On irrationality and transcendency of infinite sums of rational numbers

R. Tijdeman

Abstract

In this survey paper we consider infinite sums $\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$, $\sum_{n=1}^{\infty} (-1)^n \frac{P(n)}{Q(n)}$, and $\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N} Q(n)}$ where $P(x), Q(x) \in \mathbb{Z}[x]$. We give conditions under which such a sum is irrational and under which it is transcendental.

1 Introduction

It is well known that for every positive integer k the quotient $\zeta(2k)/\pi^{2k}$ is rational and therefore the sum $\sum_{n=1}^{\infty} n^{-2k}$ is transcendental. On the other hand, the arithmetic character of $\zeta(2k+1) = \sum_{n=1}^{\infty} n^{-2k-1}$ for $k = 1, 2, ...$ is a mystery. In 1979 Apéry [Ap] succeeded in showing that $\zeta(3)$ is irrational, cf. Beukers [Be1]. It is not known whether it is transcendental, nor whether $\zeta(2k+1)$ is irrational for $k = 2, 3, ...$, cf. Zudilin [Zu]. We summarize what is known about the irrationality and transcendency of sums $\sum_{n=1}^{\infty} \alpha^n f(n)$ where $\alpha \in \mathbb{Q}, f(x) \in \mathbb{Q}(x)$ such that $f(n)$ is defined for all positive integers n and the sum converges. In Section 2 we deal with the case that the poles of f are simple and rational. In Section 3 we consider the case of general f, but only for $\alpha = \pm 1$.

Apéry observed that $\zeta(3)$ can be written as a sum

$$\frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

This sum equals $\frac{5}{4} \sum_{N=1}^{\infty} \left(1/\prod_{n=2}^{N} Q(n)\right)$ where $Q(x) = -\frac{2x^2(2x-1)}{(x-1)^3}$. It is therefore interesting to study sums $\sum_{N=1}^{\infty} \left(P(N)/\prod_{n=1}^{N} Q(n)\right)$ where $P(x) \in \mathbb{Z}[x], Q(x) \in \mathbb{Z}(x)$. As a modest step into this direction we shall consider such sums with $Q(x) \in \mathbb{Z}[x]$ in Sections 4 and 5. First we remark that the sum is rational if Q is constant. Then we show that the sum is transcendental if Q is linear. Subsequently we discuss the case that Q has only rational zeros. In Section 5 we make some observations on the general case. In Section 6 we consider some applications and related results.

My own contributions to the subject are in cooperation with S.D. Adhikari, N. Saradha, and T.N. Shorey for Section 2 and with J. Hančl for the Sections 4-6. I thank D.W. Masser, T. Matala-aho and T. Hessami Pilehrood for their valuable remarks. I am much indebted to F. Beukers for his many additions and corrections in the paper.

2 Rational functions with simple rational poles

Let $a \in \mathbb{Q}, f(x) \in \mathbb{Q}(x)$. We consider the arithmetic character of $S := \sum_{n=0}^{\infty} \alpha^n f(n)$. Without loss of generality we can write $f(x) = \frac{P(x)}{Q(x)}$ with $P(x), Q(x) \in \mathbb{Z}[x]$ and consider $\sum_{n=0}^{\infty} \alpha^n \frac{P(n)}{Q(n)}$. We assume that Q is non-constant, that P is non-trivial and that the sum is convergent.

2.1 Impossibility of irrational algebraic values

If the zeros of Q are simple and rational, then we can decompose $\frac{P(x)}{Q(x)}$ into partial fractions and apply the theory on linear forms in logarithms. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)} = 2\sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2n+1}\right) = 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots\right) = 2 - 2\log 2,$$

and, denoting the Fibonacci numbers by F_n with $F_0 = 0, F_1 = 1$ and the golden ratio $\frac{1}{2} + \frac{1}{2}\sqrt{5}$ by τ,

$$\sum_{n=1}^{\infty} \frac{F_n}{n2^n} = \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{1}{n} \left((\frac{\tau}{2})^n) - (-\frac{1}{2\tau})^n \right) = \frac{1}{\sqrt{5}} \log(1+\tau) - \frac{1}{\sqrt{5}} \log(2-\tau).$$

In general, after decomposing into partial fractions we get sums of the form

$$\sum_{n=0}^{\infty} \alpha^n \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu n + r_\mu}$$

where $c_1, \dots, c_m \in \mathbb{Q}$ and $k_1, \dots, k_m, r_1, \dots, r_m$ are integers. By adapting the summation we can secure that $0 < r_\mu \leq k_\mu$ for $\mu = 1, \dots, m$ at the cost of an additive rational constant. (So there is no constant term if all the zeros of Q are in the interval $[-1, 0)$.) If $\alpha = 1$, then we can apply Lemma 5 of [AdSST] based on a result of D.H. Lehmer [Le]. It says that if the double sum converges and $0 < r_\mu \leq k_\mu$ for $\mu = 1, \dots, m$, then

$$(1) \qquad \sum_{n=0}^{\infty} \sum_{\mu=1}^{m} \frac{c_\mu}{k_\mu n + r_\mu} = \sum_{\mu=1}^{m} \sum_{j=1}^{k_\mu - 1} \frac{c_\mu}{k_\mu} (1 - \zeta_\mu^{-jr_\mu}) \log(1 - \zeta_\mu^j)$$

where ζ_μ is some k_μ-th root of unity for $\mu = 1, \dots, m$. If $\alpha \neq 1$, then there is a slightly more complicated formula (cf. the proof of Theorem 4 of [AdSST]). Next we apply the celebrated result of A. Baker [Ba].

If $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}} \setminus \{0\}$, $\beta_1, \dots, \beta_n \in \overline{\mathbb{Q}}$ and $\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$, then $\Lambda = 0$ or Λ is transcendental.

Here, as usual, $\overline{\mathbb{Q}}$ denotes the set of algebraic numbers. Furthermore, log should be read as the principal value of the logarithm, e.g. with the argument in $(-\pi, \pi]$. The combination of the above and some additional arguments yields the following result.

Theorem 1 ([AdSST], Theorem 4 and Corollary 4.1) *Let $P(x) \in \overline{\mathbb{Q}}[x]$ and $\alpha \in \overline{\mathbb{Q}}$. Let $Q(x) \in \mathbb{Q}[x]$ have simple rational zeros and no others. If*

$$S := \sum_{n=1}^{\infty} \alpha^n \frac{P(n)}{Q(n)}$$

converges, then $S \in \mathbb{Q}$ or $S \notin \overline{\mathbb{Q}}$. If all the zeros of $Q(x)$ are in $[-1, 0)$, then $S = 0$ or $S \notin \overline{\mathbb{Q}}$.

In particular, for every integer $k > 1$ the sum

$$\frac{1}{1 \times 2 \times \cdots \times k} + \frac{1}{(k+1)(k+2)\cdots(2k)} + \frac{1}{(2k+1)(2k+2)\cdots(3k)} + \cdots$$

is transcendental, since the corresponding zeros $-\frac{1}{k}, -\frac{2}{k}, \dots, -\frac{k}{k}$ are all simple, distinct and in the interval $[-1, 0)$, and the sum is nonzero. On the other hand the sum

$$\frac{1}{1 \times 2 \times \cdots \times k} + \frac{1}{2 \times 3 \times \cdots \times (k+1)} + \frac{1}{3 \times 4 \times \cdots \times (k+2)} + \cdots$$

is rational, as the corresponding zeros $-1, -2, \dots, -k$ are distinct and all equal modulo 1. In fact we may use induction on N to prove that

$$\sum_{n=0}^{N} \frac{1}{(n+1)\cdots(n+k)} = \frac{1}{(k-1)! \times (k-1)} - \frac{(N+1)!}{(k+N)! \times (k-1)}$$

so that

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)\cdots(n+k)} = \frac{1}{(k-1)! \times (k-1)}.$$

This formula can be used to prove that the two given expressions for $\zeta(3)$ are equal.

In fact Theorem 1 has been proved in case $P(x)$ is an exponential polynomial $\sum_{\lambda=1}^{l} P_\lambda(x)\alpha_\lambda^x$. Moreover, approximation measures have been given in case S is transcendental.

Theorem 1 excludes that S is an algebraic irrational number, but it leaves the question open whether S is rational. The sum $S = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}$ with deg(Q) - deg(P) ≥ 2 is certainly rational if all zeros of Q are rational, simple and equal modulo 1. This can be easily seen by telescoping and using the fact that the sum of the residues of $\frac{P(x)}{Q(x)}$ is zero. However, there are other cases of rationality, for example,

(2)
$$\sum_{n=0}^{\infty} \frac{16n^2+12n-1}{(4n+1)(4n+2)(4n+3)(4n+4)} = \sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{3}{4n+2} + \frac{1}{4n+3} + \frac{1}{4n+4} \right) = 0.$$

The question "Is S rational?" will be the subject of the next subsections.

2.2 Small degrees

First we consider $S = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}$ in case deg(Q) $= 2$. In order to have convergence P should be a constant. Hence S can be written as

$$\sum_{n=0}^{\infty} \frac{b}{(qn+s_1)(qn+s_2)}$$

where b, q, s_1, s_2 are integers with $q > 0,\ s_1 < s_2$. Hence

$$S = \frac{b}{s_2 - s_1} \sum_{n=0}^{\infty} \left(\frac{1}{qn+s_1} - \frac{1}{qn+s_2} \right).$$

We can write S as

$$S = q_0 + \frac{b}{s_2 - s_1} \sum_{n=0}^{\infty} \left(\frac{1}{qn+s'_1} - \frac{1}{qn+s'_2} \right)$$

with $q_0 \in \mathbb{Q},\ 0 < s'_1 \leq q,\ 0 < s'_2 \leq q,\ s'_1 \equiv s_1 (\mathrm{mod} q),\ s'_2 \equiv s_2 (\mathrm{mod}\ q)$. If $s_1 \equiv s_2$ (mod q), then $s'_1 = s'_2$ and $S = q_0 \in \mathbb{Q}$. If $s_1 \not\equiv s_2$ (mod q), then $s'_1 \not\equiv s'_2$ (mod q). Note that the sum cannot vanish, since the sum is alternating with strictly decreasing terms. Hence the sum is nonzero and therefore transcendental by the second part of Theorem 1. We conclude that if P is nontrivial, then $S \in \mathbb{Q}$ if and only if $s_1 \equiv s_2$ (mod q).

Now suppose that P is nontrivial and deg(Q) $= 3$. Then, by the convergence of $\sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}$, we have deg($P$) ≤ 1. Hence S can be written as

$$\sum_{n=0}^{\infty} \frac{an+b}{(qn+s_1)(qn+s_2)(qn+s_3)}.$$

There are two obvious reasons why S would be rational:

(3) $$s_1 \equiv s_2 \equiv s_3 (\mathrm{mod}\ q),$$

(4) $as_1 = bq, s_2 \equiv s_3 (\text{mod } q)$ or $as_2 = bq, s_3 \equiv s_1 (\text{mod } q)$ or $as_3 = bq, s_1 \equiv s_2 (\text{mod } q)$.

In case of (3) after decomposing into partial fractions and adapting the summation, the infinite parts cancel because of the convergence of the sum and a finite sum of rationals remains. Hence $S \in \mathbb{Q}$. In case of (4) we can divide numerator and denominator by a linear factor after which we are back in the case of $\deg(Q) = 2$ in which A is rational.

By using a criterion of Okada [Ok] it is proved in [SaT] that in all other cases S is transcendental.

Theorem 2 ([SaT], Theorem 2) *Suppose that*

$$S = \sum_{n=0}^{\infty} \frac{\alpha n + \beta}{(qn + s_1)(qn + s_2)(qn + s_3)}$$

with $\alpha, \beta \in \mathbb{Q},\ s_1, s_2, s_3 \in \mathbb{Z},\ q \in \mathbb{Z}_{\geq 0}$ *such that* $qn + s_1, qn + s_2, qn + s_3$ *do not vanish for* $n \geq 0$. *Assume that* s_1, s_2, s_3 *are distinct and not in the same residue class modulo* q. *Further, let*

(5)

$s_1 \not\equiv s_2 (\text{mod } q)$ *if* $\alpha s_3 = \beta q$; $s_2 \not\equiv s_3 (\text{mod } q)$ *if* $\alpha s_1 = \beta q$; $s_3 \not\equiv s_1 (\text{mod } q)$ *if* $\alpha s_2 = \beta q$.

Then S *is transcendental.*

By a complicated criterion of Okada [Ok] rationality can be decided in all cases. The example (2) shows that Theorem 2 cannot easily be extended. In this case the corresponding linear form (1) reads $-\log(1-i) + \log(1-(-1)) - \log(1+i)$ which equals 0.

2.3 Alternating series with small degrees

If we consider sums $S = \sum_{n=0}^{\infty} (-1)^n \frac{P(n)}{Q(n)}$ with $P(x), Q(x) \in \mathbb{Z}[x]$ and Q has only simple rational zeros, it is possible that $\deg(Q) = 1$. Then $\deg(P) = 0$ and we have a sum of the form $\sum_{n=0}^{\infty} (-1)^n \frac{b}{qn+s}$. We assume $b \neq 0$. We can rewrite it as a sum $q_0 + \sum_{n=0}^{\infty} (-1)^n \frac{b}{qn+s'}$ with $q_0 \in \mathbb{Q}$ and $0 < s' \leq q$. From the proof of [AdSST] Theorem 4 with $c_{\lambda,\mu} = b, \alpha_\lambda = -1, k_\mu = q, r_\mu = s'$ we find

$$S = q_0 - \frac{b}{q} \sum_{j=0}^{q-1} \beta^{-s'} \zeta^{-js'} \log(1 - \beta^{2j+1}) \tag{6}$$

where ζ is a primitive q-th root of unity and β a primitive $2q$-th root of unity. By Baker's result $S = q_0$ or $S \notin \overline{\mathbb{Q}}$. However, $(\frac{1}{qn+s'})_{n=0}^{\infty}$ is monotone decreasing so that $\sum_{n=0}^{\infty} \frac{(-1)^n}{qn+s'} \neq 0$. Thus $S \in \mathbb{Q}$ if and only if $b = 0$, and otherwise $S \notin \overline{\mathbb{Q}}$.

If $\deg(Q) = 2$, then we have $\deg(P) \leq 1$. So we consider sums

$$\sum_{n=0}^{\infty} (-1)^n \frac{an+b}{(qn+s_1)(qn+s_2)}$$

with a, b, q, s_1, s_2 integers subject to $|a| + |b| > 0, q > 0$ and $s_1 \neq s_2$. By decomposing into fractional parts we obtain

$$S = \frac{1}{q(s_2 - s_1)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{qb - as_1}{qn + s_1} - \frac{qb - as_2}{qn + s_2} \right).$$

After rearranging the terms we find a rational number q_0 such that

$$S = q_0 + \frac{1}{q(s_2 - s_1)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{qb - as_1}{qn + s_1'} - \frac{qb - as_2}{qn + s_2'} \right),$$

where $s_1' \equiv s_1 \pmod{2q}$, $s_2' \equiv s_2 \pmod{2q}$, $0 < s_1' \leq 2q$, $0 < s_2' \leq 2q$. If $s_1' = s_2'$, then we obtain $S = q_0 + \frac{a}{q} \sum_{n=0}^{\infty} (-1)^n \frac{1}{qn+s_1'}$. As we have seen before, the infinite sum is transcendental. Hence $S \in \mathbb{Q}$ if and only if $a = 0$. If $s_1' - s_2' = \pm q$, then after a further rearrangement we find rational numbers q_1, q_2 and a pair $(i, j) = (1, 2)$ or $(2, 1)$ such that

$$S = q_1 \pm \frac{1}{q(s_2 - s_1)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{qb - as_i}{qn + s_i'} - \frac{qb - as_j}{qn + (s_i' + q)} \right) =$$

$$q_2 \pm \frac{1}{q(s_2 - s_1)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{qb - as_i}{qn + s_i'} + \frac{qb - as_j}{qn + s_i'} \right) =$$

$$q_2 \pm \frac{2qb - as_1 - as_2}{q(s_2 - s_1)} \sum_{n=0}^{\infty} (-1)^n \frac{1}{qn + s_i'}.$$

As we have seen before, the infinite sum is transcendental. Hence $S \in \mathbb{Q}$ if and only if $a(s_1 + s_2) = 2bq$. If $s_1' \not\equiv s_2' \pmod{q}$, then it can be derived from Okada's criterion that S is transcendental.

Theorem 3 ([SaT], Theorem 2) *Suppose*

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha n + \beta)}{(qn + s_1)(qn + s_2)}$$

with $\alpha, \beta \in \overline{\mathbb{Q}}$, $s_1, s_2 \in \mathbb{Z}$, $q \in \mathbb{Z}_{>0}$ *such that* $s_1 \neq s_2$ *and* $qn + s_1, qn + s_2$ *do not vanish for* $n \geq 0$. *If* $s_1 \equiv s_2 \pmod{2q}$, *or* ($s_1 \equiv s_2 + q \pmod{2q}$ *and* $a(s_1 + s_2) = 2bq$), *then* S *is rational, otherwise* S *is transcendental.*

Remark 1. In case $s_1 \equiv s_2 + q \pmod{2q}$ the condition is different from Theorem 2 in [SaT]. The necessary correction of the original paper was brought to our attention by T. Hessami Pilehrood.

Remark 2. There are non-trivial cases with rational sum if deg $(Q) > 2$. For example,

$$\sum_{n=0}^{\infty}(-1)^n \frac{18n^2+15n+1}{(3n+1)(3n+2)(3n+3)} = 0, \text{ hence } \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)(3n+3)(3n+5)} = \frac{1}{16}.$$

After combining pairs of terms Okada's criterion can again be used, but there does not seem to be a straightforward extension of Theorem 3 to higher degrees.

3 General rational function values

If we drop the restriction that all the zeros of Q are simple and rational, no general transcendence methods are available. Of course, it can again happen that by changing the summation the sum $S := \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}$ reduces to a finite sum and is therefore rational. For example,

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = 1.$$

If there is not an obvious reason that S is rational, we only can hope to find an explicit expression for the sum S which is known to be transcendental. For example, since, for every positive integer k, it is known that $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ is a nonzero rational multiple of π^{2k}, we know that the sum is transcendental. I owe to D.W. Masser the remark that there is such a useful expression for $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$, viz. (cf. [Sp]. p. 189)

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{2} \cdot \frac{e^{2\pi}+1}{e^{2\pi}-1} + \frac{1}{2}.$$

It follows from a celebrated result of Nesterenko [Ne] that the numbers π and e^{π} are algebraically independent. Thus $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ is transcendental. In fact Nesterenko also proved that the numbers π and $e^{\pi\sqrt{3}}$ are algebraically independent. F. Beukers gave the following advice to me: check whether all the zeros of the denominator Q are located in some quadratic number field $\mathbb{Q}(\sqrt{-D})$ for $D \in \{1,3\}$. If so, try to find an explicit expression for S in terms of π and $e^{\pi\sqrt{D}}$. If you succeed, apply the result of Nesterenko to conclude the transcendence of the number. Here are some examples of such transcendental numbers (cf. [Sp], p. 189, p. 199, p. 196, p. 196, respectively):

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3} = \frac{\pi}{2\sqrt{3}} \cdot \frac{e^{2\pi\sqrt{3}}+1}{e^{2\pi\sqrt{3}}-1} + \frac{1}{6}, \quad \sum_{n=0}^{\infty} \frac{1}{n^4+n^2+1} = \frac{\pi\sqrt{3}}{6} \cdot \frac{e^{\pi\sqrt{3}}-1}{e^{\pi\sqrt{3}}+1} + \frac{1}{2}$$

$$\sum_{n=0}^{\infty} \frac{1}{n^4+4} = \frac{\pi}{8} \cdot \frac{e^{4\pi}-1}{e^{4\pi}-e^{2\pi}+1} + \frac{1}{8}, \quad \sum_{n=0}^{\infty} \frac{1}{(n^2+1)^2} = \frac{\pi}{4} \cdot \frac{e^{2\pi}+1}{e^{2\pi}-1} + \frac{\pi^2}{4} \cdot \frac{e^{2\pi}}{(e^{2\pi}-1)^2} + \frac{1}{2},$$

and for alternating series (cf. [Ra] (9.3), [Ti] p. 113, [Sp] p.197, respectively):

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2p+1}} \in \mathbb{Q}\pi^{2p+1} \quad (p = 0, 1, \dots),$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{n^2+n+1} = \frac{32\pi}{e^{\frac{1}{2}\pi\sqrt{3}} + e^{-\frac{1}{2}\pi\sqrt{3}}}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{\pi}{e^{2\pi}-1} + \frac{\pi+1}{2}.$$

After I had written this paper I received a preprint by Kh. and T. Hessami Pilehrood [HP] in which they elaborate the above remarks. Their general results have some remarkable corollaries, such as that if $a, b \in \mathbb{Z}, 4b > a^2$, then both the sums

$$\sum_{n=0}^{\infty} \frac{1}{n^2+an+b} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+an+b}$$

are transcendental.

I do not know any case where the sum S is irrational and algebraic. We have seen in Section 2 that this is never the case if the roots of the polynomial Q are simple and rational. We formulate the extension of this result as a conjecture.

Conjecture *Let $f(x) \in \mathbb{Z}(x)$ be a rational function such that $S := \sum_{n=0}^{\infty} f(n)$ is well defined. If S is algebraic, then it is rational.*

4 Polynomial Cantor sums with Q having only simple rational zeros

In Sections 4 and 5 we consider sums of the form

$$S = \sum_{N=0}^{\infty} \frac{P(N)}{\prod_{n=0}^{N} Q(n)}$$

where $P(x) \in \mathbb{Z}[x], Q(x) \in \mathbb{Z}[x], P \neq 0$, the denominators do not vanish, and the sum converges. We write

(7)

$$P(x) = A_0(x) + A_1(x)Q(x) + A_2(x)Q(x)Q(x-1) + \dots + A_t(x)Q(x)Q(x-1)\dots Q(x-t)$$

with $A_j(x) \in \mathbb{Q}[x]$, $\deg(A_j) < \deg(Q)$ for $j = 0, 1, \dots, t$ and $A_t \neq 0$. Hence, for suitable $q_0, q_1 \in \mathbb{Q}$,

(8) $$S = q_0 + \sum_{N=t}^{\infty} \sum_{j=0}^{t} \frac{A_j(N)}{\prod_{n=0}^{N-j} Q(n)} = q_1 + \sum_{N=0}^{\infty} \frac{\sum_{j=0}^{t} A_j(N+j)}{\prod_{n=0}^{N} Q(n)}.$$

4.1 The case Q is constant.

If $\deg(Q) = 0$, then $S = \sum_{n=0}^{\infty} a^n P(n)$ for some $a \in \mathbb{Q}$. By the convergence of the sum we have $|a| < 1$. Observe that on the one hand

$$\left(\sum_{n=0}^{\infty} x^{n+k}\right)^{(k)}_{x=a} = \left(\frac{x^k}{1-x}\right)^{(k)}_{x=a} \in \mathbb{Q}$$

and on the other hand

$$\left(\sum_{n=0}^{\infty} x^{n+k}\right)^{(k)} = \sum_{n=0}^{\infty} (n+k)\dots(n+1)x^n.$$

So writing $P(x)$ as $\sum_{k=0}^{t} A_k(x+1)\dots(x+k)$ with $A_k \in \mathbb{Z}$ for $k = 0, 1, \dots, t$, we obtain

$$\sum_{n=0}^{\infty} a^n P(n) = \sum_{k=0}^{t} A_k \sum_{n=0}^{\infty} a^n (n+1)\dots(n+k) = \sum_{k=0}^{t} A_k \left(\frac{x^k}{1-x}\right)^{(k)}_{x=a} \in \mathbb{Q}.$$

Thus $S \in \mathbb{Q}$. From now on we assume that Q is non-constant.

4.2 The case Q is linear

If Q is linear, then the coefficients A_j in (7) become rational numbers and (8) becomes

$$S = q_1 + \sum_{N=0}^{\infty} \frac{\sum_{j=0}^{t} A_j}{\prod_{n=0}^{N} Q(n)} = q_1 + \sum_{j=0}^{t} A_j \sum_{N=0}^{\infty} \frac{1}{\prod_{n=0}^{N} Q(n)}.$$

It follows from a general theorem of Shidlovskii [Sh1] that $\sum_{N=0}^{\infty} \frac{1}{\prod_{n=0}^{N}(an+b)} = \phi_{b/a}(\frac{1}{a}) \notin \overline{\mathbb{Q}}$ and therefore $S \notin \overline{\mathbb{Q}}$. Here $\phi_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{(\lambda+1)\dots(\lambda+n)}$. Thus $S \in \mathbb{Q}$ if and only if $\sum_{j=0}^{t} A_j = 0$ and otherwise $S \notin \overline{\mathbb{Q}}$.

In the special case that $Q(x) \equiv x$, we get $S = q_1 + (\sum_{j=0}^{t} A_j)e$ as has been observed by Klazar [Kl]. Hančl and the author have worked out this particular case. According to [HaT2] Cor. 3.1, if $P(x) = \sum_{i=0}^{T} a_i x^i$, then

$$\sum_{N=0}^{\infty} \frac{P(N)}{N!} \in \mathbb{Q} \Longleftrightarrow \sum_{i=0}^{T} a_i \sum_{k=0}^{i} S(i,k) = 0,$$

where $S(i,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^i$ is the Stirling number of the second kind. It is known that $S(i,k) = 0$ if $i < k$, $S(k,k) = 1$ and $S(i,k) \in \mathbb{Z}_{>0}$ if $i > k > 0$, cf. [Ai] Sec. III.2. In particular $\sum_{n=0}^{\infty} \frac{P(n)}{n!} \notin \overline{\mathbb{Q}}$ if $a_i \geq 0$ for $i = 0, 1, \dots, T-1$ and $a_T > 0$.

4.3 The case Q has only rational zeros

If Q is monic and has only rational roots, one can try to apply the theory on hypergeometric E-functions (cf. [FN], Ch.5, par 3). A generalized hypergeometric function is defined as

$$ {}_pF_q \left(\begin{matrix} a_1, a_2, \dots a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{z^n}{n!} $$

where $(\alpha)_0 = 1, (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n = 1, 2, \dots$ and $b_j \neq 0, -1, -2, \dots$ for all j (cf. [FN], p. 127). If $t := m - l \geq 1$, then the function

$$ {}_{l+1}F_m \left(\begin{matrix} 1, a_1, \dots a_l \\ b_1, b_2, \dots, b_m \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_l)_n}{(b_1)_n \dots (b_m)_n} \cdot z^n $$

is an entire function of order $1/t$ (cf. [FN], p. 212). Around 1920 Siegel proved that if $a_i, b_j \in \mathbb{Q}$ for every i and j, then

$$ {}_{l+1}F_m \left(\begin{matrix} 1, a_1, \dots a_l \\ b_1, b_2, \dots, b_m \end{matrix} ; \left(\frac{z}{t}\right)^t \right) $$

is an E-function (cf. [FN], p. 212). Such a function is called a hypergeometric E-function.

Suppose we consider the sum $S = \sum_{N=0}^{\infty} \frac{P(N)}{\prod_{n=0}^{N} Q(n)}$ where P and Q are polynomials with integer coefficients and only rational roots and Q is monic. Write $P(z) = A(x + a_1)\dots(x+a_l), Q(z) = (x+b_1)\dots(x+b_m)$, multiple roots permitted. Then

$$ S = Aa_1 \cdots a_l \times {}_{l+1}F_{l+m} \left(\begin{matrix} 1, a_1+1, \dots, a_l+1 \\ a_1, \dots, a_l, b_1+1, \dots, b_m+1 \end{matrix} ; 1 \right). $$

Thus $S/(Aa_1 \cdots a_l)$ is the value of a hypergeometric E-function for $z = t = m$.

Shidlovskii and many other mathematicians have proved results on the algebraic independence of values of hypergeometric E-functions. If we define

$$ E_Q(x) = \sum_{N=0}^{\infty} \frac{x^N}{\prod_{n=0}^{N} Q(n)}, $$

then S is a linear combination, with constant coefficients, of E_Q and its derivatives. In Shidlovskii's book [Sh2] Ch. 8, Thm. 11 there are some very general statements about algebraic independence of values $E_Q^{(r)}(\xi), \xi \in \overline{\mathbb{Q}}^*$. If both P and Q have only rational zeros, one can also try to apply one of the theorems from [FN] Ch. 5 directly. For example, for any nonzero rational (even algebraic) number α and every positive integer k the number $\sum_{n=0}^{\infty} \frac{\alpha^{kn}}{(n!)^k}$ is transcendental (cf. [FN] Thm. 5.21). This result corresponds with the choices $P(x) = 1, Q(x) = (x/\alpha)^k$. As can be seen from this example the requirement that Q is monic is not essential. This is true as well for the restriction that P has rational zeros. If not, P can be written as a linear combination of polynomials each with rational zeros. If this can be done in such a way that the corresponding sums are algebraically independent (linearly independent, respectively) over $\mathbb{Q}$ according to some theorem, then it would immediately follow that the original sum S is transcendental (irrational, respectively).

5 The general case of Polynomial Cantor sums

Let again

$$S = \sum_{N=0}^{\infty} \frac{P(N)}{\prod_{n=0}^{N} Q(n)},$$

with $P(x) \in \mathbb{Z}[x], Q(x) \in \mathbb{Z}[x], P \neq 0, Q$ non-constant, be well-defined. This is a special case of a Cantor sum $T = \sum_{N=0}^{\infty} \frac{b_N}{\prod_{n=0}^{N} a_n}$ where (a_n) and (b_n) are integer sequences with $a_n \neq 0$ for all n and such that the sum converges. We assume that $b_n \neq 0$ for infinitely many n. It follows from Theorems 4 and 8 of Oppenheim [Op] that if $a_n \to \infty$ and $\frac{b_n}{a_n} \to 0$ as $n \to \infty$, then $T \notin \mathbb{Q}$. Hence $S \notin \mathbb{Q}$ if $\deg(P) < \deg(Q) =: t$.

It is not hard to verify the mentioned result of Oppenheim. Suppose $S = r/s$ with $r, s \in \mathbb{Z}$. Then, for every large N,

$$ra_1 \dots a_N - s \sum_{n=0}^{N} b_n a_{n+1} \dots a_N = s \sum_{n=N+1}^{\infty} \frac{b_n}{a_{N+1} \dots a_n}.$$

The left-hand side is an integer, the right-hand side tends to 0 as $N \to \infty$. Hence there is an N_0 such that the right-hand side equals 0 if $N \geq N_0$. It follows that $b_n = 0$ for $N > N_0$.

If deg $(P) \geq \deg(Q)$, then we can use (7) and (8) to derive conditions on the coefficients of P and Q under which the sum S is rational. Suppose that $\sum_{j=0}^{t} A_j(N+j) \in \mathbb{Z}$ for every integer N. Then in view of $\deg\left(\sum_{j=0}^{t} A_j(x+j)\right) < \deg(Q)$, by Oppenheim's result, $S \in \mathbb{Q}$ if and only if $\sum_{j=0}^{t} A_j(x+j) \equiv 0$.

Example. Let $P(x) = ax^3 + bx^2 + cx + d \in \mathbb{Z}[x],\ Q(x) = gx^2 + h \in \mathbb{Z}[x]$ with $a \neq 0$. We have $gP(x) = (ax+b)Q(x) + ((cg-ah)x + dg - bh)$. Hence, by Oppenheim's criterion, applied to (8),

$$gS \in \mathbb{Q} \iff a(x+1) + b + (cg-ah)x + (dg-bh) = 0.$$

Thus

$$S \in \mathbb{Q} \iff cg = a(h-1) \wedge a + dg = b(h-1).$$

Remark. The above example with $b = c = 0$ occurs in Hančl and Tijdeman [HaT1], Appl. 2. We note that in this paper the distinction should not have been between $a > 1$ and $a = 1$, but between a does not divide b and $a|b$. Hence the condition in that paper should have been (in the present terminology) $h = 1 \wedge a + dg = 0$.

We can proceed in the general case as in the example. Then we compute $A_0(x), A_1(x), \ldots, A_t[X] \in \mathbb{Z}[x]$ of degree $< \deg(Q)$ such that for a suitable integer g

$$gP(x) = A_0(x)+A_1(x)Q(x)+A_2(x)Q(x)Q(x-1)+\cdots+A_t(x)Q(x)Q(x-1)\ldots Q(x-t+1).$$

Hence, as in (8),

$$gS = g\sum_{N=0}^{\infty} \frac{P(N)}{\prod_{n=0}^{N} Q(n)} = q_2 + \sum_{N=0}^{\infty} \frac{\sum_{j=0}^{t} A_j(N+j)}{\prod_{n=0}^{N} Q(N)}$$

for some $q_2 \in \mathbb{Q}$. By Oppenheim's result,

$$g\sum_{N=0}^{\infty} \frac{P(N)}{\prod_{n=0}^{N} Q(n)} \in \mathbb{Q} \iff \sum_{j=0}^{t} A_j(N+j) = 0$$

for $N > N_0$. Thus $S \in \mathbb{Q} \iff \sum_{j=0}^{t} A_j(x+j) \equiv 0$.

The case of an alternating sum $\sum_{N=0}^{\infty} (-1)^N \frac{P(N)}{\prod_{n=0}^{N} Q(n)}$ can be reduced to the above sum by replacing Q by $-Q$.

6 Some variations

6.1 Values of L-functions

In the paper [AdSST] (see Cor. 1.1) treated in Section 1 the authors also proved that if χ is a Dirichlet character mod q which is not the principal character mod q, then $L(1,\chi)$ is transcendental. The proof is a combination of the adjusted proof of Theorem 1, which implies that the value is either 0 or transcendental, and the well known result of Dirichlet that $L(1,\chi) \neq 0$. In fact the former argument is based on the identities

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \sum_{n=0}^{\infty}\sum_{\mu=1}^{q} \frac{\chi(\mu)}{nq+\mu} = -\sum_{j=1}^{q-1} \beta_j \sum_{n=1}^{\infty} \frac{\zeta_q^{jn}}{n} \left(= \sum_{j=1}^{q-1} \beta_j \log(1-\zeta_q^j) \right),$$

where β_j is an algebraic number for $j = 1, \ldots q-1$.

We can use the approach from Subsection 2.2 to say something on the arithmetic character of $L(k,\chi)$ where k is any positive integer. Analogously to the above identities we can derive in a similar, but much simpler way that

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^k} = \sum_{n=0}^{\infty} \sum_{\mu=1}^{q} \frac{\chi(\mu)}{(nq+\mu)^k} = \sum_{j=1}^{q-1} \beta_{j,k} \sum_{n=1}^{\infty} \frac{\zeta_q^{jn}}{n^k}$$

where $\beta_{j,k}$ is an algebraic number for each pair j, k. If χ is an even character, then we can express the right-hand side as

$$\sum_{j=1}^{q-1} \beta_{j,k} \sum_{n=1}^{\infty} \frac{\cos(2\pi jnt/q)}{n^k}$$

for some rational number t. It follows from a formula involving Bernoulli polynomials (see e.g. [Ra], (8.62)) that $\sum_{n=1}^{\infty} \frac{\cos(2\pi njt/q)}{n^k}$ is a rational multiple of π^k if k is even. Similarly, if χ is an odd character, then we can express the right-hand side as

$$\sum_{j=1}^{q-1} \beta_{j,k} \sum_{n=1}^{\infty} \frac{\sin(2\pi jnt/q)}{n^k}$$

which according to [Ra], (8.61) is a rational multiple of π^k if k is odd. We conclude that if χ is a Dirichlet character which is not a principal character and k is a positive integer such that $L(k,\chi) \neq 0$, then $L(k,\chi)$ is transcendental for even values of k if χ is even and for odd values of k if χ is odd. In the special case $k = 3, q = 4$ we get the formula

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$$

which of course is transcendental (cf. [Ra] (9.3)).

The corresponding question for the second part of this paper is whether $\sum_{n=1}^{\infty} \frac{\chi(n)}{n!}$ is transcendental. This a consequence of the Lindemann-Weierstrass Theorem as it is a special case of [FN], Corollary 2.4 which says:

Let $c_0, c_1, c_2, \ldots$ be a non-trivial periodic sequence of algebraic numbers. Then the value of the series $\sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$ at any nonzero algebraic point z is transcendental.

6.2 Rational values of an E-function

Matala-aho brought a recent paper of Beukers [Be2] under my attention. This paper enables one to prove the transcendence of functions of the form $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ for algebraic values of z. We give an example.

It is easy to check that the E-function $f(z) := (z-1)e^z = \sum_{n=0}^{\infty} \frac{n-1}{n!} z^n$ is rational for $z = 0$ and $z = 1$. This function satisfies the minimal differential equation $(z-1)f'(z) = zf(z)$. Beukers' Theorem 1.3 implies that there are no other algebraic values of z for which $f(\alpha)$ is algebraic.

6.3 Rounded polynomial values

In Section 4 the case was considered that the numerator attains consecutive polynomial values. Hančl and the author [HaT2] dealt with the more general situation that the sequence of numerators $(g(n))_{n=1}^{\infty}$ is an integer sequence such that $g(n) = P(n) + o(1)$ as $n \to \infty$ for some $P(x) \in \mathbb{Q}[x]$. They proved that if $\sum_{n=1}^{\infty} \frac{g(n)}{n!} \in \mathbb{Q}$, then $g(x) - P(x)$ is constant. In particular, if $\sum_{n=1}^{\infty} \frac{[P(n)]}{n!} \in \mathbb{Q}$, then P is a constant a_0 with $0 \leq a_0 < 1$. This implies the remarkable fact that for every positive integer k the strictly increasing function $\sum_{n=1}^{\infty} \frac{[\beta n^k]}{n!}$ of $\beta > 0$ does not attain any rational value.

6.4 The use of G-functions

The procedure from Subsection 4.3 can also be applied to sums of the form

$$\sum_{N=0}^{\infty} \frac{P_1(N)}{Q_1(N)} \prod_{n=0}^{N} \frac{P_2(N)}{Q_2(N)}$$

where P_1, P_2, Q_1, Q_2 are suitable polynomials. We recall that

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}} = \frac{5}{4} \sum_{N=1}^{\infty} \prod_{n=2}^{N} \left(-\frac{(n-1)^3}{2n^2(2n-1)} \right).$$

Therefore

$$\zeta(3) = {}_4F_3 \left(\begin{array}{c} 1,1,1,1 \\ 2,2,2 \end{array} ; 1 \right) = \frac{5}{4} \cdot {}_4F_3 \left(\begin{array}{c} 1,1,1,1 \\ 2,2,3/2 \end{array} ; -\frac{1}{4} \right)$$

which provides a representation in which the ratio of consecutive terms tends to $-1/4$. Hypergeometric functions

$${}_{m+1}F_m \left(\begin{array}{c} 1, a_1, a_2, \ldots a_m \\ b_1, b_2, \ldots, b_m \end{array} ; z \right)$$

belong to the class of G-functions (cf. [FN] Ch. 5, par. 7). Such functions are convergent on a finite disc. It is an open problem to obtain new transcendence results using the theory of G-functions. Beukers and Wolfart [BW] have shown that also irrational algebraic values can be obtained, e.g.

$${}_2F_1 \left(\begin{array}{c} 1/12, 5/12 \\ 1/2 \end{array} ; \frac{1323}{1331} \right) = \frac{3}{4}(11)^{1/4}.$$

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Mathematisch Instituut
Leiden University
Postbus 9512
2300 RA Leiden
The Netherlands