Stability of Travelling Waves on Exponentially Long Timescales in Stochastic Reaction-Diffusion Equations

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Abstract

In this paper we establish the meta-stability of travelling waves for a class of reaction-diffusion equations forced by a multiplicative noise term. In particular, we show that the phase-tracking technique developed in [23, 24] can be maintained over timescales that are exponentially long with respect to the noise intensity. This is achieved by combining the generic chaining principle with a mild version of the Burkholder-Davis-Gundy inequality to establish logarithmic supremum bounds for stochastic convolutions in the critical regularity regime.

AMS 2010 Subject Classification: 35K57, 35R60 .

Key words: travelling waves, stochastic forcing, nonlinear stability, stochastic phase shift.

1 Introduction

In this paper we focus on the stochastic Nagumo equation

$$dU = [\rho \partial_{xx} U + f(U)]\, dt + \sigma g(U) dW_t^Q, \tag{1.1}$$

in which we take $U = U(x,t)$ with $x \in \mathbb{R}$ and $t \geq 0$. The nonlinearities are given by

$$f(u) = u(1-u)(u-a), \qquad\qquad g(u) = u(1-u)\chi(u) \tag{1.2}$$

for a parameter $a \in (0,1)$ and a smooth cut-off function $\chi(u)$ that forces g to be bounded and globally Lipschitz continuous on $\mathbb{R}$. The stochastic forcing is generated by the cylindrical Q-Wiener process W_t^Q characterized by the convolution operator

$$Q : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad\qquad [Qv](x) = \int_{-\infty}^{\infty} e^{-(x-y)^2} v(y)\, dy. \tag{1.3}$$

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In particular, our noise satisfies the formal relation

$$E\Big[dW_s^Q(x_0)dW_t^Q(x_1)\Big] = \delta(t-s)e^{-(x_0-x_1)^2} \tag{1.4}$$

and hence is white in time but coloured and translationally invariant in space. The well-posedness of such equations has been studied extensively [34, 41] and one can construct globally defined solutions in (for example) the affine space [24, Prop. 5.2]

$$\mathcal{U}_{H^1} = H^1(\mathbb{R}) + \frac{1}{2}\big(1 - \tanh(\cdot)\big). \tag{1.5}$$

The choice for this space is motivated by the fact that is contains the well-known deterministic travelling wave solution

$$U(x,t) = \Phi_0(x - c_0 t), \qquad \Phi_0(-\infty) = 1, \qquad \Phi_0(+\infty) = 0 \tag{1.6}$$

for (1.1) with $\sigma = 0$. In [23, 24] we showed that this pair (Φ_0, c_0) can be generalized to a branch of so-called *instantaneous stochastic waves* (Φ_σ, c_σ) for (1.1) that - at onset - travel with velocity velocity c_σ and feel only stochastic forcing. These waves can be shown to satisfy

$$\|\Phi_\sigma - \Phi_0\|_{H^2} + |c_\sigma - c_0| = O(\sigma^2). \tag{1.7}$$

The key question is if one can understand the perturbations

$$V(t) = U(\cdot + \Gamma(t), t) - \Phi_\sigma \tag{1.8}$$

from these profiles, using an appropriate phase shift Γ to stochastically 'freeze' the solution U. In particular, we are interested in the behaviour of the stopping time

$$t_{\mathrm{st}}(\eta) = \inf\{t \geq 0 : \|V(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)}\|V(s)\|_{H^1}^2\, ds \leq \eta\}, \tag{1.9}$$

which measures when U exits an appropriate orbital η-neighourhood of the profile Φ_σ. Our main result states that this exit-time is (with high probability) exponentially long with respect to the parameter $1/\sigma$. As such, it establishes the meta-stability of the deterministic travelling wave (1.6) under small stochastic forcing, significantly extending our earlier results in [23, 24].

Theorem 1.1. *Pick a sufficiently large constant $K > 0$ and sufficiently small constants $\varepsilon > 0$, $\eta_0 > 0$, $\delta_0 > 0$ and $\delta_\sigma > 0$. Then for any $U(0) \in \mathcal{U}_{H^1}$ that satisfies $\|U(0) - \Phi_\sigma\|_{H^1} < \delta_0$ and any $0 \leq \sigma \leq \delta_\sigma$, there exists a scalar stochastic process Γ so that*

$$P(t_{\mathrm{st}}(\eta_0) < T) \leq K\left[\|U(0) - \Phi_\sigma\|_{H^1}^2 + \sigma\sqrt{\ln(T)}\right] \tag{1.10}$$

holds for all $2 \leq T \leq \exp[\delta_\sigma^2/\sigma^2]$,

We remark here that general 'exit-problems' have been well-studied in finite dimensional contexts [18], but much less is known in infinite dimensions [6, 20]. The recent paper by Salins and Spiliopoulous [43] discusses some of the main developments in this area, which chiefly focus on SPDEs with gradient-independent noise posed on finite domains. In this case the associated semigroups are compact, allowing tightness results to be established that lead naturally to large deviation principles [13]. Such compactness properties do not apply in the current setting and we take a completely different approach.

Stochastic waves The impact of noise on pattern formation is an important topic that has attracted significant interest from the applied community [2, 9, 19, 44, 45, 50], but for which little rigorous mathematical theory is available [8, 25, 32, 39]. The Nagumo equation is a natural starting point for such investigations, since it has served in the past as a prototypical system to analyze the interaction between two competing stable states in spatially extended domains [3, 4]. The deterministic travelling waves (1.6) represent a primary invasion mechanism by which the favourable state can spread throughout the entire domain. They are robust under perturbations, which allows them to be used as building blocks to understand the global behaviour of (1.1) in one [17, 28, 51] but also higher spatial dimensions [5, 27, 38].

The behaviour of these invasion waves under several types of stochastic forcing has been studied by various authors using a range of different techniques. The consensus emerging from a number of formal computations for (1.1) is that - to leading order in σ - the phase-shift of the wave follows a Brownian motion with a variance that can be expressed in closed form [11, 12, 19]. Various rigorous approaches have been pursued over the past five years that are able to successfully explain this diffusive behaviour on short time scales [26, 30, 46, 47]; see e.g [31] for a very recent overview and the introductions of [23, 24] for a detailed technical discussion. Several of these techniques have been extended to stochastic neural field equations [10, 33, 37] and (very recently) to the FitzHugh-Nagumo system [16].

In a recent series of papers [22–24], we pioneered a novel 'stochastic freezing' approach to rigorously analyze the behaviour of travelling fronts and pulses to a large class of reaction-diffusion equations (RDEs) - which includes (1.1) and the (fully diffusive) FitzHugh-Nagumo system. In essence, we developed a stochastic version of the freezing approach introduced by Beyn [7], which allows us to adopt the spirit behind the modern machinery for deterministic stability issues initiated by Howard and Zumbrun [52]. The power of this approach is that it leads naturally to *long-term* predictions concerning both the speed and the shape of the stochastic wave that can be computed to arbitrary order in σ. We demonstrated the accuracy of these novel predictions in [24] by performing a series of numerical experiments. As a consequence, we now have a quantitative explanation for the wave-steepening and speed-reduction phenomena that were illustrated numerically in [35] and - in a special case - derived formally in [12] using a collective coordinate approach.

Regularity issues The key novel feature of the approach in [22–24] is that the perturbation V in the decomposition (1.8) is measured in the same reference frame as the frozen profile Φ_σ. This allows the delicate interaction between the speed and shape of the wave to be untangled, but also presents several fundamental complications that need to be carefully addressed. The most important of these is that the stochastic phase shift causes extra diffusive correction terms for V that are not seen in the deterministic context, together with a multiplicative noise term that involves the derivative of V. Unlike any of the previous approaches in this area, we hence need to keep the H^1-norm of $V(t)$ under control.

To be more specific, an essential step in our stability proofs is to obtain bounds for the expression

$$E \sup_{0\le t\le T} \| \int_0^t S(t-s)B\big(V(s), \partial_x V(s)\big) dW_s^Q \|_{L^2}^2, \tag{1.11}$$

together with its integrated H^1-counterpart

$$E \sup_{0\le t\le T} \int_0^t e^{-\varepsilon(t-s)} \| \int_0^s S(s-s')B\big(V(s'), \partial_x V(s')\big) dW_{s'}^Q \|_{H^1}^2 \, ds. \tag{1.12}$$

Here S denotes the semigroup associated to the linearization of (1.1) around the deterministic travelling wave (1.6). In our previous papers [22, 23], we used the mild Burkholder-Davis-Gundy (BDG) inequality obtained by Veraar [49] to control (1.11), but the resulting bounds are unfortunately not optimal. As such, they restricted the validity range of our rigorous results to timescales of order $T \sim \sigma^{-2}$.

This shortfall is repaired by the bound in Theorem 1.1, which confirms that our phase-tracking can be maintained over the exponentially long timescales observed in the numerical results from [24]. We emphasize that our improved bound also covers regimes where the stochastic phase Γ is expected to be very far away from its deterministic counterpart. This provides a solid theoretical underpinning to the formal predictions that we made in [24] concerning the stochastic corrections to (1.6).

To understand the issues that are involved, it is highly instructive to consider the scalar Ornstein-Uhlenbeck process

$$X(t) = \int_0^t e^{-(t-s)} d\beta_s, \tag{1.13}$$

which here starts at $X(0) = 0$ and is driven by a standard Brownian motion β_t. Since $B(0,0) \neq 0$, the behaviour of X is highly comparable to that of V at lowest order in σ. Indeed, the deterministic dynamics pulls X towards zero at an exponential rate, but the stochastic forcing does not vanish there. Applying the mild Burkholder-Davis-Gundy inequality to (1.13) results in the bound

$$E \sup_{0 \le t \le T} |X(t)|^2 \le K \int_0^T 1\, ds = KT. \tag{1.14}$$

This hence fails to reproduce the well-known fact that this expectation behaves as $O\big(\ln(T)\big)$ for large T, which was originally established by examining crossing numbers [40] or analyzing explicit probability distributions [1]. Fortunately, a more general abstract approach has been developed in recent years.

Chaining A powerful modern tool to derive supremum bounds for stochastic processes is commonly referred to as 'generic chaining'; see [48] for an accessible introduction.[1] Based on contributions from a range of authors, including Kolmogorov, Dudley, Fernique and Talagrand, it uses information on the increments of a stochastic process to establish long-term supremum bounds. For instance, exploiting the fact that the Ornstein-Uhlenbeck process (1.13) is centered and Gaussian, one can obtain the tail bound

$$P(|X(t) - X(s)| > \vartheta) \le 2e^{-\frac{\vartheta^2}{2d(t,s)^2}} \tag{1.15}$$

characterized by the metric $d(t,s) = \sqrt{E\big(X(t) - X(s)\big)^2}$. An explicit computation yields the bound

$$\begin{aligned} d(t,s)^2 &= \frac{1}{2}\big(2 - e^{-2t} - e^{-2s} + 2(e^{-|t-s|} - e^{-(t+s)})\big) \\ &\le 1 - e^{-|t-s|} \\ &\le \min\{|t-s|, 1\}. \end{aligned} \tag{1.16}$$

This shows that the covering number $N(T, d, \nu)$ - which measures the minimum number of intervals of length ν or less in the metric d required to cover $[0,T]$ - can be bounded by T/ν^2 for $\nu \in (0,1]$ and by 1 for $\nu \ge 1$. The main result in [48] - see Theorem 2.7 below - now provides the Dudley bound

$$\sup_{t \in [0,T]} X_t \sim \int_0^\infty \sqrt{\ln(N(T,d,\nu))} d\nu \le \int_0^1 \sqrt{\ln\left(T/\nu^2\right)} d\nu \sim \sqrt{\ln(T)}, \tag{1.17}$$

which captures the desired logarithmic behaviour in a relatively straightforward manner.

[1] This unpublished chapter by Pollard could also be useful: `http://www.stat.yale.edu/~pollard/Books/Mini/Chaining.pdf`

Our main contribution in this paper is that we extend this technique to provide similar sharp bounds for the stochastic integrals (1.11) and (1.12). On account of the regularity issues that are involved, this is a surprisingly delicate task. In fact, we are not aware of any related results in this direction besides the factorization method developed by Da Prato, Kwapień and Zabczyk [15], which typically only provides polynomial bounds in T. Let us remark that it was not immediately clear to us how this factorization technique should be applied in the present setting, because it introduces extra singularities into integrals that cannot be readily accomodated in our critical regularity regime.

Obstructions In order to illustrate the key complications, let us consider the L^2-valued process

$$Y(t) = \int_0^t S(t-s)BdW_s^Q, \tag{1.18}$$

which can be seen as an infinite-dimensional version of (1.13). Here B is an appropriate constant Hilbert-Schmidt operator, which can be used to define the covariance operator

$$Q_\infty = \lim_{t\to\infty} \int_0^t S(t-s)BQB^*S^*(t-s)ds. \tag{1.19}$$

The analogue of the bound (1.16) is now given by[2]

$$d(t,s)^2 = E\|Y(t) - Y(s)\|_{L^2}^2 \leq 2\,\mathrm{tr}\left((I - S(t-s))Q_\infty\right), \tag{1.20}$$

but this time there is no $\alpha > 0$ for which one can extract a term of the form $|t-s|^\alpha$ from the difference $S(t-s) - I$. In principle this can be repaired by 'borrowing' some smoothness from B, but in our case this would again lead to unintegrable singularities.

In order to resolve this, it is crucial to combine the strong points of both the chaining technique and the mild Burkholder-Davis-Gundy inequality. Indeed, the former works well in the regime where $|t-s| \geq 1$ in (1.11), since here the decay and smoothening properties of the semigroup can both be put to excellent use. On the other hand, for $|t-s| \leq 1$, the H^∞-calculus underlying the mild Burkholder-Davis-Gundy inequality can resolve the critical regularity issues associated to supremum bounds without causing too much growth. The main issue is to set up an appropriate framework that allows this splitting to be achieved.

The second fundamental complication is that the integrands in (1.11) and (1.12) are time-dependent, which means that - in contrast to (1.18) - the stochastic integrals are not Gaussian. In this case, one must construct a metric such that a corresponding tail bound like (1.15) can be derived from scratch. Effectively, this requires us to control *all* the moments of the increments of (1.11). This is made possible by an effective use of stopping times in combination with a mild Itô formula.

Scope and outlook In order to make the arguments in this paper as clear and concise as possible, we chose to restrict our attention to the single specific problem (1.1). However, we emphasize that our arguments transfer immediately to the general class of (multi-component) problems considered in [23] and [24], with the single restriction that all diffusion coefficients must be equal (condition (hA) in [23]). This latter restriction can be removed by applying the spirit of [22], but this requires more complicated machinery that we will describe in an extensive forthcoming companion paper. There we will also address the long-term validity of the the perturbation results from [24].

Organization We start in §2 by introducing some basic probabilistic and deterministic concepts. The heart of this paper is contained in §3, where we provide logarithmic bounds for the stochastic integrals (1.11) and (1.12). Several supremum bounds for deterministic integrals are provided in §4, which allow for a streamlined proof of our main theorem in §5.

[2] This computation can be made rigorous using [21, §5].

Acknowledgements HJH acknowledges support from the Netherlands Organization for Scientific Research (NWO) (grant 639.032.612).

2 Preliminaries

In this section we collect several useful preliminary results that will streamline our arguments. We start in §2.1 by recalling well-known facts concerning the linearization of the Nagumo PDE around its travelling wave. We subsequently consider the relation between tail bounds and moment estimates for scalar stochastic processes in §2.2. Finally, in §2.3 we formulate the key technical tools that will allow us to apply the chaining principle to stochastic convolutions in the critical regularity regime.

2.1 Semigroup bounds

It is well-known that the Nagumo PDE (1.1) with $\sigma = 0$ admits a travelling front solution $U(x,t) = \Phi_0(x - c_0 t)$ that necessarily satisfies the travelling wave ODE

$$\rho\Phi_0'' + c_0\Phi_0' + f(\Phi_0) = 0, \qquad \Phi_0(-\infty) = 1, \qquad \Phi_0(+\infty) = 0. \tag{2.1}$$

The associated linear operators

$$\mathcal{L}_{\mathrm{tw}} v = \rho v'' + c_0 v' + Df\left(\Phi_0\right) v, \qquad \mathcal{L}_{\mathrm{tw}}^* w = \rho w'' - c_0 w' + Df\left(\Phi_0\right) w, \tag{2.2}$$

which we view as maps from $H^2(\mathbb{R})$ into $L^2(\mathbb{R})$, both admit a simple eigenvalue at $\lambda = 0$ and have no other spectrum in the half-plane $\{\mathrm{Re}\,\lambda \geq -2\beta\} \subset \mathbb{C}$ for some $\beta > 0$. Writing P_{tw} for the spectral projection onto this neutral eigenvalue for $\mathcal{L}_{\mathrm{tw}}$, we can obtain the identifications

$$\mathrm{Ker}(\mathcal{L}_{\mathrm{tw}}) = \mathrm{span}\{\Phi_0'\}, \qquad \mathrm{Ker}(\mathcal{L}_{\mathrm{tw}}^*) = \mathrm{span}\{\psi_{\mathrm{tw}}\}, \qquad P_{\mathrm{tw}} v = \langle v, \psi_{\mathrm{tw}}\rangle_{L^2}\Phi_0' \tag{2.3}$$

by writing $\psi_{\mathrm{tw}}(\xi) = \kappa e^{-\frac{c_0\xi}{\rho}}\Phi_0'(\xi)$ for some κ that we fix by the requirement $\langle \Phi_0', \psi_{\mathrm{tw}}\rangle_{L^2} = 1$.

In fact, the operator $\mathcal{L}_{\mathrm{tw}}$ is sectorial and hence generates an analytic semigroup $S(t) = e^{\mathcal{L}_{\mathrm{tw}}t}$; see [36, Prop. 4.1.4] and [23, Prop. 6.3.vi]. Upon introducing the notation

$$\begin{aligned}\mathcal{J}_{\mathrm{tw}}(t)[v,w] = \langle S(t)v, S(t)w\rangle_{L^2} &+ \frac{1}{2\rho}\langle S(t)v, (\mathcal{L}_{\mathrm{tw}} - \rho\partial_{xx})S(t)w\rangle_{L^2} \\ &+ \frac{1}{2\rho}\langle S(t)v, (\mathcal{L}_{\mathrm{tw}}^* - \rho\partial_{xx})S(t)w\rangle_{L^2},\end{aligned} \tag{2.4}$$

a short computation (see [23]) shows that

$$\langle S(t)v, S(t)w\rangle_{H^1} = \mathcal{J}_{\mathrm{tw}}(t)[v,w] - \frac{1}{2\rho}\frac{d}{dt}\langle S(t)v, S(t)w\rangle_{L^2} \tag{2.5}$$

holds for all $t > 0$ and $v, w \in L^2$. This identity allows the regularity issues that arise in §3 and §4 to be resolved.

Lemma 2.1. *Pick a sufficiently large $M \geq 1$ and write $\Pi = I - P_{\mathrm{tw}}$. Then for every $t > 0$ we have the bounds*

$$\begin{aligned}
\|S(t)\Pi\|_{\mathcal{L}(L^2,L^2)} &\leq Me^{-\beta t}, \\
\|S(t)\Pi\|_{\mathcal{L}(L^2,H^1)} &\leq Mt^{-\frac{1}{2}}e^{-\beta t}, \\
\|[\mathcal{L}_{\mathrm{tw}} - \rho\partial_{\xi\xi}]S(t)\Pi\|_{\mathcal{L}(L^2,L^2)} &\leq Mt^{-\frac{1}{2}}e^{-\beta t}, \\
\|[\mathcal{L}_{\mathrm{tw}}^* - \rho\partial_{\xi\xi}]S(t)\Pi\|_{\mathcal{L}(L^2,L^2)} &\leq Mt^{-\frac{1}{2}}e^{-\beta t}, \\
\|(S(t) - I)S(1)\|_{\mathcal{L}(L^2,L^2)} &\leq M|t|.
\end{aligned} \tag{2.6}$$

In particular, for any $t > 0$ and $v, w \in L^2$ we obtain the estimate

$$|\mathcal{J}_{\mathrm{tw}}(t)[\Pi v, \Pi w]| \le M^2 e^{-2\beta t}\left(1 + \rho^{-1}t^{-1/2}\right)\|v\|_{L^2}\|w\|_{L^2}. \tag{2.7}$$

Proof. The bounds (2.6) can be deduced from [36, Prop. 5.2.1], while (2.7) follows readily by inspecting (2.4). □

2.2 Moment estimates and tail bounds

We here briefly review the technique that we use to pass back and forth between moment estimates and tail probabilities. The former are easier to estimate, but the latter are better suited for handling maxima. Our computations are based heavily on [48, Lem. 2.2.3] and [49].

Lemma 2.2. *Consider a random variable $Z \ge 0$ and suppose that there exists a $B > 0$ so that the bound*

$$E[Z^{2p}] \le p^p B^{2p} \tag{2.8}$$

holds for all integers $p \ge 1$. Then for every $\vartheta > 0$ we have the estimate

$$P(Z > \vartheta) \le 2\exp\left[-\frac{\vartheta^2}{2eB^2}\right]. \tag{2.9}$$

Proof. For any $\nu > 0$ a formal computation shows that

$$\begin{array}{rcl} P(Z > \vartheta) & = & P(e^{\nu Z^2} > e^{\nu\vartheta^2}) \\ & \le & e^{-\nu\vartheta^2} E\left[e^{\nu Z^2}\right] \\ & \le & e^{-\nu\vartheta^2} E\left[\sum_{p=0}^{\infty} \frac{\nu^p}{p!} Z^{2p}\right] \\ & \le & e^{-\nu\vartheta^2} \sum_{p=0}^{\infty} \frac{\nu^p}{p!} B^{2p} p^p. \end{array} \tag{2.10}$$

Using $p! \ge p^p e^{-p}$ we obtain

$$P(Z > \vartheta) \quad \le \quad e^{-\nu\vartheta^2} \sum_{p=0}^{\infty} \nu^p B^{2p} e^p, \tag{2.11}$$

which leads to (2.9) by choosing $\nu = (2eB^2)^{-1}$. □

Lemma 2.3. *Fix two constants $A \ge 2$ and $B > 0$ and consider a random variable $Z \ge 0$ that satisfies the estimate*

$$P(Z > \vartheta) \le 2A\exp\left[-\frac{\vartheta^2}{2eB^2}\right] \tag{2.12}$$

for all $\vartheta > 0$. Then we have the moment bounds

$$E[Z] \le 2\sqrt{2e}B\sqrt{\ln(A)}, \qquad E[Z^2] \le 8eB^2\ln(A). \tag{2.13}$$

Proof. Starting with the second moment, we pick an arbitrary $u_0 > 0$ and compute

$$\begin{array}{rcl} E[Z^2] & = & \int_0^\infty P(Z^2 > u)\,du \\ & = & \int_0^{u_0} P(Z > \sqrt{u})\,du + \int_{u_0}^\infty P(Z > \sqrt{u})\,du \\ & \le & u_0 + \int_{u_0}^\infty 2Ae^{-u/(2eB^2)}\,du \\ & = & u_0 + 4eAB^2\exp\left(\frac{-u_0}{2eB^2}\right). \end{array} \tag{2.14}$$

Fixing $u_0 = 2eB^2 \ln(2A)$ and using $\ln(2A) > 1$, we obtain the desired estimate

$$E[Z^2] \le 2eB^2 \ln(2A) + 2eB^2 \le 8eB^2 \ln(A). \tag{2.15}$$

The bound for $E[Z]$ can now be deduced by taking square roots and applying Jensen's inequality. $\square$

By applying a crude bound for tail-probabilities, Lemmas 2.2 and 2.3 can be combined to control maximum expectations. This results in the following useful logarithmic growth estimate.

Corollary 2.4. *Consider $N \ge 2$ non-negative random variables $Y_1, Y_2, ..., Y_N$ and suppose that there exists $B > 0$ so that the bound*

$$E\left[Y_i^{2p}\right] \le p^p B^{2p} \tag{2.16}$$

holds for all integers $p \ge 1$ and each $i \in \{1, .., N\}$. Then we have the bounds

$$E \max_{i \in \{1,...,N\}} Y_i \le 2\sqrt{2e} B \sqrt{\ln(N)}, \qquad E \max_{i \in \{1,...,N\}} Y_i^2 \le 8eB^2 \ln(N). \tag{2.17}$$

Proof. For any $\vartheta > 0$ we may use Lemma 2.2 to estimate

$$P\big(\max_{i \in \{1,...,N\}} Y_i^2 > \vartheta\big) \le \sum_{i=1}^{N} P(Y_i^2 > \vartheta) \le 2N \exp\left(-\frac{\vartheta^2}{2eB^2}\right), \tag{2.18}$$

so we can directly apply Lemma 2.3. The proof for $E \max_{i \in \{1,...,N\}} Y_i$ is identical. $\square$

2.3 Supremum bounds

In this subsection we collect several key results that we will use to understand stochastic convolutions such as (1.11). In order to setup such integrals in a precise fashion, we follow the extensive discussion in [24, §5] and introduce the Hilbert space

$$L_Q^2 = L_Q^2(\mathbb{R}) = Q^{1/2}\left(L^2(\mathbb{R})\right), \tag{2.19}$$

together with the set of Hilbert-Schmidt operators

$$HS = HS(L_Q^2, L^2) = HS\big(L_Q^2(\mathbb{R}), L^2(\mathbb{R})\big) \tag{2.20}$$

that map $L_Q^2(\mathbb{R})$ into $L^2(\mathbb{R})$. Choosing an orthonormal basis (e_k) for $L^2(\mathbb{R})$, we recall that the Hilbert-Schmidt norm is given by

$$\|B\|_{HS}^2 = \sum_{k=0}^{\infty} \|B\sqrt{Q} e_k\|_{L^2}^2. \tag{2.21}$$

Fixing a complete filtered probability space $\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P}\Big)$, it turns out [29, 41, 42] that stochastic integrals against dW_t^Q are well-defined if the integrand is taken from the class

$$\begin{aligned} \mathcal{N}^2([0,T]; (\mathcal{F}_t), HS) \quad := \quad & \{B \in L^2\big([0,T] \times \Omega; dt \otimes \mathbb{P}; HS\big) : \\ & B \text{ has a progressively } (\mathcal{F}_t)\text{-measurable version}\}. \end{aligned} \tag{2.22}$$

Our previous results in [22–24] relied heavily on various versions of the Burkholder-Davis-Gundy inequality, but we only used the special case $p = 1$. The general form is stated below, where we highlight the p-dependence of the prefactors on the right-hand sides.

Lemma 2.5. *Pick*[3] *a sufficiently large $K_{\mathrm{cnv}} \geq 1$. Then for any $T > 0$, any integer $p \geq 1$ and any integrand $B \in \mathcal{N}^2\big([0,T];(\mathcal{F}_t);HS(L^2_Q,L^2)\big)$ we have the bound*

$$E \sup_{0 \leq t \leq T} \| \int_0^t B(s)\, dW^Q_s \|^{2p}_{L^2} \leq K^{2p}_{\mathrm{cnv}} p^p E \left[\int_0^T \|B(s)\|^2_{HS}\, ds \right]^p, \tag{2.23}$$

together with its mild counterpart

$$E \sup_{0 \leq t \leq T} \| \int_0^t S(t-s)B(s)\, dW^Q_s \|^{2p}_{L^2} \leq K^{2p}_{\mathrm{cnv}} p^p E \left[\int_0^T \|B(s)\|^2_{HS}\, ds \right]^p. \tag{2.24}$$

Proof. We note first that $L^2(\mathbb{R})$ is a Banach space of type 2. In particular, (2.23) follows from [49, Prop. 2.1 and Rem. 2.2]. In addition, the linear operator $\mathcal{L}_{\mathrm{tw}}$ admits a bounded H^∞-calculus [23, Lem. 9.7], which allows us to apply [49, Thm. 1.1] and obtain (2.24). □

We remark that the inequalities (2.23)-(2.24) are very strong and useful on short time intervals, but on longer scales it is no longer possible to exploit the decay properties of the semigroup. Indeed, the right-hand side of (2.24) grows linearly in time for integrands that are constant - as for the Ornstein-Uhlenbeck process. This changes if one drops the supremum.

Corollary 2.6. *Consider the setting of Lemma 2.5. Then for any $0 \leq t \leq T$ and any integer $p \geq 1$ we have the bound*

$$E\| \int_0^t S(t-s)B(s)\, dW^Q_s \|^{2p}_{L^2} \leq K^{2p}_{\mathrm{cnv}} p^p E \left[\int_0^t \|S(t-s)B(s)\|^2_{HS}\, ds \right]^p. \tag{2.25}$$

Proof. Note that

$$E\| \int_0^t S(t-s)B(s)\, dW^Q_s \|^{2p}_{L^2} \leq E \sup_{0 \leq \tilde{t} \leq t} \| \int_0^{\tilde{t}} S(t-s)B(s)\, dW^Q_s \|^{2p}_{L^2}, \tag{2.26}$$

so the result follows directly from (2.23). □

The following general result due to Talagrand [48, eq. (2.49)] is the key ingredient that will allow us to significantly improve the bound (2.24). It is based on the chaining principle, which requires us to understand the tail behaviour of the probability distribution for the temporal increments of stochastic process.

Theorem 2.7 ([48])**.** *Pick a sufficiently large $C_{\mathrm{ch}} > 0$, choose an arbitrary $T > 0$ and consider a stochastic process $X : [0,T] \to L^2$ with paths that are almost-surely continuous. Suppose furthermore that there exists a metric $d = d(\cdot,\cdot)$ on $[0,T]$ so that the increments of X satisfy the estimate*

$$P\left(\|X(t_1) - X(t_2)\|_{L^2} > \vartheta\right) \leq 2 \exp\left(- \frac{\vartheta^2}{2d(t_1,t_2)^2} \right), \tag{2.27}$$

for every $t_1, t_2 \in [0,T]$ and $\vartheta > 0$. Then we have the bound

$$E \sup_{0 \leq t \leq T} \|X(t)\|^2_{L^2} \leq C_{\mathrm{ch}} \left(\int_0^\infty \sqrt{\ln\big(N(T,d,\nu)\big)}\, d\nu \right)^2 \tag{2.28}$$

where $N(T,d,\nu)$ is the smallest number of intervals of length at most ν in the metric d required to cover $[0,T]$.

[3] Let us emphasize that all constants that appear in this paper do not depend on T.

3 Supremum bounds for stochastic integrals

In this section we develop the machinery needed to obtain bounds for two types of stochastic integrals. In particular, we introduce the L^2-valued integral

$$\mathcal{E}_B(t) = \int_0^t S(t-s)B(s)dW_s^Q \tag{3.1}$$

together with the scalar integral

$$\mathcal{I}_B^{\mathrm{s}}(t) = \int_0^t e^{-\varepsilon(t-s)} \int_0^s \langle S(s-s')V(s'), S(s-s')B(s')\, dW_{s'}^Q \rangle_{H^1}\, ds \tag{3.2}$$

and set out to obtain bounds for the quantities

$$E \sup_{0\le t\le T} \|\mathcal{E}_B(t)\|_{L^2}^2, \qquad E \max_{i\in\{1,\dots,T\}} |\mathcal{I}_B^{\mathrm{s}}(i)|. \tag{3.3}$$

We will use the first of these expressions in §5.1 to control the L^2-norm of $V(t)$, while the second term plays a crucial role in §5.2 where we bound the H^1-norm of $V(t)$ in an integrated sense. In both cases B will be replaced by a (complicated) function of V, but we make use of a generic placeholder here in order to emphasize the broad applicability of our techniques. Indeed, we only need to impose the following two general conditions on our integrands.

(hB) The process $B \in \mathcal{N}^2\left([0,T];(\mathcal{F}_t);HS(L_Q^2, L^2)\right)$ satisfies

$$\langle B(t)v, \psi_{\mathrm{tw}}\rangle_{L^2} = 0 \tag{3.4}$$

for all $t \in [0,T]$ and $v \in L_Q^2$. Furthermore, there exist $\varepsilon \in (0,\beta)$ and $\Theta_* > 0$ so that the following pathwise bounds hold for all $0 \le t \le T$:

$$\int_0^t e^{-\varepsilon(t-s)} \|B(s)\|_{HS}^2 ds \le \Theta_*^2, \qquad \|S(1)B(t)\|_{HS}^2 \le \Theta_*^2. \tag{3.5}$$

(hV) The process $V \in \mathcal{N}^2\left([0,T];(\mathcal{F}_t);H^1\right)$ satisfies

$$\langle V(t), \psi_{\mathrm{tw}}\rangle_{L^2} = 0 \tag{3.6}$$

for all $t \in [0,T]$. Furthermore, there exists a $\Lambda_* > 0$ so that the pathwise bound

$$\|V(t)\|_{L^2} \le \Lambda_* \tag{3.7}$$

holds for all $0 \le t \le T$.

We remark that (3.4) and (3.6) imply that B and V do not feel the neutral mode of the semigroup. This allows us to use the decay rates from Lemma 2.1 and establish our main result below. In particular, we obtain (3.8) in §3.1 and (3.9) in §3.2.

Proposition 3.1. *Fix a sufficiently large constant $K > 0$. Then for any integer $T \ge 2$ and any pair of processes (B,V) that satisfies (hB) and (hV), we have the supremum bound*

$$E \sup_{0\le t\le T} \|\mathcal{E}_B(t)\|_{L^2}^2 \le K\Theta_*^2 \ln(T) \tag{3.8}$$

together with its counterpart

$$E \max_{i\in\{1,\dots,T\}} |\mathcal{I}_B^s(i)| \le K\Lambda_*\Theta_*\sqrt{\ln(T)}. \tag{3.9}$$

3.1 Estimates for $\mathcal{E}_B$

Motivated by the considerations in the introduction, we make the splitting

$$\mathcal{E}_B(t) = \mathcal{E}_B^{\mathrm{lt}}(t) + \mathcal{E}_B^{\mathrm{sh}}(t), \tag{3.10}$$

in which the short time (sh) and long time (lt) contributions are respectively given by

$$\mathcal{E}_B^{\mathrm{lt}}(t) = \int_0^{t-1} S(t-s)B(s)\, dW_s^Q, \qquad \mathcal{E}_B^{\mathrm{sh}}(t) = \int_{t-1}^{t} S(t-s)B(s)\, dW_s^Q, \tag{3.11}$$

where we interpret the bound $t-1$ as $\max\{t-1, 0\}$ if necessary. Both these terms need to be handled using separate techniques.

Short time bounds Remembering that T is an integer, we introduce the function

$$\Upsilon_B^{(i)} \quad = \quad \sup_{0 \le s \le 1} \| \int_i^{i+s} S(i+s-s')B(s')\, dW_{s'}^Q \|_{L^2} \tag{3.12}$$

for any $i \in \{0, \dots, T-1\}$. An elementary computation allows us to bound $\mathcal{E}_B^{\mathrm{sh}}(t)$ in terms of at most two of this finite set of quantities.

Lemma 3.2. *Pick any integer $T \ge 2$ and assume that (hB) holds. Then for all $0 \le t \le T$ we have the bound*

$$\|\mathcal{E}_B^{\mathrm{sh}}(t)\|_{L^2} \le 2M\Upsilon_B^{(\lfloor t \rfloor - 1)} + \Upsilon_B^{(\lfloor t \rfloor)}. \tag{3.13}$$

Proof. Since the estimate is immediate for $0 \le t < 1$, we pick $t \ge 1$. Splitting the integral yields

$$\begin{aligned} \|\mathcal{E}_B^{\mathrm{sh}}(t)\|_{L^2} &\le \| \int_{t-1}^{\lfloor t \rfloor} S(t-s)B(s)dW_s^Q \|_{L^2} + \| \int_{\lfloor t \rfloor}^{t} S(t-s)B(s)dW_s^Q \|_{L^2} \\ &\le \| \int_{\lfloor t \rfloor - 1}^{\lfloor t \rfloor} S(t-s)B(s)dW_s^Q \|_{L^2} + \| \int_{\lfloor t \rfloor - 1}^{t-1} S(t-s)B(s)dW_s^Q \|_{L^2} \\ &\quad + \| \int_{\lfloor t \rfloor}^{t} S(t-s)B(s)dW_s^Q \|_{L^2}. \end{aligned} \tag{3.14}$$

Using Lemma 2.1 we obtain the estimate

$$\begin{aligned} \| \int_{\lfloor t \rfloor - 1}^{\lfloor t \rfloor} S(t-s)B(s)dW_s^Q \|_{L^2} &\le \|S(t - \lfloor t \rfloor)\|_{\mathcal{L}(L^2, L^2)} \| \int_{\lfloor t \rfloor - 1}^{\lfloor t \rfloor} S(\lfloor t \rfloor - s)B(s)dW_s^Q \|_{L^2} \\ &\le M \| \int_{\lfloor t \rfloor - 1}^{\lfloor t \rfloor} S(\lfloor t \rfloor - s)B(s)dW_s^Q \|_{L^2} \\ &\le M\Upsilon_B^{(\lfloor t \rfloor - 1)}, \end{aligned} \tag{3.15}$$

together with

$$\begin{aligned} \| \int_{\lfloor t \rfloor - 1}^{t-1} S(t-s)B(s)dW_s^Q \|_{L^2} &\le \|S(1)\|_{\mathcal{L}(L^2, L^2)} \| \int_{\lfloor t \rfloor - 1}^{t-1} S(t-1-s)B(s)dW_s^Q \|_{L^2} \\ &\le M\Upsilon_B^{(\lfloor t \rfloor - 1)}, \end{aligned} \tag{3.16}$$

from which the desired bound readily follows. □

Corollary 3.3. *Pick any integer $T \geq 2$ and assume that (hB) holds. Then we have the pathwise bound*

$$\sup_{0 \leq t \leq T} \|\mathcal{E}_B^{\mathrm{sh}}(t)\|_{L^2}^2 \leq 9M^2 \max_{i \in \{0,\ldots,T-1\}} \left(\Upsilon_B^{(i)}\right)^2. \tag{3.17}$$

The expectation of the right-hand side of (3.17) can be controlled using Corollary 2.4. We hence require moment bounds on $\Upsilon_B^{(i)}$, which can be obtained by applying the mild Burkholder-Davis-Gundy inequality. Here we use the crucial fact that $\mathcal{L}_{\mathrm{tw}}$ admits an H^∞-calculus.

Lemma 3.4. *Pick any integer $T \geq 2$ and assume that (hB) holds. Then for any integer $p \geq 1$ and any $i \in \{0,\ldots,T-1\}$ we have the bound*

$$E\left[\Upsilon_B^{(i)}\right]^{2p} \leq K_{\mathrm{cnv}}^{2p} p^p e^{\varepsilon p} \Theta_*^{2p}. \tag{3.18}$$

Proof. Applying Lemma 2.5, we readily compute

$$\begin{array}{rcl} E\left[\Upsilon_B^{(i)}\right]^{2p} & \leq & K_{\mathrm{cnv}}^{2p} p^p E\left[\int_i^{i+1} \|B(s)\|_{HS}^2 \, ds\right]^p \\ & \leq & K_{\mathrm{cnv}}^{2p} p^p e^{\varepsilon p} E\left[\int_0^{i+1} e^{-\varepsilon(i+1-s)} \|B(s)\|_{HS}^2 \, ds\right]^p, \end{array} \tag{3.19}$$

which implies the stated bound on account of (3.5). □

Long-term bounds The goal here is to apply the chaining result from Theorem 2.7 to the long-term integral $\mathcal{E}_B^{\mathrm{lt}}$. To achieve this, we will use Lemma 2.2 to turn moment bounds for the increments of $\mathcal{E}_B^{\mathrm{lt}}$ into the desired tail bounds for the associated probability distribution.

For any pair $0 \leq t_1 \leq t_2 \leq T$, we split this increment into two parts

$$\mathcal{E}_B^{\mathrm{lt}}(t_1) - \mathcal{E}_B^{\mathrm{lt}}(t_2) = \mathcal{I}_1(t_1,t_2) + \mathcal{I}_2(t_1,t_2) \tag{3.20}$$

that are defined by

$$\begin{array}{rcl} \mathcal{I}_1(t_1,t_2) & = & \int_0^{t_1-1} [S(t_2-s) - S(t_1-s)] \, B(s) \, dW_s^Q, \\ \mathcal{I}_2(t_1,t_2) & = & \int_{t_1-1}^{t_2-1} S(t_2-s) B(s) \, dW_s^Q. \end{array} \tag{3.21}$$

The first of these can be analyzed by exploiting the regularity of the semigroup $S(t-s)$ for $t-s \geq 1$, while the second requires a supremum bound on the 'smoothened' process $S(1)B$, hence explaining the assumption in equation (3.5).

Lemma 3.5. *Pick any integer $T \geq 2$ and assume that (hB) holds. Then for any $1 \leq t_1 \leq t_2 \leq T$ and any integer $p \geq 1$ we have the bound*

$$E\|\mathcal{I}_1(t_1,t_2)\|_{L^2}^{2p} \leq p^p K_{\mathrm{cnv}}^{2p} M^{4p} \Theta_*^{2p} |t_2 - t_1|^{2p}. \tag{3.22}$$

Proof. Observe first that

$$\begin{array}{rcl} E\|\mathcal{I}_1(t_1,t_2)\|_{L^2}^{2p} & \leq & \|[S(t_2-t_1) - I]S(1)\|_{\mathcal{L}(L^2,L^2)}^{2p} E\|\int_0^{t_1-1} S(t_1-1-s)B(s) \, dW_s^Q\|_{L^2}^{2p} \\ & \leq & M^{2p}|t_2-t_1|^{2p} E\|\int_0^{t_1-1} S(t_1-1-s)B(s) \, dW_s^Q\|_{L^2}^{2p}. \end{array} \tag{3.23}$$

Applying (2.25) with $T = t_1 - 1$, we find

$$\begin{aligned} E\|\mathcal{I}_1(t_1,t_2)\|_{L^2}^{2p} &\leq p^p K_{\rm cnv}^{2p} M^{2p} |t_2 - t_1|^{2p} E\left[\int_0^{t_1-1} \|S(t_1 - 1 - s)B(s)\|_{HS}^2\, ds\right]^p \\ &\leq p^p K_{\rm cnv}^{2p} M^{4p} |t_2 - t_1|^{2p} E\left[\int_0^{t_1-1} e^{-2\beta(t_1-1-s)}\|B(s)\|_{HS}^2\, ds\right]^p, \end{aligned} \tag{3.24}$$

which yields the stated bound in view of (3.8). □

Lemma 3.6. *Pick any integer $T \geq 2$ and assume that (hB) holds. Then for any $1 \leq t_1 \leq t_2 \leq T$ and any integer $p \geq 1$ we have the bound*

$$E\|\mathcal{I}_2(t_1,t_2)\|_{L^2}^{2p} \leq p^p K_{\rm cnv}^{2p} M^{2p} \Theta_*^{2p} |t_2 - t_1|^p. \tag{3.25}$$

Proof. It suffices to compute

$$\begin{aligned} E\|\mathcal{I}_2(t_1,t_2)\|_{L^2}^{2p} &= E\left[\|\int_{t_1-1}^{t_2-1} S(t_2 - 1 - s)S(1)B(s)\, dW_s^Q\|_{L^2}\right]^{2p} \\ &\leq p^p K_{\rm cnv}^{2p} E\left[\int_{t_1-1}^{t_2-1} \|S(t_2-1-s)\|_{\mathcal{L}(L^2,L^2)}^2 \|S(1)B(s)\|_{HS}^2\, ds\right]^p \\ &\leq p^p K_{\rm cnv}^{2p} M^{2p} |t_2 - t_1|^p E\left[\sup_{t_1-1\leq s\leq t_2-1} \|S(1)B(s)\|_{HS}^2\right]^p \end{aligned} \tag{3.26}$$

and apply (3.8). □

The previous two results were tailored to handle small increments $|t_2 - t_1| \leq 1$. For larger increments one can exploit the decay of the semigroup to show that $\mathcal{E}_B^{\rm lt}$ remains bounded in expectation.

Lemma 3.7. *Pick any integer $T \geq 2$ and assume that (hB) holds. Then for any $0 \leq t \leq T$ and any integer $p \geq 1$ we have the bound*

$$E\|\mathcal{E}_B^{\rm lt}(t)\|_{L^2}^{2p} \leq p^p K_{\rm cnv}^{2p} M^{2p} \Theta_*^{2p}. \tag{3.27}$$

Proof. Using Corollary 2.6, we find

$$\begin{aligned} E\|\mathcal{E}_B^{\rm lt}(t)\|_{L^2}^{2p} &\leq p^p K_{\rm cnv}^{2p} E\left[\int_0^{t-1} \|S(t-s)\|_{\mathcal{L}(L^2,L^2)}^2 \|B(s)\|_{HS}^2\, ds\right]^p \\ &\leq p^p K_{\rm cnv}^{2p} M^{2p} E\left[\int_0^{t-1} e^{-2\beta(t-1-s)}\|B(s)\|_{HS}^2 ds\right]^p \\ &\leq p^p K_{\rm cnv}^{2p} M^{2p} \Theta_*^{2p}. \end{aligned} \tag{3.28}$$

□

Corollary 3.8. *Pick any integer $T \geq 2$ and assume that (hB) holds. Then for any $0 \leq t_1 \leq t_2 \leq T$ and any integer $p \geq 1$ we have the bound*

$$E\|\mathcal{E}_B^{\rm lt}(t_1) - \mathcal{E}_B^{\rm lt}(t_2)\|_{L^2}^{2p} \leq 2^{2p} p^p K_{\rm cnv}^{2p} M^{4p} \Theta_*^{2p} \min\{|t_2 - t_1|^{1/2}, 1\}^{2p}. \tag{3.29}$$

Proof. This follows from the standard inequality $(a+b)^{2p} \leq 2^{2p-1}(a^{2p} + b^{2p})$ and a combination of the estimates from Lemmas 3.5-3.7. □

Lemma 3.9. *Fix a sufficiently large constant $K_{\rm lt} \geq 1$. Then for any integer $T \geq 2$ and any process B that satisfies (hB), we have the supremum bound*

$$E \sup_{0\leq t\leq T} \|\mathcal{E}_B^{\rm lt}(t)\|_{L^2}^2 \leq K_{\rm lt} \Theta_*^2 \ln(T). \tag{3.30}$$

Proof. Upon writing $d_{\max} = 2\sqrt{e}K_{\mathrm{cnv}}M^2\Theta_*$ together with

$$d(t_1,t_2) = d_{\max}\min\{\sqrt{|t_2-t_1|},1\}, \tag{3.31}$$

an application of Lemma 2.2 to Corollary 3.8 provides the bound

$$P\left(\|\mathcal{E}^{\mathrm{lt}}_B(t_1)-\mathcal{E}^{\mathrm{lt}}_B(t_2)\|_{L^2} > \vartheta\right) \quad\leq\quad 2\exp\left[-\frac{\vartheta^2}{2d(t_1,t_2)^2}\right]. \tag{3.32}$$

Turning to the packing number $N(T,d,\nu)$ introduced in Theorem 2.7, we note that $N(T,d,\nu)=1$ whenever $\nu \geq d_{\max}$, while for smaller ν we have

$$N(T,d,\nu) \leq \frac{Td^2_{\max}}{\nu^2}. \tag{3.33}$$

In particular, the Dudley entropy integral can be bounded by

$$\begin{aligned}\int_0^\infty \sqrt{\ln(N(T,d,\nu))}\,d\nu &\leq \int_0^{d_{\max}} \sqrt{\ln(Td^2_{\max}/\nu^2)}\,d\nu\\ &= \int_0^{d_{\max}} \sqrt{-2\ln\left(\nu/(d_{\max}\sqrt{T})\right)}\,d\nu\\ &= d_{\max}\sqrt{T}\int_0^{1/\sqrt{T}} \sqrt{-2\ln(\nu)}\,d\nu\\ &= d_{\max}\left(\sqrt{2\ln(T)}+\sqrt{\pi}\sqrt{T}\mathrm{erfc}\big(\sqrt{\ln(T)}\big)\right).\end{aligned} \tag{3.34}$$

Since the function $\sqrt{T}\mathrm{erfc}(\sqrt{\ln(T)})$ is uniformly bounded for $T \geq 2$, the desired estimate now follows directly from Theorem 2.7. □

Proof of (3.8) in Proposition 3.1. Applying Corollary 2.4 to the estimates (3.17)-(3.18), we directly find

$$E\sup_{0\leq t\leq T}\|\mathcal{E}^{\mathrm{sh}}_B(t)\|^2_{L^2} \leq 108M^2eK^2_{\mathrm{cnv}}e^{\varepsilon}\Theta^2_*\ln(T). \tag{3.35}$$

Combining this with the analogous long-term estimate (3.30) readily yields the result. □

3.2 Estimates for $\mathcal{I}^s_B$

Our strategy here for controlling $\mathcal{I}^s_B$ is to appeal to Corollary 2.4, which requires us to obtain moment bounds on $\mathcal{I}^{\mathrm{s}}_B(i)$. As a preparation, we switch the order of integration to find

$$\begin{aligned}\mathcal{I}^{\mathrm{s}}_B(i) \quad&=\quad \int_0^i e^{-\varepsilon(i-s)}\int_0^s \langle S(s-s')V(s'),S(s-s')B(s')\cdot\rangle_{H^1}\,dW^Q_{s'}\,ds\\ &=\quad \int_0^i\int_{s'}^i e^{-\varepsilon(i-s)}\langle S(s-s')V(s'),S(s-s')B(s')\cdot\rangle_{H^1}\,ds\,dW^Q_{s'}.\end{aligned} \tag{3.36}$$

Corollary 2.6 hence yields

$$E[\mathcal{I}^{\mathrm{s}}_B(i)]^{2p} \leq p^pK^{2p}_{\mathrm{cnv}}E\left[\int_0^i\sum_k \mathcal{K}^{(i)}_k(s')^2ds'\right]^p, \tag{3.37}$$

in which we have introduced the expression[4]

$$\mathcal{K}_k^{(i)}(s') = \int_{s'}^{i} e^{-\varepsilon(i-s)} \langle S(s-s')V(s'), S(s-s')B(s')\sqrt{Q}e_k\rangle_{H^1} ds. \tag{3.38}$$

Motivated by (2.5), we split $\mathcal{K}_k^{(i)}$ into the two parts

$$\begin{aligned} \mathcal{K}_{I;k}^{(i)}(s') &= \int_{s'}^{i} e^{-\varepsilon(i-s)} \mathcal{J}_{\mathrm{tw}}[V(s'), B(s')\sqrt{Q}e_k] ds, \\ \mathcal{K}_{II;k}^{(i)}(s') &= -\frac{1}{2\rho}\int_{s'}^{i} e^{-\varepsilon(i-s)} \frac{d}{ds}\langle S(s-s')V(s'), S(s-s')B(s')\sqrt{Q}e_k\rangle_{L^2} ds. \end{aligned} \tag{3.39}$$

Performing an integration by parts, the second of these integrals can be further decomposed into the three terms

$$\begin{aligned} \mathcal{K}_{IIa;k}^{(i)}(s') &= -\frac{\varepsilon}{2\rho}\int_{s'}^{i} e^{-\varepsilon(i-s)} \langle S(s-s')V(s'), S(s-s')B(s')\sqrt{Q}e_k\rangle_{L^2} ds, \\ \mathcal{K}_{IIb;k}^{(i)}(s') &= -\frac{1}{2\rho}\langle S(i-s')V(s'), S(i-s')B(s')\sqrt{Q}e_k\rangle_{L^2}, \\ \mathcal{K}_{IIc;k}^{(i)}(s') &= \frac{1}{2\rho}e^{-\varepsilon(i-s')}\langle V(s'), B(s')\sqrt{Q}e_k\rangle_{L^2}. \end{aligned} \tag{3.40}$$

Lemma 3.10. *Pick a sufficiently large constant $K_{\mathcal{K}} > 0$. Then for any integer $T \geq 2$, any pair of processes (B, V) that satisfies (hB) and (hV) and any $i \in \{1, \ldots, T\}$, we have the bound*

$$\sum_k \mathcal{K}_{\#;k}^{(i)}(s')^2 \leq K_{\mathcal{K}} e^{-2\varepsilon(i-s')} \|V(s')\|_{L^2}^2 \|B(s')\|_{HS}^2 \tag{3.41}$$

for all $0 \leq s' \leq i$ and each of the symbols $\# \in \{I, IIa, IIb, IIc\}$.

Proof. Upon introducing the expression

$$K(\varepsilon, \beta) = e^{\varepsilon(i-s')}\int_{s'}^{i} e^{-\varepsilon(i-s)} e^{-2\beta(s-s')}\left(1 + \rho^{-1}(s-s')^{-1/2}\right) ds \tag{3.42}$$

we may exploit (2.7) to obtain the bound

$$\begin{aligned} \sum_k \mathcal{K}_{I;k}^{(i)}(s')^2 &\leq M^2 e^{-2\varepsilon(i-s')}\sum_k \|V(s')\|_{L^2}^2 \|B(s')\sqrt{Q}e_k\|_{L^2}^2 K(\varepsilon,\beta)^2 \\ &= M^2 e^{-2\varepsilon(i-s')}\|V(s')\|_{L^2}^2 \|B(s')\|_{HS}^2 K(\varepsilon,\beta)^2. \end{aligned} \tag{3.43}$$

The estimate for $\# = I$ hence follows from the computation

$$K(\varepsilon,\beta) \leq \int_0^\infty e^{(\varepsilon-2\beta)s}\left(1+\rho^{-1}s^{-1/2}\right) ds = \frac{1}{2\beta-\varepsilon} + \frac{1}{\rho}\sqrt{\frac{\pi}{2\beta-\varepsilon}}. \tag{3.44}$$

The estimate for $\mathcal{K}_{IIa;k}^{(i)}$ can be obtained in the same fashion, but here the $(s-s')^{-1/2}$ term in (3.42) is not required. Finally, the estimates for $\mathcal{K}_{IIb;k}^{(i)}$ and $\mathcal{K}_{IIc;k}^{(i)}$ are immediate from Lemma 2.1 and the choice $\beta > \varepsilon$. $\square$

[4] Note that this integral is an improper integral, as the integrand is not defined for the lower bound $s = s'$. In [23] we show how this problem can be circumvented by replacing s by $s + \delta$ and subsequently sending $\delta \to 0$.

Proof of (3.9) in Proposition 3.1. Applying Young's inequality to the decomposition above, we obtain the pathwise bound

$$\int_0^i \sum_k \mathcal{K}_k^{(i)}(s')^2 ds' \quad \leq \quad 16 K_{\mathcal{K}} \int_0^i e^{-2\varepsilon(i-s')} \|V(s')\|_{L^2}^2 \|B(s')\|_{HS}^2 \, ds' \quad \leq \quad 16 K_{\mathcal{K}} \Lambda_*^2 \Theta_*^2. \tag{3.45}$$

In view of (3.37) this implies

$$E[\mathcal{I}_B^{\mathrm{s}}(i)]^{2p} \quad \leq \quad 2^{4p} p^p K_{\mathcal{K}}^p K_{\mathrm{cnv}}^{2p} \Lambda_*^{2p} \Theta_*^{2p}, \tag{3.46}$$

which leads to the desired bound by exploiting Corollary 2.4. □

4 Deterministic supremum bounds

Our goal here is to obtain pathwise bounds on the deterministic integrals

$$\begin{aligned} \mathcal{I}_F(t) &= \int_0^t e^{-\varepsilon(t-s)} \int_0^s \langle S(s-s')V(s'), S(s-s')F(s')\rangle_{H^1} \, ds' \, ds, \\ \mathcal{I}_B^{\mathrm{d}}(t) &= \int_0^t e^{-\varepsilon(t-s)} \int_0^s \|S(s-s')B(s')\|^2_{HS(L^2_Q, H^1)} \, ds' \, ds. \end{aligned} \tag{4.1}$$

We are using the process F in the first integral as a placeholder for various linear and nonlinear expressions in V that we will encounter in §5. The second integral arises as the Itô correction term coming from the integrated H^1-norm of V; see Lemma 5.5. Besides the assumptions (hB) and (hV) introduced in §5, we impose the following condition on the new function F.

(hF) The process $F : [0, T] \times \Omega \to L^2$ has paths in $L^1([0, T]; L^2)$ and satisfies

$$\langle F(t), \psi_{\mathrm{tw}} \rangle_{L^2} = 0 \tag{4.2}$$

for all $t \in [0, T]$.

In contrast to the stochastic setting of §3, pathwise bounds for the expressions (4.1) can be easily used to control their supremum expectations. Indeed, we do not need to use the Burkholder-Davis-Gundy inequalities, which allows us to take a far more direct approach to establish our two main results below. Notice that we are making no a priori assumptions on the size of F. This will be useful in §5 to obtain sharp estimates for the nonlinear terms.

Proposition 4.1. *Fix a sufficiently large constant $K > 0$. Then for any $T > 0$ and any pair of processes (F, V) that satisfies (hF) and (hV), we have the supremum bound*

$$E \sup_{0 \leq t \leq T} |\mathcal{I}_F(t)| \leq K \Lambda_* E \sup_{0 \leq t \leq T} \int_0^t e^{-\varepsilon(t-s)} \|F(s)\|_{L^2} \, ds. \tag{4.3}$$

Proposition 4.2. *Fix a sufficiently large constant $K > 0$. Then for any $T > 0$ and any process B that satisfies (hB), we have the supremum bound*

$$E \sup_{0 \leq t \leq T} \mathcal{I}_B^d(t) \leq K \Theta_*^2. \tag{4.4}$$

4.1 Estimates for $\mathcal{I}_F$ and $\mathcal{I}_B^d$

Introducing the expressions

$$\begin{aligned}\mathcal{K}_F(t,s') &= \int_{s'}^{t} e^{-\varepsilon(t-s)}\langle S(s-s')V(s'), S(s-s')F(s')\rangle_{H^1} ds,\\ \mathcal{K}_{B;k}^d(t,s') &= \int_{s'}^{t} e^{-\varepsilon(t-s)}\langle S(s-s')B(s')\sqrt{Q}e_k, S(s-s')B(s')\sqrt{Q}e_k\rangle_{H^1} ds,\end{aligned} \tag{4.5}$$

we may reverse the order of integration to find

$$\mathcal{I}_F(t) = \int_0^t \mathcal{K}_F(t,s')ds', \qquad \mathcal{I}_B^d(t) = \int_0^t \sum_k \mathcal{K}_{B;k}^d(t,s')ds'. \tag{4.6}$$

Lemma 4.3. *Pick a sufficiently large constant $K_F > 0$. Then for any $T > 0$, any pair of processes (V, F) that satisfies (hV) and (hF) and any $t \in [0, T]$, we have the bound*

$$\mathcal{K}_F(t,s') \quad \leq \quad K_F e^{-\varepsilon(t-s')}\|V(s')\|_{L^2}\|F(s')\|_{L^2} \tag{4.7}$$

for all $0 \leq s' \leq t$.

Proof. Observe that $\mathcal{K}_F(t,s')$ is identical to (3.38) after making the replacement $B(s')\sqrt{Q}e_k \mapsto F(s')$. We can hence use the same decomposition as in §3.2 and follow the proof of Lemma 3.10 to obtain the stated bound. □

Lemma 4.4. *Pick a sufficiently large constant $K_B^d > 0$. Then for any $T > 0$, any process B that satisfies (hB) and any $t \in [0, T]$, we have the bound*

$$\sum_k \mathcal{K}_{B;k}^d(t,s') \quad \leq \quad K_B^d\, e^{-\varepsilon(t-s')}\|B(s')\|_{HS}^2 \tag{4.8}$$

for all $0 \leq s' \leq t$.

Proof. Observe that $\mathcal{K}_B^d(t,s')$ is identical to (3.38) after making the replacement $V(s')\mapsto B(s')\sqrt{Q}e_k$. We can hence use the same decomposition as in §3.2 and follow the proof of Lemma 3.10 to obtain the stated bound. □

Proof of Proposition 4.1. Combining the identity (4.6) with the bound (4.7), we readily obtain the pathwise bound

$$|\mathcal{I}_F(t)| \leq K_F\Lambda_* \int_0^t e^{-\varepsilon(t-s)}\|F(s)\|_{L^2} ds. \tag{4.9}$$

The result hence follows by taking the expectation of the supremum. □

Proof of Proposition 4.2. Combining the identity (4.6) with the estimate (4.8), we readily obtain the pathwise bound

$$|\mathcal{I}_B^d(t)| \leq K_B^d\, \Theta_*^2, \tag{4.10}$$

which of course survives taking the expectation of the supremum. □

5 Nonlinear stability

With the results from the previous sections under our belt, we now set out to prove the estimates in Theorem 1.1 and hence establish the stochastic stability of the travelling wave on exponentially long timescales. Our starting point will be the computations in [23, 24], which use a time transformation to construct a mild integral equation for the perturbation $V(t)$ that contains no dangerous second-order derivatives.

The arguments in [23, Lem. 6.4] and [24, §6.1] indicate that this time transformation only affects the constants in the final estimate (1.10). For presentation purposes, we therefore simply reuse t for the transformed time and leave the definition (1.9) for the stopping time $t_{\mathrm{st}}(\eta)$ intact. The mild representation for V can now be written in the generic form

$$V(t) = V(0) + \int_0^t S(t-s)[\sigma^2 F_{\mathrm{lin}}\big(V(s)\big) + F_{\mathrm{nl}}\big(V(s)\big)]ds + \sigma \int_0^t S(t-s)B\big(V(s)\big)dW_s^Q, \qquad (5.1)$$

where the maps

$$F_{\mathrm{lin}} : H^1 \to L^2, \qquad F_{\mathrm{nl}} : H^1 \to L^2, \qquad B : H^1 \to HS(L_Q^2, L^2) \qquad (5.2)$$

satisfy the bounds

$$\begin{aligned} \|F_{\mathrm{lin}}(v)\|_{L^2} &\leq K_{\mathrm{lin}}\|v\|_{H^1}, \\ \|F_{\mathrm{nl}}(v)\|_{L^2} &\leq K_{\mathrm{nl}}\|v\|_{H^1}^2(1+\|v\|_{L^2}^3), \\ \|B(v)\|_{HS} &\leq K_B(1+\|v\|_{H^1}), \\ \|S(1)B(v)\|_{HS} &\leq K_B M(1+\|v\|_{L^2}). \end{aligned} \qquad (5.3)$$

In addition, whenever $\|v\|_{L^2}$ is sufficiently small, we have the identities

$$\langle \sigma^2 F_{\mathrm{lin}}(v) + F_{\mathrm{nl}}(v), \psi_{\mathrm{tw}} \rangle_{L^2} = 0, \qquad \langle B(v)[w], \psi_{\mathrm{tw}} \rangle_{L^2} = 0 \qquad (5.4)$$

for every $w \in L_Q^2$. Notice that $v = V(t)$ automatically satisfies this condition for $t \leq t_{\mathrm{st}}(\eta)$ provided that $\eta < \eta_0$ for some sufficiently small η_0.

In order to state the main result of this section, we write

$$N(t) = \|V(t)\|_{L^2}^2 + \int_0^t e^{-\varepsilon(t-s)}\|V(s)\|_{H^1}^2\, ds \qquad (5.5)$$

for the size of the solution V, which also features in the definition (1.9) for the stopping time $t_{\mathrm{st}} = t_{\mathrm{st}}(\eta)$. The various supremum bounds derived in §3 and §4 can now be used to obtain a similar bound for $N(t)$. This result can be seen as a significantly sharpened version of its counterpart [23, Prop. 9.1], which allows Theorem 1.1 to be established in a standard fashion.

Proposition 5.1. *Pick a constant $0 < \varepsilon < \beta$, together with two sufficiently small constants $\delta_\eta > 0$ and $\delta_\sigma > 0$. Then there exists a constant $K > 0$ so that for any integer $T \geq 2$, any $0 < \eta < \delta_\eta$ and any $0 \leq \sigma \leq \delta_\sigma \ln(T)^{-1/2}$ we have the bound*

$$E\left[\sup_{0\leq t\leq t_{\mathrm{st}}} N(t)\right] \leq K\left[\|V(0)\|_{L^2}^2 + \sigma^2\ln(T) + \sigma\sqrt{\eta}\sqrt{\ln(T)}\right]. \qquad (5.6)$$

Proof of Theorem 1.1. The arguments in [23, Cor. 9.3] and the proof of [23, Thm. 2.4] can be used almost verbatim to derive (1.10) from (5.6). □

5.1 Supremum bounds in L^2

In this subsection we establish the following bound on the supremum of the L^2-norm of $V(t)$. Notice that we are imposing less restrictions on σ and η here, but $N(t)$ still appears on the right-hand side of our estimate.

Lemma 5.2. *Pick a constant $0 < \varepsilon < \beta$. Then there exists a constant $K > 0$ so that for any integer $T \geq 2$, any $0 < \eta < \eta_0$ and any $0 \leq \sigma \leq 1$ we have the bound*

$$E \sup_{0 \leq t \leq t_{\rm st}} \|V(t)\|_{L^2}^2 \leq K\Big[\|V(0)\|_{L^2}^2 + \sigma^2 \ln(T) + (\sigma^4 + \eta)E \sup_{0 \leq t \leq t_{\rm st}} N(t)\Big]. \tag{5.7}$$

In order to streamline the proof, we recall the definition of Π from 2.1 and define the functions

$$\begin{aligned}
\mathcal{E}_0(t) =& S(t)V(0), \\
\mathcal{E}_{\rm lin}(t) =& \int_0^t S(t-s)\Pi F_{\rm lin}\big(V(s)\big)\mathbf{1}_{s \leq t_{\rm st}}\,ds, \\
\mathcal{E}_{\rm nl}(t) =& \int_0^t S(t-s)\Pi F_{\rm nl}\big(V(s)\big)\mathbf{1}_{s \leq t_{\rm st}}\,ds, \\
\mathcal{E}_B(t) =& \int_0^t S(t-s)B\big(V(s)\big)\mathbf{1}_{s \leq t_{\rm st}}\,dW_s^Q.
\end{aligned} \tag{5.8}$$

The three deterministic expressions can be controlled in a direction fashion, while the final stochastic integral was analyzed in §4.

Lemma 5.3. *For any $0 < \eta < \eta_0$, any $0 \leq \sigma \leq 1$ and any $T > 0$, we have the bounds*

$$\begin{aligned}
E \sup_{0 \leq t \leq T} \|\mathcal{E}_0(t)\|_{L^2}^2 \leq& M^2 \|V(0)\|_{L^2}^2, \\
E \sup_{0 \leq t \leq T} \|\mathcal{E}_{\rm lin}(t)\|_{L^2}^2 \leq& M^2 K_{\rm lin}^2 E \sup_{0 \leq t \leq t_{\rm st}} \|V(t)\|_{L^2}^2, \\
E \sup_{0 \leq t \leq T} \|\mathcal{E}_{\rm nl}(t)\|_{L^2}^2 \leq& M^2 K_{\rm nl}^2 (1+\eta^3)^2 \eta E \sup_{0 \leq t \leq t_{\rm st}} \int_0^t e^{-\varepsilon(t-s)} \|V(t)\|_{H^1}^2\,ds.
\end{aligned} \tag{5.9}$$

Proof. These results follow directly from Lemmas 9.8-9.11 in [23], where they were established using straightforward direct norm estimates. □

Lemma 5.4. *Pick a constant $0 < \epsilon < \beta$. Then there is a $K > 0$ such that the bound*

$$E \sup_{0 \leq t \leq T} \|\mathcal{E}_B(t)\|_{L^2}^2 \leq K \ln(T) \tag{5.10}$$

holds for any integer $T \geq 2$, any $0 < \eta < \eta_0$ and any $0 \leq \sigma \leq 1$.

Proof. We will prove this by appealing to Proposition 3.1. In order to verify (hB), we simply compute

$$\begin{aligned}
\int_0^t e^{-\varepsilon(t-s)} \|B\big(V(s)\big)\mathbf{1}_{s \leq t_{\rm st}}\|_{HS}^2\,ds \leq& K_B^2 \int_0^t e^{-\varepsilon(t-s)} (1 + \|V(s)\|_{H^1}^2)\mathbf{1}_{s \leq t_{\rm st}}\,ds \\
\leq& K_B^2 \left(\varepsilon^{-1} + \int_0^{\min\{t, t_{\rm st}\}} e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2\,ds\right) \\
\leq& K_B^2 (\varepsilon^{-1} + \eta),
\end{aligned} \tag{5.11}$$

together with

$$\|S(1)B\big(V(s)\big)\mathbf{1}_{s\leq t_{\rm st}}\|_{HS}^2 \leq M^2K_B^2(1+\|V(s)\mathbf{1}_{s\leq t_{\rm st}}\|_{L^2}^2) \leq M^2K_B^2(1+\eta), \tag{5.12}$$

which allows us to take $\Theta_*^2 = M^2K_B^2(\varepsilon^{-1}+\eta)$. $\square$

Proof of Lemma 5.2. We directly find that

$$E\sup_{0\leq t\leq t_{\rm st}}\|V(t)\|_{L^2}^2 \leq 4E\sup_{0\leq t\leq T}\big[\|\mathcal{E}_0(t)\|_{L^2}^2+\sigma^4\|\mathcal{E}_{\rm lin}(t)\|_{L^2}^2+\|\mathcal{E}_{\rm nl}(t)\|_{L^2}^2+\sigma^2\|\mathcal{E}_B(t)\|_{L^2}^2\big]. \tag{5.13}$$

Collecting the results from Lemmas 5.3 and 5.4 now proves the result. $\square$

5.2 Supremum bounds in H^1

In this subsection we control the H^1-norm of V by establishing a supremum bound for the integrated expression

$$\mathcal{I}(t) = \int_0^t e^{-\varepsilon(t-s)}\|V(s)\|_{H^1}^2\,ds. \tag{5.14}$$

In particular, we set out to obtain the following estimate.

Lemma 5.5. *Pick a constant $0<\varepsilon<\beta$. Then there exists a constant $K>0$ so that for any integer $T\geq 2$, any $0<\eta<\eta_0$ and any $0\leq\sigma\leq 1$ we have the bound*

$$E\sup_{0\leq t\leq T}\mathcal{I}(t) \leq K\Big[\|V(0)\|_{H^1}^2+\eta\sigma^2+\sqrt{\eta}E\big[\sup_{0\leq t\leq t_{\rm st}}N(t)\big]+\sigma^2+\sigma\sqrt{\eta}\sqrt{\ln(T)}\Big]. \tag{5.15}$$

Compared to §5.1 and [23, §9], our approach here is rather indirect. First of all, we exploit the fact that T is an integer to compute

$$\begin{aligned}\sup_{0\leq t\leq T}\mathcal{I}(t) &= \max_{i\in\{1,\ldots,T\}}\sup_{i-1\leq t\leq i}\int_0^t e^{-\varepsilon(t-s)}\|V(s)\|_{H^1}^2\,ds\\ &\leq \max_{i\in\{1,\ldots,T\}} e^{\varepsilon}\int_0^i e^{-\varepsilon(i-s)}\|V(s)\|_{H^1}^2\,ds\\ &= e^{\varepsilon}\max_{i\in\{1,\ldots,T\}}\mathcal{I}(i).\end{aligned} \tag{5.16}$$

We continue by applying a mild Itô formula [14] to $\|V(s)\|_{H^1}^2$, which yields

$$\begin{aligned}\|V(s)\|_{H^1}^2 =& \|S(s)V(0)\|_{H^1}^2+2\sigma^2\int_0^s\langle S(s-s')V(s'),S(s-s')F_{\rm lin}(V(s'))\rangle_{H^1}\,ds'\\ &+2\int_0^s\langle S(s-s')V(s'),S(s-s')F_{\rm nl}\big(V(s')\big)\rangle_{H^1}\,ds'\\ &+2\sigma\int_0^s\langle S(s-s')V(s'),S(s-s')B\big(V(s')\big)dW_{s'}^Q\rangle_{H^1}\\ &+\sigma^2\int_0^s\|S(s-s')B\big(V(s')\big)\|_{HS(L_Q^2,H^1)}^2\,ds'.\end{aligned} \tag{5.17}$$

In particular, upon introducing the components

$$
\begin{aligned}
\mathcal{I}_0(t) &= \int_0^t e^{-\varepsilon(t-s)} \|S(s)V(0)\|_{H^1}^2 ds, \\
\mathcal{I}_{\mathrm{lin}}(t) &= \int_0^t e^{-\varepsilon(t-s)} \int_0^s \langle S(s-s')V(s'), S(s-s')\Pi F_{\mathrm{lin}}\big(V(s')\big)\mathbf{1}_{s'\le t_{\mathrm{st}}}\rangle_{H^1} ds' ds, \\
\mathcal{I}_{\mathrm{nl}}(t) &= \int_0^t e^{-\varepsilon(t-s)} \int_0^s \langle S(s-s')V(s'), S(s-s')\Pi F_{\mathrm{nl}}\big(V(s')\big)\mathbf{1}_{s'\le t_{\mathrm{st}}}\rangle_{H^1} ds' ds, \\
\mathcal{I}_B^s(t) &= \int_0^i e^{-\varepsilon(t-s)} \int_0^s \langle S(s-s')V(s'), S(s-s')B\big(V(s')\big)\mathbf{1}_{s'\le t_{\mathrm{st}}} dW_{s'}^Q\rangle_{H^1} ds, \\
\mathcal{I}_B^d(t) &= \int_0^i e^{-\varepsilon(t-s)} \int_0^s \|S(s-s')B\big(V(s')\big)\mathbf{1}_{s'\le t_{\mathrm{st}}}\|_{HS(L_Q^2,H^1)}^2 ds' ds,
\end{aligned}
\tag{5.18}
$$

we obtain the bound

$$
\begin{aligned}
E \sup_{0\le t\le t_{\mathrm{st}}} \mathcal{I}(t) \le & e^{\varepsilon} E \max_{i\in\{1,\dots,T\}} \big[\mathcal{I}_0(i) + 2\sigma^2 \mathcal{I}_{\mathrm{lin}}(i) + 2\mathcal{I}_{\mathrm{nl}}(i) + 2\sigma \mathcal{I}_B^{\mathrm{s}}(i) + \sigma^2 \mathcal{I}_B^{\mathrm{d}}(i)\big] \\
\le & e^{\varepsilon} E \sup_{0\le t\le T} \big[\mathcal{I}_0(t) + 2\sigma^2 \mathcal{I}_{\mathrm{lin}}(t) + 2\mathcal{I}_{\mathrm{nl}}(t) + \sigma^2 \mathcal{I}_B^{\mathrm{d}}(t)\big] + 2e^{\varepsilon}\sigma E \max_{i\in\{1,\dots,T\}} \mathcal{I}_B^{\mathrm{s}}(i).
\end{aligned}
\tag{5.19}
$$

This decomposition highlights the fact that supremum bounds over deterministic integrals are easily obtained, while the stochastic integral needs to be handled with care.

Lemma 5.6. *Pick a constant $0 < \varepsilon < \beta$. Then there exists a constant $K > 0$ so that for any integer $T \ge 2$, any $0 < \eta < \eta_0$ and any $0 \le \sigma \le 1$ we have the bounds*

$$
\begin{aligned}
E \sup_{0\le t\le T} \mathcal{I}_{\mathrm{lin}}(t) \le & K\eta, \\
E \sup_{0\le t\le T} \mathcal{I}_{\mathrm{nl}}(t) \le & K\sqrt{\eta} E \sup_{0\le t\le T} \int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2 \mathbf{1}_{s\le t_{\mathrm{st}}} ds.
\end{aligned}
\tag{5.20}
$$

Proof. In order to exploit Proposition 4.1, we first note that the orthogonality conditions (3.6) and (4.2) hold true by virtue of the stopping time. In particular, (hF) and (hV) are both satisfied, with $\Lambda_* = \sqrt{\eta}$. The stated bounds can hence be obtained by using the computation

$$
\begin{aligned}
\int_0^t e^{-\varepsilon(t-s)} \|F_{\mathrm{lin}}(V(s))\mathbf{1}_{s\le t_{\mathrm{st}}}\|_{L^2} ds \le & K_{\mathrm{lin}} \int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1} \mathbf{1}_{s\le t_{\mathrm{st}}} ds \\
\le & K_{\mathrm{lin}} \frac{1}{\varepsilon} \sqrt{\int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2 \mathbf{1}_{s\le t_{\mathrm{st}}} ds} \\
\le & K_{\mathrm{lin}} \frac{\sqrt{\eta}}{\varepsilon},
\end{aligned}
\tag{5.21}
$$

together with

$$
\begin{aligned}
\int_0^t e^{-\varepsilon(t-s)} \|F_{\mathrm{nl}}(V(s))\mathbf{1}_{s\le t_{\mathrm{st}}}\|_{L^2} ds \le & K_{\mathrm{nl}} \int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2 (1 + \|V(s)\|_{L^2}^3) \mathbf{1}_{s\le t_{\mathrm{st}}} ds \\
\le & K_{\mathrm{nl}} (1+\eta^3) \int_0^t e^{-\varepsilon(t-s)} \|V(s)\|_{H^1}^2 \mathbf{1}_{s\le t_{\mathrm{st}}} ds
\end{aligned}
\tag{5.22}
$$

to evaluate the right-hand side of (4.3). □

Lemma 5.7. *Pick a constant $0 < \varepsilon < \beta$. Then there exists a constant $K > 0$ so that for any integer $T \geq 2$, any $0 < \eta < \eta_0$ and any $0 \leq \sigma \leq 1$ we have the bounds*

$$
\begin{aligned}
E \sup_{0 \leq t \leq T} \mathcal{I}_B^d(t) &\leq K, \\
E \max_{i \in \{1,\dots,T\}} \mathcal{I}_B^s(t) &\leq K \sqrt{\eta} \sqrt{\ln(T)}.
\end{aligned}
\tag{5.23}
$$

Proof. Recall from the proof of Lemma 5.2 that (hB) holds with $\Theta_*^2 = M^2 K_B^2(\varepsilon^{-1} + \eta)$. The first estimate now follows directly from Proposition 4.2, while the second can be obtained from Proposition 3.1 using the fact that (hV) is satisfied with $\Lambda_* = \sqrt{\eta}$. □

Proof of Lemma 5.5. The bound follows immediately from the decomposition (5.19) and Lemmas 5.6-5.7. □

Proof of Proposition 5.1. Summing the estimates from Lemmas 5.2 and 5.5 yields the bound

$$
E\big[\sup_{0 \leq t \leq t_{\mathrm{st}}} N(t)\big] \leq K\Big[\|V(0)\|_{L^2}^2 + \sigma^2 \ln(T) + \sigma^2 + \sigma\sqrt{\eta}\sqrt{\ln(T)} + (\sigma^4 + \eta + \sqrt{\eta}) E\big[\sup_{0 \leq t \leq t_{\mathrm{st}}} N(t)\big]\Big]. \tag{5.24}
$$

Upon restricting the size of $\sigma^4 + \eta + \sqrt{\eta}$, the result readily follows. □

References

[1] L. Alili, P. Patie and J. L. Pedersen (2005), Representations of the first hitting time density of an Ornstein-Uhlenbeck process. *Stochastic Models* **21**(4), 967–980.

[2] J. Armero, J. Sancho, J. Casademunt, A. Lacasta, L. Ramirez-Piscina, and F. Sagués (1996), External fluctuations in front propagation. *Physical review letters* **76**(17), 3045.

[3] D. G. Aronson and H. F. Weinberger (1975), Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In: *Partial differential equations and related topics*. Springer, pp. 5–49.

[4] D. G. Aronson and H. F. Weinberger (1978), Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.* **30**(1), 33–76.

[5] H. Berestycki, F. Hamel and H. Matano (2009), Bistable traveling waves around an obstacle. *Comm. Pure Appl. Math.* **62**(6), 729–788.

[6] N. Berglund and B. Gentz (2013), Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers' law and beyond. *Electron. J. Probab.* **18**, 58 pp.

[7] W.-J. Beyn and V. Thümmler (2004), Freezing solutions of equivariant evolution equations. *SIAM Journal on Applied Dynamical Systems* **3**(2), 85–116.

[8] L. A. Bianchi, D. Blömker and P. Wacker (2017), Pattern size in Gaussian fields from spinodal decomposition. *SIAM Journal on Applied Mathematics* **77**(4), 1292–1319.

[9] G. Birzu, O. Hallatschek and K. S. Korolev (2018), Fluctuations uncover a distinct class of traveling waves. *Proceedings of the National Academy of Sciences* **115**(16), E3645–E3654.

[10] P. C. Bressloff and Z. P. Kilpatrick (2015), Nonlinear Langevin equations for wandering patterns in stochastic neural fields. *SIAM Journal on Applied Dynamical Systems* **14**(1), 305–334.

[11] P. C. Bressloff and M. A. Webber (2012), Front propagation in stochastic neural fields. *SIAM Journal on Applied Dynamical Systems* **11**(2), 708–740.

[12] M. Cartwright and G. A. Gottwald (2019), A collective coordinate framework to study the dynamics of travelling waves in stochastic partial differential equations. *Physica D: Nonlinear Phenomena.*

[13] S. Cerrai and M. Röckner (2004), Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipshitz reaction term. *The Annals of Probability* **32**(1B), 1100–1139.

[14] G. Da Prato, A. Jentzen and M. Röckner (2019), A mild Itô formula for SPDEs. *Transactions of the American Mathematical Society.*

[15] G. Da Prato, S. Kwapień and J. Zabczyk (1988), Regularity of solutions of linear stochastic equations in Hilbert spaces. *Stochastics: An International Journal of Probability and Stochastic Processes* **23**(1), 1–23.

[16] K. Eichinger, M. V. Gnann and C. Kuehn (2020), Multiscale analysis for traveling-pulse solutions to the stochastic FitzHugh-Nagumo equations. *arXiv preprint arXiv:2002.07234.*

[17] P. C. Fife and J. B. McLeod (1977), The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Ration. Mech. Anal.* **65**(4), 335–361.

[18] M. I. Freidlin and A. D. Wentzell (1998), Random perturbations. In: *Random perturbations of dynamical systems.* Springer, pp. 15–43.

[19] J. García-Ojalvo and J. Sancho (2012), *Noise in spatially extended systems.* Springer Science & Business Media.

[20] E. Gautier (2005), Uniform large deviations for the nonlinear Schrödinger equation with multiplicative noise. *Stochastic processes and their applications* **115**(12), 1904–1927.

[21] M. Hairer (2009), An Introduction to Stochastic PDEs. `http://www.hairer.org/notes/SPDEs.pdf`.

[22] C. H. S. Hamster and H. J. Hupkes (2018), Stability of Travelling Waves for Systems of Reaction-Diffusion Equations with Multiplicative Noise. *To Appear in SIAM Journal on Mathematical Analysis.*

[23] C. H. S. Hamster and H. J. Hupkes (2019), Stability of Traveling Waves for Reaction-Diffusion Equations with Multiplicative Noise. *SIAM Journal on Applied Dynamical Systems* **18**(1), 205–278.

[24] C. H. S. Hamster and H. J. Hupkes (2020), Travelling waves for reaction–diffusion equations forced by translation invariant noise. *Physica D: Nonlinear Phenomena* **401**, 132233.

[25] E. Hausenblas, T. A. Randrianasolo and M. Thalhammer (2020), Theoretical study and numerical simulation of pattern formation in the deterministic and stochastic Gray–Scott equations. *Journal of Computational and Applied Mathematics* **364**, 112335.

[26] J. Inglis and J. MacLaurin (2016), A general framework for stochastic traveling waves and patterns, with application to neural field equations. *SIAM Journal on Applied Dynamical Systems* **15**(1), 195–234.

[27] T. Kapitula (1997), Multidimensional Stability of Planar Travelling Waves. *Trans. Amer. Math. Soc.* **349**, 257–269.

[28] T. Kapitula and K. Promislow (2013), *Spectral and dynamical stability of nonlinear waves.* Springer.

[29] A. Karczewska (2005), Stochastic integral with respect to cylindrical Wiener process. *arXiv preprint math/0511512*.

[30] J. Krüger and W. Stannat (2017), A multiscale-analysis of stochastic bistable reaction–diffusion equations. *Nonlinear Analysis* **162**, 197–223.

[31] C. Kuehn (2019), Travelling Waves in Monostable and Bistable Stochastic Partial Differential Equations. *Jahresbericht der Deutschen Mathematiker-Vereinigung* pp. 1–35.

[32] R. Kuske, C. Lee and V. Rottschäfer (2017), Patterns and coherence resonance in the stochastic Swift-Hohenberg equation with Pyragas control: The Turing bifurcation case. *Physica D: Nonlinear Phenomena*.

[33] E. Lang and W. Stannat (2016), L^2-stability of traveling wave solutions to nonlocal evolution equations. *Journal of Differential Equations* **261**(8), 4275–4297.

[34] W. Liu and M. Röckner (2010), SPDE in Hilbert space with locally monotone coefficients. *Journal of Functional Analysis* **259**(11), 2902–2922.

[35] G. Lord and V. Thümmler (2012), Computing stochastic traveling waves. *SIAM Journal on Scientific Computing* **34**(1), B24–B43.

[36] L. Lorenzi, A. Lunardi, G. Metafune and D. Pallara (2004), Analytic semigroups and reaction-diffusion problems. In: *Internet Seminar*, Vol. 2005. p. 127.

[37] J. N. MacLaurin and P. C. Bressloff (2020), Wandering bumps in a stochastic neural field: A variational approach. *Physica D: Nonlinear Phenomena* p. 132403.

[38] H. Matano, Y. Mori and M. Nara (2019), Asymptotic behavior of spreading fronts in the anisotropic Allen–Cahn equation on $\mathbb{R}^n$. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* **36**(3), 585 – 626.

[39] C. Mueller and R. B. Sowers (1995), Random travelling waves for the KPP equation with noise. *Journal of Functional Analysis* **128**(2), 439–498.

[40] J. Pickands (1969), Asymptotic properties of the maximum in a stationary Gaussian process. *Transactions of the American Mathematical Society* **145**, 75–86.

[41] C. Prévôt and M. Röckner (2007), *A concise course on stochastic partial differential equations*, Vol. 1905. Springer.

[42] D. Revuz and M. Yor (2013), *Continuous martingales and Brownian motion*, Vol. 293. Springer Science & Business Media.

[43] M. Salins and K. Spiliopoulos (2019), Metastability and exit problems for systems of stochastic reaction-diffusion equations. *arXiv preprint arXiv:1903.06038*.

[44] L. Schimansky-Geier and C. Zülicke (1991), Kink propagation induced by multiplicative noise. *Zeitschrift für Physik B Condensed Matter* **82**(1), 157–162.

[45] T. Shardlow (2005), Numerical simulation of stochastic PDEs for excitable media. *Journal of computational and applied mathematics* **175**(2), 429–446.

[46] W. Stannat (2013), Stability of travelling waves in stochastic Nagumo equations. *arXiv preprint arXiv:1301.6378*.

[47] W. Stannat (2014), Stability of travelling waves in stochastic bistable reaction-diffusion equations. *arXiv preprint arXiv:1404.3853*.

[48] M. Talagrand (2006), *The generic chaining: upper and lower bounds of stochastic processes.* Springer Science & Business Media.

[49] M. Veraar and L. Weis (2011), A note on maximal estimates for stochastic convolutions. *Czechoslovak mathematical journal* **61**(3), 743.

[50] J. Viñals, E. Hernández-García, M. San Miguel and R. Toral (1991), Numerical study of the dynamical aspects of pattern selection in the stochastic Swift-Hohenberg equation in one dimension. *Physical Review A* **44**(2), 1123.

[51] A. I. Volpert, V. A. Volpert and V. A. Volpert (1994), *Traveling wave solutions of parabolic systems*, Vol. 140. American Mathematical Soc.

[52] K. Zumbrun and P. Howard (1998), Pointwise semigroup methods and stability of viscous shock waves. *Indiana University Mathematics Journal* **47**(3), 741–872.