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Rigorous numerical approximation of Ruelle–Perron–Frobenius operators and topological pressure of expanding maps

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Abstract
It is well known that for different classes of transformations, including the class of piecewise C^2 expanding maps $T : [0, 1] \circlearrowleft$, Ulam's method is an efficient way to numerically approximate the absolutely continuous invariant measure of T. We develop a new extension of Ulam's method and prove that this extension can be used for the numerical approximation of the Ruelle–Perron–Frobenius operator associated with T and the potential $\phi_\beta = -\beta \log |T'|$, where $\beta \in \mathbb{R}$. In particular, we prove that our extended Ulam's method is a powerful tool for computing the topological pressure $P(T, \phi_\beta)$ and the density of the equilibrium state.

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1. Introduction

Let $(X, \mathfrak{B})$ be a measurable space and $T : X \circlearrowleft$ a measurable transformation. Let $\mathcal{M}(X, T)$ denote the set of all T-invariant probability measures and $h_\mu(T)$ denote the metric entropy of T with respect to μ. An invariant probability measure $\mu_\phi \in \mathcal{M}(X, T)$ is said to be an equilibrium state for a continuous potential $\phi : X \to \mathbb{R}$ if it satisfies the variational principle, i.e. $P(T, \phi) := h_{\mu_\phi}(T) + \int_X \phi \, \mathrm{d}\mu_\phi = \sup_{\mu \in \mathcal{M}(X,T)} (h_\mu(T) + \int_X \phi \, \mathrm{d}\mu)$ where $P(T, \phi)$ is the topological pressure associated with ϕ and T (see for example [21]).

Within the mathematical framework of the thermodynamical formalism [21], a key ingredient in obtaining analytical expressions for the topological pressure $P(T, \phi)$ and related thermodynamic quantities is the Ruelle–Perron–Frobenius (RPF) operator $\boldsymbol{L}_\phi : B(X) \circlearrowleft$, where $B(X)$ is the space of all measurable bounded functions on X, defined as $\boldsymbol{L}_\phi f(x) = \sum_{y \in T^{-1}(x)} \mathrm{e}^{\phi(y)} f(y)$. Ruelle [19] proved that the equilibrium state of a finite state topologically mixing Markov shift is given by $\mu_\phi = h\nu_\phi$, where ν_ϕ is a probability measure and h is a density satisfying $\boldsymbol{L}_\phi h = \lambda h$, $\boldsymbol{L}_\phi^* \nu_\phi = \lambda\nu_\phi$ and $\log(\lambda) = P(T, \phi)$. Later on, with some extra conditions on the potential ϕ, these results were extended to some other classes of transformations (see for instance [3, 10, 14, 20, 22, 26, 27]; see also [1] for other references). In all these settings the equilibrium measure $\mu_\phi = h\,\mathrm{d}\nu_\phi$ is absolutely continuous w.r.t. the *conformal measure* (possibly non-Lebesgue) ν_ϕ (see [4] for background on conformal measures).

Common choices of the potential are: $\phi_\beta = -\beta\log|T'|$, $\beta \in \boldsymbol{R}$, yielding the operator $\boldsymbol{L}_\beta f(x) = \sum_{y\in T^{-1}(x)} \frac{f(y)}{|T'(y)|^\beta}$, which is used to study the existence of phase transitions in certain classes of transformations (e.g. [17, 23]) and $\phi = -\log|T'|$ which yields the well-known Perron–Frobenius (PF) operator $\boldsymbol{L}f(x) = \sum_{y\in T^{-1}(x)} \frac{f(y)}{|T'(y)|}$. The densities of absolutely continuous (w.r.t. Lebesgue) invariant measures are fixed points of $\boldsymbol{L}$.

It is well known that for different classes of transformations, including the class of expanding maps of the unit interval, Ulam's method (see section 5 for details) gives good estimates of the PF operator and thus of the absolutely continuous T invariant measure [2, 5, 7, 12]. In this work we show that Ulam's method can be used to approximate the leading eigenvalue and corresponding eigenfunction of the RPF operator $\boldsymbol{L}_\beta$ for expanding, piecewise monotonic maps $T : [0, 1] \circlearrowleft$ with a finite number of monotonicity intervals. More importantly, we show that the approximated eigenfunction is exactly the density of the equilibrium state and that its associated eigenvalue gives the value of $P(T, \phi_\beta)$, where $\phi_\beta = -\beta\log(|T'|)$. Our approach has also been successfully used to study non-uniformly expanding maps that exhibit phase transitions [8].

The outline of the paper is as follows. In the first part, we develop a suitable Lasota–Yorke (LY) inequality that allows us to prove that a normalized version of $\boldsymbol{L}_\beta$ preserves a cone of non-negative functions in L^1. Related inequalities have been produced in [14] in terms of a limiting measure ν that is not explicitly known. To our knowledge the explicit BV–L^1 form of the LY inequality developed in section 3 below has not been previously published. Next, we prove that $\boldsymbol{L}_\beta$ has a positive eigenfunction h, establish that the positive eigenvalue associated with h satisfies $\lambda_\beta = \mathrm{e}^{P(T,\phi_\beta)}$ and that h is the density of the equilibrium state for (T, ϕ_β) with respect to the corresponding conformal measure. Finally, we recall Ulam's method and state our main result on the numerical approximation of the density h and of the topological pressure $P(T, \phi_\beta)$.

2. Class of transformations considered

Let I be the unit interval $[0, 1]$ and let $T : I \circlearrowleft$ be a piecewise C^2 transformation. Let $\wp = \{I_a\}$ be a finite partition of I such that I_a are closed intervals, $I = \bigcup_a I_a$ and $\mathrm{int}(I_a) \bigcap \mathrm{int}(I_{a'}) = \emptyset, \forall a \neq a'$. The restriction of T to I_a, $T_a = T \mid I_a : I_a \to T(I_a)$ is assumed to be strictly monotone. $T_a^{-1} : T(I_a) \to I_a$ represents the inverse branches of T. The nth iterate of T is defined by $T^n_{a^{(n)}} = T \mid I_{a^{(n)}} : I_{a^{(n)}} \to I$ where $I_{a^{(n)}} \in \bigvee_{i=0}^n T^{-i}\wp$ and its inverse is defined by $T^{-n}_{a^{(n)}} : T^n(I_{a^{(n)}}) \to I_{a^{(n)}}$. Where necessary, we define $T'(x)$ at the endpoints of I_a by taking an appropriate one-sided derivative. We assume that there exists $\alpha > 1$ such that

$$|T'(x)| \geqslant \alpha, \qquad \forall\, x \in I. \tag{1}$$

Note that under the above assumptions, $T'(x)$ is finite and bounded away from zero for all $x \in I$. Thus, there exists $s \geqslant 0$ such that

$$\frac{|T''(x)|}{|T'(x)|^2} \leqslant s, \qquad \forall\, x \in I. \tag{2}$$

From (1) and (2) we have that there exists $D \geqslant 0$ such that

$$\frac{|(T_{a^{(n)}}^{-n})'(x)|}{|(T_{a^{(n)}}^{-n})'(y)|} \leqslant D \leqslant e^{\frac{s\alpha}{\alpha-1}}, \qquad \forall n \geqslant 1, \quad \forall x, y \in I. \tag{3}$$

We further assume that T is *covering* (see [14, 13]), i.e. for each $n \in \mathbf{N}$ there exists $N(n) > 1$ such that $T^{N(n)}(I_{a^{(n)}}) = [0, 1], \forall I_{a^{(n)}} \in \bigvee_{i=0}^{n} T^{-i}\wp$. Under the above assumptions, we choose $c' > 0$ and $c_{N(0)} > 0$ such that

$$m(T(I_a)) \geqslant c', \qquad \forall\, I_a \in \wp, \tag{4}$$

$$m(I_{a^{(N(0))}}) \geqslant c_{N(0)}, \qquad \forall\, I_{a^{(N(0))}} \in \bigvee_{i=0}^{N(0)} T^{-i}\wp, \tag{5}$$

where m is Lebesgue measure.

For $\beta \in \mathbb{R}$ we consider the potential $\phi_\beta : I \to \mathbb{R}$ defined as $\phi_\beta(x) = -\beta \log(|T'(x)|)$ and the corresponding weight $g_\beta : I \to (0, 1)$, $g_\beta(x) = \exp(\phi_\beta(x))$. In this setting, conditions (1) and (2) are enough to guarantee that $\phi_\beta : I \to \mathbb{R}$ (and consequently $g_\beta : I \to (0, 1)$) is a function of finite variation, i.e. $V_I(\phi_\beta) < \infty$ where $V_I(\phi_\beta) = \sup\{\sum_{i=1}^{k} |\phi_\beta(x_i) - \phi_\beta(x_{i-1})| : k \geqslant 1, x_0 < \cdots < x_k, x_i \in I\}$.

Notation. Throughout the paper $\|\ .\ \|_1$ will stand for the L^1 norm and $f \in L^1$ will refer to functions f that are Lebesgue integrable. $BV(I)$ is the space of functions of bounded variation acting on I, i.e. $BV(I) = \{f : I \to \mathbb{R} : V_I(f) < \infty\}$ and is endowed with the norm $||f||_{BV} = V_I(f) + ||f||_\infty$.

3. Lasota–Yorke inequalities and cones for $\boldsymbol{L}_\beta$

Cone techniques have been used to establish the existence of the invariant density of T as a fixed point of the PF operator [13] and to obtain the density of the equilibrium measure (possibly not absolutely continuous w.r.t. Lebesgue) as an eigenfunction of the more general RPF operator [14]. The rough idea behind this technique is to choose a cone[1] of functions, typically defined via a LY-type inequality on which the operator is a contraction. In section 4 we develop a convex set of BV functions that is compact in L^1 and apply standard fixed point theorems to establish the existence of the required L^1 eigenfunction of the RPF operator. This approach may be viewed as an extension of [15], which showed that the standard PF operator associated with transformations similar to the ones introduced in section 2 preserves a suitable cone of L^1 functions and used this to prove convergence of Ulam's approximation.

Our aim for the rest of this section is to build a LY inequality for $\boldsymbol{L}_\beta$ associated with the transformations introduced in section 2 in terms of BV functions in L^1. Because $\boldsymbol{L}_\beta$ is not a Markov operator (see lemma 4), we need to treat the $\beta < 1$ and $\beta \geqslant 1$ cases separately. Also, for technical reasons that will become obvious in the proofs we need to treat the $\beta < 0$ situation as a third separate case.

3.1. Properties of $\boldsymbol{L}_\beta$

We collect some properties of $\boldsymbol{L}_\beta$ that will be used later to obtain the cone contraction. Under the assumptions of the previous section we write

$$\boldsymbol{L}_\beta f = \sum_a (g_\beta \circ T_a^{-1})(f \circ T_a^{-1})\chi_{T(I_a)} = \sum_a \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta}\chi_{T(I_a)}. \tag{6}$$

[1] A convex subset $\mathcal{P}$ of a real vector space X is a cone if for any $t > 0$ and for all $f \in \mathcal{P}$, $tf \in \mathcal{P}$.

Lemma 1.

(i) $\boldsymbol{L}_\beta$ *is a positive operator; that is,* $\boldsymbol{L}_\beta f \geqslant 0$ *for all* $f \in L^1$, $f \geqslant 0$.
(ii) $\boldsymbol{L}_\beta : L^1(I) \to L^1(I)$ *is a bounded operator.*

Proof. See proofs section. □

Define the cone B_k, $0 \leqslant k < \infty$ by $B_k = \{f \in L^1 : f \geqslant 0, V_I(f) \leqslant k||f||_1\}$, and note that B_k is a subset of $BV(I)$.

3.2. Lasota–Yorke inequality

Lemma 2. *Let* α, s, D, c' *and* $c_{N(0)}$ *be given as in (1), (2), (3), (4) and (5), respectively.*

(i) When $\beta \geqslant 1$ *, for all* $f \in B_k$

$$V_I(\boldsymbol{L}_\beta f) \leqslant \frac{2}{\alpha^\beta} V_I(f) + M_1 \parallel f \parallel_1 \leqslant \left(\frac{2}{\alpha^\beta} k + M_1\right) \parallel f \parallel_1,$$

where $M_1 = \frac{2}{\alpha^{\beta-1}}(s\beta + \frac{1}{c'})$.

(ii) When $0 \leqslant \beta < 1$ *for all* $f \in B_k$

$$V_I(\boldsymbol{L}_\beta f) \leqslant \frac{2}{\alpha^\beta} V_I(f) + M_2 \parallel f \parallel_1 \leqslant \left(\frac{2}{\alpha^\beta} k + M_2\right) ||f||_1,$$

where $M_2 = 2\left(\frac{D}{c_{N(0)}}\right)^{1-\beta}(s\beta + \frac{1}{c'})$.

(iii) When $\beta < 0$ *for all* $f \in B_k$

$$V_I(\boldsymbol{L}_\beta f) \leqslant 2\left(\frac{c_{N(0)}}{D}\right)^\beta V_I(f) + M_3 \parallel f \parallel_1 \leqslant \left(2\left(\frac{c_{N(0)}}{D}\right)^\beta k + M_3\right) ||f||_1,$$

where $M_3 = 2\left(\frac{D}{c_{N(0)}}\right)^{1-\beta}(s|\beta| + \frac{1}{c'})$.

Proof. See proofs section. □

By choosing k large enough, we can ensure that $\boldsymbol{L}_\beta B_k \subseteq B_k$. However, in order to obtain a fixed point, we need to consider a normalized operator.

4. A normalized operator and a fixed point theorem

In this section, we obtain an eigenfunction of $\boldsymbol{L}_\beta$ by demonstrating the existence of a fixed point of a normalized operator in a suitable convex set. Below we briefly summarize the method of proof. The normalized operator we consider is $\boldsymbol{L}'_\beta : H \circlearrowleft$, where $H = \{f \in L^1 : f \geqslant 0, \parallel f \parallel_1 = 1\}$, defined as

$$\boldsymbol{L}'_\beta f = \frac{\boldsymbol{L}_\beta f}{||\boldsymbol{L}_\beta f||_1}. \tag{7}$$

We prove that for some suitable k, the operator $\boldsymbol{L}'_\beta$ becomes a contraction for the convex set

$$B'_k = B_k \cap H = \{f \in L^1, f \geqslant 0 : V_I(f) \leqslant k, \parallel f \parallel_1 = 1\}. \tag{8}$$

In this sense we first establish the following lemma.

Lemma 3. *For each* $0 < k < \infty$, B'_k *is compact in* L^1.

Proof. Let f_n be a sequence in B'_k. Then $V_I(f_n) \leqslant k$ and $||f_n||_\infty \leqslant k+1$. By Helly's selection principle, there exists n_k s.t. $f_{n_k} \to f^*$ everywhere. Thus $\| f_{n_k} - f^* \|_{L^1} \leqslant \| f_{n_k} - f^* \|_\infty \to 0$ as $n_k \to \infty$. It is easily checked that $f^* \in B'_k$. □

We obtain the fixed point of $\boldsymbol{L}'_\beta$ via a standard fixed point theorem. We start by collecting some properties of $\boldsymbol{L}'_\beta$. To do so we use the following lemma that describes basic properties on the relative sizes of $||f||_1$ and $||\boldsymbol{L}_\beta f||_1$.

Lemma 4. *For all $f \in L^1$, $f \neq 0$ the following hold:*

(i) When $\beta \geqslant 1$, $\dfrac{\| f \|_1}{\| \boldsymbol{L}_\beta f \|_1} \leqslant \left(\dfrac{D}{c_{N(0)}}\right)^{\beta-1}$

(ii) When $\beta < 1$, $\dfrac{\| f \|_1}{\| \boldsymbol{L}_\beta f \|_1} \leqslant \dfrac{1}{\alpha^{1-\beta}}$.

Proof. See proofs section. □

Lemma 5. *For all $\beta \in \mathbb{R}$, $\boldsymbol{L}'_\beta : H \circlearrowleft$ is*

(i) well defined and
(ii) continuous.

Proof. Follows immediately from lemma 1, the definition of $\boldsymbol{L}'_\beta$ and lemma 4. □

We can now obtain explicit bounds for the variation of $\boldsymbol{L}'_\beta f$.

Lemma 6. *Let B'_k be as defined in (8). For all $f \in B'_k$ we have*

(i) When $\beta \geqslant 1$,

$$V_I(\boldsymbol{L}'_\beta f) \leqslant \left(2\frac{k}{\alpha^\beta} + M_1\right)\left(\frac{D}{c_{N(0)}}\right)^{\beta-1}, \tag{9}$$

where $M_1 = \frac{2}{\alpha^{\beta-1}}(s\beta + \frac{1}{c'})$.

(ii) When $0 \leqslant \beta < 1$,

$$V_I(\boldsymbol{L}'_\beta f) \leqslant \left(2\frac{k}{\alpha} + M_2\frac{1}{\alpha^{1-\beta}}\right), \tag{10}$$

where $M_2 = 2\left(\frac{D}{c_{N(0)}}\right)^{1-\beta}(s\beta + \frac{1}{c'})$.

(iii) When $\beta < 0$,

$$V_I(\boldsymbol{L}'_\beta f) \leqslant \left(\frac{1}{\alpha}\right)^{1-\beta}\left(2\left(\frac{c_{N(0)}}{D}\right)^{\beta} k + M_3\right), \tag{11}$$

where $M_3 = 2\left(\frac{D}{c_{N(0)}}\right)^{1-\beta}(s|\beta| + \frac{1}{c'})$.

Proof. The result follows immediately from lemma 2(i) and lemma 4(i) when $\beta \geqslant 1$, lemma 2(ii) and lemma 4(ii) when $0 \leqslant \beta < 1$ and lemma 2(iii) together with lemma 4(ii) when $\beta < 0$. □

We can now show that for suitably large k, B'_k is invariant under the action of $\boldsymbol{L}'_\beta$.

Lemma 7. *Let B'_k be as introduced in (8). Then*

(a) For each $\beta \geqslant 1$, if $\frac{\alpha^\beta}{2} > (\frac{D}{c_{N(0)}})^{\beta-1}$, then

$$\boldsymbol{L}'_\beta B'_k \subseteq B'_k, \forall k \geqslant k(\beta), \text{ where } k(\beta) = \frac{M_1(D/c_{N(0)})^{\beta-1}}{1-(2/\alpha^\beta)(D/c_{N(0)})^{\beta-1}}. \quad (12)$$

(b) For each $0 \leqslant \beta < 1$,

$$\boldsymbol{L}'_\beta B'_k \subseteq B'_k, \forall k \geqslant k(\beta), \text{ where } k(\beta) = \frac{M_2}{1-(2/\alpha)}. \quad (13)$$

(c) For each $\beta < 0$, if $\alpha^{1-\beta} > 2(c_{N(0)}/D)^\beta$,

$$\boldsymbol{L}'_\beta B'_k \subseteq B'_k, \forall k \geqslant k(\beta), \text{ where } k(\beta) = \frac{M_3}{\alpha^{1-\beta} - 2(c_{N(0)}/D)^\beta}. \quad (14)$$

Proof. Follows directly from (9)–(11). □

As T is covering we can choose $N(0) > 1$ such that $T^{N(0)}(I_a) = [0,1], \forall I_a \in \wp$ and prove the existence of lower bounds for $\boldsymbol{L}_\beta^{N(0)} f$, $f \in B'_k$.

Lemma 8. *For all $f \in B'_k$ there exist $M(k) > 0$ such that $\boldsymbol{L}_\beta^{N(0)} f > M(k)$.*

Proof. See proofs section. □

This allows us to demonstrate positivity of the eigenfunction of $\boldsymbol{L}_\beta$ in the main result of this section, which we state below:

Theorem 9. *For $k > k(\beta)$, with $k(\beta)$ defined as in (12)–(14), $\boldsymbol{L}_\beta$ has a positive eigenfunction h in B_k with a positive eigenvalue λ_β.*

Proof. For $k > k(\beta)$, we apply the Schauder theorem to the continuous operator $\boldsymbol{L}'_\beta$ and the compact, convex set B'_k to conclude that there exists $h \in B'_k$ with $\boldsymbol{L}'_\beta h = h$. This fixed point equation yields: there exists $h \in B'_k$ such that $\boldsymbol{L}_\beta h = ||\boldsymbol{L}_\beta h||_1 h$. By lemma 5(i) we know that $\lambda_\beta = ||\boldsymbol{L}_\beta h||_1 > 0$. The fact that h is positive follows immediately from lemma 8 and positivity of λ_β. □

5. Topological pressure, equilibrium measure for (T, ϕ_β)

So far we have demonstrated that for our class of interval maps, under the conditions of theorem 9, the operator $\boldsymbol{L}_\beta$ has a positive eigenvalue and a corresponding positive L^1 eigenfunction. In this section, we verify that the logarithm of this eigenvalue is equal to the topological pressure $P(T, \phi_\beta)$, and obtain the equilibrium measure for (T, ϕ_β). Moreover, we show that the eigenfunction h corresponding to λ_β is the only eigenfunction of $\boldsymbol{L}_\beta$ in B'_k. We recall the following:

(i) The map T is covering. This has been dealt with in section 2.
(ii) The potential $\phi_\beta = \log(1/|T'|^\beta)$ is *contracting* (see Def. 3.4 in [14]).

Lemma 10. *Under the conditions of theorem 9, the eigenvalue λ_β of $\boldsymbol{L}_\beta$ can be identified with the exponential of the pressure, i.e. $\lambda_\beta = \mathrm{e}^{P(T,\phi_\beta)}$. Moreover, h is the only eigenfunction of $\boldsymbol{L}_\beta$ in B_k and is a multiple of the density of the unique equilibrium state.*

Proof. Let the functional ν be defined as in [14] and let h_* denote the density of the (unique) equilibrium state $\mu = h_*\nu$; the existence of h_* is guaranteed by lemma 4.8 in [14]. Then, a direct application of theorem 3.2 in [14] (in particular, of footnote 5) implies that $\|\exp(n(\log\lambda_\beta - P(T,\phi_\beta)))h - \nu(h)h_*\|_\infty \to 0$ as $n \to \infty$. Thus $\log\lambda_\beta - P(T,\phi_\beta) = 0$ and $h = \nu(h)h_*$. Therefore, h is the unique (up to scalar multiples) eigenfunction for $\boldsymbol{L}_\beta$ in B_k and $h\nu$ is the unique equilibrium state for suitably scaled h. □

6. Approximating $\boldsymbol{L}_\beta$ by Ulam's method

We begin by briefly recalling Ulam's method in its original setting, the approximation of the Perron–Frobenius operator $\boldsymbol{L} := \boldsymbol{L}_1$, obtained by setting $\beta = 1$. A problem in ergodic theory that is still relevant today is the numerical approximation of absolutely continuous invariant measures (acims). If f is a fixed point of $\boldsymbol{L}$, then f is the density of an acim. The approach suggested by Ulam [24] was to build a finite-dimensional approximation of $\boldsymbol{L}$ and solve a linear system to obtain an approximation for f. Convergence of the approximate acim to the true acim, including error bounds in some cases, has been proved in a variety of settings [2, 5, 7, 12, 16].

We extend the Ulam construction to RPF operators $\boldsymbol{L}_\beta$ and prove convergence of (i) the leading numerical Ulam eigenvalues to $\mathrm{e}^{P(T,\phi_\beta)}$ and (ii) the corresponding numerical Ulam eigenfunctions to the density of the equilibrium state. In contrast to the standard Ulam approach, the leading eigenvalue of $\boldsymbol{L}_\beta$ is unknown; moreover, the nature of the action of $\boldsymbol{L}_\beta$ varies with β. Our method of proof proceeds as follows: we implicitly approximate the normalized operator $\boldsymbol{L}'_\beta$ introduced in section 4 and demonstrate the existence of approximate fixed points of $\boldsymbol{L}'_\beta$. We then extract a limit of these approximate fixed points and using the results of section 5 show that this limit is unique. Finally, this limit is identified with an eigenfunction of $\boldsymbol{L}_\beta$ and the eigenvalue convergence is demonstrated. In practical terms, all that is required is the relatively straightforward construction of a matrix approximation of $\boldsymbol{L}_\beta$.

Let $\xi^n = \{A_1, A_2, \ldots, A_n\}$ be a finite partition of $I = [0,1]$ into intervals and define $\Delta_n = \left\{f \in L^1 : f = \sum_{i=1}^n a_i\chi_{A_i}, a_i \in \mathbb{R}\right\}$. We will shortly consider a sequence of partitions $\{\xi\}_{n=n_0}^\infty$, and will assume that as $n \to \infty$, the maximal length of any interval in ξ^n approaches zero.

Define $\Pi_n f = \sum_{i=1}^n \frac{1}{m(A_i)}\left(\int_{A_i} f\,dm\right)\chi_{A_i}$ as the canonical projection of L^1 onto Δ_n, and consider the projected operator $\boldsymbol{L}_{\beta,n} := \Pi_n \circ \boldsymbol{L}_\beta : \Delta_n \circlearrowleft$. The following lemma states that the action of $\boldsymbol{L}_{\beta,n}$ on Δ_n is described by a matrix $L_{\beta,n,ij}$.

Lemma 11.

$$\boldsymbol{L}_{\beta,n}\left(\sum_{i=1}^n a_i\chi_{A_i}\right) = \sum_{j=1}^n\left(\sum_{i=1}^n a_i L_{\beta,n,ij}\right)\chi_{A_j} \tag{15}$$

where $L_{\beta,n,ij} = \frac{1}{m(A_j)}\int_{A_i\cap T^{-1}A_j}\frac{1}{|T'(y)|^{\beta-1}}\,\mathrm{d}y$.

Proof. Straightforward modification of lemma 2.3 in [12]. □

Let $vL_{\beta,n} = \lambda_{\beta,n}v$, where $\lambda_{\beta,n}$ is the largest eigenvalue of $L_{\beta,n}$. Our idea is that $\lambda_{\beta,n}$ approximates $\mathrm{e}^{P(T,\phi_\beta)}$ and the corresponding eigenfunction $h_n = \sum_{i=1}^n v_i\chi_{A_i}$ approximates a suitably normalized version of the density of the equilibrium state for (T, ϕ_β). We now state our main result, formalizing these ideas.

Theorem 12. *Assume that the hypotheses of theorem 9 hold. Let $\lambda_{\beta,n}$ be the largest magnitude positive eigenvalue of $L_{\beta,n}$ and h_n the corresponding eigenfunction. Then*

(i) as $n \to \infty$ the sequence $\{h_n\}$ converges to h, a multiple of the density of the unique equilibrium state for the pair (T, ϕ_β) and
(ii) $\lim_{n\to\infty} \lambda_{\beta,n} = \lambda_\beta = e^{P(T,\phi_\beta)}$.

Proof. See proofs section. □

The remainder of this section outlines the main steps required in the proof of the above theorem. In order to employ a fixed point theorem, we need to consider an approximate version of the normalized operator from section 4. Define $\boldsymbol{L}'_{\beta,n} : (\Delta_n \cap \{f : f \geqslant 0, ||f||_1 = 1\}) \circlearrowleft$ by

$$\boldsymbol{L}'_{\beta,n} f = \frac{\boldsymbol{L}_{\beta,n} f}{\| \boldsymbol{L}_{\beta,n} f \|_1}.$$

Analogous to lemma 5 we have the following lemma.

Lemma 13. *For all $\beta \in \mathbb{R}$, $\boldsymbol{L}'_{\beta,n} : (\Delta_n \cap \{f : f \geqslant 0, ||f||_1 = 1\}) \circlearrowleft$ is*

(i) well defined and
(ii) continuous.

Proof. Follows immediately from lemma 5, the definition of $\boldsymbol{L}'_{\beta,n}$ and the fact that for all $f \in L^1$, $f \geqslant 0$, $\beta \in \mathbb{R}$, $||\boldsymbol{L}_{\beta,n} f||_1 = ||\boldsymbol{L}_\beta f||_1$. This latter result is a consequence of the fact that for all $f \in L^1$, $f \geqslant 0$, $||\Pi_n f||_1 = ||f||_1$ (see [12]). □

The variation of functions under the action of our approximate normalized operator is no greater than that of the original normalized operator.

Lemma 14. *For all $f \in L^1$, $f \neq 0$, $f \geqslant 0$, and $\beta \in \mathbb{R}$, $V_I(\boldsymbol{L}'_{\beta,n} f) \leqslant V_I(\boldsymbol{L}'_\beta f)$.*

Proof. We begin by noting that for all $f \in L^1$, $\beta \in \mathbb{R}$, $V_I(\boldsymbol{L}_{\beta,n} f) \leqslant V_I(\boldsymbol{L}_\beta f)$, which is a consequence of the fact that for all $f \in L^1$, $V_I(\Pi_n f) \leqslant V_I(f)$ (see [12]). This, together with the property that for all $f \in L^1$, $f \geqslant 0$, $\beta \in \mathbb{R}$, $||\boldsymbol{L}_{\beta,n} f||_1 = ||\boldsymbol{L}_\beta f||_1$ yields

$$V_I(\boldsymbol{L}'_{\beta,n} f) = V_I\left(\frac{\boldsymbol{L}_{\beta,n} f}{||\boldsymbol{L}_{n,\beta} f||_1}\right) = \frac{V_I(\boldsymbol{L}_{\beta,n} f)}{||\boldsymbol{L}_{\beta,n} f||_1} = \frac{V_I(\boldsymbol{L}_{\beta,n} f)}{||\boldsymbol{L}_\beta f||_1} \leqslant \frac{V_I(\boldsymbol{L}_\beta f)}{||\boldsymbol{L}_\beta f||_1} = V_I(\boldsymbol{L}'_\beta f). \qquad \square$$

We can now establish the existence of a fixed point for our approximate normalized operator in analogy to theorem 9.

Lemma 15. *For $k > k(\beta)$, with $k(\beta)$ defined in (12)–(14), each $\boldsymbol{L}'_{\beta,n}$ has a fixed point $h_n \in B'_k$.*

Proof. Lemma 14 and $\|\boldsymbol{L}'_{\beta,n}\|_1 = 1$ imply that if $\boldsymbol{L}'_\beta$ preserves B'_k then $\boldsymbol{L}'_{\beta,n}$ also preserves B'_k. Thus, by lemma 7, $\boldsymbol{L}'_{\beta,n}$ preserves B'_k for all $k \geqslant k(\beta)$. From lemma 3 we know that B'_k is convex and compact. From lemma 13 we know that $\boldsymbol{L}'_{\beta,n} : (\Delta_n \cap \{f : f \geqslant 0, ||f||_1 = 1\}) \circlearrowleft$ is continuous. The result follows by Schauder's theorem. □

Strong convergence of $\boldsymbol{L}'_{\beta,n}$ to $\boldsymbol{L}'_{\beta}$, as an action on positive $f \in L^1$, is straightforward to establish.

Lemma 16. *For all $f \in L^1$, $f \neq 0$, $f \geqslant 0$, and $\beta \in \mathbb{R}$, $||\boldsymbol{L}'_{\beta,n} f - \boldsymbol{L}'_{\beta} f||_1 \to 0$ as $n \to \infty$.*

Proof. We first note that because $||f - \Pi_n f||_1 \to 0$ as $n \to \infty$ (see [12]) we have that for all $f \in L^1$, $\beta \in \mathbb{R}$, $||\boldsymbol{L}_{\beta} f - \boldsymbol{L}_{\beta,n} f||_1 \to 0$ as $n \to \infty$. As $||\boldsymbol{L}_{\beta,n} f||_1 = ||\boldsymbol{L}_{\beta} f||_1$ for all $f \geqslant 0$, $\beta \in \mathbb{R}$, one has that for all $f \in L^1$, $f \neq 0$, $f \geqslant 0$ and $\beta \in \mathbb{R}$

$$||\boldsymbol{L}'_{\beta} f - \boldsymbol{L}'_{\beta,n} f||_1 = \left|\left| \frac{\boldsymbol{L}_{\beta} f}{||\boldsymbol{L}_{\beta} f||_1} - \frac{\boldsymbol{L}_{\beta,n} f}{||\boldsymbol{L}_{\beta,n} f||_1} \right|\right|_1 = \frac{1}{||\boldsymbol{L}_{\beta} f||_1} ||\boldsymbol{L}_{\beta} f - \boldsymbol{L}_{\beta,n} f||_1$$

which goes to 0 as $n \to \infty$. □

Lemma 16 together with relative compactness of the sequence of fixed points of $\boldsymbol{L}'_{\beta,n}$ leads to the following lemma.

Lemma 17. *Let h_n be a fixed point of $\boldsymbol{L}'_{\beta,n}$. Then $h_n \to h$ in L^1, as $n \to \infty$, where h is the unique fixed point of $\boldsymbol{L}'_{\beta}$.*

Proof. Since $h_n \in B'_k$ and B'_k is compact in L^1, the sequence $\{h_n\}$ is relatively compact in L^1. Let $\tilde{h}$ be a limit point of this sequence and $\{h_{n_j}\}$ be the corresponding convergent subsequence: $||\tilde{h} - h_{n_j}||_1 \to 0$ as $n_j \to \infty$. But

$$||\tilde{h} - \boldsymbol{L}'_{\beta}\tilde{h}||_1 \leqslant ||\tilde{h} - h_{n_j}||_1 + ||\boldsymbol{L}'_{\beta,n_j} h_{n_j} - \boldsymbol{L}'_{\beta,n_j}\tilde{h}||_1 + ||\boldsymbol{L}'_{\beta,n_j}\tilde{h} - \boldsymbol{L}'_{\beta}\tilde{h}||_1. \tag{16}$$

Because $||\boldsymbol{L}'_{\beta,n_j} h_{n_j} - \boldsymbol{L}'_{\beta,n_j}\tilde{h}||_1 \leqslant ||\boldsymbol{L}'_{\beta,n_j}|| \cdot ||\tilde{h} - h_{n_j}||_1 = ||\tilde{h} - h_{n_j}||_1$, the second term of equation (16) goes to zero as n_j goes to infinity. Moreover, by lemma 16, $||\boldsymbol{L}'_{\beta,n_j}\tilde{h} - \boldsymbol{L}'_{\beta}\tilde{h}||_1 \to 0$ as $n_j \to \infty$. Thus, $\boldsymbol{L}'_{\beta}\tilde{h} = \tilde{h}$.

Since by lemma 10 we know that $\boldsymbol{L}_{\beta}$ has a unique eigenfunction $h \in B'_k$, $\boldsymbol{L}'_{\beta}$ has a unique fixed point $h \in B'_k$ and thus $\tilde{h}$ must be a multiple of h. Thus, the sequence $\{h_n\}$ has only one limit point, which is a multiple of h. We therefore must have that $\lim_{n\to\infty} h_n = \tilde{h}$. □

7. Discussion

The rigorous estimation of topological pressure for interval maps is a difficult problem in ergodic theory and thermodynamics. For specific maps, specialized techniques have been developed (e.g. [6, 11, 17, 18, 25]). However, to our knowledge, the results presented here represent the first rigorous numerical approach to estimating pressure for a reasonably broad class of interval maps. We close by remarking that numerical experiments reported in [8] demonstrate that our method is simple to implement, extremely efficient in terms of computing time and is a very practical way to detect phase transitions with respect to the weight functions $\phi_{\beta} = -\beta \log|T'|$ when they exist. Future work will include the extension of the rigorous results presented here to transformations that exhibit phase transitions.

8. Proofs section

8.1. Proof of lemma 1

(i) is obvious from the $\boldsymbol{L}_\beta$ definition—see equation (6). To prove (ii) we consider the following cases:

When $\beta \geqslant 1$, for all $f \in L^1$,

$$
\begin{aligned}
||\boldsymbol{L}_\beta f||_1 &\leqslant \sum_a \int_I \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \chi_{T(I_a)} \right| \\
&= \sum_a \int_{T(I_a)} \left| \frac{1}{|T_a' \circ T_a^{-1}|^{\beta-1}} \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|} \right| \\
&\leqslant \left(\frac{1}{\alpha}\right)^{\beta-1} \sum_a \int_{I_a} |f| = \left(\frac{1}{\alpha}\right)^{\beta-1} ||f||_1.
\end{aligned}
$$

When $\beta < 1$ we recall that T is covering. Let $c_{N(0)}$ be given as in (5). The mean value theorem together with equation (3) gives

$$
\frac{1}{|T^{N(0)'} \circ T_a^{-N(0)}(y)|} \geqslant \frac{m(I_{a^{(N(0))}})}{D} \geqslant \frac{c_{N(0)}}{D}, \forall y \in I_{a^{(N(0))}}. \tag{17}
$$

Now for the class of transformation considered here $\frac{1}{|T'(x)|} > \frac{1}{|T^{N(0)'}(y)|}, \forall x \in I_a, \forall y \in I_{a^{(N(0))}}$, which together with (17) implies

$$
\frac{1}{|T'(x)|} > \frac{1}{|T^{N(0)'}(y)|} \geqslant \frac{m(I_{a^{(N(0))}})}{D} \geqslant \frac{c_{N(0)}}{D}, \forall x \in I_a, \forall y \in I_{a^{(N(0))}}. \tag{18}
$$

Raising (18) to $\beta - 1$ (which is negative, since $\beta < 1$) implies

$$
\frac{1}{|T_a' \circ T_a^{-1}(x)|^{\beta-1}} < \frac{1}{|T^{N(0)'} \circ T_a^{-N(n)}(x)|^{\beta-1}} \leqslant \left(\frac{D}{c_{N(0)}}\right)^{1-\beta}, \forall x \in I_a, \forall y \in I_{a^{(N(0))}}. \tag{19}
$$

Therefore, when $\beta < 1$, for all $f \in L^1$ we have (similarly to the $\beta \geqslant 1$ case)

$$
\begin{aligned}
||\boldsymbol{L}_\beta f||_1 &\leqslant \sum_a \int_{T(I_a)} \left| \frac{1}{|T_a' \circ T_a^{-1}|^{\beta-1}} \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|} \right| \\
&\leqslant \left(\frac{D}{c_{N(0)}}\right)^{1-\beta} \sum_a \int_{I_a} |f| = \left(\frac{D}{c_{N(0)}}\right)^{1-\beta} ||f||_1.
\end{aligned}
$$

8.2. Proof of lemma 2

Because $\boldsymbol{L}_\beta f \in BV(I), \forall f \in B_k \subset BV(I)$, we may write

$$
V_I(\boldsymbol{L}_\beta f) = \int_I \mathrm{d}(\boldsymbol{L}_\beta f) := \sup \left\{ \int_I \boldsymbol{L}_\beta f \cdot g' : g \in C^1(I), |g|_\infty \leqslant 1 \right\},
$$

where $\mathrm{d}(\boldsymbol{L}_\beta f)$ is the generalized derivative (see e.g. [9]). Thus

$$
V_I(\boldsymbol{L}_\beta f) = \int_I \mathrm{d}\left(\sum_a \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right) \leqslant \int_I \sum_a \left| \mathrm{d} \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right| = \sum_a \int_I \left| \mathrm{d} \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right|. \tag{20}
$$

Let $T(I_a) = [b_a, b'_a]$ and recall from equation (4) that $|b_a - b'_a| = m(T(I_a)) \geqslant c'$. A straightforward modification of the proof of lemma 3.1.2 in [15] implies that for all β:

$$\int_{I_a} \left| \mathrm{d} \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right| \leqslant 2 \int_{T(I_a)} \left| \mathrm{d} \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right| + \frac{2}{c'} \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right|.$$

The above inequality together with (20) leads to

$$V_I(\boldsymbol{L}_\beta f) \leqslant 2 \sum_a \int_{T(I_a)} \left| \mathrm{d} \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right| + \frac{2}{c'} \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right|. \tag{21}$$

(i) $\beta \geqslant 1$. Since (21) holds for every β, to prove case (i) of lemma 2, we only need to analyse each term on the right-hand side of the inequality (21) for $\beta \geqslant 1$. With respect to the first term we have

$$\begin{aligned} 2 \sum_a \int_{T(I_a)} \left| \mathrm{d} \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right| &\leqslant 2 \sum_a \int_{T(I_a)} \left| \frac{(\mathrm{d} f) \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^{\beta+1}} \right| + 2s\beta \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right| \\ &\leqslant \frac{2}{\alpha^\beta} \sum_a \int_{I_a} |\mathrm{d} f| + 2 \frac{s\beta}{\alpha^{\beta-1}} \sum_a \int_{I_a} |f| \\ &= \frac{2}{\alpha^\beta} \int_I |\mathrm{d} f| + 2 \frac{s\beta}{\alpha^{\beta-1}} \int_I |f|. \end{aligned}$$

With respect to the second term of the right-hand side of the inequality (21) we have $\frac{2}{c'} \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right| \leqslant \frac{2}{c'} \frac{1}{\alpha^{\beta-1}} \int_I |f|$. Then the result follows from (21) and the last two inequalities.

(ii) $0 \leqslant \beta < 1$. Proceeding as in the proof of (i), since (21) holds for every β, we analyse each term on the right-hand side of the inequality (21), this time for $0 < \beta < 1$. With respect to the first term we write

$$\begin{aligned} 2 \sum_a \int_{T(I_a)} \left| \mathrm{d} \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right| &\leqslant \frac{2}{\alpha^\beta} \sum_a \int_{I_a} |\mathrm{d} f| + 2s\beta \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right| \\ &= \frac{2}{\alpha^\beta} \int_I |\mathrm{d} f| + 2s\beta \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right|. \end{aligned} \tag{22}$$

Now we need to look at $\sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right|$. Using equation (19) we write

$$\begin{aligned} \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|^\beta} \right| &\leqslant \left(\frac{D}{c_{N(0)}} \right)^{1-\beta} \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T'_a \circ T_a^{-1}|} \right| \\ &= \left(\frac{D}{c_{N(0)}} \right)^{1-\beta} \sum_a \int_{I_a} |f|. \end{aligned} \tag{23}$$

From (21) to (23) we have

$$V_I(\boldsymbol{L}_\beta f) \leqslant \frac{2}{\alpha^\beta} \int_I |\mathrm{d} f| + 2 \left(\frac{D}{c_{N(0)}} \right)^{1-\beta} \left(s\beta + \frac{1}{c'} \right) \int_I |f|$$

and we are done with the proof of (ii).

(iii) $\beta < 0$. We first observe that by raising (18) to β (which is negative in this case) we obtain an upper bound for $1/|T_a' \circ T_a^{-1}|^\beta$ as

$$\frac{1}{|T_a' \circ T_a^{-1}(x)|^\beta} < \frac{1}{|T^{N(0)'} \circ T_a^{-N(0)}(x)|^\beta} \leqslant \left(\frac{c_{N(0)}}{D}\right)^\beta, \quad \forall x \in I_a, \forall y \in I_{a^{(N(0))}}.$$

Thus

$$2\sum_a \int_{T(I_a)} \left| \mathrm{d} \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right| \leqslant 2\left(\frac{c_{N(0)}}{D}\right)^\beta \sum_a \int_{I_a} |\mathrm{d}f| + 2s|\beta| \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right|$$

$$= 2\left(\frac{c_{N(0)}}{D}\right)^\beta \int_I |\mathrm{d}f| + 2s|\beta| \sum_a \int_{T(I_a)} \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \right|.$$

Then the proof of (iii) goes exactly the same as the proof of (ii).

8.3. Proofs of lemma 4 and lemma 8

Proof of lemma 4. When $\beta \geqslant 1$, by raising equation (18) to $\beta - 1$ we have that $\forall x \in I_a$, $\forall y \in I_{a^{(N(0))}}$

$$\left(\frac{1}{|T'(x)|}\right)^{\beta-1} \geqslant \left(\frac{1}{|T^{N(0)'}(y)|}\right)^{\beta-1} \geqslant \left(\frac{m(I_{a^{(N(0))}})}{D}\right)^{\beta-1} \geqslant \left(\frac{c_{N(0)}}{D}\right)^{\beta-1}. \tag{24}$$

Thus, since $f \geqslant 0$,

$$||\boldsymbol{L}_\beta f||_1 = \sum_a \int_I \left| \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|^\beta} \chi_{T(I_a)} \right|$$

$$= \sum_a \int_{T(I_a)} \left| \frac{1}{|T_a' \circ T_a^{-1}|^{\beta-1}} \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|} \right|$$

$$\geqslant \left(\frac{c_{N(0)}}{D}\right)^{\beta-1} \sum_a \int_{I_a} |f| = \left(\frac{c_{N(0)}}{D}\right)^{\beta-1} ||f||_1,$$

and (i) follows under the assumption that $f \neq 0$.

When $\beta < 1$, we only need to observe $\frac{1}{|T'|^{\beta-1}} \geqslant \frac{1}{\alpha^{\beta-1}}$, which implies that

$$||\boldsymbol{L}_\beta f||_1 = \sum_a \int_{I_a} \left| \frac{1}{|T_a' \circ T_a^{-1}|^{\beta-1}} \frac{f \circ T_a^{-1}}{|T_a' \circ T_a^{-1}|} \right| \geqslant \left(\frac{1}{\alpha}\right)^{\beta-1} \sum_a \int_{I_a} |f| = \left(\frac{1}{\alpha}\right)^{\beta-1} ||f||_1.$$

Thus, (ii) follows under the same assumption $f \neq 0$.

Proof of lemma 8. For $f \in B_k$ let

$$\tilde{f} = \sum_{a^{(N(0))}} \left(\operatorname{ess} \inf_{I_{a^{(N(0))}}} f \right) \chi_{I_{a^{(N(0))}}}.$$

By lemma 3.2.1 in [15], $||\tilde{f}||_1 \geqslant ||f||(1 - \alpha^{-N(0)}k)$ and thus for all $f \in B_k'$,

$$||\tilde{f}||_1 \geqslant (1 - \alpha^{-N(0)}k). \tag{25}$$

From equation (17) we know that $\frac{1}{|T^{N(0)'}\circ T_a^{-N(0)}(x)|} \geqslant \frac{m(I_{a(N(0))})}{D} \geqslant \frac{c_{N(0)}}{D}, \forall x \in I_{a^{(N(0))}}$. This together with (25) gives

$$\begin{aligned} L_\beta^{N(0)} f &= \sum_{a^{(N(0))}} \frac{f \circ T^{-1}_{a^{(N(0))}}}{|(T^{N(0)})' \circ T^{-1}_{a^{(N(0))}}|^\beta} \\ &\geqslant \sum_{a^{(N(0))}} \left(\operatorname{ess}\inf_{I_{a^{(N(0))}}} f\right) \frac{1}{|(T^{N(0)})' \circ T^{-1}_{a^{(N(0))}}|^{\beta-1}} \frac{1}{|(T^{N(0)})' \circ T^{-1}_{a^{(N(0))}}|} \\ &\geqslant M \sum_{a^{(N(0))}} \tilde{f}_{a^{(N(0))}} m(I_{a^{(N(0))}})/D = M||\tilde{f}||_1/D, \end{aligned}$$

where $M = (c_{N(0)}/D)^{\beta-1}\cdot(c_{N(0)}/D) = (c_{N(0)}/D)^\beta$ if $\beta \geqslant 1$ and $M = (1/\alpha^{\beta-1})\cdot(c_{N(0)}/D)$ if $\beta < 1$. This choice of M is motivated by equation (24) when $\beta \geqslant 1$ and by the fact that $\frac{1}{|T'|^{\beta-1}} \geqslant \frac{1}{\alpha^{\beta-1}}$ when $\beta < 1$.

To complete choose $M(k) = M(1-\alpha^{-N(0)}k)$.

8.4. *Proof of theorem 12*

Proof. Let $\lambda_{\beta,n}$ be an eigenvalue of $L_{\beta,n}$ (as defined in lemma 11) and h_n the corresponding eigenfunction normalized so that $\|h_n\|_1 = 1$. By lemma 11 we know that any eigenvalue, eigenfunction pair of $L_{\beta,n}$ is an eigenvalue, eigenfunction pair of $\boldsymbol{L}_{\beta,n}$. Since we also know that any normalized eigenfunction of $\boldsymbol{L}_{\beta,n}$ is a fixed point of $\boldsymbol{L}'_{\beta,n}$, lemma 17 implies that $\{h_n\}$ converges to the unique fixed point of $\boldsymbol{L}'_\beta$ as $n \to \infty$. Furthermore, lemma 10 implies that this unique fixed point is a multiple of the density of the unique equilibrium state for the pair (T, ϕ_β).

We now prove (ii). Recall that $\lambda_{\beta,n} = ||\boldsymbol{L}_{\beta,n}h_n||_1$ and $\lambda_\beta = ||\boldsymbol{L}_\beta h||_1 = ||\boldsymbol{L}_\beta h_n||_1$. Thus using the reverse triangle inequality $|\lambda_{\beta,n} - \lambda_\beta| = |\|\boldsymbol{L}_{\beta,n}h_n||_1 - \|\boldsymbol{L}_\beta h||_1| = |\|\boldsymbol{L}_\beta h_n||_1 - ||\boldsymbol{L}_\beta h||_1| \leqslant ||\boldsymbol{L}_\beta||_1\cdot||h_n - h||_1$. From lemma 1 we know that $||\boldsymbol{L}_\beta||_1$ is bounded. By lemma 17 we know that $||h_n - h||_1 \to 0$ as $n \to \infty$. Thus, $|\lambda_{\beta,n} - \lambda| \to 0$ as $n \to \infty$. The desired result now follows by lemma 10. □

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