# Improved mixing rates for infinite measure preserving systems

Dalia Terhesiu

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**Abstract**

In this work, we introduce a new technique for operator renewal sequences associated with dynamical systems preserving an infinite measure that improves the results on mixing rates obtained in Melbourne and Terhesiu [21]. Also, this technique allows us to offer a very simple proof of the key result in [21] that provides first order asymptotic of operator renewal sequences associated with dynamical systems with infinite measure. Moreover, combining techniques used in this work with techniques used in [21], we obtain first order asymptotic of operator renewal sequences under some relaxed assumption on the first return map.

## 1 Introduction and main results

Let $(X, \mu)$ be a measure space (finite or infinite), and $f : X \to X$ a conservative, ergodic measure preserving map. Fix $Y \subset X$ with $\mu(Y) \in (0, \infty)$. Let $\varphi : Y \to \mathbb{Z}^+$ be the first return time $\varphi(y) = \inf\{n \geq 1 : f^n y \in Y\}$ (finite almost everywhere by conservativity). Let $L : L^1(X) \to L^1(X)$ denote the transfer operator for $f$ and define

$$T_n = 1_Y L^n 1_Y, \;\; n \geq 0, \qquad R_n = 1_Y L^n 1_{\{\varphi = n\}}, \;\; n \geq 1.$$

Thus $T_n$ corresponds to general returns to $Y$ and $R_n$ corresponds to first returns to $Y$. The relationship $T_n = \sum_{j=1}^{n} T_{n-j} R_j$ generalizes the notion of scalar renewal sequences (see [10, 6] and references therein).

Operator renewal sequences were introduced by Sarig [24] to study lower bounds for mixing rates associated with finite measure preserving systems, and this technique was substantially extended and refined by Gouëzel [12, 15].

In the infinite mean setting a crucial ingredient for the asymptotics of renewal sequences is that

(H) $\mu(y \in Y : \varphi(y) > n) = \ell(n)n^{-\beta}$ where $\ell$ is slowly varying[1] and $\beta \in (0, 1)$.

For the necessity of this assumption in the setting of scalar renewal sequences we refer to Garsia and Lamperti [11] and Erickson [9]. Under the same assumption, Melbourne and Terhesiu [21] developed a theory of renewal operator sequences for dynamical systems with infinite measure, generalizing the results of [11, 9] to the operator case. Imposing suitable assumptions on the first return map $T^\varphi$, [21] shows that for a ('sufficiently regular') function $v$ supported on $Y$ and a constant $d_0 = \frac{1}{\pi}\sin\beta\pi$, the following hold: i) when $\beta \in (\frac{1}{2}, 1]$ then $\lim_{n\to\infty} \ell(n)n^{1-\beta}T_n v = d_0 \int_Y v\, d\mu$, uniformly on $Y$; ii) if $\beta \in (0, \frac{1}{2}]$ and $v \geq 0$ then $\liminf_{n\to\infty} \ell(n)n^{1-\beta}T_n v = d_0 \int_Y v\, d\mu$, pointwise on $Y$ and iii) if $\beta \in (0, \frac{1}{2})$ then $T_n v = O(\ell(n)n^{-\beta})$. As shown in [21], the above results on $T_n$ extend to similar results on $L^n$ associated with a large class of systems preserving an infinite measure.

The apparently weaker results for the case $\beta < 1/2$ are in fact optimal under the general assumption $\mu(\varphi > n) = \ell(n)n^{-\beta}$ (see [11]). Under the *additional assumption* $\mu(\varphi = n) = O(\ell(n)n^{-(\beta+1)})$, Gouëzel [14] obtains first order asymptotic for $L^n v$ for *all* $\beta \in (0, 1)$.

Apart from first order theory, the method developed in [21] also yields mixing rates and *higher* order asymptotics of $T_n$, which for 'nice' maps leads to the desired asymptotic for $L^n$. By higher order asymptotics of $T_n v$, for some $v$ in a suitable function space with norm $\|\|$, we mean the existence of real constants $d_0, d_1, \ldots, d_q$, $q \geq 1$ such that $T_n v = \left(d_0 n^{\beta-1} + d_1 n^{2(\beta-1)} + \cdots + d_q n^{q(\beta-1)}\right) \int v d\mu + D$, where $\|D\| = o(n^{q(\beta-1)})$. The weaker notion of mixing rates refers to the existence of an upper bound for $\|n^{1-\beta}Tnv - d_0 \int_Y v\, d\mu\|$. As shown in [21], higher order asymtptotics/mixing rates can be obtained by exploiting higher order expansions of $\mu(\varphi > n)$. Exploiting a tail condition of the form $\mu(\varphi > n) = cn^{-\beta} + O(n^{-2\beta})$, for some $c > 0$, [21] obtains mixing rates for $\beta > 1/2$ and higher order asymptotics for $\beta > 3/4$.

In this work, we introduce a new technique for *operator* renewal sequences associated with dynamical systems preserving an infinite measure that allows us to explore higher order expansions of $\mu(\varphi > n)$; for a precise formulation of such an expansion we refer to assumption (H3) below. As far as we understand, the advantage of tail expansion of the type (H3) below cannot be exploited using the techniques in [21](see the explanatory Remark 1.2). Along the way, we obtain improved higher order theory for *scalar* renewal sequences (see Remark 1.4). Previous results on higher order theory for scalar renewal sequences are contained in [21].

Moreover, combining the techniques used in this work with the techniques in [21, 11], we obtain first order asymptotics for $T_n$ under the general assumption $\mu(\varphi > n) = \ell(n)n^{-\beta}$ and some relaxed (weaker than in [21]) assumption on the first return map. For a precise formulation of our main results we need to provide a description

[1] We recall that a measurable function $\ell : (0, \infty) \to (0, \infty)$ is slowly varying if $\lim_{x\to\infty} \ell(\lambda x)/\ell(x) = 1$ for all $\lambda > 0$. Good examples of slowly varying functions are the asymptotically constant functions and the logarithm.

of the abstract framework and main assumptions for operator renewal sequences.

## 1.1 Main assumptions and general setup

Throughout we assume that (H) holds.

Recall that the transfer operator $R : L^1(Y) \to L^1(Y)$ for the first return map $F : Y \to Y$ is defined via the formula $\int_Y Rv\, w\, d\mu = \int_Y v\, w \circ F\, d\mu$, $w \in L^\infty(Y)$. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$. Given $z \in \bar{\mathbb{D}}$, we define $R(z) : L^1(Y) \to L^1(Y)$ to be the operator $R(z)v = R(z^\varphi v)$. Also, for each $n \ge 1$, we define $R_n : L^1(Y) \to L^1(Y)$, $R_n v = R(1_{\{\varphi = n\}} v)$. It is easily verified that $R(z) = \sum_{n=1}^\infty R_n z^n$.

As in [21, 22], our assumptions on the first return map $F : Y \to Y$ are functional-analytic. We assume that there is a function space $\mathcal{B} \subset L^2(Y)$ containing constant functions, with norm $\| \;\|$ satisfying $\|v\|_{L^2} \le \|v\|$ for $v \in \mathcal{B}$, such that for some constant $C > 0$:

(H1) For all $n \ge 1$, $R_n : \mathcal{B} \to \mathcal{B}$ is a bounded linear operator satisfying $\sum_{j>n} \|R_j\| \le C\ell(n)n^{-\beta}$, with $\beta$ and $\ell$ as in (H).

For several comments on the assumption $\mathcal{B} \subset L^2(Y)$ we refer to Remark 1.3.

As shown in [22] the assumption $\sum_{n=1}^\infty \|R_n\| < \infty$ turns out to be sufficient for several results on the average operator $\sum_{j=0}^{n-1} L^n$. In this work we show the following stronger version is enough for first order of $T_n$ (this is the content of Theorem 1.5).

(H1′) $\sum_{j>n} \|R_j\| < n^{-\tau}$, where $\tau$ is such that $\tau > \max\{1-\beta, \beta/2\}$ for $1/2 < \beta < 1$.

We notice that $z \mapsto R(z)$ is a continuous family of bounded linear operators on $\mathcal{B}$ for $z \in \bar{\mathbb{D}}$. Since $R(1) = R$ and $\mathcal{B}$ contains constant functions, 1 is an eigenvalue of $R(1)$. Throughout we assume:

(H2) (i) The eigenvalue 1 is simple and isolated in the spectrum of $R(1)$.

(ii) For $z \in \bar{\mathbb{D}} \setminus \{1\}$, the spectrum of $R(z)$ does not contain 1.

In particular, we note that $z \mapsto (I - R(z))^{-1}$ is an analytic family of bounded linear operators on $\mathcal{B}$ for $z \in \mathbb{D}$. Define $T_n : L^1(Y) \to L^1(Y)$ for $n \ge 0$ and $T(z) : L^1(Y) \to L^1(Y)$ for $z \in \bar{\mathbb{D}}$ by setting

$$T_n v = 1_Y L^n(1_Y v), \qquad T(z) = \sum_{n=0}^\infty T_n z^n.$$

(Here, $T_0 = I$.) We have the usual relation $T_n = \sum_{j=1}^n T_{n-j} R_j$ for $n \ge 1$. An induction argument on $n$ together with the boundedness of $R_j$ (see (H1) above) shows that $\|T_n\|$ grows at most exponentially. Hence, $T(z)$ is well defined for $z$ in a small disk around 0. Furthermore, $T(z) = I + T(z)R(z)$ on $\mathbb{D}$ and thus, the renewal equation $T(z) = (I - R(z))^{-1}$ holds for $z \in \mathbb{D}$. It follows that $T(z) = \sum_{n=0}^{\infty} T_n z^n$ can be analytically extended to the whole of $\mathbb{D}$.

As shown in [21], the higher order expansion of $T(e^{i\theta})$, as $\theta \to 0$ is an essential ingredient in obtaining higher order asymptotics of $T_n$. In [22], the higher order expansion of $T(z)$ for $z \in \mathbb{D}$ is essential for higher order expansion of $\sum_{j=1}^{n} T_j$.

Our main idea is to exploit the analyticity of $T(z)$ for $z \in \mathbb{D}$. As it will be made clear in Sections 3, 4 and 5, good estimates on $\frac{d}{d\theta}T(z)$ (at least for $z$ outside a small neighborhood of 1) allow us to obtain good estimates on the coefficients $T_n$ of $T(z)$.

In Section 2 we obtain a higher order expansion of $\frac{d}{d\theta}T(z)$, under the following assumption (stronger than the ones used in [21, 22]):

(H3) Let $\beta \in (1/2, 1)$. Assume that $\mu(\varphi > n) = cn^{-\beta} + b(n) + H(n)$, for some positive constant $c$, some function $b$ such that $nb(n)$ has bounded variation and $b(n) = O(n^{-2\beta})$, and some function $H$ such that $H(n) = O(n^{-\gamma})$ with $\gamma > 2$.

In this work we show that (H3) is enough to improve the results on mixing rates obtained in [21] for the range $\beta \in (2/3, 1)$ (this is the content of Theorem 1.1 below).

## 1.2 Main results

To state the first result we need to specify the following

**Notation**. Let $c$ , $b(n)$ and $H(n)$ be as given in (H3). Let $H_1(x) = c([x]^{-\beta} - x^{-\beta}) + b([x]) + H([x])$, where by [.] stands for the ceiling function. With the convention $0^{-\beta} = 0$, the function $H_1(x)$ is well defined in $[0, 1)$ and we set $c_H = \int_0^{\infty} H_1(x)\, dx$.

Let $q = \max\{j \geq 0 : (j+1)\beta - j > 0\}$. Set $C_H = -c_H c^{-1}\Gamma(1-\beta)^{-1}$. For $p = 0, \ldots, q$ define $d_p = (C_H)^p c^{-1}\Gamma(1-\beta)^{-1}\Gamma((p+1)\beta - p)^{-1}$.

In what follows, $P$ denotes the spectral projection corresponding to the eigenvalue 1 for $R(1)$. So, we can write $Pv(y) \equiv \int_Y v\, d\mu$. With these specified we can state

**Theorem 1.1** *Assume (H1), (H2) and (H3). Then*

$$T_n = d_0 n^{\beta-1}P + d_1 n^{2(\beta-1)}P + d_2 n^{3(\beta-1)}P + \cdots + d_q n^{q(\beta-1)}P + D.$$

*where* $\|D\| = O(n^{-\beta})$.

**Remark 1.2** Theorem 1.1 improves the error term of Theorem [21, Theorem 9.1]; the error term in Theorem [21, Theorem 9.1] is $O(n^{-1/2})$. Thus, Theorem [21, Theorem 9.1] provides second order asymptotic for $T_n$ for $\beta > 3/4$, while Theorem 1.1 provides second order asymptotic for $T_n$ for $\beta > 2/3$.

As explained below, the proof of Theorem 1.1 is based on a good understanding of the asymptotic behavior of $\frac{d}{d\theta}T(z)$, $z \in \mathbb{D}$. We do not know how to prove Theorem 1.1 using the techniques in [21], which require a careful analysis of $T(z)$ on the circle $\{z \in \mathbb{C} : |z| = 1\}$.

**Remark 1.3** We believe that the proofs of the abstract results on higher order theory in [21, 22] can also be adjusted to the requirement $\mathcal{B} \subset L^2(Y)$ instead of the stronger requirement $\mathcal{B} \subset L^\infty(Y)$ used in these works. However, the assumption $\mathcal{B} \subset L^\infty(Y)$ is important for uniform convergence results (see Subsection 1.3 below).

For first order theory under (H1), as pointed to me by Ian Melbourne in one discussion, the assumption $\mathcal{B} \subset L^1(Y)$ suffices (see Section 5).

**Remark 1.4** Along the proof of Theorem 1.1, we obtain that Theorem 1.1 holds for scalar renewal sequences as well (see Remark 3.1). To make this explicit we recall the required terminology. Let $(X_i)_{i\geq 1}$ be a sequence of positive integer-valued independent identically distributed random variables with probabilities $P(X_i = j) = f_j$. Define the partial sums $S_n = \sum_{j=1}^n X_j$, and set $u_0 = 1$ and $u_n = \sum_{j=1}^n f_j u_{n-j}$, $n \geq 1$. Then it is easy to see that $u_n = \sum_{j=0}^n P(S_j = n)$. The sequences $(u_n)_{n\geq 0}$ are called *renewal sequences*. The assumption on $\mu(\varphi > n)$ in the dynamical systems setting corresponds to an equivalent assumption $\sum_{j>n} f_j$ in the scalar setting and the scalar renewal sequences $u_n$ correspond to the operator renewal sequences $T_n$.

As far as we understand, the technique in [11] (generalized to the operator case in [21]) does not allow one to prove the scalar version of Theorem 1.1.

Our second result reads as follows

**Theorem 1.5** *Suppose that $\mu(\varphi > n) = \ell(n)n^{-\beta}$, where $\ell(n)$ is a slowly varying function. Assume (H1′) and (H2) . Then*

$$\lim_{n\to\infty} \ell(n)n^{1-\beta}T_n = d_0 P.$$

At the moment we do not have an interesting example where [21, Theorem 2.1] applies, but Theorem 1.1 does not. However, there are indications that the above result might apply to the type of maps considered in [17]. More importantly, since $\mathcal{B} \subset L^2(Y)$, both Theorem 1.1 and Theorem 1.5 might be applied to two dimensional maps with $\mathcal{B} = BV(Y)$, where $BV$ is the usual notation for the space of functions of bounded variation.

So far we said that the techniques considered here improve the results on mixing rates obtained in [21]. As a matter of completeness, in Section 5, we show that our techniques offer a very simple proof of the key [21, Theorem 2.1], which provides first order asymptotics of $T_n$, under the general assumption $\mu(\varphi > n) = \ell(n)n^{-\beta}$.

**Strategy of the proofs** By the arguments in [21] (equivalently, by the simplified argument in [22]), the Fourier coefficients of $T(z)$, $z \in \mathcal{S}^1$ coincide with the coefficients of $T(z))$, $z \in \mathbb{D}$. Hence, first and higher order of $T_n$ can be obtained by estimating either the Fourier or Taylor coefficients of $T(z)$, $z \in \bar{\mathbb{D}}$.

As already mentioned, the techniques in [21] require a careful analysis of $T(z)$ on the circle $\{z \in \mathbb{C} : |z| = 1\}$, which lead to the desired estimates for the Fourier coefficients of $T(z)$, $z \in \mathcal{S}^1$. In this paper, we shift the computation in the open unit disk $\mathbb{D}$, where we can exploit the analyticity of $T(z)$.

In order to prove Theorem 1.1, we need a higher order asymptotic of $\frac{d}{d\theta}(T(z))$ for both $z \in \bar{\mathbb{D}} \cap B_\epsilon(1)$ and $z \in \bar{\mathbb{D}} \setminus B_\epsilon(1)$. The asymptotic of $\frac{d}{d\theta}(T(z))$, $z \in \bar{\mathbb{D}} \cap B_\epsilon(1)$ (see Section 2) is obtained using the strenght of (H3). The symptotic of $\frac{d}{d\theta}(T(z))$ for $z \in \bar{\mathbb{D}} \setminus B_\epsilon(1)$ is much easier to obtain (see Section 2) and it only requires (H1). The precise asymptotic of $\frac{d}{d\theta}(T(z))$, $z \in \mathbb{D}$ allows us to obtain higher order of $T_n$ (in Section 3) integrating by parts on a well chosen contour.

For the proof of Theorem 1.5, it is still very important that we can do the computation on a well chosen contour in $\mathbb{D}$. This allows us to exploit un upper bound of $\frac{d}{d\theta}(T(z))$, $z \in \bar{\mathbb{D}} \setminus B_\epsilon(1)$, for which (H1') suffices. To control the asymptotic of $T(z)$, $z \in \bar{\mathbb{D}} \cap B_\epsilon(1)$ we use techniques similar to the ones in [21], although the required estimates are much trickier. For details we refer to Section 4.

## 1.3 Application of Theorem 1.1 to intermittency maps of the interval

An important class of maps to which Theorem 1.1 is given by the family of Pomeau-Manneville intermittency maps [23]. These are interval maps with indifferent fixed points; that is, they are uniformly expanding except for an indifferent fixed point at 0. To fix notation, we recall the version studied by Liverani *et al.* [20]:

$$fx = \begin{cases} x(1+2^\alpha x^\alpha), & 0 < x < \frac{1}{2} \\ 2x-1, & \frac{1}{2} < x < 1 \end{cases}. \tag{1.1}$$

It is well known that in this case $\mu(\varphi = n) = O(n^{-(\beta+1)})$ with $\beta = 1/\alpha$. We also recall that for $\alpha \geq 1$, we are in the situation of infinite ergodic theory; there exists a unique (up to scaling) $\sigma$-finite, absolutely continuous invariant measure $\mu$.

In the special case of (1.1), we exploit the fact that $\mu(\varphi > n) = cn^{-\beta} + c_1 n^{-2\beta} + c_2 n^{-3\beta} + \hat{c}n^{-(\beta+1)} + \tilde{c}(\log n)n^{-(\beta+1)} + H(n)$ where $H(n) = O((\log n)^2 n^{-(\beta+2)} + n^{-4\beta})$ and $c, c_1, c_2, \hat{c}, \tilde{c}$ are real constants (see Proposition B.1 in Appendix B). Hence, assumption (H3) is satisfied and we state

**Proposition 1.6** *Assume the setting of* (1.1). *Let* $v : [0,1] \to \mathbb{R}$ *be a Hölder or bounded variation observable supported on a compact subset of* $(0,1]$. *Let* $q = \max\{j \geq 0 : (j+1)\beta - j > 0\}$. *If* $\beta \in (1/2, 1)$ *then there exist positive constants* $d_0, \ldots, d_q$ *(that depend only on* $f$*) such that*

$$L^n v = \left(d_0 n^{\beta-1} + d_1 n^{2(\beta-1)} + d_2 n^{3(\beta-1)} + \cdots + d_q n^{q(\beta-1)}\right) \int v d\mu + O(n^{-\beta}),$$

*uniformly on compact subsets of* $(0,1]$.

**Proof** The conclusion follows immediately from Theorem 1.1, Proposition B.1 and the fact that in this case the Banach space $\mathcal{B}$ (of either Hölder or bounded variation supported on $Y$) is embeded in $L^\infty(Y)$. As noted in [21], the fact that $\mathcal{B} \subset L^\infty(Y)$ implies almost sure convergence at a uniform rate on $Y$. Redifinig sequences on a set of measure zero, we obtain uniform convergene on $Y$. ∎

Proposition 1.6 (via Theorem 1.1) improves the error term of [21, Corollary 11.13] by, essentially, improving the error term in [21, Theorem 9.1] which provides second asymptotics order for $T_n$. The error term in [21, Theorem 9.1] is $O(n^{-1/2})$. Thus, while [21, Theorem 9.1/ Corollary 11.13] provide second order asymptotics for $T_n$ and $L^n$ for $\beta > 3/4$, Theorem 1.1/ Proposition 1.6 provide second order asymptotics for $T_n$ and $L^n$ for $\beta > 2/3$.

We also note that using Proposition 1.6, one can obtain second order asymptotics of $L^n v$ for observables $v$ supported on the whole $[0,1]$ (the corresponding statement and proof are identical to the statement and proof [21, Theorem 11.4]) with the improved error term $O(n^{-\beta})$.

As shown in [21], higher order asymptotics/mixing rates can be used to obtain error rates for some limit laws (such as arcsine laws) associated with systems preserving an infinite measure. For typical limit laws for infinite measure preserving transformations we refer to [1, 2, 5, 26, 28, 30] (see also [7, 19, 10, 6] and references therein for the setting of Markov chains). An interesting consequence of Theorem 1.1 is an improved convergence rate in the Dynkin-Lamperti arcsine law for waiting times. Proposition 1.7 below improves the convergence rate obtained in [21, Corollary 9.10]. It is known that the arcsine law holds for a large class of interval maps with indiferent fixed points (not necessarily Markov) for all $\beta$ [30]. See also [27, 28] for more general transformations.

To state our result we recall the following. For $x \in \bigcup_{j=0}^n f^{-j}Y$, $n \geq 1$, let

$$Z_n(x) = \max\{0 \leq j \leq n : f^j x \in Y\},$$

denote the time of the last visit of the orbit of $x$ to $Y$ during the time interval $[0,n]$. Let $\zeta_\beta$ denote a random variable distributed according to the $\mathcal{B}(1-\beta,\beta)$ distribution:

$$\mathbb{P}(\zeta_\beta \leq t) = d_0 \int_0^t \frac{1}{u^{1-\beta}} \frac{1}{(1-u)^\beta}\, du, \quad t \in [0,1],$$

where $d_0 = \frac{1}{\pi} \sin \beta\pi$.

**Proposition 1.7** *Assume the setting of* (1.1). *Suppose* $\beta \in (1/2, 1)$ *and let* $\gamma = \min\{1-\beta, 2\beta - 1\}$.

*Let* $\nu$ *be an absolutely continuous probability measure on* $Y$ *with density* $g \in \mathcal{B}$. *Then there is a constant* $C > 0$ *independent of* $\nu$ *such that*

$$\left|\nu\{\tfrac{1}{n} Z_n \leq t\} - \mathbb{P}(\zeta_\beta \leq t)\right| \leq C\|g\| n^{-\gamma}.$$

**Proof** The proof goes exactly as the proof of Corollary [21, Corollary 9.10], except for the use of Proposition 1.6 instead of [21, Theorem 11.4]. ∎

**Remark 1.8** Corollary [21, Corollary 9.10] differs from the above result in the fact that $\gamma = \min\{1-\beta, \beta - 1/2\}$. Thus, Corollary [21, Corollary 9.10] is optimal for $\beta > 3/4$, while the above result is optimal for $\beta > 2/3$.

Finally, we note that one can obtain a higher order tail expansion (better than the estimate obtained in Proposition B.1). In this sense, a natural question is whether one can improve Proposition 1.6. In fact, given the results of [14], it seems natural to seek for a method that can provide mixing rates for all $\beta$. The techniques used here are not suitable for this task (see the explanatory Remark 3.2). Mixing rates for all $\beta$ in the special case of (1.1), where we exploit the stronger, but much harder to take advantage of, assumption on the *small* tail $\mu(\varphi = n) = O(n^{-(\beta+1)})$ is work in progress.

The rest of the paper is organized as follows. Sections 2, 3 are devoted to the proof of Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.5. In Section 5, we offer a short proof of the [21, Theorem 2.1]. Appendix A contains the proof of several technical results used in Section 2. In appendix B we improve the estimate on the tail sequence $\mu(\varphi > n)$ associated with ( 1.1) obtained in [22, Proposition C2]; this is the content of Proposition B.1.

**Notation** We use "big O" and $\ll$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ as $n \to \infty$ if there is a constant $C > 0$ such that $a_n \le C b_n$ for all $n \ge 1$.

# 2 Asymptotics of $\frac{d}{d\theta}T(z)$

Throughout this section we assume (H1), (H2) and (H3).

For $p = 0, \ldots, q$ define $C_p = (C_H)^p((p+1)\beta - p)$. The main result of this section reads as follows

**Lemma 2.1** *Write $z = e^{-u+i\theta}, u > 0$. Choose $\epsilon > 0$ such that $\lambda(z)$ is well defined for $z = e^{-u+i\theta}$, $u \in (0,\epsilon)$ and $\theta \in (-\epsilon, \epsilon)$. Then, for all $u \in (0,\epsilon)$ and all $\theta \in (-\epsilon,\epsilon)$,*

$$c\Gamma(1-\beta)\frac{d}{d\theta}T(z) = i\sum_{p=0}^{q} C_p(u - i\theta)^{(p-1)-(p+1)\beta}P + E(z),$$

*where $\|E(z)\| = O(|u - i\theta|^{-1}) + O(u^{\beta-1}|u - i\theta|^{-\beta})$.*

*Also, for all $u > 0$ and all $\theta \in [\epsilon, \pi]$, $\|\frac{d}{d\theta}T(z)\| = O(u^{\beta-1})$.*

Below we collect some results that will be instrumental in the proof of Lemma 2.1. By (H1′) and (H2), there exist $\epsilon > 0$ and a continuous family of simple eigenvalues of $R(z)$, namely $\lambda(z)$ for $z \in \bar{\mathbb{D}} \cap B_\epsilon(1)$ with $\lambda(1) = 1$. Let $P(z) : \mathcal{B} \to \mathcal{B}$ denote the corresponding family of spectral projections with $P(1) = P$ and complementary projections $Q(z) = I - P(z)$. Also, let $v(z) \in \mathcal{B}$ denote the corresponding family of eigenfunctions normalized so that $\int_Y v(z)\, d\mu = 1$ for all $z$. In particular, $v(1) \equiv 1$.

Then we can write

$$T(z) = (1 - \lambda(z))^{-1} P(z) + (I - R(z))^{-1} Q(z), \tag{2.1}$$

for $z \in \bar{\mathbb{D}} \cap B_\epsilon(1)$, $z \neq 1$.

**Proposition 2.2** *Assume (H1′) and (H2). There exists $\epsilon, C > 0$ such that $\|(I - R(z))^{-1} Q(z)\| \leq C$ for $z \in \bar{\mathbb{D}} \cap B_\epsilon(1)$, $z \neq 1$ and $\|T(z)\| \leq C$ for $z \in \bar{\mathbb{D}} \setminus B_\epsilon(1)$.* ∎

The following consequence of (H1') (and thus (H1)) is standard (see, for instance, step 1 of the proof of [12, Lemma 3.1])).

**Proposition 2.3** *Assume $\sum_{j>n} \|R_j\| \ll n^{-\rho}\ell(n)$ with $\ell$ is slowly varying and $\rho \in (0,1)$. Then there is a constant $C > 0$ such that for all $u \geq 0$, $\theta \in [0, 2\pi)$ and $h > 0$, $\|R(e^{-u+i\theta}) - R(e^{-u+i(\theta-h)})\| \leq C\ell(1/h)h^\rho$ and $\|R(e^{-u+i\theta}) - R(1)\| \leq C\ell(1/|u-i\theta|)|u-i\theta|^\rho$.* ∎

**Corollary 2.4** *The estimates for $R(z)$ in Proposition 2.3 are inherited by the families $P(z)$, $Q(z)$, $\lambda(z)$ and $v(z)$, where defined.* ∎

**Proposition 2.5** *Suppose $\sum_{j>n} \|R_j\| \ll n^{-\rho}\ell(n)$ where $\rho \in (0,1)$ and $\ell$ is slowly varying. Write $z = e^{-u+i\theta}$, $u > 0$. Then,*

$$\|\frac{d}{d\theta} R(z)\| \ll u^{\rho-1}\ell(1/u).$$

*Moreover, these estimates are inherited by the families $\frac{d}{d\theta}P(z)$, $\frac{d}{d\theta}Q(z)$, $\frac{d}{d\theta}\lambda(z)$ and $\frac{d}{d\theta}v(z)$, where defined.*

**Proof** The result follows by standard computations. We provide the argument for completeness. Put $S_j = \sum_{q=j+1}^\infty \|R_q\|$ and note that $S_j \ll j^{-\rho}\ell(j)$. Compute that

$$\begin{aligned}\|\frac{d}{d\theta} R(z)\| &\ll \|\sum_{j=1}^\infty j\|R_j\| e^{-(j-1)u} \ll \sum_{j=1}^\infty j(S_{j-1} - S_j)e^{-(j-1)u} = \sum_{j=1}^\infty S_{j-1}e^{-(j-1)u} \\ &+ \sum_{j=1}^\infty \Big((j-1)S_{j-1} - jS_j\Big)e^{-(j-1)u} = \sum_{j=1}^\infty S_{j-1}e^{-(j-1)u} + (1 - e^u)\sum_{j=1}^\infty jS_j e^{-uj}\end{aligned}$$

Hence,

$$\begin{aligned}\|\frac{d}{d\theta} R(z)\| &\ll \sum_j j^{-\rho}\ell(j)e^{-uj} + u\sum_j j^{1-\rho}\ell(j)e^{-uj} \ll \frac{1}{u}u^\rho \ell(1/u)\int_0^\infty \sigma^{-\rho}\ell(\sigma/u)\ell^{-1}(1/u)e^{-\sigma}d\sigma \\ &+ u^{\rho-1}\ell(1/u)\int_0^\infty \sigma^{1-\rho}e^{-\sigma}\ell(\sigma/u)\ell^{-1}(1/u)d\sigma \ll u^{\rho-1}\ell(1/u).\end{aligned}$$

The estimates for the families $\frac{d}{d\theta}P(z)$, $\frac{d}{d\theta}Q(z)$, $\frac{d}{d\theta}\lambda(z)$ and $\frac{d}{d\theta}v(z)$ follow from the above estimate and their corresponding definitions (see, for instance, [18]). ∎

Before stating the next result, we recall the meaning of the parameters in (H3): $\mu(\varphi > n) = cn^{-\beta} + b(n) + H(n)$ for $c > 0$, $b(n) = O(n^{-2\beta})$, and $H(n) = O(n^{-\gamma})$, $\gamma > 2$. The other assumptions on the function $b(n)$ will be recalled in the proof below.

**Proposition 2.6** *Assume the conditions of Lemma 2.1. Choose $\epsilon > 0$ such that $\lambda(z)$ is well defined for $z = e^{-u+i\theta}$, $u \in (0, \epsilon)$ and $\theta \in (-\epsilon, \epsilon)$. Then,*

$$1 - \lambda(z) = c\Gamma(1-\beta)(u - i\theta)^{\beta} + c_H(u - i\theta) + D(z),$$

*where $D(z)$ is such that $|D(z)| = O(|u - i\theta|^{2\beta})$ and $|\frac{d}{d\theta}D(z)| = O(|u - i\theta|^{2\beta-1} + u^{\beta-1}|u - i\theta|^{\beta})$.*

**Proof** The proof below is a refinement of the proof of [22, Lemma A.4] for the case $\beta > 1/2$.

Recall that $v(z) \in \mathcal{B}$ denotes the family of eigenfunctions of $R(z)$ corresponding to the eigenvalue $\lambda(z)$, normalized so that $\int_Y v(z)\, d\mu = 1$ for all $z$. In particular, $v(1) \equiv 1$. Also, under (H3) Corollary 2.4 applies (with $\rho = \beta$ and $\ell$ bounded), yielding $\|v(z) - 1\| \ll |u - i\theta|^{\beta}$. For a short argument for the desired estimate , we follow the formalism in [13] (a simplification of [3]) and write

$$1-\lambda(z) = 1-\int_Y \lambda(z)v(z)d\mu = 1-\int_Y R(e^{(-u+i\theta)\phi}v(z))d\mu = \int_Y (1-e^{(-u+i\theta)\phi})\, d\mu - V(z), \tag{2.2}$$

where $V(z) = \int_Y (R(z) - R(1))(v(z) - v(1))d\mu$. Since $\mathcal{B} \subset L^2(Y)$,

$$\begin{aligned}|V(z)| \le \Big(\int_Y (R(z) - R(1))^2\, d\mu\Big)^{1/2} \Big(\int_Y (v(z) - v(1))^2\, d\mu\Big)^{1/2} &\ll \|v(z) - v(1)\|\, \|R(z) - R(1)\| \\ &\ll |u - i\theta|^{2\beta}.\end{aligned}$$

where the last inequality was obtained using Proposition 2.3 (with $\rho = \beta$ and $\ell$ bounded).

Write

$$\begin{aligned}\frac{d}{d\theta}(V(z)) &= \int_Y \frac{d}{d\theta}(R(z))(v(z) - v(1))d\mu + \int_Y (R(z) - R(1))\frac{d}{d\theta}(v(z))d\mu \\ &= V_1(z) + V_2(z).\end{aligned}$$

The fact that$\mathcal{B} \subset L^2(Y)$ together with Proposition 2.5 and Corollary 2.4 (with $\rho = \beta$) yields

$$|V_1(z))| \ll \Big(\int_Y \Big(\frac{d}{d\theta}(R(z)\Big)^2\, d\mu\Big)^{1/2} \Big(\int_Y (v(z) - v(1))^2\, d\mu\Big)^{1/2} \ll u^{\beta-1}|u - i\theta|^{\beta}.$$

Similarly, $|V_2(z))| \ll u^{\beta-1}|u-i\theta|^\beta$ and thus, $|\frac{d}{d\theta}(V(z))| \ll u^{\beta-1}|u-i\theta|^\beta$.

It remains to estimate $\int_Y(1-e^{-(u-i\theta)\varphi})\,d\mu$. We proceed as in the proof of [22, Lemma A.4 ]. Define the distribution function $G(x)=\mu(\varphi\le x)$. Then $\int_Y(1-e^{-(u-i\theta)\varphi})\,d\mu=\int_0^\infty(1-e^{-(u-i\theta)x)})\,dG(x)$, where $1-G(x)=cx^{-\beta}+H_1(x)$. We recall that $H_1(x)=c([x]^{-\beta}-x^{-\beta})+b([x])+H([x])$, where $b$ is such that $xb(x)$ has bounded variation and $b(x)=O(x^{-2\beta})$ and $H(x)=O(x^{-\gamma})$, $\gamma>2$. So, $H_1(x)=O(x^{-2\beta})$.

Integrating by parts,

$$\begin{aligned}\int_0^\infty(1-e^{-(u-i\theta)x)})\,dG(x)&=c(u-i\theta)^\beta\int_0^\infty\frac{e^{-(u-i\theta)x}}{[(u-i\theta)x]^\beta}(u-i\theta)\,dx+(u-i\theta)\int_0^\infty e^{-(u-i\theta)x}H_1(x)\,dx\\&=c(u-i\theta)^\beta I+(u-i\theta)M(u,\theta).\end{aligned}$$

By [22, Proposition B1], $I=\Gamma(1-\beta)$. By Proposition A.1 (a) (with $M(x)=H_1(x)$ and $\rho=\gamma$) we have $(u-i\theta)M(u,\theta)-c_H(u-i\theta)=O(|u-i\theta|^\gamma)$. Hence, $\int_0^\infty(1-e^{(-u+i\theta)x)})\,dG(x)=c\Gamma(1-\beta)(u-i\theta)^\beta+c_H(u-i\theta)+O(|u-i\theta|^\gamma)$.

Define $D(z)=\Big((u-i\theta)M(u,\theta)-c_H(u-i\theta)\Big)+V(z)$ and note that $|D(z)|=O(|u-i\theta|^\gamma,|u-i\theta|^{2\beta})$.

We continue with the estimate for $\frac{d}{d\theta}D(z)$. Since, we already know $|\frac{d}{d\theta}(V(z))|\ll u^{\beta-1}|u-i\theta|^\beta$, we just have to estimate $\frac{d}{d\theta}\Big((u-i\theta)M(u,\theta)-c_H(u-i\theta)\Big)$.

Put $\Delta(x)=[x]^{-\beta}-x^{-\beta}$. Recall $H_1(x)=c\Delta(x)+b([x])+H([x])$ and write

$$\begin{aligned}M(u,\theta)&=c\int_0^\infty e^{-(u-i\theta)x}\Delta(x)\,dx+\int_0^\infty e^{-(u-i\theta)x}b([x])\,dx+\int_0^\infty e^{-(u-i\theta)x}H([x])\,dx\\&=W(u,\theta)+J_b(u,\theta)+J_H(u,\theta).\end{aligned}$$

Put $C_\Delta=\int_0^\infty\Delta(x)\,dx$. By Proposition A.4 (a), $(u-i\theta)W(u,\theta)=C_\Delta(u-i\theta)+O(|u-i\theta|^{\beta+1})$. By Proposition A.4 (b), $\Big|\frac{d}{d\theta}\Big((u-i\theta)W(u,\theta)-C_{\Delta_\ell}(u-i\theta)\Big)\Big|=O(|u-i\theta|^\beta)$.

Recall that $b(x)=O(x^{-2\beta})$, $2\beta>1$. Put $C_b=\int_0^\infty b([x])\,dx$. By Proposition A.1 (a) (with $M(x)=b([x])$ and $\rho=2\beta$), $(u-i\theta)J_b(u,\theta)=C_b(u-i\theta)+O(|u-i\theta|^\gamma)$. Next, recall that $xb(x)$ has bounded variation. By Proposition A.1 (b), $\Big|\frac{d}{d\theta}\Big((u-i\theta)J_b(u,\theta)-C_b(u-i\theta)\Big)\Big|=O(|u-i\theta|^{2\beta-1})$.

Finally, recall that $H(x)=O(x^{-\gamma})$, $\gamma>2$ and put $C_h=\int_0^\infty H([x])\,dx$. By Proposition A.3 (a) (with $M(x)=H([x])$ and $\rho=\gamma$), $(u-i\theta)J_H(u,\theta)=C_h(u-i\theta)+O(|u-i\theta|^\gamma)$. By Proposition A.3 (b) , $\Big|\frac{d}{d\theta}\Big((u-i\theta)J_H(u,\theta)-C_h(u-i\theta)\Big)\Big|=O(|u-i\theta|^{\gamma-1})$. The conclusion follows by putting all these estimates together. ∎

**Proof of Lemma 2.1** Recall $C_H=-c_Hc^{-1}\Gamma(1-\beta)^{-1}$. By Proposition 2.6 we know that for all $u\in(0,\epsilon)$, $\theta\in(-\epsilon,\epsilon)$,

$$1-\lambda(z)=c\Gamma(1-\beta)(u-i\theta)^\beta\Big(1-\Big(C_H(u-i\theta)^{1-\beta}+(u-i\theta)^{-\beta}D(z)\Big)\Big),$$

where $|D(z)| = O(|u - i\theta|^{2\beta})$ and $|\frac{d}{d\theta}D(z)| \ll u^{\beta-1}|u - i\theta|^{\beta}$.

Recall $q = \max\{j \geq 0 : (j+1)\beta - j > 0\}$ and compute that

$$c\Gamma(1-\beta)(1-\lambda(z))^{-1} = \sum_{p=0}^{q}(C_H)^p(u-i\theta)^{p-(p+1)\beta} + O(1).$$

Furthermore, by Proposition 2.6 we also know that for all $u \in (0,\epsilon)$, $\theta \in (-\epsilon,\epsilon)$,

$$(c\Gamma(1-\beta))^{-1}\frac{d}{d\theta}(\lambda(z)) = i\beta(u-i\theta)^{\beta-1} - iC_H c\Gamma(1-\beta) + O(|u-i\theta|^{2\beta-1}, u^{\beta-1}|u-i\theta|^{\beta}).$$

Note that

$$\frac{d}{d\theta}\Big((1-\lambda(z))^{-1}\Big) = ie^{-u+i\theta}\frac{d}{dz}(\lambda(z))(1-\lambda(z))^{-2} = \frac{d}{d\theta}(\lambda(z))(1-\lambda(z))^{-2}.$$

The asymptotic expansions of $(1-\lambda(z))^{-1}$ and $\frac{d}{d\theta}(\lambda(z))$ above together with standard calculations show that for all $u \in (0,\epsilon)$, $\theta \in (-\epsilon,\epsilon)$,

$$c\Gamma(1-\beta)\frac{d}{d\theta}\Big((1-\lambda(z))^{-1}\Big) = i\sum_{p=0}^{q}C_p(u-i\theta)^{(p-1)-(p+1)\beta} + E(z),$$

where $\|E(z)\| \ll |u-i\theta|^{\gamma-2\beta-1} + |u-i\theta|^{-\beta} + |u-i\theta|^{-1} + u^{\beta-1}|u-i\theta|^{-\beta} \ll |u-i\theta|^{-1} + u^{\beta-1}|u-i\theta|^{-\beta}$.

Next, for $z = e^{-u+i\theta}$, $u \in (0,\epsilon)$ and $\theta \in (-\epsilon,\epsilon)$ we write

$$T(z) = (1-\lambda(z))^{-1}P + (1-\lambda(z))^{-1}(P(z)-P) + (I-R(z))^{-1}Q(z).$$

By Proposition 2.5, $\|\frac{d}{d\theta}(P(z)-P)\| \ll u^{\beta-1}$. We already know $|\frac{d}{d\theta}(1-\lambda(z))^{-1})| \ll |u-i\theta|^{-(\beta+1)} + |u-i\theta|^{-1} + u^{\beta-1}|u-i\theta|^{-\beta}$. These last two estimates together with Corollary 2.4 imply that $|\frac{d}{d\theta}\Big((1-\lambda(z))^{-1}(P(z)-P)\Big)| \ll |u-i\theta|^{-1} + u^{\beta-1}|u-i\theta|^{-\beta}$.

Note that $(I-R(z))^{-1}Q(z)$ is well defined for all $z = e^{-u+i\theta}$, $u \in (0,\epsilon)$ and $\theta \in (-\epsilon,\epsilon)$. Since $\frac{d}{d\theta}(I-R(z))^{-1} = ie^{-u+i\theta}(I-R(z)^{-1}\frac{d}{dz}R(z)(I-R(z))^{-1}$, differentiating (in $\theta$) yields

$$\frac{d}{d\theta}((I-R(z))^{-1}Q(z)) = ie^{-u+i\theta}\Big((I-R(z))^{-1}\frac{d}{dz}R(z)(I-R(z))^{-1}Q(z) + (I-R(z))^{-1}\frac{d}{dz}R(z)\Big).$$

We already know $\|(I-R(z))^{-1}\| \ll |u-i\theta|^{-\beta}$. This together with Proposition 2.2 and Proposition 2.5 implies that $\|\frac{d}{d\theta}((I-R(z))^{-1}Q(z))\| \ll u^{\beta-1}|u-i\theta|^{-\beta}$ for all $z = e^{-u+i\theta}$, $u \in (0,\epsilon)$ and $\theta \in (-\epsilon,\epsilon)$. Putting all these estimates together we obtain the desired estimate for $\frac{d}{d\theta}T(z)$ for $z = e^{-u+i\theta}$, $u > 0$ and $\theta \in (-\epsilon,\epsilon)$.

To conclude, we just need to recall that by Proposition 2.2, $\|(I-R(z))^{-1}\| \leq C$ for $z = e^{-u+i\theta}$, $u > 0$ and $\theta \in [\epsilon,\pi)$. This together with Proposition 2.5 implies $\|\frac{d}{d\theta}(I-R(z))^{-1}\| \ll u^{\beta-1}$ for $z = e^{-u+i\theta}$, $u > 0$ and $\theta \in [\epsilon,\pi)$, ending the proof. ∎

**Remark 2.7** For use below, we note that within the proof above we obtained higher order asymptotic of $\Psi(e^{-(u-i\theta)\varphi}) := \int_Y (1-e^{-(u-i\theta)\varphi})\,d\mu$ for all $u>0$, $\theta\in(-\pi,\pi]$. More precisely,

$$1-\Psi(e^{-(u-i\theta)\varphi}) = c\Gamma(1-\beta)(u-i\theta)^\beta + c_H(u-i\theta) + D(z),$$

where $D(z)$ is such that $|D(z)| = O(|u-i\theta|^{2\beta})$ and $|\frac{d}{d\theta}D(z)| = O(|u-i\theta|^{2\beta-1} + u^{\beta-1}|u-i\theta|^\beta)$.

# 3 Proof of Theorem 1.1

The proof of Theorem 1.1 will be completed once we estimate the coefficients of $T(z)$ for $z\in\mathbb{D}$.

**Proof of Theorem 1.1** Put $\Gamma = \{e^{-u}e^{i\theta} : -\pi\le\theta<\pi\}$ with $e^{-u}=e^{-1/n}$, $n\ge 1$. Write

$$T_n = \frac{1}{2\pi i}\int_\Gamma \frac{T(z)}{z^{n+1}}dz = \frac{e}{2\pi}\int_{-\pi}^{\pi} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta = \frac{i}{n}\frac{e}{2\pi}\int_{-\pi}^{\pi} T(e^{-1/n}e^{i\theta})\frac{d}{d\theta}(e^{-in\theta})d\theta.$$

Put $a=[-\pi,-\epsilon]\cup[\epsilon,\pi]$. Integration by parts together with Lemma 2.1 imply

$$\begin{aligned}\frac{2\pi}{e}c\Gamma(1-\beta)T_n &= -\frac{i}{n}\int_{-\pi}^{\pi}\frac{d}{d\theta}(T(e^{-1/n}e^{i\theta}))e^{-in\theta}d\theta = \frac{1}{n}\sum_{p=0}^{q}C_p\int_{-\epsilon}^{\epsilon}(\frac{1}{n}-i\theta)^{(p-1)-(p+1)\beta}e^{-in\theta}d\theta\Big)P \\ &\quad -\frac{i}{n}\int_{-\epsilon}^{\epsilon}E(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta - \frac{i}{n}\int_a\frac{d}{d\theta}(T(e^{-1/n}e^{i\theta}))e^{-in\theta}d\theta \\ &= \sum_{p=0}^{q}C_pI_pP + J + J'.\end{aligned}$$

By Lemma 2.1, for all $\theta\in[\epsilon,\pi]$, $\frac{d}{d\theta}(T(e^{-1/n}e^{i\theta}))\| \ll n^{1-\beta}$ . Hence, $|J'|\ll n^{-\beta}$. Also, by Lemma 2.1 we have $\|E(e^{-1/n}e^{i\theta})\| = O(|\frac{1}{n}-i\theta|^{-1}) + O((\frac{1}{n})^{\beta-1}|\frac{1}{n}-i\theta|^{-\beta})$. Thus,

$$\begin{aligned}|J| &\ll \frac{1}{n}\Big(\int_{-\epsilon}^{\epsilon}|\frac{1}{n}-i\theta|^{-1}d\theta + n^{1-\beta}\int_{-\epsilon}^{\epsilon}|\theta|^{-\beta}d\theta\Big) \ll \frac{1}{n}\Big(1+\int_{1/n}^{\epsilon}\theta^{-1}d\theta\Big) + \frac{n^{1-\beta}}{n} \\ &\ll \frac{\log n}{n} + n^{-\beta} \ll n^{-\beta}.\end{aligned}$$

Next, by [22, Corollary B.3](with $\rho=(p+1)\beta-p$ for $p=0,\dots,q$) we have

$$\frac{1}{n}\sum_{p=0}^{q}C_p\int_{-\epsilon}^{\epsilon}\frac{e^{-in\theta}}{(\frac{1}{n}-i\theta)^{(p+1)\beta-p+1}}d\theta = \frac{2\pi}{e}\sum_{p=0}^{q}\frac{C_p}{\Gamma((p+1)\beta-p+1))}n^{(p+1)(\beta-1)} + O(1/n).$$

Finally, note that

$$\frac{C_p}{\Gamma((p+1)\beta-p+1)}=\frac{(C_H)^p((p+1)\beta-p)}{\Gamma((p+1)\beta-p+1)}=\frac{(C_H)^p}{\Gamma((p+1)\beta-p)}=c\Gamma(1-\beta)d_p.$$

■

**Remark 3.1** We recall that $\mu(\varphi>n)$ in the dynamical systems setting corresponds to an equivalent assumption $\sum_{j>n} f_j$ in the scalar setting and the scalar renewal sequences $u_n$ correspond to the operator renewal sequences $T_n$ (see Remark 1.4). Replacing $T(z)$ in the proof above with $\Psi(z)$ defined in Remark 2.7, we obtain the scalar version of Theorem 1.1, that is higher oder asymptotic for $u_n$.

**Remark 3.2** The present techniques do not allow one to improve Theorem 1.1. Although, assuming a better tail expansion (better than (H3), for instance), one can obtain very good estimates on first and second derivatives of $\int_Y (1-e^{(-u+i\theta)\varphi})\,d\mu$, one cannot hope for good enough estimates for $\frac{d}{d\theta}V(z)$. Assuming a good enough tail expansion and that the invariant density is smooth enough, the best one can hope for is that the term $E(z)$ in Lemma 2.1 can be improved to $u^{\beta-1}|u-i\theta|^{q\beta}$ for some $q\geq 1$. Clearly, this improvement is useless when estimating coefficients: due to the presence of $u^{\beta-1}$, the error term in the expansion of $T_n$ is $n^{-\beta}$ and this cannot be improved using the current techniques.

# 4 Proof of Theorem 1.5

The results below will be instrumental in the proof of Theorem 1.5.

**Proposition 4.1** *Suppose that $\mu(\varphi>n)=\ell(n)n^{-\beta}$. Assume (H1′) and (H2)(i). Write $z=e^{-u+i\theta}$. Then as $u,\theta\to 0$,*

$$1-\lambda(e^{-u+i\theta})=\ell(1/|u-i\theta|)(u-i\theta)^{\beta}(1+o(1))$$

**Proof** By equation (2.2),

$$1-\lambda(z)=\int_Y(1-e^{(-u+i\theta)\varphi})\,d\mu-V(z),$$

where $V(z)=\int_Y(R(z)-R(1))(v(z)-v(1))d\mu$. Since $\mathcal{B}\subset L^2(Y)$,

$$|V(z)|\leq\Big(\int_Y(R(z)-R(1))^2\,d\mu\Big)^{1/2}\Big(\int_Y(v(z)-v(1))^2\,d\mu\Big)^{1/2}\ll\|v(z)-v(1)\|\,\|R(z)-R(1)\|$$
$$\ll|u-i\theta|^{2\tau}.$$

where the last inequality was obtained using Corollary 2.4 (with $\rho = \tau$). As shown in [22, Proof of Lemma A.4] (a generalization of the arguments used, for instance, in [11]), under (H) we have

$$\int_Y (1 - e^{(-u+i\theta)\varphi})\, d\mu = 1 - d_\beta \ell(1/|u - i\theta|)(u - i\theta)^\beta (1 + o(1)), \tag{4.1}$$

as $u, \theta \to 0$.

Putting together the last two equations,

$$1 - \lambda(e^{-u+i\theta}) = \ell(1/|u - i\theta|)(u - i\theta)^\beta (1 + G(e^{-u+i\theta})),$$

where $|G(e^{-u+i\theta})| \ll |u - i\theta|^{2\tau-\beta}$. The conclusion follows immediately since under (H1′), $2\tau > \beta$. ∎

By (2.1), $T(z) = (1-\lambda(z))^{-1}P + (1-\lambda(z))^{-1}(P(z)-P) + (I-R(z))^{-1}Q(z)$. Thus, an immediate consequence of Proposition 4.2, Corollary 2.4 and Proposition 2.2 is that as $u, \theta \to 0$,

$$T(z) = \ell(1/|u - i\theta|)(u - i\theta)^\beta (1 + o(1)) \tag{4.2}$$

**Proposition 4.2** *Suppose that $\mu(\varphi > n) = \ell(n)n^{-\beta}$. Assume (H1′) and (H2)(i).*

*Choose $\epsilon > 0$ such that $\lambda(z)$ is well defined for $z = e^{-u+i\theta}$, $u \in (0, \epsilon)$ and $\theta \in (-\epsilon, \epsilon)$. Then for any $h > 0$, $h \le \min\{|\theta|, u\}$ we have*

$$|\lambda(e^{-u+i\theta}) - \lambda(e^{-u+i(\theta-h)})| \ll h^\beta \ell(1/h) + h^\tau |u - i\theta|^\tau.$$

**Proof** Put $\Delta_\lambda = \lambda(e^{-u+i\theta}) - \lambda(e^{-u+i(\theta-h)})$. Using eq.(2.2) we write

$$\Delta_\lambda = \int_Y (e^{(-u+i(\theta-h))\varphi} - e^{(-u+i\theta)\varphi}) d\mu + \int_Y (R(e^{-u+i(\theta-h)}) - R(e^{-u+i\theta}))(v(e^{-u+i(\theta-h)}) - 1) d\mu$$
$$+ \int_Y (R(e^{-u+i\theta}) - R)(v(e^{-u+i(\theta-h)}) - v(e^{-u+i\theta})) d\mu$$

By Proposition 2.3 and Corollary 2.4, $\|R(e^{-u+i(\theta-h)}) - R(e^{-u+i\theta})\| \ll h^\tau$ and $\|v(e^{-u+i(\theta-h)}) - v(e^{-u+i\theta})\| \ll h^\tau$. Also, $\|R(e^{-u+i(\theta-h)}) - R\| \ll |u - i(\theta - h)|^\tau \ll |u - i\theta|^\tau$ and $\|v(e^{-u+i(\theta-h)}) - 1\| \ll |u - i(\theta - h)|^\tau \ll |u - i\theta|^\tau$.

Using the fact that $\mathcal{B} \subset L^2(Y)$ and reasoning as in the proof of Proposition 4.1,

$$|\int_Y (R(e^{-u+i(\theta-h)}) - R(e^{-u+i\theta}))(v((e^{-u+i(\theta-h)}) - 1) d\mu| \ll h^\tau |u - i\theta|^\tau$$

and

$$|\int_Y (R(e^{-u+i\theta}) - R)(v(e^{-u+i(\theta-h)}) - v(e^{-u+i\theta})) d\mu| \ll h^\tau |u - i\theta|^\tau.$$

Let $G(x) = \mu(\varphi \le x)$. Proceeding as in the proof of Proposition 2.6, $\int_Y (e^{(-u+i(\theta-h))\varphi} - e^{(-u+i\theta)\varphi}) d\mu = \int_0^\infty e^{-(u-i\theta)x}(e^{ihx} - 1) dG(x)$. The estimate $|\int_0^\infty e^{-(u-i\theta)x}(e^{ihx} - 1) dG(x)| \ll h^\beta \ell(1/h)$ follows by the argument used in the proof of [11, Lemma 3.3.2], ending the proof. ∎

**Corollary 4.3** *Assume the setting of Proposition 4.2. Let $u \in (0, \epsilon)$, $\theta \in (-\epsilon, \epsilon)$ and $h > 0$, $h \le \min\{|\theta|, u\}$. Then*

$$\|T(e^{-u}e^{i\theta}) - T(e^{-u}e^{i(\theta-h)})\| \ll \ell(1/|u-i\theta|)^{-2} h^\tau |u-i\theta|^{\tau-2\beta} + \ell(1/|u-i\theta|)^{-2} \ell(1/h) h^\beta |u-i\theta|^{-2\beta}.$$

**Proof** Let $\Delta_T = T(e^u e^{i\theta}) - T(e^{-u}e^{i(\theta-h)})$. Set

$$\Delta_{\lambda,P} = (1 - \lambda(e^{-u}e^{i\theta}))^{-1} P(e^{-u}e^{i\theta}) - (1 - \lambda(e^{-u}e^{i(\theta-h)}))^{-1} P(e^{-u}e^{i(\theta-h)})$$

and

$$\Delta_{(I-R)^{-1}Q} = (I - R(e^{-u}e^{i\theta}))^{-1} Q(e^{-u}e^{i\theta}) - (I - R(e^{-u}e^{i(\theta-h)}))^{-1} Q(e^{-u}e^{i(\theta-h)}).$$

So, we can write $\Delta_T = \Delta_{\lambda,P} + \Delta_{(I-R)^{-1}Q}$. Next,

$$\begin{aligned} \|\Delta_{\lambda,P}\| \ll\ & \|(1 - \lambda(e^{-u}e^{i\theta}))^{-1} (P(e^{-u}e^{i\theta}) - P(e^{-u}e^{i(\theta-h)}))\| \\ & + \|P(e^{-u}e^{i(\theta-h)}) \Big( (1 - \lambda(e^{-u}e^{i\theta}))^{-1} - (1 - \lambda(e^{-u}e^{i(\theta-h)}))^{-1} \Big)\|. \end{aligned}$$

By Proposition 4.1, $|(1 - \lambda(e^{-u+i\theta}))^{-1}| \ll \ell(1/|u - i\theta|)^{-1} |u - i\theta|^{-\beta}$. This together with Corollary 2.4 yields $\|(1 - \lambda(e^{-u}e^{i\theta}))^{-1}(P(e^{-u}e^{i\theta}) - P(e^{-u}e^{i(\theta-h)}))\| \ll \ell(1/|u - i\theta|)^{-1} \ell(1/h) h^\tau |u - i\theta|^{-\beta}$. Also, by Proposition 4.1 and Proposition 4.2,

$$\begin{aligned} \|P(e^{-u}e^{i(\theta-h)}) \Big( (1 - \lambda(e^{-u}e^{i\theta}))^{-1} - (1 - \lambda(e^{-u}e^{i(\theta-h)}))^{-1} \Big)\| \ll\ & \ell(1/|u - i\theta|)^{-2} \ell(1/h) h^\beta |u - i\theta|^{-2\beta} \\ & + \ell(1/|u - i\theta|)^{-2} h^\tau |u - i\theta|^{\tau-2\beta}. \end{aligned}$$

Thus,

$$\begin{aligned} \|\Delta_{\lambda,P}\| \ll\ & \ell(1/|u - i\theta|)^{-1} \ell(1/h) h^\tau |u - i\theta|^{-\beta} + \ell(1/|u - i\theta|)^{-2} \ell(1/h) h^\beta |u - i\theta|^{-2\beta} \\ & + \ell(1/|u - i\theta|)^{-2} h^\tau |u - i\theta|^{\tau-2\beta}. \end{aligned}$$

To estimate $\|\Delta_{(I-R)^{-1}Q}\|$, we compute that

$$\begin{aligned} \|\Delta_{(I-R)^{-1}Q}\| \ll\ & \|(I - R(e^{-u}e^{i\theta}))^{-1} (Q(e^{-u}e^{i\theta}) - Q(e^{-u}e^{i(\theta-h)}))\| \\ & + \|(I - R(e^{-u}e^{i\theta}))^{-1} (R(e^{-u}e^{i\theta}) - R(e^{-u}e^{i\theta-h}))(I - R(e^{-u}e^{i(\theta-h)}))^{-1} Q(e^{-u}e^{i(\theta-h)})\|. \end{aligned}$$

By (4.2), $\|(I - R(e^{-u}e^{i\theta}))^{-1}\| \ll \ell(1/|u - i\theta|)^{-1} |u - i\theta|^{-\beta}$ and $\|(I - R(e^{-u}e^{i\theta-h}))^{-1}\| \ll \ell(1/|u - i(\theta-h)|)^{-1} |u - i(\theta-h)|^{-\beta} \ll \ell(1/|u - i(\theta-h)|)^{-1} |u - i\theta|^{-\beta}$. This together with Corollary 2.4 and Proposition 2.2 implies that

$$\|\Delta_{(I-R)^{-1}Q}\| \ll \Big( \ell(1/|u - i\theta|)^{-1} + \ell(1/|u - i(\theta-h)|)^{-1} \Big) h^\tau |u - i\theta|^{-\beta}.$$

Putting these together,

$$\|\Delta_T\| \ll \ell(1/|u-i\theta|)^{-2}h^\tau|u-i\theta|^{\tau-2\beta} + \ell(1/|u-i\theta|)^{-2}\ell(1/h)h^\beta|u-i\theta|^{-2\beta}$$
$$+ \ell(1/|u-i\theta|)^{-1}\ell(1/h)h^\tau|u-i\theta|^{-\beta} + \Big(\ell(1/|u-i\theta|)^{-1} + \ell(1/|u-i(\theta-h)|)^{-1}\Big)h^\tau|u-i\theta|^{-\beta}.$$

Under (H1′), $\beta > 1/2$ and $\tau > 1-\beta$. Assume without loss, $\tau < \beta$. Hence, $2\beta - \tau > \beta$ and thus

$$\ell(1/|u-i\theta|)^{-1}\ell(1/h)h^\tau|u-i\theta|^{-\beta} + \Big(\ell(1/|u-i\theta|)^{-1} + \ell(1/|u-i(\theta-h)|)^{-1}\Big)h^\tau|u-i\theta|^{-\beta}$$
$$\ll \ell(1/|u-i\theta|)^{-2}h^\tau|u-i\theta|^{\tau-2\beta},$$

which ends the proof. ∎

**Proposition 4.4** *[22, Proposition B.2] Let $\rho \in (0,1)$. Then*

$$\int_{-\infty}^{\infty} \frac{e^{-i\sigma}}{(1-i\sigma)^\rho}\, d\sigma = \frac{2\pi}{e}\frac{1}{\Gamma(\rho)}.$$

**Remark 4.5** The above estimate is actually contained in the proof of [22, Proposition B.2], which establishes $\int_{-\infty}^{\infty} e^{-i\sigma}(1-i\sigma)^{-(\rho+1)}\, d\sigma = \frac{2\pi}{e}\frac{1}{\Gamma(1+\rho)}$.

**Proposition 4.6** *Suppose that $\mu(\varphi > n) = \ell(n)n^{-\beta}$. Assume (H1′) and (H2). Choose $\epsilon > 0$ such that $\lambda(z)$ is well defined for $z = e^{-u+i\theta}$, $u \in (0,\epsilon)$ and $\theta \in (-\epsilon,\epsilon)$. Then for $n \geq 1$ and $b \in (0,\epsilon n)$,*

$$\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta = \frac{2\pi}{e}\frac{1}{\Gamma(1-\beta)\Gamma(\beta)}P.$$

**Proof** The proof below is standard (see, for instance, [21, Proof of Lemma 5.2]. We recall the argument for completeness.

By (4.2),

$$\Gamma(1-\beta)T(e^{-1/n}e^{i\theta}) = \ell(1/|\frac{1}{n}-i\theta|)^{-1}(\frac{1}{n}-i\theta)^{-\beta}g(n,\theta),$$

where $\lim_{n\to\infty,\theta\to 0} g(n,\theta) \to P$. Hence,

$$\Gamma(1-\beta)\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta = \int_{-b/n}^{b/n} \ell(1/|\frac{1}{n}-i\theta|)^{-1}(\frac{1}{n}-i\theta)^{-\beta}g(n,\theta)e^{-in\theta}d\theta$$
$$= n^{\beta-1}\ell^{-1}(n)\int_{-b}^{b} \frac{\ell(n)}{\ell(n/|1-i\sigma|)}\frac{1}{(1-i\sigma)^\beta}g(n,\sigma/n)e^{-i\sigma}d\sigma.$$

Hence,

$$n^{1-\beta}\ell(n)\Gamma(1-\beta)\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta = \Big(\int_{-b}^{b}\frac{e^{-i\sigma}}{(1-i\sigma)^{\beta}}d\sigma\Big)P.$$

For $b$ fixed, the DCT applies and thus,

$$\lim_{n\to\infty} n^{1-\beta}\ell(n)\Gamma(1-\beta)\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta = \Big(\int_{-b}^{b}\frac{e^{-i\sigma}}{(1-i\sigma)^{\beta}}d\sigma\Big)P.$$

By Proposition 4.4 (with $\rho = \beta$), $\lim_{b\to\infty}\int_{-b}^{b}\frac{e^{-i\sigma}}{(1-i\sigma)^{\beta}}d\sigma = \frac{2\pi}{e}\frac{1}{\Gamma(\beta)}$, which ends the proof. ∎

**Proof of Theorem 1.5** Let $\Gamma = \{e^{-u}e^{i\theta} : -\pi \le \theta < \pi\}$ with $e^{-u} = e^{-1/n}$, $n \ge 1$. Choose $\epsilon > 0$ such that $\lambda(e^{-1/n}e^{i\theta})$ is well defined for $\theta \in (-\epsilon, \epsilon)$. Let $b \in (0, \epsilon n)$. Let $A = [-\pi, -\epsilon] \cup [\epsilon, \pi]$.

With the above specified, we proceed to estimate $T_n$.

$$\begin{aligned} T_n &= \frac{1}{2\pi i}\int_{\Gamma}\frac{T(z)}{z^{n+1}}dz = \frac{e}{2\pi}\int_{-\pi}^{\pi} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta = \frac{e}{2\pi}\Big(\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta \\ &+ \int_{-\epsilon}^{-b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta + \int_{b/n}^{\epsilon} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta + \int_{A} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta\Big) \\ &= \frac{e}{2\pi}\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta + \frac{e}{2\pi}(I_\epsilon + I_{-\epsilon} + I_A). \end{aligned}$$

By Proposition 4.6, $\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)\Gamma(1-\beta)\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta = \frac{2\pi}{e}\frac{1}{\Gamma(\beta)}P$. Hence, the conclusion will follow once we show that $I_A = o(1)$ and $\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)(I_\epsilon + I_{-\epsilon}) = 0$.

We first estimate $I_A$. Compute that

$$I_A = \frac{i}{n}\int_A T(e^{-1/n}e^{i\theta})\frac{d}{d\theta}(e^{-in\theta})d\theta = \frac{1}{in}\int_a \frac{d}{d\theta}(T(e^{-1/n}e^{i\theta}))e^{-in\theta}d\theta + E(n),$$

where $E(n) \ll n^{-1}(\|T(e^{-1/n}e^{i\epsilon b/n}) + T(e^{-1/n}e^{i\pi})\|)$. By Proposition 2.2, $\|T(e^{-1/n}e^{i\theta})\| = O(1)$ for all $\theta \in A$. Hence $E(n) = O(n^{-1})$. Also, note that $\frac{d}{d\theta}(T(e^{-1/n}e^{i\theta})) = T(e^{-1/n}e^{i\theta})\frac{d}{d\theta}R(e^{-1/n}e^{i\theta})T(e^{-1/n}e^{i\theta})$. By Proposition 2.5, $\|\frac{d}{d\theta}R(e^{-1/n}e^{i\theta})\| \ll n^{1-\tau}$. Since $\|T(e^{-1/n}e^{i\theta})\| = O(1)$ for all $\theta \in a$, we obtain that $n^{-1}\|\frac{d}{d\theta}(T(e^{-1/n}e^{i\theta}))\| \ll n^{-\tau}$. Putting these together, $|I_A| \ll n^{-\tau} + n^{-1} \ll n^{-\tau}$. Since $\tau > 1-\beta$, we have $n^{1-\beta}\ell(n)|I_A| = o(1)$.

Next, we estimate $I_\epsilon$. The estimate for $I_{-\epsilon}$ follows by a similar argument. Proceeding as in the proof of [21, Lemma 5.1](see also [11]), we write

$$I_\epsilon = \int_{b/n}^{\epsilon} T(e^{-1/n}e^{i\theta})e^{-in\theta}\,d\theta = -\int_{(b+\pi)/n}^{\epsilon+\pi/n} T(e^{-1/n}e^{i(\theta-\pi/n)})e^{-in\theta}\,d\theta.$$

Hence

$$2I_\epsilon = \int_{b/n}^{\epsilon} T(e^{-1/n}e^{i\theta})e^{-in\theta}\,d\theta - \int_{(b+\pi)/n}^{\epsilon+\pi/n} T(e^{-1/n}e^{i(\theta-\pi/n)})e^{-in\theta}\,d\theta = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{\epsilon}^{\epsilon+\pi/n} T(e^{-1/n}e^{i(\theta-\pi/n)})e^{-in\theta}\,d\theta, \qquad I_2 = \int_{b/n}^{(b+\pi)/n} T(e^{-1/n}e^{i(\theta-\pi/n)})e^{-in\theta}\,d\theta,$$
$$I_3 = \int_{(b+\pi)/n}^{\epsilon} \{T(e^{-1/n}e^{i\theta}) - T(e^{-1/n}e^{i(\theta-\pi/n)})\}e^{-in\theta}\,d\theta.$$

By Proposition 2.2, $|I_1| \ll 1/n$. By (4.2), $\|(I - R(e^{-u}e^{i\theta}))^{-1}\| \ll \ell(1/|u - i\theta|)^{-1}|u - i\theta|^{-\beta}$. Thus, $|I_2| \ll n^{\beta-1}\ell(n)n^{\beta-1}b^{-(\beta-\delta)}$, for any $0 < \delta < \beta$. Putting the above together, $n^{1-\beta}\ell(n)I_\epsilon = n^{1-\beta}\ell(n)I_3 + O(b^{-(\beta-\delta)})$.

Next, we estimate $I_3$. By Corollary 4.3, for all $\theta \in ((b+\pi)/n, \epsilon)$

$$\|T(e^{-1/n}e^{i\theta}) - T(e^{-1/n}e^{i(\theta-\pi/n)})\| \ll \ell(n/|1-in\theta|)^{-2}n^{-\tau}|\frac{1}{n}-i\theta|^{\tau-2\beta} + \ell(n)\ell(n/|1-in\theta|)^{-2}n^{-\beta}|\frac{1}{n}-i\theta|^{-2\beta}.$$

Hence,

$$\begin{aligned}
|I_3| &\ll n^{-\tau}\int_{(b+\pi)/n}^{\epsilon} \ell(n/|1-in\theta|)^{-2}\theta^{\tau-2\beta}d\theta + n^{-\beta}\ell(n)\int_{(b+\pi)/n}^{\epsilon} \ell(n/|1-in\theta|)^{-2}\theta^{-2\beta}d\theta \\
&\ll \ell(n)^{-2}n^{-\tau}\int_{(b+\pi)/n}^{\epsilon} \theta^{-(2\beta-\tau)}\frac{\ell(n)^2}{\ell(n/|1-in\theta|)^2}d\theta + \ell(n)^{-1}n^{-\beta}\int_{(b+\pi)/n}^{\epsilon} \theta^{-2\beta}\frac{\ell(n)^2}{\ell(n/|1-in\theta|)^2}d\theta \\
&= \ell(n)^{-2}n^{-\tau}I_3^1 + \ell(n)^{-1}n^{-\beta}I_3^2. \qquad (4.3)
\end{aligned}$$

Using Potter's bounds (see, for instance, [6]), for any $\delta > 0$,

$$I_3^2 = n^{2\beta-1}\int_{b+\pi}^{n\epsilon} \sigma^{-2\beta}\frac{\ell(n)^2}{\ell(n/|1-i\sigma|)^2}\,d\sigma \ll n^{(2\beta-1)}\int_{b+\pi}^{n\epsilon} \sigma^{-(2\beta-\delta)}d\sigma.$$

Taking $0 < \delta < 2\beta - 1$,

$$\ell(n)^{-1}n^{-\beta}|I_3^2| \ll \ell(n)^{-1}n^{-\beta}n^{2\beta-1}n^{\delta}b^{2\beta-\delta-1} = \ell(n)^{-1}n^{\beta-1}b^{-(2\beta-1-\delta)}. \qquad (4.4)$$

In what follows, we continue to estimate $I_3^1$ considering each of the two possible cases: i) $\tau \geq 2\beta - 1$ and ii) $\tau < 2\beta - 1$. Recall $\beta > 1/2$ and $\tau > 1 - \beta$. When $\beta > 2/3$, we assume again without loss, that $\tau \in (1-\beta, 2\beta-1)$. When $\beta \in (1/2, 3/2]$, we assume without loss that $\tau \in [2\beta - 1, \beta/2)$.

We note that cases i) and ii) above correspond to: i) $\beta \in (2/3, 1)$ and $\tau \in (1-\beta, 2\beta-1)$; ii) $\beta \in (1/2, 2/3]$ and $\tau \in [2\beta-1, \beta/2)$.

In case i), using Potter's bounds, we obtain that for any $\delta' > 0$,

$$I_3^1 \ll \int_{(b+\pi)/n}^{\epsilon} \theta^{-(2\beta-\tau)}\Big((\theta n)^{\delta'} + (\theta n)^{-\delta'}\Big)\,d\theta \ll n^{\delta'}\int_{(b+\pi)/n}^{\epsilon} \theta^{-(2\beta-\tau-\delta')}\,d\theta.$$

If $2\beta-\tau<1$, taking $\delta'<1-(2\beta-\tau)$, we have $I_3^1 \ll n^{\delta'}$. If $\tau=2\beta-1$ we have $I_3^1 \ll n^{2\delta'}$, for any $\delta'>0$. Thus, $\ell(n)^{-2}n^{-\tau}I_3^1 \ll \ell(n)^{-2}n^{-(\tau-2\delta')}$. This together with (4.3) and (4.4) yields

$$\ell^{-2}(n)n^{-\tau}|I_3| \ll \ell^{-2}(n)n^{-(\tau-\delta')}+\ell(n)^{-1}n^{\beta-1}b^{-(2\beta-1-\delta)},$$

for any $0<\delta<2\beta-1$, $0<\delta'<1-(2\beta-\tau)$ if $2\beta-\sigma<1$ and any $\delta'>0$ if $\tau=2\beta-1$. Since $\tau>1-\beta$, $\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)I_3^1=0$.

It remains to estimate $I_3^1$ in case ii). Note that in this case $2\beta-\sigma>1$. Using Potter's bounds, for any $\delta'>0$ we have

$$I_3^1 = n^{2\beta-\tau-1}\int_{b+\pi}^{n\epsilon}\frac{1}{\sigma^{2\beta-\tau}}\frac{\ell(n)^2}{\ell(n/|1-i\sigma|)^2}\,d\sigma \ll n^{2\beta-\tau-1}\int_{b+\pi}^{n\epsilon}\frac{1}{\sigma^{2\beta-\tau-\delta'}}.$$

Taking $\delta'<2\beta-\tau-1$, we have $\ell(n)^{-2}n^{-\tau}I_3^1 \ll \ell(n)^{-2}n^{-(2\tau-2\beta+1)}b^{-(2\beta-\tau-1-\delta')}$. This together with (4.3) and (4.4) implies that for any $0<\delta<2\beta-1$ and any $0<\delta'<2\beta-\tau-1$,

$$|I_3| \ll \ell(n)^{-2}n^{-(2\tau-2\beta+1)}b^{-(2\beta-\tau-1-\delta')}+\ell(n)^{-1}n^{\beta-1}b^{-(2\beta-1-\delta)}.$$

Hence, $n^{1-\beta}\ell(n)|I_3| \ll \ell(n)^{-1}n^{-(\tau-(1-\beta)-\gamma(2\beta-\tau-1)-\delta'(1-\gamma))}+n^{-(2\beta-1-\delta)(1-\gamma)}$. Recalling $\tau\in[2\beta-1,\beta/2)$, we have $\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)I_3=0$.

To conclude, recall that $n^{1-\beta}\ell(n)I_\epsilon = n^{1-\beta}\ell(n)I_3+O(b^{-(\beta-\delta)})$. Since $\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)I_3=0$, we have $\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)|I_\epsilon|=0$. By a similar argument, $\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)|I_{-\epsilon}|=0$, ending the proof. ∎

# 5 A simple proof of [21, Theorem 2.1]

The main advantage of our proof below is that it bypasses the issue of identifying the Fourier coefficients in [21]. However, we recall that the coefficients of $T(z)$, $z\in\mathbb{D}$ coincide with the Fourier coefficients of $z\in\mathcal{S}^1$ (see the last paragraph of Subsection 1.2).

As in the setting of [21, 22], for first order theory under (H1), the assumption $\mathcal{B}\subset L^1(Y)$ suffices. In this sense we note that

**Proposition 5.1** *Suppose that $\mathcal{B}\subset L^1(Y)$. Assume (H), (H1) and (H2)(i). Write $z=e^{-u+i\theta}$. Then as $u,\theta\to 0$,*

$$T(z)=\ell(1/|u-i\theta|)(u-i\theta)^\beta(1+o(1))$$

**Proof** As we have already seen under (H) the asymptotic of $T(z)$ is given by the asymptotic of $1-\lambda(z)$.

By (2.2),$1-\lambda(z) = \int_Y (1-e^{(-u+i\theta)\varphi})\,d\mu - V(z)$, where $V(z) = \int_Y (R(z)-R(1))(v(z)-v(1))d\mu = \int_Y (z^\varphi - 1)(v(z)-v(1))d\mu$. By (4.1), $\int_Y (1-e^{(-u+i\theta)\varphi})\,d\mu = 1 - d_\beta \ell(1/|u-i\theta|)(u-i\theta)^\beta(1+o(1))$, as $u,\theta\to 0$.

Using the fact that $\mathcal{B}\subset L^1(Y)$, the asymptotic of $\ell(1/|u-i\theta|)^{-1}|u-i\theta|^{-\beta}V(z)$ can be obtained by the dominated convergence theorem. Write

$$\ell(1/|u-i\theta|)^{-1}|u-i\theta|^{-\beta}V(z) = \int_Y \ell(1/|u-i\theta|)^{-1}|u-i\theta|^{-\beta}(v(z)-v(1))(e^{(-u+i\theta))\varphi}-1)\,d\mu.$$

By Corollary 2.4 (with $\rho=\beta$), $\|v(z)-v(1)\| \ll \ell(1/|u-i\theta|)|u-i\theta|^\beta$. Hence, the integrand is bounded by $C|1-e^{(-u+i\theta)\varphi}|$ for some $C>0$, which goes to zero pointwise as $u,\theta\to 0$. Also, since $z\to v(z)$ is $C^\beta$ in $L^1$, the infinity norm of the integrand is bounded by $C|z^\phi - 1|_\infty \le 2C$, for some $C>0$. By the dominated convergence theorem, as $u,\theta\to 0$,

$$1-\lambda(e^{-u+i\theta}) = \ell(1/|u-i\theta|)(u-i\theta)^\beta(1+o(1)).$$

Recall $T(z) = (1-\lambda(z))^{-1}P + (1-\lambda(z))^{-1}(P(z)-P) + (I-R(z))^{-1}Q(z)$. The conclusion follows immdeiately from the last displayed equation, Corollary 2.4 and Proposition 2.2. ∎

**Proposition 5.2** *[21, Theorem 2.1] Suppose that $\mathcal{B}\subset L^1(Y)$ and that (H) holds with $\beta>1/2$. Assume (H1) and (H2). Then*

$$\lim_{n\to\infty} n^{1-\beta}T_n = d_0 P.$$

**Proof** Put $\Gamma = \{e^{-u}e^{i\theta} : -\pi\le\theta<\pi\}$ with $e^{-u}=e^{-1/n}$, $n\ge 1$. Let $b\in(0,\epsilon n)$. Write

$$T_n = \frac{1}{2\pi i}\int_\Gamma \frac{T(z)}{z^{n+1}}dz = \frac{e}{2\pi}\int_{-\pi}^{\pi} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta = \frac{e}{2\pi}\Big(\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta$$
$$+\int_{-b/n}^{-\pi} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta + \int_{b/n}^{\pi} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta\Big) = \frac{e}{2\pi}I + \frac{e}{2\pi}(I^- + I^+).$$

By Proposition 4.6, $\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)\Gamma(1-\beta)\int_{-b/n}^{b/n} T(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta = \frac{2\pi}{e}\frac{1}{\Gamma(\beta)}P$.

Hence, the conclusion will follow once we show that $\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)(I^- + I^+) = 0$. Now,

$$I^+ = \frac{i}{n}\int_A T(e^{-1/n}e^{i\theta})\frac{d}{d\theta}(e^{-in\theta})d\theta = \frac{1}{in}\int_{b/n}^{\pi}\frac{d}{d\theta}(T(e^{-1/n}e^{i\theta}))e^{-in\theta}d\theta + E(n),$$

where $\|E(n)\| \ll n^{-1}\|T(e^{-1/n}e^{ib/n})\| + n^{-1}\|T(e^{-1/n}e^{in\pi})\|$. By Proposition 5.1, $\|T(e^{-1/n}e^{i\theta})\| \ll \ell^{-1}(1/|1/n - i\theta|)|1/n - i\theta|^{-\beta} \ll \ell^{-1}(1/|1/n - i\theta|)|\theta|^{-\beta}$ for all $\theta \in [-\pi, \pi)$. Hence,

$$\|E(n)\| \ll \ell(n/|1-ib|)b^{-\beta}n^{\beta-1} \ll \ell^{-1}(n)n^{\beta-1}b^{-\beta}\ell^{-1}(n/|1-ib|)\ell(n)) \ll \ell^{-1}(n)n^{\beta-1}b^{-\beta+\delta},$$

for any $\delta > 0$. The last estimate above was obtained using Potter's bounds.

Next, $\frac{d}{d\theta}(T(e^{-1/n}e^{i\theta})) = T(e^{-1/n}e^{i\theta})\frac{d}{d\theta}R(e^{-1/n}e^{i\theta})T(e^{-1/n}e^{i\theta})$. By Proposition 2.5, $\|\frac{d}{d\theta}R(e^{-1/n}e^{i\theta})\| \ll \ell(n)n^{1-\beta}$. This together with the above estimate for $E(n)$ and Potter's bounds implies that for any $\delta, \delta' > 0$,

$$\begin{aligned}|I^+| &\ll \ell(n)n^{-\beta}\int_{b/n}^{\pi} |u - i\theta|^{-2\beta}\ell^{-2}(1/|u - i\theta|)\,d\theta + \ell^{-1}(n)n^{\beta-1}b^{-\beta+\delta}\\ &\ll \ell^{-1}(n)n^{\beta-1}\int_b^{n\pi} \sigma^{-2\beta}\frac{\ell^2(n)}{\ell^2(n/|1-i\sigma|)}\,d\sigma + \ell^{-1}(n)n^{\beta-1}b^{-\beta+\delta}\\ &\ll \ell^{-1}(n)n^{\beta-1}b^{-(2\beta-1-\delta')} + \ell^{-1}(n)n^{\beta-1}b^{-\beta+\delta}.\end{aligned}$$

Hence, $n^{1-\beta}\ell(n)|I^+| \ll b^{-(2\beta-1-\delta')} + b^{-\beta+\delta}$, for any $\delta > 0$ and $\delta' > 0$. It follows that $\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)I^+ = 0$. By the same argument $\lim_{b\to\infty}\lim_{n\to\infty} n^{1-\beta}\ell(n)I^- = 0$, ending the proof. ∎

# A Proof of several results used in the proof of Lemma 2.1

Proposition A.1 and Proposition A.4 below provide similar results for different regimes of $\rho > 1$.

**Proposition A.1** *Let $M : [0,\infty] \to \mathbb{R}$ be such that $M(x) = M \in \mathbb{R}$ for $x \in [0,1)$ and $M(x) = O(x^{-\rho})$, for all $x \geq 1$ and some $\rho \in (1,2)$. For $u > 0$, $\theta \in (-\pi,\pi)$, define*

$$J(u,\theta) := \int_0^\infty e^{-(u-i\theta)x}M(x)\,dx$$

*Put $c_M = \int_0^\infty M(x)\,dx$. Then*

*(a) $(u - i\theta)J(u,\theta) = c_M(u - i\theta) + O(|u - i\theta|^\rho)$.*

*(b) Moreover, if $xM(x)$ has bounded variation then $\frac{d}{d\theta}\Big((u - i\theta)J(u,\theta) - c_M(u - i\theta)\Big) = O(|u - i\theta|^{\rho-1})$.*

**Remark A.2** We notice that since $u > 0$, $\frac{d^k}{d\theta^k}Integrand(J(u,\theta))$ is bounded for any $k \geq 1$, so we can move the derivative(s) inside the integral. This type of argument will be used in the proofs of several results below without further explanation.

**Proof** Item (a) follows by an argument used in the proof of [22, Lemma A.4](in estimating $I_2$ there).

**(b)**. Compute that

$$\frac{d}{d\theta}\Big((u-i\theta)J(u,\theta)-c_M(u-i\theta)\Big) = -i(J(u,\theta)-c_M)+i(u-i\theta)\int_0^\infty e^{-(u-i\theta)x}xM(x)\,dx$$
$$= -i(J(u,\theta)-c_M)+i(u-i\theta)J_1(u,\theta). \tag{A.1}$$

By (a), $J(u,\theta)-c_M = O(|u-i\theta|^{\rho-1})$. Also, the desired estimate for $(u-i\theta)J_1(u,\theta)$ follows by an argument used in the proof of [22, Lemma A.4], which we provide below for completeness.

We assume $\theta>0$ (since the case $\theta<0$ follows similarly) and consider separately each of two possible cases: (i) $0\le\theta\le u$, (ii) $0\le u\le\theta$. In case (i),

$$|(u-i\theta)J_1(u,\theta)| \ll u\int_0^\infty e^{-ux}x^{-(\rho-1)}\,dx = u^{\rho-1}\int_0^\infty e^{-\sigma}\sigma^{-(\rho-1)}\,d\sigma \ll |u-i\theta|^{\rho-1}.$$

We turn to case (*ii*). By assumption $\tilde M(x) := xM(x) = O(x^{-(\rho-1)})$ has bounded variation. Hence, we can write $\tilde M(x) = \tilde M_1(x)-\tilde M_2(x)$, where $\tilde M_1(x)$, $\tilde M_2(x)$ are positive decreasing. Put $y=u/\theta$, so $y\le 1$. Substituting $\sigma=\theta x$,

$$\theta J_1(u,\theta) = \int_0^\infty e^{-\sigma y}e^{i\sigma}\tilde M_1(\sigma/\theta)\,d\sigma - \int_0^\infty e^{-\sigma y}e^{i\sigma}\tilde M_2(\sigma/\theta)\,d\sigma.$$

Since for $j=1,2$, $\sigma\mapsto e^{-\sigma y}\tilde M_j(\sigma/\theta)$ is decreasing for each fixed value of $u$ and $\theta$, we have

$$\int_0^\infty e^{-\sigma y}\cos\sigma\,\tilde M_j(\sigma/\theta)\,d\sigma \le \int_0^{\pi/2} e^{-\sigma y}\cos\sigma\,\tilde M_j(\sigma/\theta)\,d\sigma \ll \theta^{\rho-1}\int_0^{\pi/2}\sigma^{-(\rho-1)}d\sigma \ll \theta^{\rho-1}.$$

The integral with cos replaced by sin is treated similarly. Hence, $|(u-i\theta)J_1(u,\theta)| \ll |u-i\theta|^{\rho-1}$, which ends the proof of (b). ∎

**Proposition A.3** *Let $M:[0,\infty]\to\mathbb{R}$ be such that $M(x)=M\in\mathbb{R}$ for $x\in[0,1)$ and $M(x)=O(x^{-\rho})$, for all $x\ge 1$ and some $\rho>2$. For $u>0$, $\theta\in(-\pi,\pi)$, define $J(u,\theta) := \int_0^\infty e^{-(u-i\theta)x}M(x)\,dx$. Put $c_M=\int_0^\infty M(x)\,dx$. Then*

*(a)* $(u-i\theta)J(u,\theta) = c_M(u-i\theta)+O(|u-i\theta|^2)$ *and*

*(b)* $\frac{d}{d\theta}\Big((u-i\theta)J(u,\theta)-c_M(u-i\theta)\Big) = O(|u-i\theta|^{\rho-1}+|u-i\theta|^2)$.

**Proof** Item (a), (b) follow by an argument used in the proof of [22, Lemma A.4] (in estimating $I_2$ there in the case $q>1$) together with the fact that $\rho>2$. ∎

The content of the next result is similar to the content of Proposition A.1. However, the assumption of bounded variation made in Proposition A.1, (b) is not satisfied and thus, it requires a different proof.

**Proposition A.4** *For $\rho \in (0,1)$, set $\Delta(x) = ([x]^{-\rho} - x^{-\rho})$. For $u > 0$, $\theta \in (-\pi, \pi)$ define*

$$W(u,\theta) := \int_0^\infty e^{-(u-i\theta)x}\Delta(x)\,dx.$$

*Put $c_\Delta = \int_0^\infty \Delta(x)\,dx$. Then for all $u > 0$ and for all $\theta \neq 0$,*

*(a)* $(u - i\theta)W(u,\theta) = c_\Delta(u - i\theta) + O(|u - i\theta|^{\rho+1})$ *and*

*(b)* $\left|\frac{d}{d\theta}\Big((u - i\theta)W(u,\theta) - c_\Delta(u - i\theta)\Big)\right| = O(|u - i\theta|^\rho)$.

**Proof** As in the previous two propositions, item (a) follows by the argument used in the proof of [22, Lemma A.4](in estimating $I_2$ there).

**(b)**. Compute that

$$\begin{aligned}\frac{d}{d\theta}\Big((u - i\theta)W(u,\theta) - c_\Delta(u - i\theta)\Big) &= -i(W(u,\theta) - c_\Delta) + i(u - i\theta)\int_0^\infty e^{-(u-i\theta)x}x\Delta([x])\,dx \\ &= -i(W(u,\theta) - c_\Delta) + (u - i\theta)\tilde{W}(u,\theta).\end{aligned}$$

By (a), $W(u,\theta) - c_\Delta = O(|u - i\theta|^\rho)$. Next, we estimate $\tilde{W}(u,\theta)$. With the convention $0^{-\beta} = 0$, $\Delta(x) = x^{-\rho}$ for $x \in [0,1)$. Hence,

$$(u-i\theta)\tilde{W}(u,\theta) = (u-i\theta)\int_1^\infty e^{-(u-i\theta)x}x\Delta([x])\,dx + O(|u-i\theta|) = (u-i\theta)\tilde{W}_1(u,\theta) + O(|u-i\theta|).$$

Put $\{x\} = x - [x]$ and note that

$$x^{-\rho} - [x]^{-\rho} = x^{-\rho} - x^{-\rho}\Big(1 - \frac{\{x\}}{x}\Big)^{-\rho} = -\rho\frac{\{x\}}{x^{\rho+1}} + O\Big(\frac{\{x\}^2}{x^{\rho+2}}\Big).$$

Hence,

$$\begin{aligned}(u - i\theta)\tilde{W}_1(u,\theta) &= -\rho(u - i\theta)\int_1^\infty e^{-(u-i\theta)x}\frac{\{x\}}{x^\rho}\,dx + O\Big(|u - i\theta|\int_1^\infty e^{-ux}\{x\}x^{-(\rho+1)}\,dx\Big) \\ &= -\rho(u - i\theta)I_1 + O(|(u - i\theta)|J).\end{aligned}$$

Clearly, $J = O(1)$ and thus $|u - i\theta|J \ll |u - i\theta|$. It remains to estimate $(u - i\theta)I_1$. Write $-(u - i\theta)I_1 = -(u - i\theta)\int_1^{1/|u-i\theta|} - (u - i\theta)\int_{1/|u-i\theta|}^\infty = -(u - i\theta)L_1 - (u - i\theta)L_2$. With respect to the first term,

$$|(u - i\theta)L_1| \ll |u - i\theta|\int_1^{1/|u-i\theta|} x^{-\rho}\,dx \ll |u - i\theta|^\rho.$$

We continue with the estimate for $(u - i\theta)L_2$. Recall that $\{x\} = x - j$ for $x \in [j, j+1]$. Hence,

$$-(u - i\theta)L_2 = -(u - i\theta)\sum_{j>1/|u-i\theta|}\int_j^{j+1} e^{-(u-i\theta)x}x^{-\rho}(x - j)\,dx = -\sum_{j>1/|u-i\theta|}(u - i\theta)L^j.$$

Compute that

$$
\begin{aligned}
-\sum_{j>1/|u-i\theta|}(u-i\theta)L^j &= \sum_{j>1/|u-i\theta|}\int_j^{j+1}\frac{d}{dx}(e^{-(u-i\theta)x})x^{-\rho}(x-j))\,dx = \sum_{j>1/|u-i\theta|}\frac{e^{-(u-i\theta)(j+1)}}{(j+1)^\rho}\\
&\quad -\sum_{j>1/|u-i\theta|}\int_j^{j+1}\frac{e^{-(u-i\theta)x}}{x^\rho}\,dx+\rho\sum_{j>1/|u-i\theta|}\int_j^{j+1}\frac{e^{-(u-i\theta)x}}{x^{\rho+1}}(x-j)\,dx\\
&=\sum_{j>1/|u-i\theta|}\frac{e^{-(u-i\theta)(j+1)}}{(j+1)^\rho}-\sum_{j>1/|u-i\theta|}\int_j^{j+1}\frac{e^{-(u-i\theta)x}}{x^\rho}\,dx+O(|u-i\theta|^\rho).
\end{aligned}
$$

Write

$$
\begin{aligned}
\sum_{j>1/|u-i\theta|}\frac{e^{-(u-i\theta)(j+1)}}{(j+1)^\rho}-\sum_{j>1/|u-i\theta|}\int_j^{j+1}\frac{e^{-(u-i\theta)x}}{x^\rho}\,dx &= \sum_{j>1/|u-i\theta|}\Big(\frac{e^{-(u-i\theta)(j+1)}}{(j+1)^\rho}-\int_{j+1}^{j+2}\frac{e^{-(u-i\theta)x}}{[x]^\rho}\,dx\Big)\\
&\quad -\sum_{j>1/|u-i\theta|}\int_j^{j+1}e^{-(u-i\theta)x}\Big(\frac{1}{x^\rho}-\frac{e^{u-i\theta}}{[x-1]^\rho}\Big)\,dx.
\end{aligned}
$$

But, $[x-1]^{-\rho}=x^{-\rho}+b(x)$, where $b(x)=O((x-1)^{-(\rho+1)})$. Thus,

$$
\begin{aligned}
\sum_{j>1/|u-i\theta|}\int_j^{j+1}e^{-(u-i\theta)x}\Big(\frac{1}{x^\rho}-\frac{e^{u-i\theta}}{[x-1]^\rho}\Big)\,dx &= \sum_{j>1/|u-i\theta|}\int_j^{j+1}\frac{e^{-(u-i\theta)x}}{x^\rho}(1-e^{u-i\theta})\,dx\\
&\quad+\sum_{j>1/|u-i\theta|}\int_j^{j+1}e^{-(u-i\theta)(x-1)}b(x)\,dx\\
&=(1-e^{-(u-i\theta)})Q1+Q2.
\end{aligned}
$$

Clearly, $|Q2|\ll|u-i\theta|^\rho$. By the argument used in the proof of [22, Lemma A.4](in estimating $I_2$ there), $|Q1|\ll|u-i\theta|^{\rho-1}$. Hence, $|(1-e^{u-i\theta})Q1|\ll|u-i\theta|^\rho$ and thus,

$$
\Big|\sum_{j>1/|u-i\theta|}\int_j^{j+1}e^{-(u-i\theta)x}\Big(\frac{1}{x^\rho}-\frac{e^{-(u-i\theta)}}{[x-1]^\rho}\Big)\,dx\Big|\ll|u-i\theta|^\rho.
$$

Altogether,

$$
-\sum_{j>1/|u-i\theta|}(u-i\theta)L^j=\sum_{j>1/|u-i\theta|}\Big(\frac{e^{-(u-i\theta)(j+1)}}{(j+1)^\rho}-\int_{j+1}^{j+2}\frac{e^{-(u-i\theta)x}}{[x]^\rho}\,dx\Big)+O(|u-i\theta|^\rho).
$$

Now,

$$\sum_{j>1/|u-i\theta|}\Big(\frac{e^{-(u-i\theta)(j+1)}}{(j+1)^\rho}-\int_{j+1}^{j+2}\frac{e^{-(u-i\theta)x}}{[x]^\rho}\,dx\Big)=-\sum_{j>1/|u-i\theta|}\frac{e^{-(u-i\theta)(j+1)}}{(j+1)^\rho}\int_{j+1}^{j+2}(e^{-(u-i\theta)(x-j-1)}-1)\,dx$$
$$=-\sum_{j>1/|u-i\theta|}\frac{e^{-(u-i\theta)(j+1)}}{(j+1)^\rho}\int_0^1(e^{-(u-i\theta)x}-1)\,dx$$
$$=g(u,\theta)\sum_{j>1/|u-i\theta|}\frac{e^{-(u-i\theta)(j+1)}}{(j+1)^\rho}$$

where $g(u,\theta)=O(|u-i\theta|)$.

Finally, we use the oscillatory nature of the sums above. As in the proof of Proposition A.1, we assume without loss $\theta>0$ and consider separately each of the two possible cases: (i) $0\le\theta\le u$, (ii) $0\le u\le\theta$.

In case (i),

$$\Big|g(u,\theta)\sum_{j>1/|u-i\theta|}\frac{e^{-(u-i\theta)(j+1)}}{(j+1)^\rho}\Big|\ll|u-i\theta|\sum_{j>1/|u-i\theta|}e^{-uj}j^{-\rho}\ll|u-i\theta|\int_{1/|u-i\theta|}^\infty e^{-ux}x^{-\rho}\,dx$$
$$=|u-i\theta|u^{\rho-1}\int_0^\infty e^{-\sigma}\sigma^{-\rho}\,d\sigma\ll|u-i\theta|^\rho$$

In case (ii), we recall that [31, Theorem 2.4] (see also [6, Theorem 4.3.2] for an improved version) shows that $\sum_1^\infty\cos(\theta j)j^{-\rho}=C_\rho\theta^{\rho-1}(1+o(1))$, where $C_\rho$ is a positive constant that depends only on $\rho$. By the same argument, $\sum_{j>1/|u-i\theta|}e^{-uj}\cos(\theta j)j^{-\rho}\ll\theta^{\rho-1}$. Hence,

$$\Big|g(u,\theta)\sum_{j>1/|u-i\theta|}e^{-uj}\cos(\theta j)j^{-\rho}\,dx\Big|\ll|u-i\theta|\theta^{\rho-1}\ll|u-i\theta|^\rho.$$

The sum with cos replaced with sin is treated similarly, ending the proof. ∎

# B Tail sequence for (1.1)

The following proposition is an improved version of [22, Proposition C1]. Recall that $h$ denotes the density for the measure $\mu$.

**Proposition B.1** *Suppose that $f:[0,1]\to[0,1]$ is given as in (1.1) with $\beta=1/\alpha\in(0,1)$. Let $C$ be a compact subset of $(0,1]$. Then there exists $Y\subset(0,1]$ compact with $C\subset Y$, such that the following hold*

*(i) The first return function $\varphi:Y\to\mathbb{Z}^+$ satisfies $\mu(\varphi>n)=cn^{-\beta}+c_1n^{-2\beta}+c_2n^{-3\beta}+\hat{c}n^{-(\beta+1)}+\tilde{c}(\log n)n^{-(\beta+1)}+H(n)$ where $H(n)=O((\log n)^2n^{-(\beta+2)})+O(n^{-4\beta})$ and $c,c_1,c_2,\hat{c},\tilde{c}$ are real constants.*

*(ii) The first return map $F = f^\varphi : Y \to Y$ satisfies hypotheses (H1) and (H2) with $\mathcal{B}$ taken to consist of either Hölder or BV observables.*

**Proof** Let $Y = [\frac{1}{2}, 1]$. Let $x_n \in (0, \frac{1}{2}]$ be the sequence with $x_1 = \frac{1}{2}$ and $x_n = f x_{n+1}$ so $x_n \to 0$. It is well known (see for instance [20]) that $x_n \sim \frac{1}{2}\beta^\beta n^{-\beta}$ and moreover that $x_n = \frac{1}{2}\beta^\beta n^{-\beta} + O((\log n) n^{-(\beta+1)})$. First, we claim that $x_n = \frac{1}{2}\beta^\beta n^{-\beta} + \hat{C} n^{-(\beta+1)} + \tilde{C}(\log n) n^{-(\beta+1)} + b(n)$, where $\hat{C}, \tilde{C}$ are some constants (to be specified below) that depends only on $\beta$ and $b(n)$ is such that $b(n) = O((\log n) n^{-(\beta+2)})$. The proof of the claim builds upon previous calculations such as [16].

Put $g(x) = 2^\alpha x^\alpha$. Since $\beta = 1/\alpha$, $x_n = \frac{1}{2}\beta^\beta n^{-\beta}(1 + m(x_n))$, where $m(x_n) = O((\log n)/n)$. So, $g(x_n) = 2^\alpha x_n^\alpha = \beta/n(1 + M(x_n))$, where $M(x_n) = O((\log n)/n)$.

Next, put $d(x) = 1/g(x)$ and compute that

$$d(x_n) = \frac{1}{2^\alpha x_n^\alpha} = d(x_{n+1})(1 + g(x_{n+1}))^{-\alpha} = d(x_{n+1})\Big(1 - \alpha g(x_{n+1}) + \frac{\alpha(\alpha+1)}{2!} g(x_{n+1})^2\Big)$$
$$+ O(g(x_{n+1})^3) = d(x_{n+1}) - \alpha + \frac{\alpha(\alpha+1)}{2} g(x_{n+1}) + O(n^{-2}). \qquad \text{(B.1)}$$

It follows that $d(x_{n+1}) - d(x_n) = \alpha - \frac{\alpha(\alpha+1)}{2} g(x_{n+1}) + O(n^{-2})$. Summing from $j = 1$ to $n - 1$,

$$d(x_n) = 1 + (n-1)\alpha - \frac{(\alpha+1)}{2} \sum_{j=1}^{n-1} \frac{1}{j} + O(1).$$

Since $\sum_{j=1}^{n-1} \frac{1}{j} = \log n + \gamma + O(1/n)$, where $\gamma$ is the Euler constant (see, for instance, [8]), we have

$$d(x_n) = (2x_n)^{-\alpha} = n\alpha - \frac{(\alpha+1)}{2}(\log n) + O(1) = n\alpha\Big(1 - \frac{(\alpha+1)}{2\alpha}\frac{\log n}{n} + O(1/n)\Big).$$

Hence,

$$g(x_n) = \frac{1}{d(x_n)} = \frac{1}{n\alpha}\Big(1 - \frac{(\alpha+1)}{2\alpha}\frac{\log n}{n} + O(1/n)\Big)^{-1} = \frac{1}{n\alpha} + \frac{(\alpha+1)}{2\alpha^2}\frac{\log n}{n^2} + O(n^{-2}).$$

Plugging the above estimate back into (B.1) we have

$$d(x_{n+1}) - d(x_n) = \alpha - \frac{(\alpha+1)}{2}\frac{1}{n} - \frac{(\alpha+1)^2}{4\alpha}\frac{\log n}{n^2} + O(n^{-2}).$$

Summing from $j = 1$ to $n - 1$,

$$d(x_n) = 1 + (n-1)\alpha - \frac{(\alpha+1)}{2}\sum_{j=1}^{n-1}\frac{1}{j} - \frac{(\alpha+1)^2}{4\alpha}\sum_{j=1}^{n-1}\Big(\frac{\log j}{j^2} + b(j)\Big),$$

where $b(j) = O(j^{-2})$. Put $C = \frac{(\alpha+1)^2}{4\alpha}\sum_{j=1}^{\infty}\frac{\log j}{j^2} + b(j)$. By Karamata's theorem, $\sum_{n-1}^{\infty}\frac{\log j}{j^2}(1+o(1)) = \frac{\log n}{n}(1+o(1))$. Recall $\sum_{j=1}^{n-1}\frac{1}{j} = \log n + \gamma + O(1/n)$. Thus,

$$d(x_n) = 1 + (n-1)\alpha - \frac{(\alpha+1)}{2}(\log n + \gamma) - C + O\Big(\frac{\log n}{n}\Big).$$

Put $C' = 1 - \alpha - \frac{(\alpha+1)\gamma}{2} - C$. Note that

$$d(x_n) = n\alpha - \frac{(\alpha+1)}{2}\log n + C' + O\Big(\frac{\log n}{n}\Big) = n\alpha\Big(1 - \frac{(\alpha+1)}{2\alpha}\frac{\log n}{n} + \frac{C'}{n\alpha} + O\Big(\frac{(\log n)}{n^2}\Big)\Big).$$

Since $\beta = 1/\alpha$ and $d(x_n) = (2x_n)^{-\alpha}$,

$$\begin{aligned} x_n &= \frac{1}{2}\beta^\beta n^{-\beta}\Big(1 - \frac{(\beta+1)}{2}\frac{\log n}{n} + \frac{C'\beta}{n} + O\Big(\frac{\log n}{n^2}\Big)\Big)^{-\beta} \\ &= \frac{1}{2}\beta^\beta\frac{1}{n^\beta} + \frac{\beta+1}{2^2}\beta^{\beta-1}\frac{\log n}{n^{\beta+1}} - \frac{\beta^\beta C'}{2}\frac{1}{n^{\beta+1}} + O\Big(\frac{(\log n)^2}{n^{\beta+2}}\Big). \end{aligned}$$

To end the proof of the claim put $\hat{C} = -\frac{1}{2}\beta^\beta C'$ and $\tilde{C} = \frac{\beta+1}{2^2}\beta^{\beta-1}$.

It is known that the density $h \in C^3$ (this follows, for instance, from the argument of [26, Lemma 2]) . Hence for $x \in [\frac{1}{2}, 1]$ we can write $h(x) = h(\frac{1}{2}) + h'(\frac{1}{2})(x - \frac{1}{2}) + \frac{1}{2!}h''(\frac{1}{2})(x-\frac{1}{2})^2 + O\Big((x-\frac{1}{2})^3\Big)$.

Set $y_n = \frac{1}{2}(x_n + 1)$ (so $fy_n = x_n$). Then $\varphi = n$ on $[y_n, y_{n-1}]$, hence $\{\varphi > n\} = [\frac{1}{2}, y_n]$. It follows that

$$\begin{aligned} \mu(\varphi > n) &= \int_{1/2}^{y_n} h(x)\,dx = \frac{1}{2}h(\frac{1}{2})x_n + \frac{1}{8}h'(\frac{1}{2})x_n^2 + \frac{1}{2^5}h''(\frac{1}{2})x_n^3 + O((x_n)^4) \\ &= cn^{-\beta} + c_1 n^{-2\beta} + c_2 n^{-3\beta} + \hat{c}n^{-(\beta+1)} + \tilde{c}(\log n)n^{-(\beta+1)} + H(n), \end{aligned}$$

where $c = \frac{1}{4}\beta^\beta h(1/2)$, $H(n) = O((\log n)^2 n^{-(\beta+2)}) + O(n^{-4\beta})$ and $c_1, c_2, \hat{c}, \tilde{c}$ are real constants that depend only on $\beta$ and $h^{(j)}(1/2)$, $j = 0, 1, 2$. This ends the proof of $(i)$ for the choice $Y = [\frac{1}{2}, 1]$. Item $(i)$ follows since the same estimates are obtained by inducing on $Y = [x_q, 1]$ for any fixed $q \geq 0$.

Finally, it is well-known that hypotheses (H1) and (H2) are satisfied on such sets $Y$ (see for example [21, Section 11]) and item $(ii)$ follows. ∎

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## References

[1] J. Aaronson. Random $f$-expansions. *Ann. Probab.* **14** (1986) 1037–1057.

[2] J. Aaronson. *An Introduction to Infinite Ergodic Theory.* Math. Surveys and Monographs **50**, Amer. Math. Soc., 1997.

[3] J. Aaronson and M. Denker. Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps. *Stoch. Dyn.* **1** (2001) 193–237.

[4] J. Aaronson, M. Denker, O. Sarig and R. Zweimüller. Aperiodicity of cocycles and conditional local limit theorems. *Stoch. Dyn.* **4** (2004) 31–62.

[5] J. Aaronson, M. Denker and M. Urbański. Ergodic theory for Markov fibred systems and parabolic rational maps. *Trans. Amer. Math. Soc.* **337** (1993) 495–548.

[6] N. H. Bingham, C. M. Goldie and J. L. Teugels. *Regular variation.* Encyclopedia of Mathematics and its Applications **27**, Cambridge University Press, Cambridge, 1987.

[7] D. A. Darling and M. Kac. On occupation times for Markoff processes. *Trans. Amer. Math. Soc.* **84** (1957) 444–458.

[8] D. W. DeTemple. A quicker convergence to the Euler constant. *The Amer. Math. Monthly* **100** (1993) 468–470.

[9] K. B. Erickson. Strong renewal theorems with infinite mean. *Trans. Amer. Math. Soc.* **151** (1970) 263–291.

[10] W. Feller. *An Introduction to Probability Theory and its Applications, II.* Wiley, New York, 1966.

[11] A. Garsia and J. Lamperti. A discrete renewal theorem with infinite mean. *Comment. Math. Helv.* **37** (1962/1963) 221–234.

[12] S. Gouëzel. Sharp polynomial estimates for the decay of correlations. *Israel J. Math.* **139** (2004) 29–65.

[13] S. Gouëzel. Characterization of weak convergence of Birkhoff sums for Gibbs-Markov maps. *Israel J. Math.* **180** (2010) 1–41.

[14] S. Gouëzel. Correlation asymptotics from large deviations in dynamical systems with infinite measure. *Colloquium Math.***125**, (2011) 193–212

[15] S. Gouëzel. Berry-Esseen theorem and local limit theorem for non uniformly expanding maps. *Ann. Inst. H. Poincaré Probab. Statist.* **41** (2005) 997–1024.

[16] M. Holland. Slowly mixing systems and intermittency maps. *Ergodic Theory Dynam. Systems* **25** (2005) 133–159.

[17] H. Hu, S. Vaienti. Absoltutely continuous invariant measures for non-uniformly hyperbolic maps. *Ergodic Theory Dynam. Systems* **29** (2009) 1185–1215.

[18] T. Kato. Perturbation theory of linear operators. *Grundlehren* **132**, Springer New York (1976).

[19] J. Lamperti. Some limit theorems for stochastic processes. *J. Math. Mech.* **7** (1958) 433–448.

[20] C. Liverani, B. Saussol and S. Vaienti. A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems* **19** (1999) 671–685.

[21] I. Melbourne, D. Terhesiu. *Operator renewal theory and mixing rates for dynamical systems with infinite measure*, to appear in *Invent. Math.*

[22] I. Melbourne, D. Terhesiu. *First and higher order uniform ergodic theorems for dynamical systems with infinite measure*, to appear in *Isarel J.*

[23] Y. Pomeau and P. Manneville. Intermittent transition to turbulence in dissipative dynamical systems. *Comm. Math. Phys.* **74** (1980) 189–197.

[24] O. M. Sarig. Subexponential decay of correlations. *Invent. Math.* **150** (2002) 629–653.

[25] M. Thaler. Transformations on [0,1] with infinite invariant measures. *Isr. J. Math.* **46**(1983) 67–96

[26] M. Thaler. A limit theorem for the Perron-Frobenius operator of transformations on $[0, 1]$ with indifferent fixed points. *Israel J. Math.* **91** (1995) 111–127.

[27] M. Thaler. The Dynkin-Lamperti arc-sine laws for measure preserving transformations. *Trans. Amer. Math. Soc.* **350** (1998) 4593–4607.

[28] M. Thaler and R. Zweimüller. Distributional limit theorems in infinite ergodic theory. *Probab. Theory Related Fields* **135** (2006) 15–52.

[29] R. Zweimüller. Ergodic structure and invariant densities of non-Markovian interval maps with indifferent fixed points. *Nonlinearity* **11** (1998) 1263–1276.

[30] R. Zweimüller. Ergodic properties of infinite measure-preserving interval maps with indifferent fixed points. *Ergodic Theory Dynam. Systems* **20** (2000) 1519–1549.

[31] A. Zygmund. *Trigonometric series*, vol. I, second edition, CUP, 1968.