Reduction of hyperelliptic curve equations

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`http://bit.ly/sdhyp`

Sage Days: Algorithms in Arithmetic Geometry
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Elliptic curves

Elliptic curve (over k, for simplicity $\mathrm{char}(k) \nmid 6$):

$$E : y^2 = x^3 + ax + b, \qquad (4a^3 + 27b^2 \neq 0).$$

j-invariant:

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2},$$

$$E \cong_{\overline{k}} E' \iff j(E) = j(E').$$

Given $j \in k$, let

$$E_j : y^2 = x^3 + ax + a, \qquad \text{where } a = \frac{27}{4} \frac{j}{(1728 - j)},$$

$$E_0 : y^2 = x^3 - 1, \quad E_{1728} : y^2 = x^3 + x,$$

then

$$j(E_j) = j.$$

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Example

$$j(E) = 1728\frac{4a^3}{4a^3 + 27b^2}$$

$$E_j : y^2 = x^3 + ax + b \qquad a = b = \frac{27}{4}\frac{j}{(1728 - j)}$$

$$E : y^2 = x^3 + 2x + 3$$
$$\rightsquigarrow j(E) = 55296/275$$
$$\rightsquigarrow E_j : y^2 = x^3 + 8/9x + 8/9,$$

$$E \to E_j \quad : \quad (x, y) \mapsto (\frac{2}{3}x, \sqrt{\frac{2}{3}}y).$$

```
sage: EllipticCurve_from_j(55296/275)
Elliptic Curve defined by y^2 = x^3 + 2*x - 3 over
 Rational Field
```

Motivation

genus 1

- There exist exactly 13 elliptic curves over $\mathbf{Q}$ with complex multiplication (CM).
- To compute them: evaluate j on imaginary quadratic integers, then compute E with given j.

genus 2

- van Wamelen (1999): numerical search for curves of genus two over $\mathbf{Q}$ with CM.
- Found 19, but all with very special CM-field (Galois C_4).
- For the generic case (Weyl CM-field, quartic D_4), there is a theoretical obstruction, need curves over quadratic fields.

Hyperelliptic curves (char$(k) \neq 2$)

- A hyperelliptic curve C of genus $g \geq 2$ over k is (a smooth curve birational to) a curve of the form $y^2 = f(x)$ with $f(x) \in k[x]$ separable of degree $2g+1$ or $2g+2$.
- Every curve of genus two is hyperelliptic.
- $C = C_F : Y^2 = F(X,Z) \subset \mathbf{P}^{(1,g+1,1)}$, where

$$F(X,Z) = Z^{2g+2} f(X/Z)$$

  is a binary form of degree $2g+2$.
- $f(x) = F(x,1)$
- Fractional-linear transformations: for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(k)$,

$$F \circ A = F(aX + bZ, cX + dZ) \qquad f \cdot A = f\left(\frac{ax+b}{cx+d}\right).$$

- $C_F \cong C_{F'} \quad \Leftrightarrow \quad F' \sim_{\mathrm{GL}_2(k)} F$

Igusa invariants (char$(k) \neq 2, 3, 5$)

General g:

- Over $\overline{k}$, $C \cong y^2 = x(x-1)(x-\lambda_1)\cdots(x-\lambda_{2g-1})$.
- Need a $2g-1$-dimensional space instead of the j-line.

$g = 2$:

- Igusa-Clebsch invariants: polynomials I_2, I_4, I_6, I_{10} in the coefficients of F.
- e.g. $I_{10}(C) = \Delta(F) \neq 0$.
- $C \cong_{\overline{k}} C' \quad \Leftrightarrow \quad I_n(C) = u^n I_n(C')$ for some $u \in \overline{k}^*$.
- Given (x_2, x_4, x_6, x_{10}) with $x_{10} \neq 0$, $\exists C/\overline{k}$ with $I_n(C) = x_n$.
- Moduli space $\mathcal{M}_2 : (x_{10} \neq 0) \subset \mathbf{P}^{(2,4,6,10)}$ (three-dimensional)

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- Absolute Igusa invariants ($\exists$ many choices, this one by Kohel)

$$i_1 = \frac{I_4 I_6}{I_{10}}, \quad i_2 = \frac{I_2^3 I_4}{I_{10}}, \quad i_3 = \frac{I_2^2 I_6}{I_{10}}$$

  generate the function field of $\mathcal{M}_2$.
- We use "better" absolute invariants from arXiv:0903.4766

Just a little bit more complicated than the j-invariant

Given $f = a_6 \prod_{i=1}^{6}(X - \alpha_i)$, let

$$I_{10} = a_6^{10} \prod_{i<j} (\alpha_i - \alpha_j)^2 = \Delta(F).$$

Use (ij) to denote $(\alpha_i - \alpha_j)$ and take sums over the S_6-orbit of the given expression in $\mathbf{Q}[\alpha_i]$:

$$I_2 = a_6^2 \sum_{15 \text{ terms}} (12)^2(34)^2(56)^2 = a_6 \sum_{\substack{\{f_1,f_2,f_3\} \\ f = a_6 f_1 f_2 f_3 \\ \text{monic quadratic}}} \Delta(f_1)\Delta(f_2)\Delta(f_3),$$

$$I_4 = a_6^4 \sum_{10 \text{ terms}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2 = \sum_{\substack{\{f_1,f_2\} \\ f = a_6 f_1 f_2 \\ \text{monic cubic}}} \Delta(f_1)\Delta(f_2),$$

$$I_6 = a_6^6 \sum_{60 \text{ terms}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2,$$

Sage (fortunately)

```
sage: P.<x> = QQ[]
sage: C = HyperellipticCurve(x^6 + 25*x^2 + 7*x + 2013)
sage: ic = C.igusa_clebsch_invariants(); ic
(-7729920, 1680707527680, -4005339745316290560, -161890
29906296894815815559808)
sage: C.a
C.absolute_igusa_invariants_kohel    C.ambient_space
C.absolute_igusa_invariants_wamelen  C.an_element
C.affine_patch                       C.arithmetic_genus
C.algebra
sage: C.absolute_igusa_invariants_kohel()
(21399830693782118373636600/5146354009722676218 61,
 246772053462700544993280000/514635400972267621861,
 760795947676807865729280000/514635400972267621861)
```

From invariants to curves

- Given $j \in k$, there exists an elliptic curve E_j/k with $j(E_j) = j$:

$$E_j : y^2 = x^3 + ax + b \qquad a = b = \frac{27}{4}\frac{j}{(1728 - j)}$$

- Given $p = (x_2 : x_4 : x_6 : x_{10}) \in \mathcal{M}_2(k)$, there exists a hyperelliptic curve of genus two $C/\overline{k}$ with $(I_n(C))_n = p$. Over k? Can we construct it?
- Answer: Mestre's algorithm (1991)
  Assume $\mathrm{Aut}(C_{\overline{k}}) = \{1, \iota\}$, where $\iota : (x, y) \mapsto (x, -y)$
  (Otherwise yes and yes by Cardona-Quer (2005)).

From invariants to curves

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  (Otherwise yes and yes by Cardona-Quer (2005)).
- Given p, Mestre constructs a conic M/k and an effective divisor $\{P_1, \ldots, P_6\}$ over k of degree 6 on it such that
  - If $M(k) = \emptyset$, then C has no model over k.
  - If $\varphi : M \to \mathbf{P}^1$, let $f \in k[x]$ be the polynomial with roots $\phi(P_i) \in \overline{k} \subset \mathbf{P}^1(\overline{k})$. Then C is given by $y^2 = f(x)$ over k.
- End of story?
  Theoretically yes, practically no: horrible coefficients.
- van Wamelen solved this problem for his 19 curves, but not good enough for ours.

Horrible coefficients

Recall $C : y^2 = x^6 + 25x^2 + 7x + 2013$.
Apply Mestre's algorithm to its invariants to get

$$\begin{aligned} y^2 = &- 1972203849035705412586930258501404091374954717860141395699843878352911882861 63x^6 \\ &+ 7705096870070252252649995584659824094013182570280377424236459475407947725467909x^5 \\ &- 12548035615983590686979690415918344006067750716794600249390800630870915935531 5380x^4 \\ &+ 10903218830722891238527500202715331554871107169030234713155025081481950839341 13770x^3 \\ &- 53313845497441997644252885680774644799827883669038986051240531767740110429229 52195x^2 \\ &+ 13909648993550394785876798246155505974982100319395126989813089011028042418855639505x \\ &- 15129442990476714375403330597363303309852007272814366545312402484436587044868908694 \end{aligned}$$

But wait those coefficients have a gcd...

Horrible coefficients

Recall $C : y^2 = x^6 + 25x^2 + 7x + 2013$.
Apply Mestre's algorithm to its invariants to get

$$\begin{aligned} y^2 = &-6091327792665873x^6 \\ &+ 237978800887088439x^5 \\ &- 3875572909381249980x^4 \\ &+ 33675565497741734670x^3 \\ &- 164664575100209805345x^2 \\ &+ 429611936626468175355x \\ &- 467286364036379202674 \end{aligned}$$

Goal: given a hyperelliptic curve C/k, find a "small" hyperelliptic curve C'/k with $C \cong_{\overline{k}} C'$.
Note: this example is very tame, conic-solving works much better over $\mathbf{Q}$ than over number fields!

Models over k and transformations over $\overline{k}$

Recall $C_F : y^2 = F(X, Z)$.
Given $[A, u] = [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), u] \in \mathrm{GL}_2(k) \times k^*$, let

$$F' = F \cdot [A, u] := u \cdot F \circ A.$$

Then $C_{F'} \cong_{\overline{k}} C_F$
(i.e., $y^2 = F(X, Z) \quad \cong \quad y^2 = uF(aX + bZ, cX + dZ)$)

In fact (assuming $\mathrm{Aut}(C_{\overline{k}}) = \{1, \iota\}$), given any pair $F, F' \in k[X, Z]$, we have

$$C_{F'} \cong_{\overline{k}} C_F \quad \Longleftrightarrow \quad F' \in F \cdot (\mathrm{GL}_2(k) \times k^*).$$

So
Goal: given F, find a "small" element of its $(\mathrm{GL}_2(k) \times k^*)$-orbit.

Define "small"....

For k a number field, "small" means

1. coefficients in the integers $\mathcal{O}_k$,
2. ...
3. coefficient height ($\max\{|a_i|_v : i, \text{arch. val.} v\}$) "small" (sorry).

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- For simplicity: class number one.
- Then
  1. is achieved by scaling,
  2. we deal with locally,
- Given 1. and 2., the only freedom left is the action of the subgroup $\mathrm{GL}_2(\mathcal{O}_k) \times 1$, which will be step 3.

2. Local reduction of Δ

- Let k be a field with discrete valuation v.
- Given F, we want a v-integral element of the orbit $F \cdot (\mathrm{GL}_2(k) \times k^*)$ with $v(\Delta)$ minimal.
- If $v(\Delta) > 0$, then there is a multiple root in $\mathbf{P}^1$ mod v. $\rightsquigarrow$ generic p-adic stuff, or...

Proposition (B-S)? Suppose f is integral, degree $2g+1$ or $2g+2$, with $\mathrm{Aut}((C_f)_{\overline{k}}) = \{1, \iota\}$. Let π be uniformizer of v.
Then F is *non*-minimal at $v \iff$ one of the following holds

1. f is non-primitive, i.e., $h := f/\pi$ is integral.
2. (f mod π) has a $(g+2)$-fold root $\overline{t}$ mod π. Moreover, for every lift t, $h := f(\pi X + t)/\pi^{g+2}$ is integral.
3. $\deg(f \bmod \pi) \leq g$ Moreover, $h := f(x/\pi)\pi^g$ is integral.

In each case, we have $v(\Delta(h)) < v(\Delta(f))$.

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In each case, we have $v(\Delta(h)) < v(\Delta(f))$.

$$\gcd(\overline{f}, \overline{f}', \overline{f}'', \ldots, \overline{f}^{(g+1)}) = \begin{cases} (x - \overline{t})^n & 1 \leq n \leq g+1, \text{ if } \overline{t} \text{ exists,} \\ 1 & \text{otherwise.} \end{cases}$$

- $\overline{t}$ is $-n^{-1}$ times the coefficient of x^{n-1}.
- 1., 2., 3., easy recursion!

2. Global reduction of Δ

Let k be a number field. If we know a factorization of $\Delta(F)$ into principal ideals, then the local algorithm yields a global algorithm. But...

- If there are non-principal ideals?
- Factoring

2. Global reduction of Δ

Let k be a number field. If we know a factorization of $\Delta(F)$ into principal ideals, then the local algorithm yields a global algorithm. But...

- If there are non-principal ideals?
  Then just make $\Delta(F)$ "almost minimal".
- Factoring seems to be essential:
  - Let p and q be large unknown primes. Given $n = p^2 q$, take
    $$F = n^2 X^6 + XY^5 + Y^6$$
    with $\Delta(F) = (5^5 - 6^6 n^2) n^8$.
  - Let $G(X, Y) = F(X, pY)/p^4 = q^2 X^6 + pXY^5 + p^2 Y^6$
    with $\Delta(G) = (5^5 - 6^6 p^4 q^2) p^6 q^8 = \Delta(F)/p^{10}$.
    For most p and q, G is the minimal model.
  - If we can compute the minimal model G, then we can find $p = \sqrt[10]{\Delta(F)/\Delta(G)}$ and factor n.

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  - Let $G(X, Y) = F(X, pY)/p^4 = q^2 X^6 + pXY^5 + p^2 Y^6$ with $\Delta(G) = (5^5 - 6^6 p^4 q^2) p^6 q^8 = \Delta(F)/p^{10}$. For most p and q, G is the minimal model.
  - If we can compute the minimal model G, then we can find $p = \sqrt[10]{\Delta(F)/\Delta(G)}$ and factor n.
- So we spent hours factoring using Magma, GMP-ECM, and Cado-NFS,

2. Global reduction of Δ

- Factoring seems to be essential: (in worst case!)
  - Let p and q be large unknown primes. Given $n = p^2 q$, take

    $$F = n^2 X^6 + XY^5 + Y^6$$

    with $\Delta(F) = (5^5 - 6^6 n^2) n^8$.
  - Let $G(X, Y) = F(X, pY)/p^4 = q^2 X^6 + pXY^5 + p^2 Y^6$ with $\Delta(G) = (5^5 - 6^6 p^4 q^2) p^6 q^8 = \Delta(F)/p^{10}$. For most p and q, G is the minimal model.
  - If we can compute the minimal model G, then we can find $p = \sqrt[10]{\Delta(F)/\Delta(G)}$ and factor n.
- So we spent hours factoring using Magma, GMP-ECM, and Cado-NFS,
- but we did not need to!

Without factoring

- All we need to do is, given f and $\mathfrak{a}$
  - make $\Delta(f)$ smaller at the primes dividing $\mathfrak{a}$ without affecting other primes,
  - find a non-trivial factor of $\mathfrak{a}$, or
  - prove f is minimal at all primes dividing $\mathfrak{a}$.
- We can't: we don't know how to factor $n = p^2 q$.
- Still....
  - Trial division
  - Recognize pure powers
  - Pretend $\mathfrak{a}$ is prime:
    - Recall $\gcd(\overline{f}, \overline{f}', \overline{f}'', \ldots, \overline{f}^{(g+1)}) = (x - \overline{t})^n$
    - Euclid, division $\rightsquigarrow \overline{t}$ or non-trivial factor of $\mathfrak{a}$
    - If $\mathfrak{a} = (\pi)$ is square-free, use $f(\pi x + t)/\pi^{g+2}$.
- Works very well in practice.

2. Example

$f =$

$-6091327792665873x^6 + 237978800887088439x^5 - 3875572909381249980x^4 + 33675565497741734670x^3 - 164664575100209805345x^2 + 429611936626468175355x - 467286364036379202674$

Discriminant $\Delta(f) = -2^{20}3^{61}\cdot$

$73471086592869851303644427746893403889666267546667046818392909627451877729118440209278238621067233680094429667618339707628861588789$

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$7347108659286985130364442774689340388966626754666704681839290962745187772911844020927823862106723368009442966761833970762886158878 9$

gcd of Igusa invariants: $2^8 3^{14} 72051973608026572862444559$

Finding the prime factors 2 and 3 is easy. Let's assume we have dealt with those.

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gcd of Igusa invariants: $2^8 3^{14} 720519736080265728 6244559$

Finding the prime factors 2 and 3 is easy. Let's assume we have dealt with those.

$f =$

$$-16711461708274x^6 + 309734081346717x^5 - 2393068441397265x^4 + 9865584275416110x^3 - 22888496673941935x^2 + 28335144240596681x - 14625260694982851$$

gcd of Igusa invariants: $3 \cdot 7205197360802657286244559$

2. Example

$f =$

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gcd of Igusa invariants: $3 \cdot 7205197360802657286244559$

Let $a = 7205197360802657286244559$, not a perfect power, no obvious prime factors.

Work over $R = \mathbf{Z}/a\mathbf{Z}$, let $g = (f \bmod a) \in R[x]$.
Full degree.
So we compute $\gcd(g, g')$ by Euclid's algorithm:
$g = q_1 g' + r_1$ for some r_1 of degree ≤ 4
$g' = q_2 r_1 + r_2$? Wait, the leading coefficient of r_1 has a factor 22061809 in common with a.

2. Example

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gcd of Igusa invariants: $3 \cdot 720519736080265728 6244559$

Let $a = 22061809$ (and save $7205197360802657286244559/22061809$ for later)

Perfect power: $22061809 = 4697^2$

2. Example

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gcd of Igusa invariants: $3 \cdot 72051973608026572862444559$

Let $a = 22061809$ (and save $72051973608026572862444559/22061809$ for later)

Perfect power: $22061809 = 4697^2$

Let $a = 4697$, not a perfect power, pretend for talk no obvious prime factors.

2. Example

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$$-16711461708274x^6 + 3097340813467 17x^5 - 2393068441397265x^4 + 9865584275416110x^3 - 22888496673941935x^2 + 28335144240596681x - 14625260694982851$$

gcd of Igusa invariants: $3 \cdot 72051973608026572862 44559$

Let $a = 4697$, not a perfect power, pretend for talk no obvious prime factors.

Work over $R = \mathbf{Z}/a\mathbf{Z}$, let $g = (f \bmod a) \in R[x]$.
Full degree:

$$g = 2670x^6 + 2183x^5 + 3757x^4 + 1088x^3 + 3429x^2 + 4030x + 4014$$

So we compute $\gcd(g, g')$ etc.
Get $\gcd(g, g', g'', g''') = 1004x^3 + 4161x^2 + 925x + 1831$.
So $\bar{t} = -4161/1004/3 = 1360 \in R$.
Take $f \rightsquigarrow f(ax + 1360)/a^4$.

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gcd of Igusa invariants: $3 \cdot 72051973608026572862444559$
Get $\gcd(g, g', g'', g''') = 1004x^3 + 4161x^2 + 925x + 1831$.
So $\bar{t} = -4161/1004/3 = 1360 \in R$.
Take $f \rightsquigarrow f(ax + 1360)/a^4$.

$f =$

$$-368685076318754707666x^6 - 639054061873020370731x^5 - 461538994949637577665x^4 - 177777964512478115170x^3 - 38518555013347270015x^2 - 4451032576435902303x - 214308954883203411$$

gcd of Igusa invariants: $3 \cdot 32659141237251475 1$ And these coefficients have gcd 22061809, divide by that.

$f =$

$$-16711461708274x^6 - 28966530436059x^5 - 20920269727185x^4 - 8058177120130x^3 - 1745938196335x^2 - 201752837967x - 9714024579$$

gcd of Igusa invariants: 3, so no further reduction possible.

2. If you don't want to twist

Recall:

$$
\begin{array}{llll}
1. & f(x) & \rightsquigarrow & f(x)/\pi, \\
2. & f(x) & \rightsquigarrow & f(\pi x + t)/\pi^{g+2}, \\
3. & f(x) & \rightsquigarrow & f(x/\pi)\pi^{g},
\end{array}
$$

2. If you don't want to twist

Recall:

$$\begin{array}{llll} 1. & y^2 = f(x) & \rightsquigarrow & y^2 = f(x)/\pi, \\ 2. & y^2 = f(x) & \rightsquigarrow & y^2 = f(\pi x + t)/\pi^{g+2}, \\ 3. & y^2 = f(x) & \rightsquigarrow & y^2 = f(x/\pi)\pi^{g}, \end{array}$$

- 1. is a quadratic twist by π
- 2. and 3. are quadratic twists by π if g is odd and are isomorphisms over k if g is even.
- If you want an isomorphism over k, then after finishing the reduction do

  $$f \quad \rightsquigarrow \quad \pi^k f,$$

  where $k \in \{0, 1\}$ is $\#1. + g \cdot (\#2. + \#3.)$ modulo 2.

3. van Wamelen's approach (1999)

Next, suppose $k = \mathbf{Q}$ and that we have an integral model $y^2 = F(X, Y)$ with globally minimal $\Delta(F)$.

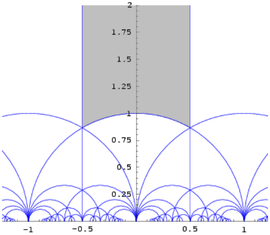
- The globally minimal models are the $\mathrm{GL}_2(\mathbf{Z})$-orbit of F, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ does not change the size of the model.
- $\mathrm{SL}_2(\mathbf{Z}) = \langle S, T \rangle$, where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- Let W be a set of small words in S, T, T^{-1}.
  Given f, find the smallest element of $F \cdot W$ (and start over with that new f).
- Works reasonably well, despite danger of local minima.
- We could not make it work for quadratic fields.

3. Stoll-Cremona reduction over $\mathbf{Q}$ (2003)

Same problem: find minimal element of $\mathrm{SL}_2(\mathbf{Z})$-orbit of F.
Idea: use a covariant $z = z(f) \in \mathcal{H}$,
i.e.,

$$z(f \cdot A^{-1}) = A \cdot z = \frac{az+b}{cz+d}, \quad \text{for all} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R}).$$

Call f reduced iff $z(f)$ is, i.e., iff $z(f) \in \mathbf{F}$.



Remark: we restrict to separable even degree binary forms over $\mathbf{R}$, they do more general arbitrary degree binary forms over $\mathbf{R}$ and $\mathbf{C}$.

3. What is z?

Multiple covariants exist. Most notably z_0 and z:

- $z_0(F)$ is the root in $\mathcal{H}$ of
$$\sum_{j=1}^{2g+2} \left|f'(\alpha_i)\right|^{-1/g} (x-\alpha_i)(x-\overline{\alpha_i}).$$
(easy to implement, fast to evaluate)
- "*The representative point $z(F)$ is the unique point in upper half-space such that the sum of its distances from all the roots of F is minimal.*" [Prop. 5.3 in Stoll-Cremona]
(very natural, yields better reduction in practice)

Stoll implemented both in Magma, we implemented z_0 in Sage and use Magma for z.

3. Example

$f = -16711461708274x^6 - 28966530436059x^5 - 20920269727185x^4 - 8058177120130x^3 - 1745938196335x^2 - 201752837967x - 9714024579$

$z_0 = -0.288883100864693 + 0.000138756004820783i$

$z_0 \mapsto -1/z_0, \quad f \mapsto f(-1/x)x^6$

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$$f = -9714024579x^6 + 201752837967x^5 - 1745938196335x^4 + 8058177120130x^3 - 20920269727185x^2 + 28966530436059x - 16711461708274$$
$$z_0 = 3.46160701785811 + 0.00166267517421363i$$

$$z_0 \mapsto z_0 - 3, \quad f \mapsto f(x+3)$$

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$$z_0 = -2.16631677508735 + 0.0078029167280281 8i$$

$$z_0 \mapsto z_0 + 2, \quad f \mapsto f(x-2)$$

3. Example

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$$f = -93896497 * x^6 - 93894768 * x^5 - 39123000 * x^4 - 8694240 * x^3 - 1086835 * x^2 - 72461 * x - 2013$$
$$z_0 = -0.166316775087345 + 0.00780291672802818i$$

$$z_0 \mapsto -1/z_0, \quad f \mapsto f(-1/x)x^6$$

3. Example

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$$z_0 = 5.99941721622790 + 0.281468618726764i$$

$$z_0 \mapsto z_0 - 6, \quad f \mapsto f(x+6)$$

3. Example

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$$z_0 \mapsto z_0 - 6, \quad f \mapsto f(x+6)$$

$$f = -2013x^6 - 7x^5 - 25x^4 - 1$$
$$z_0 = -0.000582783772097132 + 0.281468618726764i$$

3. Example

$f = -2013x^6 + 72461x^5 - 1086835x^4 + 8694240x^3 - 39123000x^2 + 93894768x - 93896497$
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$f = -2013x^6 - 7x^5 - 25x^4 - 1$
$z_0 = -0.000582783772097132 + 0.281468618726764i$

$z_0 \mapsto -1/z_0, \quad f \mapsto f(-1/x)x^6$

$f = -x^6 - 25x^2 + 7x - 2013$
$z_0 = 0.00735606612524563 + 3.55277869883883i$

z_0 is reduced, so f is z_0-reduced.

$z = 0.00731702950415161 + 3.55336225369141i$

z is also reduced, so f is (z-)reduced.

3. Stoll-Cremona reduction in general

If k is totally real of degree d, then take $\phi_1, \ldots, \phi_d : k \to \mathbf{R}$.
$z(F) = z(\phi_i(F))_i \in \mathcal{H}^d$

$$z(f \cdot A^{-1}) = A \cdot z = \left(\frac{\phi_i(a) z_i + \phi_i(b)}{\phi_i(c) z_i + \phi_i(d)} \right)_i \in \mathcal{H}^d.$$

"We call F *reduced* if $z(F)$ is in a fixed fundamental domain for the action of $\mathrm{SL}(2, \mathcal{O}_k)$." [Stoll-Cremona]

We implemented a fundamental domain if k is real quadratic of class number one.

Table 1a

DAB	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[5,5,5]$	1	$2^8 \cdot 5^5$	$x^5 - 1$
$[5,10,20]$	2^{12}	$2^{10} \cdot 5^5$	$4x^5 - 30x^3 + 45x - 22$
	$2^{12} \cdot 11^{12}$	$2^{10} \cdot 5^5$	$8x^6 + 52x^5 - 250x^3 + 321x - 131$
$[5,65,845]$	11^{12}	$2^{20} \cdot 5^5 \cdot 13^{10}$	$8x^6 - 112x^5 - 680x^4 + 8440x^3 + 28160x^2 - 55781x + 111804$
	$31^{12} \cdot 41^{12}$	$2^{20} \cdot 5^5 \cdot 13^{10}$	$-9986x^6 + 73293x^5 - 348400x^3 - 118976x - 826072$
$[5,85,1445]$	71^{12}	$2^{20} \cdot 5^5 \cdot 17^{10}$	$-73x^6 + 1005x^5 + 14430x^4 - 130240x^3 - 1029840x^2 + 760976x - 2315640$
	$11^{12} \cdot 41^{12} \cdot 61^{12}$	$2^{20} \cdot 5^5 \cdot 17^{10}$	$2160600x^6 - 8866880x^5 + 2656360x^4 - 582800x^3 + 44310170x^2 + 6986711x - 444408$
$[8,4,2]$	2^6	2^{15}	$x^5 - 3x^4 - 2x^3 + 6x^2 + 3x - 1$
$[8,20,50]$	$2^6 \cdot 7^{12} \cdot 23^{12}$	$2^{15} \cdot 5^{10}$	$-8x^6 - 530x^5 + 160x^4 + 64300x^3 - 265420x^2 - 529x$
	$2^6 \cdot 7^{12} \cdot 17^{12} \cdot 23^{12}$	$2^{15} \cdot 5^{10}$	$4116x^6 + 64582x^5 + 139790x^4 - 923200x^3 + 490750x^2 + 233309x - 9347$
$[13,13,13]$	1	$2^{20} \cdot 13^5$	$x^6 - 8x^4 - 8x^3 + 8x^2 + 12x - 8$
$[13,26,52]$	$2^{12} \cdot 3^{12} \cdot 23^{12}$	$2^{10} \cdot 13^5$	$-243x^6 - 2223x^5 - 1566x^4 + 19012x^3 + 903x^2 - 19041x - 5882$
	$2^{12} \cdot 3^{12} \cdot 23^{12} \cdot 131^{12}$	$2^{10} \cdot 13^5$	$59499x^6 - 125705x^5 - 801098x^4 + 1067988x^3 + 2452361x^2 + 707297x - 145830$
$[13,65,325]$	3^{12}	$2^{20} \cdot 5^{10} \cdot 13^5$	$36x^5 - 1040x^3 + 1560x^2 + 1560x + 1183$
	$3^{12} \cdot 53^{12}$	$2^{20} \cdot 5^{10} \cdot 13^5$	$-1323x^6 - 1161x^5 + 9360x^4 + 9590x^3 - 34755x^2 + 1091x + 32182$
$[29,29,29]$	5^{12}	$2^{20} \cdot 29^5$	$43x^6 - 216x^5 + 348x^4 - 348x^2 - 116x$
$[37,37,333]$	$3^{12} \cdot 11^{12}$	$2^{20} \cdot 37^5$	$-68x^6 + 57x^5 + 84x^4 - 680x^3 + 72x^2 - 1584x - 4536$
$[53,53,53]$	$17^{12} \cdot 29^{12}$	$2^{20} \cdot 53^5$	$-3800x^6 + 15337x^5 + 160303x^4 - 875462x^3 + 896582x^2 - 355411x + 50091$
$[61,61,549]$	$3^{24} \cdot 5^{12} \cdot 41^{12}$	$2^{20} \cdot 61^5$	$40824x^6 + 103680x^5 - 67608x^4 - 197944x^3 - 17574x^2 + 41271x + 103615$

Table 1b

DAB	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[5,15,45]$	$(2)^{12} \cdot (3)^6$	$(2a+1)_5^{10}$	$-x^6 + (-3a-3)\,x^5 + (5a+15)\,x^3 + (-15a-3)\,x - 4a + 1$
	$(2)^{12} \cdot (3)^6 \cdot (5a+2)_{31}^{12}$	$(2a+1)_5^{10}$	$(-2a+3)\,x^6 + (-9a+18)\,x^5 + (15a-70)\,x^3 + (39a+54)\,x - 52a - 1$
$[5,30,180]$	$(3a+2)_{11}^{12} \cdot (2)^{18} \cdot (3)^6 \cdot (5a+2)_{31}^{12}$	$(2a+1)_5^{10}$	$684x^6 + (390a+90)\,x^5 + (24a-3138)\,x^4 + (217a+401)\,x^3 +$ $(96a+3918)\,x^2 + (-2112a-1698)\,x + 284a + 432$
	$(3a+1)_{11}^{12} \cdot (2a-11)_{139}^{12} \cdot (4a+3)_{19}^{12}$ $\cdot (2)^{18} \cdot (3)^6 \cdot (5a+2)_{31}^{12}$	$(2a+1)_5^{10}$	$(927a+2906)\,x^6 + (5541a+18822)\,x^5 + (-33535a-124380)\,x^3 +$ $(33417a+183726)\,x + 12641a - 31928$
$[5,35,245]$	$(3a+2)_{11}^{12} \cdot (2)^{12} \cdot (a+6)_{29}^{12}$ $\cdot (7)^6 \cdot (a+9)_{71}^{12}$	$(2a+1)_5^{10}$	$(-4527a-783)\,x^6 + (6392a+7811)\,x^5 + (-4500a-17085)\,x^3 +$ $(-6948a+9783)\,x - 1687a + 39$
	$(3a+1)_{11}^{12} \cdot (11a+5)_{151}^{12}$ $\cdot (2a+15)_{191}^{12} \cdot (2)^{12}$ $\cdot (a-5)_{29}^{12} \cdot (7)^6$	$(2a+1)_5^{10}$	$(-435a-521)\,x^6 + (353a+110)\,x^5 + (131927a+189531)\,x^4 +$ $(-696187a-952511)\,x^3 + (-10094248a-15393369)\,x^2 +$ $(94869598a+145990333)\,x - 210533420a - 329328479$
$[5,105,2205]$	$(3a+1)_{11}^{12} \cdot (3)^6 \cdot (7)^6$	$(2)^{20} \cdot (2a+1)_5^{10}$	$(-5a+4)\,x^6 + (-81a+30)\,x^5 + (-135a+210)\,x^4 + (450a-210)\,x^3 +$ $(360a-1785)\,x^2 + (600a+15)\,x - 950a + 5625$
	$(a+11)_{109}^{12} \cdot (3a+2)_{11}^{12} \cdot (3)^6$ $\cdot (7)^6 \cdot (8a+3)_{79}^{12}$	$(2)^{20} \cdot (2a+1)_5^{10}$	$(-3a-260)\,x^6 + (1032a+1389)\,x^5 + (19160a+8760)\,x^3 +$ $(-16224a+163200)\,x + 162976a + 114632$
$[8,12,18]$	$(a)_2^{12} \cdot (3)^6$ $\cdot (2a-1)_7^{12} \cdot (2a+1)_7^{12}$	$(a)_2^{30}$	$(24a-54)\,x^5 + (-66a+96)\,x^4 + (-32a+220)\,x^3 + (12a-312)\,x^2 +$ $(96a+21)\,x - 5a - 16$
$[17,119,3332]$	$(2a+15)_{179}^{12} \cdot (a+2)_2^{36}$ $\cdot (a-1)_2^{12} \cdot (4a+7)_{43}^{12} \cdot (7)^6$	$(2a+1)_{17}^{10}$	$(213a+1875)\,x^6 + (8071a+4059)\,x^5 + (-1045a+58039)\,x^4 +$ $(32898a+26657)\,x^3 + (-12585a+3550)\,x^2 + (-46889a-136176)\,x$ $- 42057a - 104692$

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Table 1b, continued from previous page

DAB	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[17, 255, 15300]$	$(2a-5)_{19}^{12} \cdot (a+2)_{2}^{24}$ $\cdot (a-1)_{2}^{24} \cdot (3)^6 \cdot (2a+31)_{883}^{12}$	$(2a+1)_{17}^{10} \cdot (5)^{10}$	$(-4264a-13208)\,x^6 + (9516a-94116)\,x^5 + (331770a-503670)\,x^4 + (-1195640a+1593625)\,x^3 + (1141785a-2476410)\,x^2 + (-69927a+2540472)\,x - 301251a - 1280828$
	$(2a+3)_{13}^{12} \cdot (4a+17)_{157}^{12} \cdot (2a+7)_{19}^{12}$ $\cdot (a+2)_{2}^{12} \cdot (a-1)_{2}^{12} \cdot (3)^6$ $\cdot (4a+3)_{67}^{12} \cdot (2a-9)_{83}^{12} \cdot (2a+11)_{83}^{12}$	$(2a+1)_{17}^{10} \cdot (5)^{10}$	$(3703196a+9037010)\,x^6 + (12666396a+36366348)\,x^5 + (33133830a+56148570)\,x^4 + (35333760a+111063545)\,x^3 + (71845845a+45282705)\,x^2 + (154100103a-105860229)\,x + 81081415a - 36366223$

Table 2b

DAB	DAB^r	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[5, 11, 29]$	$[29, 7, 5]$	α^2+3	$(2)^{12} \cdot (a-1)_{5}^{12} \cdot (a+1)_{7}^{12}$	$(a+2)_{5}^{10}$	$(18a+60)\,x^6 + (-76a-246)\,x^5 + (127a+329)\,x^4 + (-77a-209)\,x^3 + (-30a+155)\,x^2 + (29a-69)\,x + 71a - 156$
			$(2)^{12} \cdot (a+6)_{23}^{12} \cdot (a-1)_{5}^{12}$	$(a+2)_{5}^{10}$	$(2a+1)\,x^6 + (-a-26)\,x^5 + (9a+38)\,x^4 + (-40a-25)\,x^3 + (-21a-37)\,x^2 + (100a+218)\,x + 102a + 268$
$[5, 13, 41]$	$[41, 11, 20]$	α^2+5	$(a-3)_{2}^{12}$	$(a+4)_{2}^{20} \cdot (2a-5)_{5}^{10}$	$(-a+3)\,x^6 + (4a-8)\,x^5 + 10x^4 + (-a+20)\,x^3 + (4a+5)\,x^2 + (a+4)\,x + 1$
$[5, 17, 61]$	$[61, 9, 5]$	α^2+4	$(a-3)_{3}^{12}$	$(2)^{20} \cdot (a-4)_{5}^{10}$	$(a+4)\,x^6 + (-8a-42)\,x^5 + (37a+117)\,x^4 + (-20a-240)\,x^3 + (56a-9)\,x^2 + (22a-114)\,x + 9a - 28$
$[5, 21, 109]$	$[109, 17, 45]$	α^2+8	$(a-5)_{3}^{12} \cdot (3a+17)_{5}^{12}$	$(2)^{20} \cdot (3a-14)_{5}^{10}$	$(-28a+53)\,x^6 + (-113a+913)\,x^5 + (-495a+1890)\,x^4 + (-746a+3308)\,x^3 + (-563a+3574)\,x^2 + (-378a+1069)\,x - 151a - 227$
$[5, 26, 149]$	$[149, 13, 5]$	α^2+6	$(a+7)_{5}^{12} \cdot (a-5)_{7}^{12}$	$(2)^{20} \cdot (a-6)_{5}^{10}$	$(-125a-875)\,x^6 + (-1375a-8575)\,x^5 + (-9090a-62160)\,x^4 + (-38862a-251798)\,x^3 + (-73257a-489843)\,x^2 + (-53235a-347403)\,x - 12896a - 86314$
$[5, 33, 261]$	$[29, 21, 45]$	$\frac{1}{3}\alpha^2+3$	$(a+5)_{13}^{12} \cdot (3)^6$	$(2)^{20} \cdot (a+2)_{5}^{10}$	$(-27a-96)\,x^5 + (-18a-51)\,x^4 + (-34a-58)\,x^3 + (-18a-36)\,x^2 - 15x - 9a - 27$
			$(3)^6 \cdot (a)_{7}^{12}$	$(2)^{20} \cdot (a+2)_{5}^{10}$	$(-3a+6)\,x^5 - 90x^4 + (-128a-136)\,x^3 + (-72a-744)\,x^2 + (-240a-240)\,x - 216$
$[5, 34, 269]$	$[269, 17, 5]$	α^2+8	$(a-7)_{11}^{12} \cdot (2a-15)_{13}^{12}$ $\cdot (a+9)_{5}^{12}$	$(2)^{20} \cdot (a-8)_{5}^{10}$	$(-283a+2246)\,x^6 + (-4563a+33800)\,x^5 + (-11932a+103166)\,x^4 + (127408a-1032304)\,x^3 + (998576a-7558008)\,x^2 + (2439792a-18969664)\,x + 2110776a - 16149072$
$[5, 41, 389]$	$[389, 37, 245]$	α^2+18	$(2a+21)_{11}^{12} \cdot (8a+83)_{17}^{12}$ $\cdot (5a+52)_{19}^{12} \cdot (3a-28)_{5}^{12}$	$(2)^{20} \cdot (3a+31)_{5}^{10}$	$(1248a-11685)\,x^6 + (-16097a+150611)\,x^5 + (37185a-349530)\,x^4 + (250806a-2359968)\,x^3 + (-972081a+9046728)\,x^2 + (-942318a+8701533)\,x + 4994791a - 46866753$
$[5, 66, 909]$	$[101, 33, 45]$	$\frac{1}{3}\alpha^2+5$	$(a-2)_{19}^{12} \cdot (3)^6$ $\cdot (2a+13)_{43}^{12} \cdot (a-4)_{5}^{12}$	$(2)^{20} \cdot (a+5)_{5}^{10}$	$(-340a-1674)\,x^6 + (-4179a-26820)\,x^5 + (-26433a-118800)\,x^4 + (-38358a-315240)\,x^3 + (-46686a-41130)\,x^2 + (40761a-15348)\,x - 13013a + 39100$

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DAB	DAB^r	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
			$(3)^6 \cdot (a+8)_{31}^{12}$ $\cdot(2a-7)_{37}^{12} \cdot (a-4)_5^{12}$	$(2)^{20} \cdot (a+5)_5^{10}$	$(-6120a - 36189)\,x^6 + (-22143a - 102375)\,x^5 +$ $(-21378a - 184140)\,x^4 + (-31356a - 65810)\,x^3 +$ $(765a - 81765)\,x^2 +$ $(-3783a + 6192)\,x$
$[8, 10, 17]$	$[17, 5, 2]$	$\alpha^2 + 2$	$(a+2)_2^6$	$(a+2)_2^{45} \cdot (a-1)_2^{20}$	$x^6 + (2a+4)\,x^5 + (3a+14)\,x^4 + (10a+8)\,x^3 +$ $(-9a+32)\,x^2 + (16a-16)\,x - 4a + 8$
$[8, 18, 73]$	$[73, 9, 2]$	$\alpha^2 + 4$	$(a-4)_2^6 \cdot (a+5)_2^{12}$ $\cdot(4a-15)_3^{12}$	$(a-4)_2^{45}$	$(a+5)\,x^6 + (28a+132)\,x^5 + (214a+1026)\,x^4 +$ $(349a+1658)\,x^3 + (259a+1242)\,x^2 +$ $(47a+222)\,x - 3a - 14$
$[8, 22, 89]$	$[89, 11, 8]$	$\alpha^2 + 5$	$(a-4)_2^{12} \cdot (a+5)_2^6$ $\cdot(4a-17)_5^{12}$	$(a+5)_2^{45}$	$(a-4)\,x^6 + (8a-36)\,x^5 + (16a-62)\,x^4 + (-13a+57)\,x^3 +$ $(-17a+73)\,x^2 + (13a-57)\,x - a + 5$
$[8, 34, 281]$	$[281, 17, 2]$	$\alpha^2 + 8$	$(42a-331)_{17}^{12} \cdot (a-8)_2^6$ $\cdot(a+9)_2^{24} \cdot (76a+675)_5^{12}$ $\cdot(8a-63)_7^{12}$	$(a-8)_2^{45}$	$(-15024a + 118185)\,x^6 +$ $(310153a - 2435026)\,x^5 + (-2658057a + 20990488)\,x^4 +$ $(12047831a - 97400942)\,x^3 + (-33280854a + 231380920)\,x^2 +$ $(34989188a - 413796872)\,x - 37610304a + 81055944$
$[8, 38, 233]$	$[233, 19, 32]$	$\alpha^2 + 9$	$(38a-271)_{13}^{12} \cdot (a+8)_2^{12}$ $\cdot(a-7)_2^6 \cdot (8a+65)_7^{12}$ $\cdot(8a-57)_7^{12}$	$(a-7)_2^{45}$	$(-166628a - 1355047)\,x^6 +$ $(-354121a - 2879769)\,x^5 + (-318274a - 2588269)\,x^4 +$ $(-153661a - 1249743)\,x^3 + (-41827a - 339754)\,x^2 +$ $(-6158a - 48444)\,x - 441a - 2400$
$[8, 50, 425]$	$[17, 25, 50]$	$\frac{1}{5}\alpha^2 + 2$	$(a+2)_2^6 \cdot (a-1)_2^{12} \cdot (5)^6$	$(a+2)_2^{45} \cdot (5)^{15}$	$(34a+80)\,x^6 + (140a+224)\,x^5 + (110a-220)\,x^4 +$ $(-455a+220)\,x^3 + (-5a+190)\,x^2 +$ $(91a-104)\,x + 254a - 395$
			$(2a+3)_{13}^{12} \cdot (2a-5)_{19}^{12}$ $\cdot(a+2)_2^6 \cdot (a-1)_2^{24} \cdot (5)^6$	$(a+2)_2^{45} \cdot (5)^{15}$	$(-1455a + 1511)\,x^6 +$ $(-1004a - 2656)\,x^5 + (-19100a + 20290)\,x^4 +$ $(-3805a - 4380)\,x^3 + (-72745a + 108600)\,x^2 +$ $(-7451a + 10748)\,x - 99295a + 155108$
$[8, 66, 1017]$	$[113, 33, 18]$	$\frac{1}{3}\alpha^2 + 5$	$(4a-19)_{11}^{12} \cdot (a+6)_2^{12}$ $\cdot(a-5)_2^6 \cdot (3)^6$ $\cdot(8a+47)_{41}^{12} \cdot (6a+35)_7^{12}$ $\cdot(6a-29)_7^{12}$	$(a-5)_2^{45}$	$(-4215a - 14698)\,x^6 +$ $(30036a + 338652)\,x^5 + (-549576a - 134610)\,x^4 +$ $(-2945519a + 22716733)\,x^3 + (12849441a - 76601511)\,x^2 +$ $(234523575a - 1115687637)\,x - 843111919a + 4054444133$
			$(a+6)_2^{12} \cdot (a-5)_2^6$ $\cdot(3)^6 \cdot (2a+13)_{31}^{12}$ $\cdot(28a+163)_{53}^{12} \cdot (6a+35)_7^{12}$	$(a-5)_2^{45}$	$(-27a - 2538)\,x^6 +$ $(7230a + 8412)\,x^5 + (-3867a - 272622)\,x^4 +$ $(121693a + 458725)\,x^3 + (-1686144a + 6014715)\,x^2 +$ $(-5324007a + 27892107)\,x + 110392412a - 532554277$
$[13, 9, 17]$	$[17, 15, 52]$	$\alpha^2 + 7$	$(a+2)_2^{12}$	$(2a-1)_{13}^{10} \cdot (a-1)_2^{20}$	$(a-2)\,x^6 + (-8a+8)\,x^5 + (14a-32)\,x^4 + (-19a+27)\,x^3 +$ $(6a-21)\,x^2 + (3a+9)\,x - 4a - 7$
$[13, 18, 29]$	$[29, 9, 13]$	$\alpha^2 + 4$	$(a-1)_5^{12}$	$(a-4)_{13}^{10} \cdot (2)^{20}$	$(9a-22)\,x^6 + (-19a+21)\,x^5 + (8a-95)\,x^4 +$ $(-70a-6)\,x^3 + (-23a-148)\,x^2 +$ $(-7a-127)\,x - 18a - 7$
$[13, 29, 181]$	$[181, 41, 13]$	$\frac{1}{3}\alpha^2 + \frac{19}{3}$	$(6a-37)_{29}^{12} \cdot (a-6)_3^{12}$ $\cdot(a+7)_3^{12} \cdot (4a+29)_5^{12}$	$(3a-19)_{13}^{10} \cdot (2)^{20}$	$(-16581a - 119826)\,x^6 +$ $(-52472a - 379062)\,x^5 + (-67729a - 508419)\,x^4 +$ $(-78876a - 162464)\,x^3 + (-44960a + 21657)\,x^2 +$ $(14402a - 144114)\,x - 21885a + 131494$

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Table 2b, continued from previous page

DAB	DAB^r	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[13, 41, 157]$	$[157, 25, 117]$	$\alpha^2 + 12$	$(3a+20)_{11}^{12} \cdot (a-7)_{17}^{12}$ $\cdot (a-6)_3^{12} \cdot (a+7)_3^{12}$	$(2a-11)_{13}^{10} \cdot (2)^{20}$	$(-1181a+7035)\,x^6 +$ $(18395a-104353)\,x^5 + (-116071a+664673)\,x^4 +$ $(386042a-2282384)\,x^3 + (-742970a+4253365)\,x^2 +$ $(784564a-4063679)\,x - 253294a + 2224205$
$[17, 5, 2]$	$[8, 10, 17]$	$\frac{1}{2}\alpha^2 + \frac{5}{2}$	1	$(3a+1)_{17}^{10} \cdot (a)_2^{30}$	$(-3a+4)\,x^5 - x^4 + (6a-2)\,x^3 + (9a-5)\,x^2 +$ $(-3a+8)\,x - 3a + 6$
$[17, 15, 52]$	$[13, 9, 17]$	$\alpha^2 + 4$	$(a)_3^{12}$	$(a-4)_{17}^{10} \cdot (2)^{20}$	$-x^6 - 2ax^5 + (3a-3)\,x^4 + (8a+4)\,x^3 + (-19a+39)\,x^2 +$ $(16a-30)\,x + 3a - 36$
$[17, 25, 50]$	$[8, 50, 425]$	$\frac{1}{10}\alpha^2 + \frac{5}{2}$	$(a)_2^{24} \cdot (2a+1)_7^{12}$	$(3a+1)_{17}^{10} \cdot (5)^{10}$	$(6a-2)\,x^6 + (-50a-64)\,x^5 + (285a+485)\,x^4 +$ $(-485a-435)\,x^3 + (-70a+90)\,x^2 +$ $(244a+92)\,x + 70a - 166$
			$(a)_2^{36} \cdot (a+7)_{47}^{12}$ $\cdot (2a+1)_7^{12}$	$(3a+1)_{17}^{10} \cdot (5)^{10}$	$(315a+422)\,x^6 + (1212a+1757)\,x^5 + (-2605a-3240)\,x^4 +$ $(-50a-625)\,x^3 + (1730a-570)\,x^2 +$ $(864a-212)\,x + 72a + 456$
$[17, 46, 257]$	$[257, 23, 68]$	$\alpha^2 + 11$	$(11, a+5)^{12} \cdot (13, a+10)^{12}$ $\cdot (2, a)^{12} \cdot (2, a+1)^{24}$ $\cdot (59, a+14)^{12}$	$(17, a+6)^{10} \cdot (2, a+1)^{20}$	$(-22a-1802)\,x^6 +$ $(3596a+11488)\,x^5 + (-30700a-354072)\,x^4 +$ $(243927a+1843299)\,x^3 + (-616892a-5576996)\,x^2 +$ $(647768a+5283496)\,x - 198146a - 1755298$
$[17, 47, 548]$	$[137, 35, 272]$	$\alpha^2 + 17$	$(14a-75)_{11}^{12} \cdot (4a+25)_{19}^{12}$ $\cdot (3a-16)_2^{12} \cdot (3a+19)_2^{24}$	$(8a+51)_{17}^{10}$	$(285a+1620)\,x^6 +$ $(-2683a-19110)\,x^5 + (13341a+76698)\,x^4 +$ $(-28642a-195577)\,x^3 + (40284a+245904)\,x^2 +$ $(-27600a-177408)\,x + 8154a + 51670$
$[29, 7, 5]$	$[5, 11, 29]$	$\alpha^2 + 5$	$(2)^{12} \cdot (2a+1)_5^{12}$	$(a-5)_{29}^{10}$	$(-4a-5)\,x^6 + (11a+37)\,x^5 + (-65a-62)\,x^4 +$ $(111a+104)\,x^3 + (-28a-189)\,x^2 +$ $(-28a+157)\,x - 19a - 76$
			$(2)^{12} \cdot (5a+3)_{31}^{12} \cdot (2a+1)_5^{12}$	$(a-5)_{29}^{10}$	$(18a+42)\,x^6 + (62a+194)\,x^5 + (-209a+31)\,x^4 +$ $(-648a-471)\,x^3 + (116a+338)\,x^2 +$ $(244a+259)\,x - 65a - 159$
$[29, 9, 13]$	$[13, 18, 29]$	$\frac{1}{4}\alpha^2 + \frac{7}{4}$	$(a)_3^{12}$	$(2)^{20} \cdot (3a+2)_{29}^{10}$	$(-25a+56)\,x^6 + (172a-39)\,x^5 + (-39a+561)\,x^4 +$ $(312a+234)\,x^3 + (73a+354)\,x^2 +$ $(76a+141)\,x + 15a + 37$
$[29, 21, 45]$	$[5, 33, 261]$	$\frac{1}{3}\alpha^2 + 5$	$(4a+1)_{19}^{12} \cdot (3)^6$	$(2)^{20} \cdot (a-5)_{29}^{10}$	$(-a+20)\,x^6 + (-87a-18)\,x^5 + (-48a+198)\,x^4 +$ $(-8a-296)\,x^3 + (384a+360)\,x^2 +$ $(-384a-480)\,x + 144a + 216$
			$(3)^6$	$(2)^{20} \cdot (a-5)_{29}^{10}$	$(-102a-165)\,x^5 + (45a+72)\,x^4 + (-174a-262)\,x^3 +$ $(36a-66)\,x^2 + (69a-144)\,x + 5a - 107$
$[29, 26, 53]$	$[53, 13, 29]$	$\alpha^2 + 6$	$(a-1)_{11}^{12} \cdot (a+1)_{13}^{12}$ $\cdot (a+6)_{17}^{12}$	$(2)^{20} \cdot (a-6)_{29}^{10}$	$(-790a+1564)\,x^6 +$ $(241a-12431)\,x^5 + (-15139a-14345)\,x^4 +$ $(-2950a-165614)\,x^3 + (-51588a-116086)\,x^2 +$ $(-58139a-53507)\,x + 12653a - 123381$
$[41, 11, 20]$	$[5, 13, 41]$	$\alpha^2 + 6$	1	$(2)^{20} \cdot (a-6)_{41}^{10}$	$(a+4)\,x^6 + (6a-2)\,x^5 + 17x^4 + (-12a-16)\,x^3 + (24a-5)\,x^2 +$ $(-54a-16)\,x + 33a + 9$
$[53, 13, 29]$	$[29, 26, 53]$	$\frac{1}{4}\alpha^2 + \frac{11}{4}$	$(a+6)_{23}^{12} \cdot (a-1)_5^{12}$ $\cdot (a)_7^{12}$	$(2)^{20} \cdot (3a+5)_{53}^{10}$	$(-31a+70)\,x^6 + (151a-322)\,x^5 + (-405a+658)\,x^4 +$ $(238a-846)\,x^3 + (3288a+2437)\,x^2 +$ $(-3262a+12157)\,x - 27420a - 58255$

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Table 2b, continued from previous page

DAB	DAB^r	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[61,9,5]$	$[5,17,61]$	$\frac{1}{3}\alpha^2+\frac{7}{3}$	1	$(2)^{20}\cdot(7a+4)_{61}^{10}$	$(a+2)\,x^6+(-2a-15)\,x^5+(36a-4)\,x^4+(72a+24)\,x^3+(8a-24)\,x^2+(-48a-80)\,x-24a-40$
$[73,9,2]$	$[8,18,73]$	$\frac{1}{2}\alpha^2+\frac{9}{2}$	$(a)_2^{24}\cdot(2a-1)_7^{12}$	$(2a-9)_{73}^{10}$	$(-12a-6)\,x^6+(8a+82)\,x^5+(-51a+92)\,x^4+(-126a-1)\,x^3+(-36a+35)\,x^2+(32a+50)\,x+10a+8$
$[73,47,388]$	$[97,94,657]$	$\frac{1}{8}\alpha^2+\frac{43}{8}$	$(20a+109)_{101}^{12}\cdot(7a+38)_2^{24}\cdot(7a-31)_2^{12}\cdot(2a-9)_3^{12}\cdot(2a+11)_3^{12}\cdot(30a+163)_{79}^{12}$	$(22a+119)_{73}^{10}$	$(23a-43)\,x^6+(-149a-1221)\,x^5+(8675a+44883)\,x^4+(-128038a-698079)\,x^3+(928849a+5037588)\,x^2+(123515a+671208)\,x+4023a+21640$
$[89,11,8]$	$[8,22,89]$	$\frac{1}{4}\alpha^2+\frac{11}{4}$	$(a)_2^{24}$	$(7a+3)_{89}^{10}$	$-x^5+(-4a+2)\,x^4+21x^3+(-16a+64)\,x^2-160x+142a-190$
$[97,94,657]$	$[73,47,388]$	$\frac{1}{3}\alpha^2+\frac{22}{3}$	$(a-4)_2^{12}\cdot(a+5)_2^{12}\cdot(14a-53)_{23}^{12}\cdot(4a-15)_3^{12}\cdot(4a+19)_3^{12}\cdot(30a+143)_{41}^{12}\cdot(10a+47)_{61}^{12}$	$(24a+115)_{97}^{10}$	$(-128252a-611298)\,x^6+(-984572a-4709700)\,x^5+(-3071730a-15394554)\,x^4+(-6889006a-20077475)\,x^3+(-39650571a+105355350)\,x^2+(174191751a-679664106)\,x+256866525a-973717416$
$[101,33,45]$	$[5,66,909]$	$\frac{1}{12}\alpha^2+\frac{9}{4}$	$(3)^6\cdot(2a+1)_5^{12}\cdot(7a+3)_{61}^{12}$	$(9a+5)_{101}^{10}\cdot(2)^{20}$	$(-216a+464)\,x^6+(-2304a-48)\,x^5+(-3984a-960)\,x^4+(-864a+3088)\,x^3+(-720a+1422)\,x^2+(-4047a-5322)\,x-818a-2423$
			$(4a+3)_{19}^{12}\cdot(4a+1)_{19}^{12}\cdot(3)^6\cdot(5a+3)_{31}^{12}\cdot(2a+1)_5^{12}$	$(9a+5)_{101}^{10}\cdot(2)^{20}$	$(-5229a+4019)\,x^6+(-6132a-6909)\,x^5+(44637a-2364)\,x^4+(53094a+58660)\,x^3+(-39159a+19266)\,x^2+(-30363a-55761)\,x-16848a-16911$
$[109,17,45]$	$[5,21,109]$	α^2+10	$(2a+1)_5^{12}$	$(a-10)_{109}^{10}\cdot(2)^{20}$	$(-8a-8)\,x^6-16x^5+(8a+72)\,x^4+(152a+184)\,x^3+(6a+84)\,x^2+(-255a-339)\,x-319a-524$
$[113,33,18]$	$[8,66,1017]$	$\frac{1}{6}\alpha^2+\frac{11}{2}$	$(3a+11)_{103}^{12}\cdot(a)_2^{24}\cdot(3)^6\cdot(4a-1)_{31}^{12}\cdot(2a-1)_7^{12}\cdot(2a+1)_7^{12}$	$(2a-11)_{113}^{10}$	$(122a+800)\,x^6+(-1509a-909)\,x^5+(36762a-85470)\,x^4+(-116871a+265713)\,x^3+(-467682a+704460)\,x^2+(-480528a+365352)\,x-7616a+226442$
			$(a)_2^{24}\cdot(3)^6\cdot(4a+1)_{31}^{12}\cdot(2a+1)_7^{12}$	$(2a-11)_{113}^{10}$	$(-418a-190)\,x^6+(1476a-660)\,x^5+(1146a+6810)\,x^4+(2145a+2175)\,x^3+(-1437a-3489)\,x^2+(-42a-2736)\,x+830a+394$
$[137,35,272]$	$[17,47,548]$	α^2+23	$(2a-5)_{19}^{12}\cdot(a+2)_2^{12}\cdot(a-1)_2^{12}$	$(6a-1)_{137}^{10}$	$(4a+6)\,x^6+(8a+36)\,x^5+(-4a+42)\,x^4+(586a+1289)\,x^3+(1066a+2808)\,x^2+4ax+25596a+65566$
$[149,13,5]$	$[5,26,149]$	$\frac{1}{4}\alpha^2+\frac{11}{4}$	$(3a+1)_{11}^{12}$	$(11a+7)_{149}^{10}\cdot(2)^{20}$	$8x^6+96x^5+(-24a+168)\,x^4+(-576a-808)\,x^3+(66a-132)\,x^2+(292a+47)\,x+86a-87$
$[157,25,117]$	$[13,41,157]$	$\frac{1}{9}\alpha^2+\frac{16}{9}$	$(a-4)_{17}^{12}\cdot(3a-1)_{23}^{12}\cdot(a)_3^{24}\cdot(a+1)_3^{12}$	$(7a+5)_{157}^{10}\cdot(2)^{20}$	$(-3328a-7633)\,x^6+(-17510a-39323)\,x^5+(-32518a-68044)\,x^4+(-17960a-66720)\,x^3+(256a-51704)\,x^2+(5184a-22864)\,x+1432a-5264$
$[181,41,13]$	$[13,29,181]$	$\frac{1}{3}\alpha^2+\frac{13}{3}$	$(a+5)_{17}^{12}\cdot(3a+2)_{29}^{12}\cdot(a)_3^{24}\cdot(a+1)_3^{12}$	$(3a-13)_{181}^{10}\cdot(2)^{20}$	$(330a+1417)\,x^6+(11102a+1701)\,x^5+(1396a+59742)\,x^4+(24016a+92792)\,x^3+(74408a+38064)\,x^2+(35248a+26160)\,x-5784a+21888$
$[233,19,32]$	$[8,38,233]$	$\frac{1}{8}\alpha^2+\frac{19}{8}$	$(a)_2^{24}\cdot(a-5)_{23}^{12}\cdot(a+5)_{23}^{12}\cdot(2a+1)_7^{12}$	$(11a+3)_{233}^{10}$	$(2348a-3554)\,x^6+(11828a-12348)\,x^5+(4498a-23598)\,x^4+(12704a+9133)\,x^3+(-3151a-14433)\,x^2+(5344a-1974)\,x+18a-604$

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Table 2b, continued from previous page

DAB	DAB^r	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[257, 23, 68]$	$[17, 46, 257]$	$\frac{1}{8}\alpha^2 + \frac{19}{8}$	$(2a+3)_{13}^{12} \cdot (a+2)_{2}^{12}$ $\cdot(a-1)_{2}^{24} \cdot (4a-3)_{43}^{12}$ $\cdot(2a+9)_{47}^{12} \cdot (4a+13)_{53}^{12}$	$(8a-19)_{257}^{10}$	$(-2809a - 7326)\,x^6 +$ $(5069a + 3572)\,x^5 + (52427a - 51416)\,x^4 +$ $(249518a + 105951)\,x^3 + (-311115a - 180355)\,x^2 +$ $(156533a - 20215)\,x - 34657a + 19003$
$[269, 17, 5]$	$[5, 34, 269]$	$\frac{1}{4}\alpha^2 + \frac{15}{4}$	$(3a+1)_{11}^{12} \cdot (2a+1)_{5}^{12}$	$(2)^{20} \cdot (15a+11)_{269}^{10}$	$(-168a - 272)\,x^6 + (960a + 1696)\,x^5 + (472a - 1008)\,x^4 +$ $(-4448a - 1552)\,x^3 + (358a + 904)\,x^2 +$ $(945a + 1690)\,x$
$[281, 17, 2]$	$[8, 34, 281]$	$\frac{1}{2}\alpha^2 + \frac{17}{2}$	$(a)_{2}^{36} \cdot (4a+1)_{31}^{12}$ $\cdot(2a-1)_{7}^{12} \cdot (2a+1)_{7}^{12}$	$(2a-17)_{281}^{10}$	$(-835a + 1960)\,x^6 + (1343a + 7589)\,x^5 + (19630a + 6428)\,x^4 +$ $(26923a + 13601)\,x^3 + (-6743a + 44228)\,x^2 +$ $(-5762a + 18262)\,x + 17138a - 23184$
$[389, 37, 245]$	$[5, 41, 389]$	$\frac{1}{5}\alpha^2 + \frac{18}{5}$	$(3a+1)_{11}^{12} \cdot (3a+2)_{11}^{12}$ $\cdot(4a+3)_{19}^{12} \cdot (4a+1)_{19}^{12}$ $\cdot(a+6)_{29}^{12} \cdot (2a+1)_{5}^{12}$	$(2)^{20} \cdot (18a+13)_{389}^{10}$	$(-22952a - 6848)\,x^6 +$ $(162272a - 61136)\,x^5 + (296568a + 208208)\,x^4 +$ $(-212600a - 959344)\,x^3 + (89874a + 1610270)\,x^2 +$ $(-428348a - 1023457)\,x + 315516a + 343397$

Theorems about our list

Correctness:

- Thanks to denominator bounds for CM Igusa invariants (Lauter-Viray, see also Goren-Lauter, Bruinier-Yang).
- We implemented the denominator bounds and used interval arithmetic to evaluate Igusa invariants in Sage.
- This proves correctness.

Completeness:

- Is a class number one problem.
- Work in progress of Pınar Kılıçer.

Work to do in Sage

- Include our current Mestre and reduction code into Sage.
- Implement the covariant z instead of only a Magma-interface.
- Twist-preserving reduction.
- More general fundamental domains
(also relevant to Hilbert modular forms).
- Hyperbolic 3-space / fields that are not totally real.
- Cardona-Quer (analogue of Mestre for $\mathrm{Aut}(C_{\overline{k}}) \not\cong \{1, \iota\}$)
- Reduction if $\mathrm{Aut}(C_{\overline{k}}) \not\cong \{1, \iota\}$.
- Characteristic 2, 3, 5.
- ...

Contact me if interested.