](\xi) + \sum_{r_j<0} A_j \hat{\phi}(\xi + r_j) \mathbb{1}_{0<\xi<-r_j} \\ &\quad + \sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j \mathcal{J}^{(1)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k) - \alpha \sum_{j=1}^{\infty} A_j \mathcal{J}^{(1)}_{\alpha;r_j}[\hat{\phi}](\xi). \end{aligned} \tag{7.116}$$

Proof. In order to compute $\mathcal{M}^2_\alpha[\hat{\phi}]'(\xi)$ for $\xi \in \mathbb{R} \setminus \mathcal{R}$ by using identities (7.68) and (7.69), we need to show that we may perform term wise differentiation. Because the derivatives in (7.68) and (7.69) are clearly continuous for $\xi \in \mathbb{R} \setminus \mathcal{R}$, we need to only show that the sums of the derivatives converge locally uniformly in $\xi \in \mathbb{R} \setminus \mathcal{R}$. This follows from Lemma 7.19 and Lemma 7.20. Hence, we conclude that we may perform term wise differentiation and obtain identity (7.114).
The remaining identities follow from direct computations. □

Lemma 7.22. *Fix $\hat{\phi} \in C([r_{\min}, 0]; \mathbb{C}^d) \times C([0, r_{\max}]; \mathbb{C}^d)$. For $r_j < 0$ the function $\mathcal{M}^2_\alpha[\hat{\phi}]'$ has the jump*

$$\mathcal{M}^2_\alpha[\hat{\phi}]'(-r_j^+) - \mathcal{M}^2_\alpha[\hat{\phi}]'(-r_j^-) = -A_j \hat{\phi}(0^-), \tag{7.117}$$

while for $r_j > 0$ we have

$$\mathcal{M}^2_\alpha[\hat{\phi}]'(-r_j^+) - \mathcal{M}^2_\alpha[\hat{\phi}]'(-r_j^-) = -A_j \hat{\phi}(0^+). \tag{7.118}$$

In addition, the discontinuity at $\xi = 0$ is given by

$$\mathcal{M}^2_\alpha[\hat{\phi}]'(0^+) - \mathcal{M}^2_\alpha[\hat{\phi}]'(0^-) = \sum_{r_j<0} A_j \hat{\phi}(r_j) + \sum_{r_j>0} A_j \hat{\phi}(r_j). \tag{7.119}$$

Proof. Identities (7.115), (7.116) and (7.121) follow from (7.114). Identities (7.117), (7.118) and (7.119) follow from (7.71), (7.73) and the fact that we may calculate the limits term-wise due the uniform convergence of the sums. □

We introduce the expression

$$\mathcal{H}_{\mathcal{M}^2_\alpha}[\hat{\phi}](\xi) = \mathcal{M}^2_\alpha[\hat{\phi}]'(\xi) - \mathrm{Lev}_\xi \mathcal{M}^2_\alpha[\hat{\phi}] \tag{7.120}$$

for $\xi \in \mathbb{R} \setminus \mathcal{R}$, Substituting $\mathcal{M}^1_\alpha[v]$ into the differential equation (2.10) yields

$$\begin{aligned} \mathcal{H}_{\mathcal{M}^2_\alpha}[\hat{\phi}](\xi) = &- \sum_{l=1}^{\infty} A_l \sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k + r_l) + 2\alpha \sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k) \\ &- \alpha^2 \sum_{j=1}^{\infty} A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi) - \sum_{j=1}^{\infty} A_j \mathrm{sign}(r_j) \hat{\phi}(\xi + r_j) \mathbb{1}_{\min\{-r_j,0\}<\xi<\max\{-r_j,0\}}, \end{aligned} \tag{7.121}$$

for all such ξ.

7.1.3 Characterization of the remaining terms $\mathcal{R}^1_\alpha$ and $\mathcal{R}^2_\alpha$.

We will now inspect the remaining terms of $\mathcal{T}_{L;\eta}$. After checking some smoothness conditions, we obtain the following similar results as in [10, Lemma 5.12].

Proposition 7.23. *Let L be as defined in (2.8) with coefficients that satisfy condition (2.4) for a certain $\tilde{\eta} > 0$. Pick $\eta \in \mathbb{R}$ with $|\eta| < \tilde{\eta}$ such that $\Delta_L(s)$ is invertible for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \eta$. Pick $\alpha > \eta$ with $|\alpha| < \tilde{\eta}$, then for any sufficiently small $\epsilon > 0$, the map*

$$\mathbb{C}^d \ni v \mapsto \mathcal{R}^1_\alpha[v] \in BC_\eta(\mathbb{R}, \mathbb{C}^d) \cap BC^-_{\eta+\epsilon}(\mathbb{C}^d) \cap BC^+_{\eta-\epsilon}(\mathbb{C}^d), \tag{7.122}$$

$$\mathbb{C}^d \ni v \mapsto \mathcal{R}^1_\alpha[v]' \in BC_\eta(\mathbb{R},\mathbb{C}^d) \cap BC^-_{\eta+\epsilon}(\mathbb{C}^d) \cap BC^+_{\eta-\epsilon}(\mathbb{C}^d), \tag{7.123}$$

where

$$\mathcal{R}^1_{L;\alpha}[v]'(\xi) = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} s e^{\xi s} R_{L;\alpha}(s) v ds \tag{7.124}$$

together with the maps

$$\mathbb{C}^d \ni v \mapsto \mathcal{R}^2_\alpha[\hat{\phi}] \in BC_\eta(\mathbb{R},\mathbb{C}^d) \cap BC^-_{\eta+\epsilon}(\mathbb{C}^d) \cap BC^+_{\eta-\epsilon}(\mathbb{C}^d), \tag{7.125}$$

$$\mathbb{C}^d \ni v \mapsto \mathcal{R}^2_\alpha[\hat{\phi}]' \in BC_\eta(\mathbb{R},\mathbb{C}^d) \cap BC^-_{\eta+\epsilon}(\mathbb{C}^d) \cap BC^+_{\eta-\epsilon}(\mathbb{C}^d), \tag{7.126}$$

where

$$\mathcal{R}^2_{L;\alpha}[v]'(\xi) = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} s e^{\xi s} R_{L;\alpha}(s) \sum_{j=1}^{\infty} A_j \int_{r_j}^{0} e^{-s(\sigma - r_j)}\hat{\phi}(\sigma) d\sigma ds \tag{7.127}$$

are well-defined and bounded.
Upon fixing $\hat{\phi} \in C([r_{\min},0];\mathbb{C}^d) \times C([0,r_{\max}];\mathbb{C}^d)$ *and* $v \in \mathbb{C}^d$*, we define*

$$\begin{aligned} \mathcal{H}_{\mathcal{R}^1_\alpha}[v](\xi) &= \mathcal{R}^1_\alpha[v]'(\xi) - \mathrm{Lev}_\xi \mathcal{R}^1_\alpha[v], \\ \mathcal{H}_{\mathcal{R}^2_\alpha}[v](\xi) &= \mathcal{R}^2_\alpha[\hat{\phi}]'(\xi) - \mathrm{Lev}_\xi \mathcal{R}^2_\alpha[\hat{\phi}], \end{aligned} \tag{7.128}$$

for any $\xi \in \mathbb{R}$*, we have the identity*

$$\begin{aligned} \mathcal{H}_{\mathcal{R}^1_\alpha}[v](\xi) = &- \sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j (\xi + r_k + r_j) e^{(\xi + r_k + r_j)\alpha} H(-(\xi + r_k + r_j)) v \\ &+ 2\alpha \left(\sum_{k=1}^{\infty} A_k (\xi + r_k) e^{\alpha(\xi + r_k)} H(-(\xi + r_k)) \right) v - \alpha^2 \xi e^{\alpha \xi} H(-\xi) v \end{aligned} \tag{7.129}$$

together with

$$\begin{aligned} \mathcal{H}_{\mathcal{R}^2_\alpha}[v](\xi) = &\sum_{k=1}^{\infty} A_k \sum_{l=1}^{\infty} A_l \sum_{j=1}^{\infty} A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k + r_l) - 2\alpha \sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k) \\ &+ \alpha^2 \sum_{j=1}^{\infty} A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi) \end{aligned} \tag{7.130}$$

for all $\xi \in \mathbb{R}$.

Proof. We first make the estimate

$$\begin{aligned} \left| \sum_{j=1}^{\infty} A_j \int_{r_j}^{0} e^{-s(\sigma - r_j)} \hat{\phi}(\sigma) d\sigma \right| &\leq \sum_{j=1}^{\infty} |A_j| e^{\eta r_j} \left| \int_{r_j}^{0} e^{-s\sigma} \hat{\phi}(\sigma) d\sigma \right| \\ &\leq \sum_{j=1}^{\infty} |A_j| e^{\mathrm{Re}(s) r_j} \int_{r_{\min}}^{r_{\max}} e^{\mathrm{Re}(s)\sigma} |\hat{\phi}(\sigma)| d\sigma \\ &\leq K \|\phi\|_{L^\infty([r_{\min}, r_{\max}],\mathbb{C}^d)}, \end{aligned} \tag{7.131}$$

which holds uniformly for $|\mathrm{Re}(s) - \eta| < \epsilon$ with certain $\epsilon > 0$.
Furthermore, according to Lemma 7.6, we have a uniform bound for $\|R_{L;\alpha}(s)\|$ and we see that $|s|\|R_{L;\alpha}(s)\|$ is in $L^1(\eta + i\mathbb{R})$ and $L^1(\eta \pm \epsilon + i\mathbb{R})$. This shows that identities (7.124) and (7.127) are simply a consequence of Lemma 3.8.

The proof for identities (7.122), (7.123), (7.125) and (7.126) is identical to [10, proof of Lemma 5.12], which is done by shifting the integration path from $\mathrm{Re}(s) = \eta$ to $\mathrm{Re}(s) = \eta \pm \epsilon$. The above estimations shows that we may indeed perform the shifting.

To obtain identity (7.129), we compute

$$\begin{aligned}&\mathcal{R}^1_\alpha[v]'(\xi) - L\mathrm{ev}_\xi \mathcal{R}^1_\alpha[v] \\ &= \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} s e^{\xi s} R_{L;\alpha}(s) v ds - \sum_{k=1}^{\infty} A_k \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{(\xi+r_k)s} R_{L;\alpha}(s) v ds.\end{aligned} \tag{7.132}$$

It is clear that we may apply Fubini and obtain

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} s e^{\xi s} R_{L;\alpha}(s) v ds - \sum_{k=1}^{\infty} A_k \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{(\xi+r_k)s} R_{L;\alpha}(s) v ds \\ &= \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} \left(s - \sum_{k=1}^{\infty} A_k e^{s r_k}\right) e^{\xi s} R_{L;\alpha}(s) v ds \\ &= \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s} \Delta_L(s) R_{L;\alpha}(s) v ds.\end{aligned} \tag{7.133}$$

As in [10, proof of Lemma 5.12], we have

$$\begin{aligned}\Delta_L(s) R_{L;\alpha}(s) &= I - \frac{s - Le^{s\cdot}}{s-\alpha} - \frac{(s - Le^{s\cdot})(se^{s\cdot} - \alpha)}{(s-\alpha)^2} \\ &= \frac{(Le^{s\cdot} - \alpha)^2}{(s-\alpha)^2}.\end{aligned} \tag{7.134}$$

Hence, we obtain

$$\begin{aligned}\mathcal{R}^1_\alpha[v]'(\xi) - L\mathrm{ev}_\xi \mathcal{R}^1_\alpha[v] &= \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s} \frac{(Le^{s\cdot} - \alpha)^2}{(s-\alpha)^2} v ds \\ &= \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s} \left(\frac{(Le^{s\cdot})^2}{(s-\alpha)^2} - 2\alpha \frac{Le^{s\cdot}}{(s-\alpha)^2} + \frac{\alpha^2}{(s-\alpha)^2}\right) v ds.\end{aligned} \tag{7.135}$$

We inspect these terms separately. By applying Fubini, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s} \frac{(Le^{s\cdot})^2}{(s-\alpha)^2} v ds &= \sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{(\xi+r_k+r_j)s} \frac{1}{(s-\alpha)^2} v ds \\ &= -\sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j (\xi + r_k + r_j) e^{(\xi+r_k+r_j)\alpha} H(-(\xi + r_k + r_j)) v.\end{aligned} \tag{7.136}$$

For the remaining terms, it can be computed that

$$\begin{aligned}&-2\alpha \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s} \frac{Le^{s\cdot}}{(s-\alpha)^2} v ds + \alpha^2 \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s} \frac{1}{(s-\alpha)^2} v ds \\ &= 2\alpha \left(\sum_{k=1}^{\infty} A_k (\xi + r_k) e^{\alpha(\xi+r_k)} H(-(\xi + r_k)) v\right) - \alpha^2 \xi e^{\alpha\xi} H(-\xi) v.\end{aligned} \tag{7.137}$$

Hence, we have

$$\begin{aligned}\mathcal{R}^1_\alpha[v]'(\xi) - L\mathrm{ev}_\xi \mathcal{R}^1_\alpha[v] = &-\sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j (\xi + r_k + r_j) e^{(\xi+r_k+r_j)\alpha} H(-(\xi + r_k + r_j)) v \\ &+ 2\alpha \left(\sum_{k=1}^{\infty} A_k (\xi + r_k) e^{\alpha(\xi+r_k)} H(-(\xi + r_k))\right) v - \alpha^2 \xi e^{\alpha\xi} H(-\xi) v.\end{aligned} \tag{7.138}$$

For identity (7.130), we examine

$$\mathcal{R}_\alpha^2[\hat{\phi}]'(\xi) - \mathrm{Lev}_\xi \mathcal{R}_\alpha^2[\hat{\phi}] = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s}\left(\frac{(Le^{s\cdot})^2}{(s-\alpha)^2} - 2\alpha\frac{Le^{s\cdot}}{(s-\alpha)^2} + \frac{\alpha^2}{(s-\alpha)^2}\right)\psi[\hat{\phi}](s)ds. \tag{7.139}$$

For the first term we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s}\frac{(Le^{s\cdot})^2}{(s-\alpha)^2}\psi[\hat{\phi}](s)ds &= \sum_{k=1}^{\infty} A_k \sum_{l=1}^{\infty} A_l \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{(\xi+r_k+r_l)s}\frac{1}{(s-\alpha)^2}\psi[\hat{\phi}](s)ds \\ &= \sum_{k=1}^{\infty} A_k \sum_{l=1}^{\infty} A_l \sum_{j=1}^{\infty} A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k + r_l).\end{aligned} \tag{7.140}$$

For the second term we have

$$-2\alpha\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s}\frac{Le^{s\cdot}}{(s-\alpha)^2}\psi[\hat{\phi}](s)ds = -2\alpha\sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k). \tag{7.141}$$

The third term is

$$\alpha^2\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\xi s}\frac{1}{(s-\alpha)^2}\psi[\hat{\phi}](s)ds = \alpha^2\sum_{j=1}^{\infty} A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi). \tag{7.142}$$

Thus, we have

$$\begin{aligned}\mathcal{R}_\alpha^2[\hat{\phi}]'(\xi) - L\mathrm{ev}_\xi \mathcal{R}_\alpha^2[\hat{\phi}] &= \sum_{k=1}^{\infty} A_k \sum_{l=1}^{\infty} A_l \sum_{j=1}^{\infty} A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k + r_l) \\ &\quad - 2\alpha\sum_{k=1}^{\infty} A_k \sum_{j=1}^{\infty} A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi + r_k) + \alpha^2\sum_{j=1}^{\infty} A_j \mathcal{J}^{(2)}_{\alpha;r_j}[\hat{\phi}](\xi)\end{aligned} \tag{7.143}$$

for all $\xi \in \mathbb{R}$. □

7.2 Characterizations of $\mathcal{T}_{L;\eta}$

We have now studied all of the terms of $\mathcal{T}_{L;\eta}$, which leads us to the following similar conclusions as in [10, Proposition 5.13]

Proposition 7.24. *Let L be as defined in (2.8) with coefficients that satisfy condition (2.4) for a certain $\tilde{\eta} > 0$. Then for any sufficiently small $\epsilon > 0$, we have for any $\hat{\phi} \in C([r_{\min}, 0]; \mathbb{C}^d) \times C([0, r_{\max}]; \mathbb{C}^d)$ and $v \in \mathbb{C}^d$, the inclusion*

$$\mathcal{T}_{L;\eta}[\hat{\phi}, v] \in BC^-_{\eta+\epsilon}(\mathbb{C}^d) \cap BC^+_{\eta-\epsilon}(\mathbb{C}^d). \tag{7.144}$$

Upon fixing $\hat{\phi} \in C([r_{\min}, 0]; \mathbb{C}^d) \times C([0, r_{\max}], \mathbb{C}^d)$ and $v \in \mathbb{C}^d$, we have the jump discontinuity

$$\mathcal{T}_{L;\eta}[\hat{\phi}, v](0^+) - \mathcal{T}_{L;\eta}[\hat{\phi}, v](0^-) = v. \tag{7.145}$$

In addition, $\mathcal{T}_{L;\eta}[\hat{\phi}, v]$ is continuously differentiable on $\mathbb{R} \setminus \mathcal{R}$. For $r_j < 0$, the derivative has the jump

$$\mathcal{T}_{L;\eta}[\hat{\phi}, v]'(-r_j^+) - \mathcal{T}_{L;\eta}[\hat{\phi}, v]'(-r_j^-) = A_j v - A_j\hat{\phi}(0^-), \tag{7.146}$$

while for $r_j > 0$, we have

$$\mathcal{T}_{L;\eta}[\hat{\phi}, v]'(-r_j^+) - \mathcal{T}_{L;\eta}[\hat{\phi}, v]'(-r_j^-) = A_j v - A_j\hat{\phi}(0^+). \tag{7.147}$$

On the other hand, we have

$$\mathcal{T}_{L;\eta}[\hat{\phi},v]'(0^+) - \mathcal{T}_{L;\eta}[\hat{\phi},v]'(0^-) = \sum_{r_j<0} A_j\hat{\phi}(r_j) + \sum_{r_j>0} A_j\hat{\phi}(r_j). \tag{7.148}$$

Furthermore, we have for $\xi \in (-\infty,0) \setminus \mathcal{R}$,

$$\mathcal{H}_{\mathcal{T}_{L;\eta}}[\hat{\phi},v](\xi) := \mathcal{T}_{L;\eta}[\hat{\phi},v]'(\xi) - Lev_\xi \mathcal{T}_{L;\eta}[\hat{\phi},v] = -\sum_{r_j>0} A_j\hat{\phi}(\xi+r_j)\mathbb{1}_{-r_j<\xi<0}, \tag{7.149}$$

while for $\xi \in (0,\infty) \setminus \mathcal{R}$, we have

$$\mathcal{H}_{\mathcal{T}_{L;\eta}}[\hat{\phi},v](\xi) = \sum_{r_j<0} A_j\hat{\phi}(\xi+r_j)\mathbb{1}_{0<\xi<-r_j}. \tag{7.150}$$

We note that $Lev_\xi \mathcal{T}_{L;\eta}[\hat{\phi},v] = \sum_{j=1}^{\infty} A_j\mathcal{T}_{L;\eta}[\hat{\phi},v](\xi+r_j)$ is well-defined due to identity (7.144). Namely, identity (7.144) implies that $\mathcal{T}_{L;\eta}[\hat{\phi},v] \in BC_\eta(\mathbb{R},\mathbb{C}^d)$. Hence

$$\sum_{j=1}^{\infty} |A_j||\mathcal{T}_{L;\eta}[\hat{\phi},v](\xi+r_j)| \leq Ke^{\xi}\sum_{j=1}^{\infty}|A_j|e^{\eta r_j} < \infty, \tag{7.151}$$

with $K>0$ that does not depends on ξ. Thus, we see that the convergence is also locally uniform in ξ.

7.3 Proof of exponential dichotomy

With these characterizations of the $\mathcal{T}_{L;\eta}$ functions, we can easily prove the results that were stated earlier in this chapter.

Proof of Proposition 7.1. This follows immediately from Proposition 7.24. □

Proof of Proposition 7.4. This is identical to [10, Proof of Proposition 5.5]. □

Proof of Proposition 7.5. The proof is identical to [10, proof of Proposition 5.6], but we will justify the following claim that was made in the proof.

Claim: We have the characterizations

$$\begin{aligned} P_L(\eta) &= \{\phi \in C([r_{\min}, r_{\max}];\mathbb{C}^d) : \phi = \mathrm{ev}_0 E_{\widehat{P}_L(\eta)}\phi\} \\ Q_L(\eta) &= \{\phi \in C([r_{\min}, r_{\max}];\mathbb{C}^d) : \phi = \mathrm{ev}_0 E_{\widehat{Q}_L(\eta)}\phi\}, \end{aligned} \tag{7.152}$$

which immediately implies that $P_L(\eta)$ and $Q_L(\eta)$ are closed.

Proof of the claim: From Corollary 7.3, it indeed follows that for $\phi \in P_L(\eta)$, and $\psi \in Q_L(\eta)$, we have $\phi = \mathrm{ev}_0 E_{\widehat{P}_L(\eta)}\phi\}$ and $\psi = \mathrm{ev}_0 E_{\widehat{Q}_L(\eta)}\psi$. By definitions, we still need to show that $E_{\widehat{P}_L(\eta)}\phi \in \mathfrak{P}_L(\eta)$ and $E_{\widehat{Q}_L(\eta)}\psi \in \mathfrak{Q}_L(\eta)$. Given identity (7.33) and Proposition 7.4, it only remains to show that for $\xi \in [0,\infty)\cap\mathcal{R}$, we have $E_{\widehat{Q}_L(\eta)}\psi'(\xi) - Lev_\xi E_{\widehat{Q}_L(\eta)}\psi = 0$, and for $\xi \in (-\infty,0]\cap\mathcal{R}$, we have $E_{\widehat{P}_L(\eta)}\phi'(\xi) - Lev_\xi E_{\widehat{P}_L(\eta)}\phi = 0$. This is done as follows.
For $\phi \in C([r_{\min}, r_{\max}];\mathbb{C}^d)$, we have that $\mathcal{T}'_{L;\eta}[\phi,\phi(0)](\xi)$ is continuous on $\mathbb{R}\setminus 0$ due to identities (7.146) and

(7.147). As a consequence, we have for $\xi \in \mathcal{R} \cap (0, \infty)$,

$$\begin{aligned}
&[E_{\widehat{Q}_L(\eta)}\phi]'(\xi) - \sum_{j=1}^{\infty} A_j [E_{\widehat{Q}_L(\eta)}\phi](\xi + r_j) \\
&= \mathcal{T}'_{L;\eta}[\phi, \phi(0)](\xi) - \sum_{j=1}^{\infty} A_j [E_{\widehat{Q}_L(\eta)}\phi](\xi + r_j) \\
&= \lim_{\xi' \to \xi} \mathcal{T}'_{L;\eta}[\phi, \phi(0)](\xi') - \sum_{j=1}^{\infty} A_j \lim_{\xi' \to \xi} [E_{\widehat{Q}_L(\eta)}\phi](\xi' + r_j) \\
&= \lim_{\xi' \to \xi} \mathcal{T}'_{L;\eta}[\phi, \phi(0)](\xi') - \lim_{\xi' \to \xi} \sum_{j=1}^{\infty} A_j [E_{\widehat{Q}_L(\eta)}\phi](\xi' + r_j) \\
&= \lim_{\xi' \to \xi} \left[\mathcal{T}'_{L;\eta}[\phi, \phi(0)](\xi') - \sum_{j=1}^{\infty} A_j [E_{\widehat{Q}_L(\eta)}\phi](\xi' + r_j) \right] \\
&= \lim_{\xi' \to \xi} 0 = 0
\end{aligned} \tag{7.153}$$

by identity (7.30) and the fact that $\mathcal{R}$ is countable. Note that we may bring the limit outside the sum due to the locally uniform convergence of $\sum_{j=1}^{\infty} |A_j| |\mathcal{T}_{L;\eta}[\phi, v](\xi + r_j)$ that was noted before. Similarly, we have for $\xi \in \mathcal{R} \cap (-\infty, 0)$, that

$$[E_{\widehat{P}_L(\eta)}\phi]'(\xi) - \sum_{j=1}^{\infty} A_j [E_{\widehat{P}_L(\eta)}\phi](\xi + r_j) = 0. \tag{7.154}$$

Identities (7.153) and (7.154) also holds for $\xi = 0$ by replacing the limits with the 0^+ and 0^- limits.

To show that $P_L(\eta)$ are $Q_L(\eta)$ are closed, we recall the bounds (7.21), (7.31) and (7.32), which shows that

$$E_{\widehat{P}_L(\eta)}(C([r_{\min}, r_{\max}]; \mathbb{C}^d)) \to BC_\eta^{\ominus}(\mathbb{C}^d), \quad E_{\widehat{Q}_L(\eta)}(C([r_{\min}, r_{\max}]; \mathbb{C}^d)) \to BC_\eta^{\oplus}(\mathbb{C}^d) \tag{7.155}$$

are bounded linear operators and therefore continuous. Hence, for a convergent sequences $\phi_n \to \phi$, we have that

$$\phi = \lim_{n \to \infty} \phi_n = \lim_{n \to \infty} \mathrm{ev}_0[E_{\widehat{P}_L(\eta)}\phi_n] = \mathrm{ev}_0 \lim_{n \to \infty} [E_{\widehat{P}_L(\eta)}\phi_n] = \mathrm{ev}_0[E_{\widehat{P}_L(\eta)} \lim_{n \to \infty} \phi_n] = \mathrm{ev}_0[E_{\widehat{P}_L(\eta)}\phi]. \tag{7.156}$$

Similarly, we have $\mathrm{ev}_0[E_{\widehat{Q}_L(\eta)}\phi] = \phi$. Note that ev_0 is just the restriction operator, which is also continuous. □

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