Exercises for week 12

Hand in four out of the following nine exercises: 10, 11, 13, 14 from § 3 of the notes and the five exercises below.

19. Determine the number of isomorphism classes of complex elliptic curves having endomorphism ring $\mathcal{O}_D$ for
(a) $D = -164$;
(b) $D = -163$.

20. (a) Let E be a complex elliptic curve with endomorphism ring $\mathbf{Z}[i]$. Show that E is isomorphic to the curve $Y^2 = X^3 + X$.
(b) Let E be a complex elliptic curve with endomorphism ring $\mathbf{Z}[\zeta_3]$, where ζ_3 is a primitive third root of unity. Show that E is isomorphic to the curve $Y^2 = X^3 + 1$.

21. (a) Determine for which values of D the order $\mathcal{O}_D$ contains an element α of norm $\alpha\bar{\alpha} = 2$.
(b) Show that for these D the only lattice (up to homothety) having $\mathcal{O}(\Lambda) \cong \mathcal{O}_D$ is $\mathcal{O}_D$ itself.
(c) Define

$$E\colon Y^2 = X(X^2 + aX + b) \quad \text{and} \quad E'\colon Y^2 = X(X^2 - 2aX + a^2 - 4b).$$

Let $\psi\colon E \to E'$ be the standard 2-isogeny from § 4 of the notes. Suppose we have an isomorphism $E' \xrightarrow{\sim} E$ of the form

$$(x, y) \mapsto (u^{-2}x, u^{-3}y).$$

Show that we have either

$$a = 0, \quad \mathrm{End}(E) \cong \mathbf{Z}[i], \quad u = \pm 1 \pm i$$

or

$$a \neq 0, \quad a^2 = 8b, \quad u = \pm\sqrt{-2}$$

(d) * Can you determine $\mathrm{End}(E)$ in the second case?

22. Define E and E' as in Exercise 21.
(a) Prove that

$$j(E) = \frac{2^8(a^2 - 3b)^3}{b^2(a^2 - 4b)} \quad \text{and} \quad j(E') = \frac{2^4(a^2 + 12b)^3}{b(a^2 - 4b)^2}.$$

For which values of a^2/b are the curves E and E' isomorphic? Compute $j(E)$ in each of these cases.
(b) Determine the endomorphism ring for each of these three isomorphism classes.