QUADRATIC FORMS AND THETA FUNCTIONS

Abstract. My notes for a talk on modular forms given on October 10, 2007 in Leiden.

Introduction

The following two facts about modular forms are not at all clear from the definition:

(1) Modular forms exist;
(2) $\dim M_k(\Gamma)$ is finite.

In fact, a stronger version of (1) is true: *interesting* modular forms exist. Think of those forms associated with elliptic curves over $\mathbb{Q}$, Galois representations, certain partition functions, and–and this is the topic of this talk–with integral quadratic forms.

The linear pigeon hole principle states that if $n+1$ pigeons flock together in an n-dimensional vector space, then they satisfy a linear relation. In this talk we will find 4 interesting modular forms, elements of the same 3-dimensional vector space.

1. $\theta(z)^8 \in M_4(\Gamma_0(4))$

Define the following function of $q = \exp(2\pi i z)$:

$$\theta(z) := \sum_{d\in\mathbb{Z}} q^{d^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + \cdots$$

Proposition. $\theta(z/(4z+1)) = \sqrt{4z+1}\cdot\theta(z)$

Lemma (Poisson summation). *$\phi : \mathbb{R} \to \mathbb{C}$ continuous and rapidly decreasing, put*

$$\hat{\phi} : \mathbb{R} \to \mathbb{C} : t \mapsto \hat{\phi}(t) := \int_{-\infty}^{\infty} \phi(x)\exp(-2\pi i t x)dt$$

its Fourier transform, then $\sum_{n\in\mathbb{Z}} \phi(n) = \sum_{k\in\mathbb{Z}} \hat{\phi}(k)$.

Proof of the Lemma. The proof is easy. Define $\psi(x) := \sum_{n\in\mathbb{Z}} \phi(x+n)$. Then $\psi(x)$ is periodic hence has a Fourier series expansion:

$$\psi(x) = \sum_{k\in\mathbb{Z}} c_k \exp(2\pi i k x) \text{ where } c_k := \int_0^1 \psi(x)\exp(-2\pi i k x)dx.$$

Now we have:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \phi(n) = \psi(0) = \sum_{k\in\mathbb{Z}} c_k &= \sum_{k\in\mathbb{Z}} \int_0^1 \psi(x)\exp(-2\pi i k x)dx \\
&= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \phi(x)\exp(-2\pi i k x)dx \\
&= \sum_{k\in\mathbb{Z}} \hat{\phi}(k)
\end{aligned}$$

□

Proof of the Proposition. Take $\phi(x) = e^{-2\pi i z x^2}$. This is 'rapidly decreasing' if the imaginary part of z is positive. Then $\hat{\phi}(t) = (-2iz)^{-1/2} e^{-\pi i t^2/2z}$. (Recall that $e^{-\pi x^2}$ is its own Fourier transform.) Poisson summation gives

$$\sum_n e^{-2\pi i z n^2} = (-2iz)^{-1/2} \sum_k e^{-\pi i k^2/2z}$$

or in other words:

$$\theta(z) = (-2iz)^{-1/2}\theta(-1/4z).$$

This gives the transformation behavior for $z \mapsto -1/4z$. Together with the invariance under $z \mapsto z+1$ we can calculate the transformation behavior for

$$z \mapsto \frac{-1}{4z} \mapsto \frac{-1-4z}{4z} \mapsto \frac{z}{4z+1}$$

and verify the correctness of the Proposition. □

The transformations $z \mapsto z+1$ and $z \mapsto \frac{z}{4z+1}$ generate the group $\Gamma_0(4)$, so it follows that:

Proposition. $\theta(z)^8 \in M_4(\Gamma_0(4))$

We have

$$\theta(z)^8 = 1 + 16q + 112q^2 + 448q^3 + 1136q^4 + \cdots$$

where the coefficient of q^n is of course the number of vectors of length square root of n in the standard eight dimensional lattice.

2. Three More Elements of $M_4(\Gamma_0(4))$

They are:

$$\begin{aligned} G_4(z) = \frac{1}{240} + \sum_{n>0}\sigma_3(n)q^n &= \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + \cdots \\ G_4(2z) &= \frac{1}{240} + q^2 + 9q^4 + \cdots \\ G_4(4z) &= \frac{1}{240} + q^4 + \cdots \end{aligned}$$

3. $\dim_{\mathbb{C}} M_4(\Gamma_0(4)) = 3$

See the picture of the fundamental domain. The quotient $Y(\Gamma)$ of H by $\Gamma_0(4)$ has genus 0 and three cusps. Also, the quotient is a *free* quotient by $P\Gamma_0(4) = \Gamma_0(4)/\{\pm 1\}$, *i.e.* there are no elliptic points. (In fact, the group $P\Gamma_0(4)$ is torsion-free, for if it would contain some nontrivial torsion element $\pm\gamma$ then the trace of γ would be 0 or ± 1, but traces of elements of $\Gamma_0(4)$ are 2 modulo 4.)

(From the freeness it follows that $P\Gamma_0(4)$ is the fundamental group of the Riemann sphere minus three points: a free group on two generators.)

The freeness of the action is *not* in contradiction with the fact that $Y(\Gamma_0(4))$ does not carry a universal elliptic curve: elliptic curves with $\Gamma_0(4)$-structure still have the nontrivial automorphism -1.

Proposition. $\dim M_4(\Gamma_0(4)) = 3$.

We could of course just plug in the details into the big formula giving the dimension of a space of modular forms, but that formula is one that I cannot remember. It is the method by which it is deduced rather that I can remember.

Sketch of proof. We identify M_4 with a subset of the set of meromorphic invariant forms of degree 2 on $\mathbb{H}$:

$$\begin{aligned} M_4(\Gamma_0(4)) &= \{f \in \mathbb{A}_4(\Gamma_0(4)) : v_P(f) \geq 0 \text{ and } v_C(f) \geq 0\} \\ &= \{\omega : v_P(f) \geq 0 \text{ and } v_C(f) \geq -2\} \\ &= \{h\omega_0 : \operatorname{div}(h) \geq -2s_1 - 2s_2 - 2s_3 - \operatorname{div}(\omega_0)\} \\ &= \{h : \operatorname{div}(h) \geq D\} \end{aligned}$$

for some D of degree -2. (Note that the degree of the divisor of a differential 1-form on the Riemann sphere is -2, hence that the degree of the divisor of any degree 2 form ω_0 is -4).

Since on the projective line all divisors of same degree are equivalent, we can take D to be -2∞ and we get the space of polynomials of degree at most 2, which is of dimension 3. □

4. An Elementary Solution of the Eight Squares Problem

The eight squares problem is the problem of finding the number of vectors of square length n in the lattice $\mathbb{Z}^8$. An elementary solution is one that uses only the five fundamental operations of arithmetic: addition, subtraction, multiplication, division and modular form (Eichler.)

Theorem. $\theta(z)^8 = 16G_4(z) - 32G_4(2z) + 256G_4(4z)$.

Proof. Apply the linear pigeon hole principle. □