FORMULAE: GENUS, DIMENSION

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Summary

Consider a subgroup Γ of $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$ of finite index. We will derive a formula for the genus of the modular curve $X(\Gamma)$, and a formula for the dimensions of the spaces of modular forms of weight k for such a group, where $k \in \mathbb{Z}$. This is done by establishing an isomorphism of vector spaces between the set of these forms and the sections of certain line bundles on the associated modular curve. These formulae depend only on very simple parameters of these groups. We will give a conceptual reason for this. For a concrete set of examples, namely $\Gamma = \Gamma_1(N)$, explicit calculations are provided. Finally, generalizations of the classical theory are briefly mentioned.

Note that all our techniques also apply for non-congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$: at this point, the theory for these groups is still the same as for congruence subgroups (this will change with the appearance of Hecke operators).

Notation

For a discrete group Γ in $\mathrm{SL}(2,\mathbb{R})$, we denote the compactification of the algebraization of the Riemann surface $\Gamma\backslash\mathfrak{H}$ by $X(\Gamma)$.

The *elliptic points* of $\Gamma\backslash\mathfrak{H}$ are the branch points of the map $\mathfrak{H} \to \Gamma\backslash\mathfrak{H}$. Equivalently, they are the image under this map of the points in $\mathfrak{H}$ with non-trivial stabilizer. The *order* of an elliptic point is the order of the stabilizer in $\mathrm{P}\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$, which is the same as the ramification index of the map $\mathfrak{H} \to \Gamma\backslash\mathfrak{H}$ in any element of the fiber of the elliptic point.

The *parabolic points* of are the points added in the process of compactification, that is, the points of $X(\Gamma)$ not already in $\Gamma\backslash\mathfrak{H}$. Careful consideration of this process reveals that these correspond to the set of orbits $\Gamma\backslash\mathbb{P}^1_{\mathbb{Q}}$ (see [3] for this). The set of these points can be visualized as the intersection of the closure of a fundamental domain with $\mathbb{P}^1(\mathbb{R})$. A fixed cusp can be sent to ∞ by an $\mathrm{SL}(2,\mathbb{R})$-transformation, so let us suppose that it in fact equals infinity. The stabilizer of this point in $\mathrm{P}\Gamma$ is then of the form $\left\langle \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right\rangle$. This h is called the *width* of the cusp: it can also be interpreted as the ramification index of the map $X(\Gamma) \to X(1)$ for the cusp in question. Around such a cusp at infinity, a local coordinate is given by $\tau \mapsto e^{2\pi i\tau/h}$. This maps a neighborhood of this cusp in the fundamental domain isomorphically to the punctured disc $\mathbb{D}^*$.

1. The curve $X(\Gamma)$ as a covering of $X(1)$; genus

Recall the following theorem:

Theorem 1.1 (Riemann-Hurwitz-Zeuthen-?). *Let $f : X \to Y$ be a non-constant morphism of curves over $\mathbb{C}$ of degree n. Then the number of points at which f ramifies is finite, and we have*

$$2g(X) - 2 = (2g(Y) - 2)n + \sum_{x \in X} e_x(f),$$

denoting the ramification index at a point $x \in X$ by $e_x(f)$.

Proof. This theorem is true for more general schemes, but over $\mathbb{C}$ there is a very simple proof: triangulate Y, making sure that the branch points of f are included among the vertices of the triangularization, then take the inverse image of this triangulation under f and compare to get the formula. □

The first step to deriving our formula is to determine the genus of the modular curve $X(\Gamma)$. The inclusion $\mathrm{P}\Gamma \subset \mathrm{PSL}(2, \mathbb{Z})$ gives a natural map

$$X(\Gamma) \xrightarrow{f_\Gamma} X(1)$$

of modular curves, of degree $[\mathrm{P}\Gamma(1) : \mathrm{P}\Gamma] =: n$ say.

Now the remarkable fact is the for the modular covering above, we can determine the ramification indices if we know

(1) the number and orders of the elliptic points of Γ;
(2) the number of parabolic points.

Indeed, consider the diagram below.

$$\begin{array}{ccc} & \mathfrak{H} & \\ \swarrow & & \searrow \\ \Gamma\backslash\mathfrak{H} & \longrightarrow & \Gamma(1)\backslash\mathfrak{H} \end{array}$$

The morphism $\mathfrak{H} \to \Gamma(1)\backslash\mathfrak{H}$ ramifies above two points only, namely the elliptic point P_2 of order 2 and the elliptic point P_3 of order 3. This means that the morphism $\Gamma\backslash\mathfrak{H} \to \Gamma(1)\backslash\mathfrak{H}$ also ramifies only above these two points. One also sees that the elliptic points of Γ have order either 2 or 3, and that if they have order i, they lie in the fiber of P_i. Moreover, a point in the fiber of P_i is unramified under this morphism if and only if it is elliptic of order i itself: otherwise it ramifies of order exactly i. So if we slightly abuse notation by denoting the number of elliptic points of order i by e_i, we have

$$\sum_{x \in \Gamma\backslash\mathfrak{H}} e_x(f_\Gamma) = \sum_{x \in f_\Gamma^{-1}(P_2)} e_x(f_\Gamma) + \sum_{x \in f_\Gamma^{-1}(P_3)} e_x(f_\Gamma) = \frac{n - e_2}{2} + \frac{2(n - e_3)}{3}.$$

Note that the argument above hinges on the facts that (1) the elliptic points of fixed order for $\Gamma(1)$ are unique, and (2) the only occuring orders are prime. These certainly hold true for $\Gamma(1)$, but or course they do not for more general Γ.

So we know the ramification behavior of $X(\Gamma) \to X(1)$ above all points except the cusp of $X(1)$. However, for that point we know the cardinality of the fiber, because it equals the number of parabolic points. This allows us to determine the remaining sum $\sum_{x \in f_\Gamma^{-1}(\mathrm{cusp})} e_x(f_\Gamma)$, (it equals $n - c$), and one obtains:

Theorem 1.2. *Let Γ be a finite index subgroup of $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$. Set $n = [\mathrm{P}\Gamma(1) : \mathrm{P}\Gamma]$, let c be the number of cusps of Γ, and for $i \in \{2,3\}$, let e_i equal the number of elliptic points of order i. Then we have*

$$g(X) = 1 + \frac{n}{12} - \frac{e_2}{4} - \frac{e_3}{3} - \frac{c}{2}.$$

2. From meromorphic modular forms to rational differential forms

This section will show another interpretations of meromorphic modular forms of even weight k. Recall that these were meromorphic functions f on $\mathfrak{H} \cup \mathbb{P}^1_{\mathbb{Q}}$ satisfying the transformation rule

$$f(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau) = (c\tau + d)^k f(\tau).$$

The action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathfrak{H}$ induces an action of $\mathrm{SL}(2,\mathbb{R})$ on the sheaf of meromorphic differential forms on $\mathfrak{H}$. The form $d\tau$ transforms by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* d\tau = (c\tau + d)^{-2} d\tau.$$

Since meromorphic differential $k/2$-form can be written as $f d\tau^{\otimes k/2}$ for a unique meromorphic function f on $\mathfrak{H}$, this means that the meromorphic modular forms of weight k correspond exactly to $k/2$-fold meromorphic differentials invariant under the action of Γ. In turn, these can be identified with the rational differential forms on the algebraic curve $X(\Gamma)$ (some work is needed to prove this near cusps and elliptic points as well). So:

Theorem 2.1. *Let $\Gamma \subset (\mathrm{P})\mathrm{SL}(2,\mathbb{R})$ be discrete, and let $k \in \mathbb{Z}$ be even. Denote by $M_k^{\mathrm{mer}}(\Gamma)$ the meromorphic modular functions of weight k with respect to Γ, and by $(\Omega^{\mathrm{rat}})^{\otimes k/2}(X(\Gamma))$ the $k/2$-fold rational differential forms on $X(\Gamma)$. Then the map*

$$\begin{array}{ccccc} M_k^{\mathrm{mer}}(\Gamma) & \longrightarrow & (\Omega^{\mathrm{rat}})^{\otimes k/2}(X(\Gamma)) & \cong & \mathcal{K} \otimes \Omega^{\otimes k/2}(X(\Gamma)) \\ f & \longmapsto & f d\tau^{\otimes k/2} & & \end{array}$$

defines a natural isomorphism of vector spaces.

3. Riemann-Roch; dimension

In this paragraph, we again consider finite index subgroups of $\Gamma(1)$. We will show that, like the genus of $X(\Gamma)$, the dimensions m_k of the vector spaces $M_n(\Gamma)$ of Γ-modular forms of weight k only depend on e_2, e_3, c, n and k (at least for even k, for odd k, there is a tiny snag). For this, we use the following

Theorem 3.1 (Riemann-Roch)**.** *Let X be a complex curve of genus g. Define the following spaces for a divisor D:*

$$\mathcal{L}(D) := \{f \text{ rational on } X \mid \mathrm{div}(f) + D \geq 0\}.$$

and set

$$l(D) := \dim \mathcal{L}(D).$$

Let K be "the" canonical divisor on K (that is, the divisor of some non-zero 1-form). Then one has

$$l(D) - l(K - D) = \deg D + 1 - g.$$

This implies $\deg K = 2g - 2$.

Theorem 3.2. *Let $k \in \mathbb{Z}$ be even. If $k > 0$, we have*

$$m_k(\Gamma) = (k-1)(g-1) + \frac{k}{2}c + \lfloor \frac{k}{4} \rfloor e_2 + \lfloor \frac{k}{3} \rfloor e_3.$$

Additionally, $m_0(\Gamma) = 1$, and $m_k(\Gamma)$ is zero for $k < 0$.

Proof. We have already seen that rational $k/2$-fold differential forms on $X(\Gamma)$ correspond to meromorphic modular forms of weight k. All that remains is a more precise analysis of the relations between the order of a modular form f at a certain point x and the order of the corresponding differential form $f dz^{\otimes k/2}$ pushed forward to $X(\Gamma)$ at the image of x.

By considering the projection morphism $\pi_\Gamma : \mathfrak{H} \to \Gamma\backslash\mathfrak{H}$ locally around x, one sees that

$$\mathrm{ord}_x(f) = e_x(\pi_\Gamma)\mathrm{ord}_{\pi_\Gamma(x)}(f d\tau^{\otimes k/2}) + \frac{k}{2}(e_x(\pi_\Gamma) - 1).$$

For the cusps, we have

$$\mathrm{ord}_x(f) = \mathrm{ord}_{\pi_\Gamma(x)}(f d\tau^{\otimes k/2}) + \frac{k}{2}.$$

This follows because the local parameters $q(\tau) = \exp(2\pi i \tau/h)$ from $\mathfrak{H} \cup \mathbb{P}^1_{\mathbb{Q}}$ to $\mathbb{D}$ used in the compactification have the property $dq = \mathrm{const} \cdot q d\tau$: because of the factor q, the orders of n-forms are increased by n under this map.

Our search, then, is for rational n-fold differentials ω on $X(\Gamma)$ with the following property:

$$\begin{array}{ll} e_x(\pi_\Gamma)\mathrm{ord}_x(\omega) + \frac{k}{2}(e_x(\pi_\Gamma) - 1) \geq 0 & \text{for } x \in \Gamma\backslash\mathfrak{H} \\ \mathrm{ord}_x(\omega) + \frac{k}{2} \geq 0 & \text{for } x \text{ a cusp} \end{array}.$$

If we fix some rational $k/2$-fold differential ω_0, then any such ω can be written as $\omega = h\omega_0$ for a unique rational function h. In terms of h, the above demands on ω are equivalent to

$$\begin{array}{ll} \mathrm{ord}_x(h) + \mathrm{ord}_x(\omega_0) + \frac{k}{2}(1 - \frac{1}{e_x(\pi_\Gamma)}) \geq 0 & \text{for } x \in \Gamma\backslash\mathfrak{H} \\ \mathrm{ord}_x(h) + \mathrm{ord}_x(\omega_0) + \frac{k}{2} \geq 0 & \text{for } x \text{ a cusp} \end{array}.$$

So we should have $\mathrm{div}(h) + D \leq 0$, where D is the integral divisor

$$D = \mathrm{div}(\omega_0) + \sum_{x \text{ a cusp}} \frac{k}{2}x + \sum_{x \in \Gamma\backslash\mathfrak{H}} \lfloor \frac{k}{2}(1 - \frac{1}{e_x(\pi_\Gamma)} \rfloor.$$

Now we use Riemann-Roch. One sees that, because Γ has a cusp, the $l(K - D)$ part disappears because the degree of $K - D$ is strictly negative, and after some calculation, we obtain our formula.

$M_0(\Gamma)$ corresponds to the ring of regular functions on $X(\Gamma)$, and these consist of the constants only.

We now quickly see that there are no non-trivial modular forms of even weight smaller than zero. Indeed, consider a form f of weight $-k$ say. Then take any non-trivial form g of positive weight l say with a zero. Then, on the one hand the modular form $f^l g^k$ should be constant, whereas on the other hand it should have a zero. This means that it is zero. So f should be zero anywhere that g is not, so almost everywhere: hence it is zero itself. □

Remark With some extra effort, one can also calculate the dimensions of the spaces of cusp forms. Exercise: note that $S_2(\Gamma)$ is isomorphic to $\Omega(X(\Gamma))$.

Remark The theorem generalizes quite easily to more general discrete subgroups of (P)SL$(2,\mathbb{R})$. In [4], page 45, this general form is given, even though Milne's preamble suggest that only Γ of finite index in $\Gamma(1)$ are considered. Sadly, the proof there is wrong (one needs to assume that Γ has a cusp, or enough elliptic points): a correct proof is given in [2].

The next step is to calculate the dimensions of the spaces of modular forms with odd weight. These spaces are non-trivial only if $-1 \notin \Gamma$. For such Γ, there are no elliptic points of order 2.

There is a technicality that has to be dealt with here, namely the concept of an irregular cusp. By definition of the width h of a cusp c, the stabilizer of c in $\mathrm{P}\Gamma$ is conjugate to some group $\left\langle \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right\rangle$. Now the stabilizer in Γ might be of the form $\left\langle -\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right\rangle$. Using the functional equation and developing f in a Fourier series in the local parameter $q = \exp 2\pi i\tau/h$, we see that f has strictly half-integral zero order at such a cusp. If this occurs, the cusp c is called *irregular*. If the stabilizer in Γ has any other form, the problem just mentioned does not occur, the order of f at the cusp is always integral, and the cusp c is called *regular*.

Our argument will depend on the existence of a non-zero meromorphic modular form for Γ. Such a form exists. Indeed, there always exists a form in $M_1^{\mathrm{mero}}(\Gamma)$ as long as $-I \notin \Gamma$. The reason for this as follows. Take a non-zero ω in $\Omega(X(\Gamma))$, and let x in $X(\Gamma)$. Then $\mathrm{div}(\omega) - 2(g-1)x$ defines an element of the $\mathrm{Jac}(X(\Gamma))$. Since this is a complex torus, there exists a degree zero divisor D and a rational function f on $X(\Gamma)$ such that

$$2D = \mathrm{div}(\omega) - 2(g-1)x + \mathrm{div}(f).$$

The differential form $f\omega$ corresponds to a modular form g, which one can check has even order at any point of $\mathfrak{H}$. Let g_1 be a function such that $g_1^2 = g$. Because g is invariant of weight 2 with respect to Γ, g_1 is a modular form of weight 1 with respect to a subgroup Γ_1 of Γ of index at most 2: the action of the complement of this subgroup will transform g_1 into $-g_1$. If this index is 1, we are done; otherwise, $\mathrm{P}\Gamma_1$ is also of index 2 in $\mathrm{P}\Gamma$ because $-I \notin \Gamma$. Then the rational function field of $X(\Gamma)$ also has index 2 in that of $X(\Gamma_1)$. By Galois theory, there is some rational function g_2 on $X(\Gamma_1)$ such that $g_1 g_2$ is invariant with respect to Γ. Indeed, this works if we choose g_2 such that the Galois group of the inclusion acts as multiplication by -1.

Now for a nice theorem.

Theorem 3.3. *Denote by c_r respectively c_i the number of regular respectively irregular cusps of Γ. Then for $k \in \mathbb{Z}_{\geq 3}$ odd we have*

$$m_k(\Gamma) = (k-1)(g-1) + \lfloor \frac{k}{3} \rfloor e_3 + \frac{k}{2} c_r + \frac{k-1}{2} c_i.$$

Furthermore, $m_n(\Gamma)$ is zero for $n < 0$. Finally, we have $m_1(\Gamma) \geq c_r/2$.

Proof. (Sketch) A similar argument to the one used in the previous theorem shows $m_n(\Gamma) = l(\lfloor \mathrm{div}(f) \rfloor)$, where f is an arbitrary meromorphic modular form for Γ. Note that $\mathrm{div}(f)$ is not necessarily an integral divisor: it might take rational values at the elliptic points and the irregular cusps. We do know that f^2 defines a differential form ω on $X(\Gamma)$, and one easily sees that

$$\operatorname{div}(f) = \frac{1}{2}\operatorname{div}(\omega) + \sum_{x \in \Gamma\backslash\mathfrak{H}} n(1 - \frac{1}{e_x(\pi_\Gamma)})x + \sum_{c_i \text{ regular}} \frac{n}{2}c_i + \sum_{c_i' \text{ irregular}} \frac{n}{2}c_i'.$$

(Note the abuse of notation.) One now has to take the entier of this divisor. The difference between the regular and the irregular cusps is the following. At the regular cusps, $\operatorname{div}(f)$ is integral, so at those points, the valuation of $\operatorname{div}(f)$ does not change under taking the entier. In contrast to this, at the irregular cusps the divisor $\operatorname{div}(f)$ has a value in $(1/2) + \mathbb{Z}$, so taking the entier decreases the valuation by $1/2$ at such a cusp. Finally, since all elliptic points have odd order, the valuation of $\operatorname{div}(\omega)$ at those points is even, so we need not round down that part of $\operatorname{div}(f)$. Using this, one finds

$$\lfloor \operatorname{div}(f) \rfloor = \frac{1}{2}\operatorname{div}(\omega) + \sum_{x \in \Gamma\backslash\mathfrak{H}} \lfloor n(1 - \frac{1}{e_x(\pi_\Gamma)}) \rfloor x + \sum_{c_i \text{ regular}} \frac{n}{2}c_i + \sum_{c_i' \text{ irregular}} \frac{n-1}{2}c_i'.$$

Applying Riemann-Roch gives the result. The other two statements of the theorem are simpler. □

Remark Note that the theorem implies that the number of regular cusps is always even.

Remark This result, too, can be generalized to other discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$.

For a less ad hoc treatment of the results in this paragraph, see [2].

4. The case $\Gamma = \Gamma_1(N)$

In this section, following [2], we will apply the previous theory to $\Gamma = \Gamma_1(N)$. Few books mention these formulae: the reason for this is that the group $\Gamma_0(N)$ is more interesting for the modularity theorem: the calculations for these groups are already in [5].

First the degree of $X_1(N) \to X(1)$. The covering $X(N) \to X(1)$ has is Galois with group $\mathrm{SL}(\mathbb{Z}/N\mathbb{Z})$. Since $\mathrm{P}\Gamma_1(N)$ has index N in $\mathrm{P}\Gamma(N)$, this means that

$$\deg(X_1(N) \to X(1)) = \frac{|\mathrm{SL}(\mathbb{Z}/N\mathbb{Z})|}{N} = \begin{cases} \frac{1}{2}N^2 \prod_{p|N}(1 - \frac{1}{p^2}) & \text{for } N > 2 \\ 6 & \text{for } N = 2 \end{cases}$$

As for the elliptic points, we know that $\Gamma(1)$ has unique elliptic points of order 2 and 3. The stabilizers of lifts of these elements are therefore all conjugate. The non-trivial elements of the stabilizers have trace 0 for the order 2 points, and trace in $\{\pm 1\}$ for the order 3 points. Elliptic points for $\Gamma_1(N)$ project to elliptic points for $\Gamma(1)$. But for $N > 3$, no elements of $\Gamma_1(N)$ can have trace in $\{-1, 0, 1\}$. Hence for $N > 3$, $\Gamma_1(N)$ has no elliptic points. For $N = 2$ and $N = 3$, there turns out to be only one elliptic point, of order 2 and 3, respectively; all non-trivial elliptic elements are conjugate to $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}$, respectively, although this is not extremely easy to see.

The cusps are in bijection with the orbit space $\Gamma_1(N)\backslash\mathbb{P}^1_{\mathbb{Q}}$. One checks that

$$\frac{a}{c} \sim_{\Gamma_1(N)} \frac{a'}{c'} \Leftrightarrow \begin{pmatrix} a \\ c \end{pmatrix} \equiv \pm \begin{pmatrix} a' \\ c' \end{pmatrix} \mod N.$$

For $N = 2$ and $N = 3$ this gives 2 cusps, for $N = 4$ it gives 4, and for bigger N the number of cusps equals $\frac{1}{2}\sum_{d|N}\varphi(d)\varphi(N/d)$. Now to see which of these cusps are regular. Irregularity at a cusp a/c can only occur if, after choosing some element α of $\mathrm{SL}(2,\mathbb{Z})$ sending ∞ to $\frac{a}{c}$, we have $\alpha^{-1}\mathrm{Stab}(\frac{a}{c})\alpha = \left\langle -\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\right\rangle$, where h is the width of the cusp: only then does the cardinality of the stabilizer equal twice the width of the cusp. Looking at the trace, we see that for an irregular cusp to occur, we need $N|4$. If $N = 2$, then $-I \in \Gamma_1(N)$ so that we have regularity, and if $N = 4$, there is one irregular cusp $\frac{1}{2}$.

In principle, we can plug in these values to obtain all the information about modular forms for $\Gamma_1(N)$ that we might want to know. Figures 3.3 and 3.4 in [2] summarize all that is known: they are not copied here.

5. Generalizations; volume

One can ask oneself whether for general discrete subgroups of (P)SL$(2,\mathbb{R})$, there are formulae similar to those in Theorems 1.2, 3.2 and 3.3. The author is not aware of the existence of such formulae in general. However, there is a class of subgroups of $\mathrm{SL}(2,\mathbb{R})$ that are very interesting and for which such formulae are available. These are the so-called *rational arithmetic subgroups*. These are more or less obtained by replacing $\mathrm{SL}(2,\mathbb{Z})$ by a maximal order in a quaternion algebra over $\mathbb{Q}$ that injects into $\mathrm{M}(2,\mathbb{R})$. Such an algebra has a discriminant D: the case $\mathrm{SL}(2,\mathbb{Z})$ corresponds to discriminant 1. For any D, there is a natural analogue $X_0^D(N)$ of the classical modular curve $X_0(N)$, and its genus is given by

$$g(X_0^D(N)) = 1 + \frac{\prod_{p|D}(p-1)\prod_{q|N}(q+1)}{12} - \frac{e_2}{4} - \frac{e_3}{3} - \frac{c}{2}.$$

Furthermore, the e_i allow expressions in terms of certain class numbers of the quaternion algebra B. Note the similarity with our genus formula. This formula can most naturally be established using theorems on reduction of of Deligne and Rapoport. For an overview of this subject, see the notes on Shimura curves at [1].

Formulae for the dimensions of the spaces of modular forms can also be derived in this case, at least for k big enough. For $N = 1$, these functions do not have q-expansions anymore, because this time, $X^D(1)$ does not have any cusps!

The most general genus formula is

$$g(X(\Gamma)) = 1 + \frac{1}{4\pi}V(\Gamma) - \sum_{n\in\mathbb{N}} \frac{n-1}{n}\frac{e_n}{2} - \frac{c}{2},$$

where $V(\Gamma)$ is the hyperbolic area of a fundamental domain for Γ: this value is in general hard to determine. Again, note the similarities with our previous genus formula.

References

[1] P.L. Clark. Shimura curves. Notes available at `http://math.uga.edu/~pete/expositions.html`.

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[5] G. Shimura. *Introduction to the Arithmetic Theory of Automorphic Functions*. Princeton University Press (or Iwami Shōten if you are Japanese), 1971.