On the irrationality of Cantor series

Jaroslav Hančl* and Robert Tijdeman

Abstract

We study conditions under which the Cantor series $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n}$ is irrational. Here $a_1, a_2, \dots$ are integers greater than one and $b_1, b_2, \dots$ are arbitrary integers. We generalize a result of Oppenheim and show that some conditions in results of Erdős and Straus can be dropped or relaxed. For example, we precisely say when $\sum_{n=1}^{\infty} \frac{b_n}{n!}$ is rational if $\{b_n\}_{n=1}^{\infty}$ is a monotonic sequence with $b_{n+1} - b_n = o(b_n)$, when $\sum_{n=1}^{\infty} \frac{p_n}{a_1 \dots a_n}$ is rational if $\{p_n\}_{n=1}^{\infty}$ is the sequence of consecutive prime numbers and $\{a_n\}_{n=1}^{\infty}$ is a monotonic sequence with $\frac{a_n}{\log n} \to \infty$ as $n \to \infty$, and when $\sum_{n=1}^{\infty} \frac{n}{a_1 \dots a_n}$ is rational if $\{a_n\}_{n=1}^{\infty}$ is a monotonic sequence. On the other hand we show that for every monotonic sequence $\{b_n\}_{n=1}^{\infty}$ there exists a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n \in \{2,3,4\}$ for every n such that $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n}$ is rational.

1 Introduction

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of integers with $a_n > 1$ for all n. It is an easy observation that

$$S := \sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n}$$

is rational if $\frac{b_n}{a_n - 1}$ is constant for n larger than some number n_0. Oppenheim [11] proved in 1954 that if $|b_n| \le a_n - 1$ with strict inequality for infinitely many n, and, moreover, for every positive integer q there is a positive integer N such that q divides $\prod_{i=1}^{N} a_i$, then S is rational only if $b_n = 0$ for $n \ge n_0$. Another theorem of Oppenheim states that the divisibility condition can be replaced by the condition $\liminf_{n\to\infty} \frac{|b_n|+1}{a_n} = 0$. For some variants see Hančl [8], [9]. By a simple example we shall show that the single condition $0 < |b_n| \le a_n - 1$ with strict inequality for infinitely many n is not sufficient to exclude rationality of S. In our example $b_n = 1$ for all n and $\{a_n\}_{n=1}^{\infty}$ is bounded. We call a Cantor series non-degenerate if there is no n for which $a_n \mid b_n$. (If $a_n \mid b_n$, then the n-th term

*Supported by the grant 201/01/0471 of the Czech Grant Agency.
Mathematical subject classification: 11J72.
Key words and phrases: Irrationality, infinite series.

can be incorporated into the $(n-1)$-th term.) In Theorem 3.2 we show that for non-degenerate series S is irrational if $\liminf_{n\to\infty} \frac{|b_n|}{a_n} = 0$ and $b_n = o(a_n a_{n-1})$ for $n \to \infty$.

Erdős and Straus [7] (Corollary 2.10) considered the condition $\lim_{n\to\infty} \frac{b_{n+1}-b_n}{a_n} \leq 0$ instead of the condition $\liminf_{n\to\infty} \frac{|b_n|+1}{a_n} = 0$. They proved that if $b_n = o(a_{n-1}a_n)$, $b_n > 0$, $\lim_{n\to\infty} \frac{b_{n+1}-b_n}{a_n} \leq 0$ and $\liminf_{n\to\infty} \frac{a_n}{b_n} = 0$, then S is irrational. We shall show that the condition $\liminf_{n\to\infty} \frac{a_n}{b_n} = 0$ can be weakened to the necessary condition that $\frac{b_n}{a_n - 1}$ is not constant from some n_0 on. Some growth condition on $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ is needed for such a result. In fact we show that for any monotonic sequence $\{b_n\}_{n=1}^{\infty}$ of positive integers there is a sequence $\{a_n\}_{n=1}^{\infty}$, with $a_n \in \{2,3,4\}$ for every n such that $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n}$ has a wanted value in a prescribed interval.

Erdős, partly in collaboration with others, considered the irrationality and linear independence of Cantor series involving multiplicative functions [1], [2], [5], [6], [7]. In 1958 he claimed [3] the irrationality of $\sum_{n=1}^{\infty} \frac{p_n^k}{n!}$ for $k = 1, 2, \dots$, where $\{p_n\}_{n=1}^{\infty}$ is the sequence of consecutive prime numbers. Unfortunately he proved only the case $k = 1$. In Corollary 4.2 we generalize the latter result as follows. Suppose $\{b_n\}_{n=1}^{\infty}$ is a monotonic sequence with $b_{n+1} - b_n = o(b_n)$. Then $\sum_{n=1}^{\infty} \frac{b_n}{n!}$ is rational if and only if $\frac{b_n}{n-1}$ is constant for n greater than some n_0. Erdős [3] also established the irrationality of $\sum_{n=1}^{\infty} \frac{p_n}{a_1 \dots a_n}$ where $\{a_n\}_{n=1}^{\infty}$ is a monotonically non-decreasing sequence such that there is $k > 0$ with $\lim_{n\to\infty} \frac{a_n (\log n)^k}{n} = \infty$. With Straus [7] he replaced this condition by the pair $p_n = o(a_n^2)$ and $\liminf_{n\to\infty} \frac{a_n}{p_n} = 0$. In Theorem 5.1 we replace the latter condition with the necessary condition that $\frac{p_n}{a_n - 1}$ is not constant from some n_0 on. Theorem 5.2 is a similar result on the irrationality of $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n}$. It implies that if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are monotonic sequences of positive integers such that $b_n = o(a_n^2)$, $\gcd(a_n - 1, b_n) = 1$ for all n and $a_{2n} b_{2n} = o(n a_n^2)$, then S is rational if and only if $\frac{b_n}{a_n - 1}$ is constant for n greater than some n_0. We show further that the condition $a_{2n} b_{2n} = o(n a_n^2)$ cannot be dropped. A consequence of Theorem 5.2 is that $\sum_{n=1}^{\infty} (\frac{p_n}{n!})^k$ is irrational for $k = 1, 2, \dots$.

Note that the condition $p_n = o(a_n^2)$ in Theorem 5.1 implies that $a_n > \sqrt{n \log n}$ for n large. It becomes more difficult to prove the irrationality of $\sum_{n=1}^{\infty} \frac{p_n}{a_1 \dots a_n}$ if $\{a_n\}_{n=1}^{\infty}$ grows more slowly. Erdős [3], [7] mentioned more than once that he could not prove the irrationality of $\sum_{n=1}^{\infty} \frac{p_n}{2^n}$. In Theorem 6.1 we prove the irrationality of $\sum_{n=1}^{\infty} \frac{p_n}{a_1 \dots a_n}$ for monotonic sequences $\{a_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} \frac{a_n}{\log n} = \infty$, unless $\frac{p_n}{a_n - 1}$ is constant from some n_0 on. It is not essential that the numerators are primes. Theorem 6.2 states that for any monotonic unbounded sequence $\{a_n\}_{n=1}^{\infty}$ the sum $\sum_{n=1}^{\infty} \frac{n}{a_1 \dots a_n}$ is rational if and only if $\frac{n}{a_n - 1}$ is constant from some n_0 on. Obviously the sum is rational if $\{a_n\}_{n=1}^{\infty}$ is monotonic and bounded. In Theorems 6.1 and 6.2 the monotonicity of $\{a_n\}_{n=1}^{\infty}$ is crucial.

We thank Pingzhi Yuan for a useful remark.

2 Notation and basic properties

Throughout the paper $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ will denote sequences of rational integers with $a_n > 1$ for all n. We study conditions under which the Cantor series

$$S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \ldots a_n} \tag{1}$$

is irrational. We do so by studying the sums

$$S_N := \sum_{n=N}^{\infty} \frac{b_n}{a_N \ldots a_n}. \tag{2}$$

The following lemma is crucial.

Lemma 2.1 *If $S = \frac{r}{q}$ for some $r \in \mathbb{Z}$, $q \in \mathbb{N}$, then $qS_N \in \mathbb{Z}$ for all N.*

Proof. We have

$$r \prod_{j=1}^{N-1} a_j = q \sum_{n=1}^{N-1} b_n \prod_{j=n+1}^{N-1} a_j + q \sum_{n=N}^{\infty} \frac{b_n}{a_N \ldots a_n}$$

and the first two terms are integers. □

Remark If q divides $\prod_{j=1}^{N-1} a_j$, then we need not multiply by q to obtain integers and can conclude that S_N itself is an integer. This observation was used by Oppenheim to obtain the first mentioned result in the introduction. The remark is applicable when S is of the form $\sum_{n=1}^{\infty} \frac{b_n}{n!}$.

We call a Cantor series S degenerate if $a_n \mid b_n$ for some n. In this case the n-th term can be contracted with the $(n-1)$-th term to $\frac{b_{n-1}+(\frac{b_n}{a_n})}{a_1 \ldots a_{n-1}}$. If $a_n \nmid b_n$ for all n, we call S non-degenerate. In a non-degenerate series we have $b_n \neq 0$ for all n.

Suppose $b_n = o(a_{n-1}a_n)$. Let $\epsilon > 0$. Let n_0 be such that $|b_n| < \frac{\epsilon}{2} a_{n-1} a_n$ for $n \geq n_0$. Then for such n

$$|S_n - \frac{b_n}{a_n}| = |\frac{b_{n+1}}{a_n a_{n+1}} + \frac{b_{n+2}}{a_n a_{n+1} a_{n+2}} + \ldots| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \ldots = \epsilon. \tag{3}$$

Hence, by Lemma 2.1, if $S = \frac{r}{q}$, then $\|q\frac{b_n}{a_n}\| < \epsilon q$ for $n \geq n_0$. Here $\|.\|$ denotes the distance to the nearest integer. Moreover, if $b_{n+1}, b_{n+2}, \ldots$ have all the same sign and are not all 0, we have $|S_n - \frac{b_n}{a_n}| \geq \frac{1}{qa_n}$.

Example 2.1 (cf. Erdős and Straus [6], Theorem 2.26). Let φ denote Euler's totient function. Then $\sum_{n=1}^{\infty} \frac{\varphi(n)}{n!} = 1 + \sum_{n=1}^{\infty} \frac{\varphi(n)-n+1}{n!}$ is irrational, since

$\varphi(p) = p - 1$ for p prime and $0 < \varphi(n) \leq n - 1$ for all n.
Let $\sigma(n)$ denote the sum of positive divisors of n. From Lemma 2.1 and (3) we obtain that $\sum_{n=1}^{\infty} \frac{\sigma(n)}{n!} = 1 + \sum_{n=1}^{\infty} \frac{\sigma(n)-n+1}{n!}$ is irrational, since $\sigma(p) = p + 1$ for p prime and $n < \sigma(n) = o(n^2)$ for all n.

3 The case $\liminf \frac{b_n}{a_n} = 0$

As a first application of Lemma 2.1 we prove the following result.

Theorem 3.1 *If $b_n > 0$, $b_{n+1} < (1+\epsilon)b_n$ for some $\epsilon < 1$ and all $n \geq n_1$, and $\liminf_{n\to\infty} \frac{b_n}{a_n} = 0$, then S is irrational.*

Proof. Suppose $S = \frac{r}{q}$ with $r \in \mathbb{Z}$, $q \in \mathbb{N}$. Choose some $N \geq n_1$, with $\frac{b_N}{a_N} < \frac{1-\epsilon}{2q}$. Then, by $b_{n+1} < (1+\epsilon)b_n$ for $n \geq N$,

$$S_N = \frac{b_N}{a_N} + \frac{b_{N+1}}{a_N a_{N+1}} + \frac{b_{N+2}}{a_N a_{N+1} a_{N+2}} + \ldots < \frac{1-\epsilon}{2q}(1 + \frac{1+\epsilon}{2} + (\frac{1+\epsilon}{2})^2 + \ldots) = \frac{1}{q}.$$

Hence by Lemma 2.1, $S_N = 0$. Thus $b_n = 0$ for $n \geq N$. This is a contradiction. $\square$

Example 3.1 For any integer $k > 0$ the sum $\sum_{n=1}^{\infty} \frac{p_n^k}{2^{p_n}}$ is irrational.
Indeed, by $p_{n+1}^k - p_n^k < \epsilon p_n^k$ and by the result of Westzynthius [12] that $\limsup_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = \infty$, we have

$$\liminf_{n\to\infty} \frac{p_n^k}{2^{p_n - p_{n-1}}} \leq \liminf_{n\to\infty} \frac{(n(\log n)^2)^k}{2^{2k\log n}} \leq \liminf_{n\to\infty} \frac{(n(\log n)^2)^k}{n^{1.2k}} = 0.$$

Thus we may apply Theorem 3.1 to $b_n = p_n^k$ and $a_n = 2^{p_n - p_{n-1}}$ where $p_0 = 0$. The case $k = 1$ is claimed by Erdős and Graham [4], page 62.

Proposition 3.1 *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of integers with $a_n \not| b_n$ for all n. If $\liminf_{N\to\infty} |S_N| = 0$, then S is irrational.*

The proof of Proposition 3.1 is based on the following lemma.

Lemma 3.1 *If M and N are positive integers with $M < N$ and $a_{N-1} \not| b_{N-1}$, then we have*

$$S_{M,N} = \sum_{n=M}^{N-1} \frac{b_n}{a_M \ldots a_n} \neq 0.$$

Proof of Lemma 3.1. Suppose $S_{M,N} = 0$. Then we obtain by bringing the fractions under common denominater that $\frac{ca_{N-1}+b_{N-1}}{a_M \ldots a_{N-1}} = 0$ for some integer c. Hence $ca_{N-1} + b_{N-1} = 0$ and so $a_{N-1} \mid b_{N-1}$. □

Proof of Proposition 3.1. Suppose $S = \frac{r}{q}$ with $r \in \mathbb{Z}$, $q \in \mathbb{N}$. By Lemma 2.1 there exist integers $M < N$ with $qS_M = qS_N = 0$. Hence

$$S_{M,N} = \sum_{n=M}^{N-1} \frac{b_n}{a_M \ldots a_n} = S_M - \frac{1}{a_M \ldots a_{N-1}} S_N = 0. \tag{4}$$

The contradiction with Lemma 3.1 shows that S is irrational. □

Corollary 3.1 *If $a_n \nmid b_n$, $|b_{n+1}| < (1+\epsilon)|b_n|$ for some ϵ with $0 < \epsilon < 1$ and all $n \geq n_0$, and $\liminf_{n\to\infty} \frac{|b_n|}{a_n} = 0$, then S is irrational.*

Proof. Following the proof of Theorem 3.1 with $|b_n|$ in place of b_n we find that $S_N = 0$ for infinitely many N. According to Proposition 3.1 this implies the irrationality of S. □

Example 3.2 Let $k \in \mathbb{N}$. Let $\{\epsilon_n\}_{n=1}^{\infty}$ be any sequence with $\epsilon_n \in \{-1, 1\}$ for all n. Then the sum $\sum_{n=1}^{\infty} \epsilon_n \frac{p_n^k}{2^{p_n}}$ is irrational.

Corollary 3.2 (Oppenheim [11]). *If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are integer sequences with $|b_n| < a_n$ for every n and $\liminf_{n\to\infty} \frac{|b_n|+1}{a_n} = 0$, then S is rational if and only if $b_n = 0$ for $n \geq n_0$.*

Proof. For every n with $b_n = 0$, we omit the n-th term and replace a_{n+1} with $a_n a_{n+1}$. Suppose $b_n \neq 0$ for infinitely many n. After rewriting the subscripts we obtain a non-degenerate series $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \ldots a_n}$ with $|b_n| < a_n$ for every n and $\liminf_{n\to\infty} \frac{|b_n|+1}{a_n} = 0$. Hence,

$$|S_N| < \frac{|b_N|}{a_N} + \sum_{n=N+1}^{\infty} \frac{a_n - 1}{a_N \ldots a_n} = \frac{|b_N| + 1}{a_N}.$$

By applying Proposition 3.1, we obtain that S is irrational. □

The following result provides a variant of Corollary 3.2.

Theorem 3.2 *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of integers such that $a_n \nmid b_n$ for every n. Suppose*

$$\liminf_{n\to\infty} \frac{|b_n|}{a_n} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{b_n}{a_{n-1}a_n} = 0.$$

Then $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \ldots a_n}$ is irrational.

Proof. Suppose $S = \frac{r}{q}$ with $r \in \mathbb{Z}$, $q \in \mathbb{N}$. Let $\epsilon > 0$. Choose N so large that

$$\frac{|b_N|}{a_N} < \epsilon \quad \text{and} \quad \frac{|b_n|}{a_{n-1}a_n} < \epsilon \quad \text{for} \quad n \geq N.$$

Then

$$|S_N| \leq \frac{|b_N|}{a_N} + \sum_{n=N+1}^{\infty} \frac{\epsilon}{a_N \ldots a_{n-2}} < \epsilon + \sum_{n=N+1}^{\infty} \frac{\epsilon}{2^{n-N-1}} = 3\epsilon.$$

Hence we may apply Proposition 3.1 and conclude that S is irrational. $\square$

Remark If $\{a_n\}_{n=1}^{\infty}$ is bounded, then it may be hard to prove that S is irrational. Suppose $b_n = 1$ for all n. Then obviously S is rational if $\{a_n\}_{n=1}^{\infty}$ is periodic. The following construction shows that S may also be rational if $\{a_n\}_{n=1}^{\infty}$ is not periodic. Note that

$$\frac{1}{2} + \frac{1}{2\cdot 6} + \frac{1}{2\cdot 6\cdot 2} + \frac{1}{2\cdot 6\cdot 2\cdot 6} + \frac{1}{2\cdot 6\cdot 2\cdot 6\cdot 2} + \frac{1}{2\cdot 6\cdot 2\cdot 6\cdot 2\cdot 6} + \ldots =$$
$$\frac{7}{12} + \frac{1}{12}\cdot\frac{7}{12} + \frac{1}{12^2}\cdot\frac{7}{12} + \ldots = \frac{7}{11} \in \mathbb{Q}.$$

Furthermore, $\frac{1}{2}+\frac{1}{2\cdot 6} = \frac{1}{3}+\frac{1}{3\cdot 2}+\frac{1}{3\cdot 2\cdot 2}$. If in the sequence $\{a_n\}_{n=1}^{\infty} = \{2,6,2,6,2,6,\ldots\}$ we replace arbitrary pairs $2,6$ with $3,2,2$ then the value of S does not change. We can do so in uncountably many ways. Only countably many among them are periodic. So we obtain uncountably many bounded sequences $\{a_n\}_{n=1}^{\infty}$ such that the corresponding series

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \ldots a_n}$$

equals $\frac{7}{11}$.

The following algorithm shows that it is even possible for any given monotonic sequence $\{b_n\}_{n=1}^{\infty}$ of positive integers to construct a sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n \in \{2,3,4\}$ for every n such that $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \ldots a_n}$ assumes any prescribed value in some interval.

Algorithm 3.1 *Let $\{b_n\}_{n=1}^{\infty}$ be any monotonically non-decreasing sequence of positive integers such that $T = \sum_{n=1}^{\infty} b_n 2^{-n}$ converges. Let $S \in (\frac{T}{3}, T]$. Put $S_1 = S$ and $T_N = \sum_{n=N}^{\infty} b_n 2^{-(n-N)}$. Define for $n = 1, 2, \ldots$*

$$a_n = \begin{cases} 2 & \text{if} \ \ \frac{1}{3}T_n < S_n \leq \frac{1}{2}T_n \\ 3 & \text{if} \ \ \frac{1}{4}T_n < S_n \leq \frac{1}{3}T_n \\ 4 & \text{if} \ \ \frac{1}{6}T_n < S_n \leq \frac{1}{4}T_n \end{cases}$$

and $S_{n+1} = a_n S_n - b_n$.

Note that $\frac{2}{3}T_n < a_n S_n \leq T_n$ and $T_n \geq 2b_n$. By induction it follows that $\frac{1}{6}T_n < S_n \leq \frac{1}{2}T_n$ and that $\sum_{n=1}^{N} \frac{b_n}{a_1 \ldots a_n} = S - \frac{S_{N+1}}{a_1 \ldots a_N}$ for all N. Since $\frac{S_{N+1}}{a_1 \ldots a_N} \leq T_{N+1} 2^{-N-1} = \sum_{n=N+1}^{\infty} b_n 2^{-n} \to 0$ as $N \to \infty$, we have $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \ldots a_n} = S$ and $a_n \in \{2,3,4\}$ for all n.

4 The case $\liminf_{n\to\infty}(\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n}) = 0$.

Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be integer sequences with $a_n > 1$ for all n.

Proposition 4.1 *If $\{S_n\}_{n=1}^\infty$ is bounded from below and for every $\epsilon > 0$ we have $S_{n+1} - S_n < \epsilon$ for $n \geq n_0(\epsilon)$, then S is rational if and only if $\frac{b_n}{a_n-1}$ is constant for $n > N_0$.*

Proof. If $\frac{b_n}{a_n-1} = c$ for $n \geq N_0$, then

$$a_1 \dots a_{N_0-1} \sum_{n=1}^\infty \frac{b_n}{a_1 \dots a_n} \in \mathbb{Z} + \sum_{n=N_0}^\infty \frac{b_n}{a_{N_0} \dots a_n} = \mathbb{Z} + c \sum_{n=N_0}^\infty \frac{a_n - 1}{a_{N_0} \dots a_n} =$$

$$\mathbb{Z} + c \sum_{n=N_0}^\infty (\frac{1}{a_{N_0} \dots a_{n-1}} - \frac{1}{a_{N_0} \dots a_n}) = \mathbb{Z} + \frac{c}{a_{N_0}} \subset \mathbb{Q}.$$

Thus $\sum_{n=1}^\infty \frac{b_n}{a_1 \dots a_n}$ is a rational number.

On the other hand, suppose $S = \frac{r}{q}$ with $r \in \mathbb{Z}$, $q \in \mathbb{N}$. By Lemma 2.1, $qS_n \in \mathbb{Z}$ for every n. Since $qS_{n+1} - qS_n < 1$ for $n \geq n_0(\frac{1}{q})$, the sequence $\{qS_n\}_{n=n_0}^\infty$ is monotonically non-increasing. Furthermore, $\{qS_n\}_{n=n_0}^\infty$ is bounded from below and integer valued. Thus $S_n = S_{n_1}$ for n greater than some n_1. Since $a_n S_n = S_{n+1} + b_n$ for all n and $S_n = S_{n+1}$ for $n \geq n_1$, it follows that $\frac{b_n}{a_n-1}$ equals the constant S_{n_1} for $n \geq n_1$. □

As an application of Proposition 4.1 we derive the following result.

Theorem 4.1 *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of integers with $a_n > 1$ for all n. If $\{\frac{b_n}{a_n}\}_{n=1}^\infty$ is bounded from below, $\lim_{n\to\infty} \frac{b_n}{a_{n-1}a_n} = 0$ and for every $\epsilon > 0$ there exists $n_0(\epsilon)$ such that $\frac{b_{n+1}}{a_{n+1}} < \frac{b_n}{a_n} + \epsilon$ for $n > n_0(\epsilon)$, then $\sum_{n=1}^\infty \frac{b_n}{a_1 \dots a_n} \in \mathbb{Q}$ if and only if $\frac{b_n}{a_{n-1}}$ is constant for $n \geq N_0$.*

Proof. By $b_n = o(a_{n-1}a_n)$ and (3) we have $|S_n - \frac{b_n}{a_n}| < \epsilon$ for $n \geq n_1(\epsilon)$. It follows that $\{S_n\}_{n=1}^\infty$ is bounded from below. Furthermore,

$$S_{n+1} - S_n = (S_{n+1} - \frac{b_{n+1}}{a_{n+1}}) + (\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n}) + (\frac{b_n}{a_n} - S_n) < 3\epsilon$$

for $n \geq \max(n_0(\epsilon), n_1(\epsilon))$. Thus we can apply Proposition 4.1. □

The following consequence of Theorem 4.1 shows that the condition $\liminf_{n\to\infty} \frac{a_n}{b_n} = 0$ in Corollary 2.10 of Erdős and Straus [7] can be relaxed.

Corollary 4.1 *Let ϵ be a positive real number. If $\{b_n\}_{n=1}^{\infty}$ is a sequence of positive integers and $\{a_n\}_{n=1}^{\infty}$ a monotonically non-decreasing sequence of integers greater than one such that $\lim_{n\to\infty} \frac{b_n}{a_n^2} = 0$ and $b_{n+1} - b_n < \epsilon a_n$ for $n \geq n_0(\epsilon)$, then $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n}$ is rational if and only if $\frac{b_n}{a_n - 1}$ is constant for every $n \geq n_1$.*

Proof. Obviously $\{\frac{b_n}{a_n}\}_{n=1}^{\infty}$ is bounded from below. Furthermore, $\frac{b_n}{a_{n-1}a_n} \leq \frac{b_{n-1}+\epsilon a_{n-1}}{a_{n-1}a_n} \leq \frac{b_{n-1}}{a_{n-1}^2} + \frac{\epsilon}{a_n} \to 0$ as $n \to \infty$. Finally, $\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} \leq \frac{b_{n+1}-b_n}{a_n} < \epsilon$ for $n \geq n_0(\epsilon)$. Thus the conditions of Theorem 4.1 are fulfilled. $\Box$

The following statement follows directly from Corollary 4.1, since $b_{n+1} - b_n = o(n)$ implies $b_n = o(n^2)$.

Corollary 4.2 *If $\{b_n\}_{n=1}^{\infty}$ is a sequence of positive integers such that $b_{n+1} - b_n = o(n)$, then $\sum_{n=1}^{\infty} \frac{b_n}{n!}$ is rational if and only if $\frac{b_n}{n-1}$ is constant for $n \geq n_1$.*

Example 4.1 Let $d(n)$ denote the number of divisors of n. By Corollary 4.2 we obtain the irrationality of $\sum_{n=1}^{\infty} \frac{d(n)}{n!}$ and of $\sum_{n=1}^{\infty} \frac{n-d(n)}{n!}$. The condition $\liminf_{n\to\infty} \frac{a_n}{b_n} = 0$ of Erdős and Straus is not fulfilled.

5 The case $b_n = o(a_{n-1}a_n)$

In 1958 Erdős [3] proved that $\sum_{n=1}^{\infty} \frac{p_n}{n!}$ is irrational, where $\{p_n\}_{n=1}^{\infty}$ is the sequence of consecutive primes. This result also follows from Corollary 4.2. Recall that $p_{n+1} - p_n = o(n^{0.55})$ by refinements of the theorem of Hoheisel e.g. due to Mozzochi [10].

Erdős and Straus [7] generalized Erdős' result by showing that if $\{a_n\}_{n=1}^{\infty}$ is a monotonic sequence of positive integers such that $\lim_{n\to\infty} \frac{p_n}{a_n^2} = 0$ and $\liminf_{n\to\infty} \frac{a_n}{p_n} = 0$, then the sum $\sum_{n=1}^{\infty} \frac{p_n}{a_1 \dots a_n}$ is irrational. The following result shows that the latter condition can be relaxed.

Theorem 5.1 *Let $\{a_n\}_{n=1}^{\infty}$ be a monotonic sequence of positive integers satisfying $p_n = o(a_n^2)$. Then $S = \sum_{n=1}^{\infty} \frac{p_n}{a_1 \dots a_n}$ is rational if and only if $\frac{p_n}{a_n - 1}$ is constant for $n \geq n_0$.*

Proof. Suppose $S = \frac{r}{q}$ with $r \in \mathbb{Z}$, $q \in \mathbb{N}$. Then, by Lemma 2.1, $qS_n \in \mathbb{Z}$ for all n. Note that $\{a_n\}_{n=1}^{\infty}$ is non-decreasing. By (3) we have $|S_n - \frac{p_n}{a_n}| < \epsilon$ for $n \geq n_1(\epsilon)$.

If $S_n < S_{n+1}$, then $\frac{p_{n+1}}{a_{n+1}} - \frac{p_n}{a_n} > \frac{1}{q} - 2\epsilon$, whence $p_{n+1} - p_n > (\frac{1}{q} - 2\epsilon)a_n$. Hence, for large N, the inequality $S_n < S_{n+1}$ occurs for at most $\frac{p_{2N}-p_N}{(\frac{1}{q}-2\epsilon)a_n}$ integers n in $[N, 2N)$. Since $p_n \ll n \log n$ and $p_n = o(a_n^2)$ implies $\frac{a_n}{\sqrt{n}} \to \infty$ as $n \to \infty$, this happens at most $O(\sqrt{N} \log N)$ times.

If $S_{n-1} = S_n < S_{n+1}$, then $(a_{n-1}-1)qS_{n-1} = qp_{n-1}$ and $p_{n+1}-p_n > (\frac{1}{q}-2\epsilon)a_n$. Since q is fixed and $a_n \to \infty$, we see that $p_{n-1} \mid (a_{n-1}-1)$ for n large. Hence $p_{n-1} < a_{n-1} \leq a_n$. Since $p_{n+1} - p_n = o(p_{n-1})$, we obtain that $S_{n-1} = S_n < S_{n+1}$ does not occur for large n.

Suppose $S_n > S_{n+1}$ occurs for more than $\frac{N}{2}$ integers $n \in [N, 2N)$, for $n_1 < n_2 < \ldots < n_m$ say. From $\frac{p_n}{a_n} - \frac{p_{n+1}}{a_{n+1}} > (\frac{1}{q} - 2\epsilon)$ we obtain

$$a_{n+1} - a_n > (\frac{1}{q} - 2\epsilon)\frac{a_n a_{n+1}}{p_n} > 0 \tag{5}$$

if $\epsilon < \frac{1}{4q}$ and n is sufficiently large. Hence, for large N, $a_{n_{[m/2]}} \geq \frac{N}{4}$, and by substituting this bound in (5) we obtain $a_{2N} \geq \frac{N}{4}(\frac{1}{q} - 2\epsilon)\frac{N^2}{16p_{2N}} \gg \frac{N^2}{\log N}$ which implies the validity of the theorem in view of Theorem 3.2. We conclude that $S_n \neq S_{n+1}$ is valid for at most $O(\sqrt{N}\log N + \frac{N}{2})$ integers $n \in [N, 2N)$ so that $S_n = S_{n+1}$ has to occur for arbitrary large n.

Suppose $\frac{p_n}{a_n - 1}$ is not constant for $n \geq n_0$. Then S_n is not constant for $n \geq n_0$. Since $S_{n-1} = S_n < S_{n+1}$ can occur only finitely many times, it happens infinitely often that $S_{n-1} = S_n > S_{n+1}$. From $S_{n-1} = S_n$ we have $(a_{n-1} - 1)S_{n-1} = p_{n-1}$ and since p_{n-1} divides $(a_{n-1} - 1)$, we see that $S_{n-1} \leq 1$. Recall that $qS_n \in \mathbb{Z}$ for all n. It is therefore impossible that $\{S_n\}_{n=N_1}^{\infty}$ is non-increasing without becoming constant from some point on. Thus, for some k with $0 < k \leq q$, the situation $S_{n-1} = S_n > S_{n+1} > \ldots > S_{n+k} < S_{n+k+1}$ occurs infinitely often. We have, for n large enough, $a_{n-1} > p_{n-1} > \frac{1}{2}n\log n$, $a_{n+i+1} - a_{n+i} \geq (\frac{1}{q} - 2\epsilon)\frac{n^2(\log n)^2}{4p_{n+i}} \geq (\frac{1}{4q} - \epsilon)n\log n$ for $i = 0, \ldots, k-1$. Hence

$$a_{n+k} \geq (\frac{1}{4q} - \epsilon)kn\log n \geq (\frac{1}{4q} - 2\epsilon)(n+k)\log(n+k).$$

It follows from $S_{n+k+1} > S_{n+k}$ that

$$p_{n+k+1} - p_{n+k} > (\frac{1}{4q} - 2\epsilon)^2(n+k)\log(n+k) > \frac{1}{20q^2}p_{n+k}.$$

This is a contradiction with $p_{n+1} - p_n = o(p_n)$. □

Remark The following example shows that the monotonicity of $\{a_n\}_{n=1}^{\infty}$ is important. If $b_n = p_n$ and $a_n = p_n + 1$ for every n, then

$$\sum_{n=1}^{N} \frac{p_n}{a_1 \ldots a_n} = 1 - \frac{1}{a_1 \ldots a_N}$$

for every N and the limit equals 1. If we replace infinitely often a pair $(a_N, a_{N+1}) = (p_N + 1, p_{N+1} + 1)$ by $(p_N + 2, \frac{p_{N+1}+1}{2})$, then

$$\sum_{n=1}^{N} \frac{p_n}{a_1 \ldots a_n} = 1 - \frac{2}{a_1 \ldots a_N}, \qquad \sum_{n=1}^{N+1} \frac{p_n}{a_1 \ldots a_n} = 1 - \frac{1}{a_1 \ldots a_{N+1}}$$

and again the limit for $N \to \infty$ equals 1.

Many variations are possible. Note that the growth order of $\{a_n\}_{n=1}^{\infty}$ does not change, but that the monotonicity is disturbed.

In Theorem 5.1 the primality of p_n can be relaxed considerably. For example, by the same method we can derive the following result.

Theorem 5.2 *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two monotonic sequences of positive integers such that $\lim_{n\to\infty} \frac{b_n}{a_n^2} = 0$ and $\lim_{n\to\infty} \frac{a_{2n}b_{2n}}{n\cdot a_n^2} = 0$. Suppose for every $\epsilon > 0$ there exists $\delta > 0$ and infinitely many N such that $\gcd(a_n - 1, b_n) < \epsilon b_n$ for at least δN integers $n \in [N, 2N)$. Then $S = \sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n}$ is irrational.*

Proof. Suppose $S = \frac{r}{q}$ with $r \in \mathbb{Z}$, $q \in \mathbb{N}$. Then $qS_n \in \mathbb{Z}$ for all n. Recall that, if $S_{n+1} = S_n$, then $(a_n - 1)qS_n = b_n q$. Applying the gcd-condition with $\epsilon = \frac{1}{q}$ we see that there exist arbitrary large N such that $S_n \neq S_{n+1}$ for a positive proportion of $n \in [N, 2N)$, since $a_n - 1$ cannot divide $b_n q$ for such n.

Let $\epsilon > 0$. Recall that, by (3), it follows from $b_n = o(a_n^2)$ that $|S_n - \frac{b_n}{a_n}| < \epsilon$ for $n \geq n_0(\epsilon)$. If $S_{n+1} > S_n$, then

$$\frac{b_{n+1} - b_n}{a_n} \geq \frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} > \frac{1}{q} - 2\epsilon.$$

Hence $b_{n+1} - b_n > (\frac{1}{q} - 2\epsilon)a_N$ if $n \in [N, 2N)$, $N \geq n_0(\epsilon)$, $S_{n+1} > S_n$. Thus the number of $n \in [N, 2N)$ with $S_{n+1} > S_n$ does not exceed $\frac{b_{2N} - b_N}{(\frac{1}{q} - 2\epsilon)a_N}$. Similarly, if $S_{n+1} < S_n$, then from $b_n(\frac{1}{a_n} - \frac{1}{a_{n+1}}) \geq \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} > \frac{1}{q} - 2\epsilon$ we obtain $a_{n+1} - a_n > (\frac{1}{q} - 2\epsilon)\frac{a_n a_{n+1}}{b_n}$. Thus for N large the number of $n \in [N, 2N)$ with $S_{n+1} < S_n$ does not exceed $\frac{b_{2N}(a_{2N} - a_N)}{(\frac{1}{q} - 2\epsilon)a_N^2}$. The total number of $n \in [N, 2N)$ with $S_{n+1} \neq S_n$ is therefore bounded by

$$\frac{b_{2N}a_N + b_{2N}(a_{2N} - a_N)}{(\frac{1}{q} - 2\epsilon)a_N^2} = \frac{a_{2N}b_{2N}}{(\frac{1}{q} - 2\epsilon)a_N^2}.$$

Since $a_{2N}b_{2N} = o(Na_N^2)$, the proportion of $n \in [N, 2N)$ with $S_{n+1} \neq S_n$ tends to 0. This yields a contradiction with the bound for the number of $n \in [N, 2N)$ with $S_{n+1} = S_n$. Note that the gcd-condition excludes that $\frac{b_n}{a_n - 1}$ is constant from some number n_0 on. □

Remark By using a similar argument as in the proof of Theorem 5.1 for the case $S_n > S_{n+1}$, the condition $a_{2N}b_{2N} = o(Na_N^2)$ may be relaxed. The following example shows, however, that this condition cannot be merely dropped. Some growth condition is necessary.

Define the recurrence sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ by $a_1 = 10$, $b_1 = 9$, $a_2 = 13$, $b_2 = 11$ and

$$a_{2n-1} = a_{2n-2} + 2, \quad b_{2n-1} = 2a_{2n-1} - 1, \quad b_{2n} = b_{2n-1} + 2, \quad a_{2n} = b_{2n} + 2.$$

Then $\gcd(a_n - 1, b_n) = \gcd(a_n, b_n) = 1$ for every n and

$$\sum_{n=1}^{N} \frac{b_n}{a_1 \dots a_n} = \begin{cases} 1 - \frac{1}{a_1 \dots a_N} & \text{if } N \text{ is odd} \\ 1 - \frac{2}{a_1 \dots a_N} & \text{if } N \text{ is even} \end{cases}$$

so that $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \dots a_n} = 1$. Furthermore, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are monotonically increasing, whereas $\{\frac{b_n}{a_n}\}_{n=1}^{\infty}$ has only limit points at 1 and 2 so that $b_n = O(a_n)$.

Example 5.1 Theorem 5.2 shows that $\sum_{n=1}^{\infty} (\frac{p_n}{n!})^k$ is irrational for any integer $k \geq 1$.

6 The case $b_{n+1} - b_n = o(a_{n-1}a_n)$.

The condition $p_n = o(a_n^2)$ in Theorem 5.1 can be considerably relaxed. Instead of requiring $\frac{a_n}{\sqrt{n \log n}} \to \infty$ we only impose $\frac{a_n}{\log n} \to \infty$. In this way we partially affirm the expectation expressed by Erdős in [3] p.99 that only the monotonicity of $\{a_n\}_{n=1}^{\infty}$ suffices. We are not able to prove the irrationality of $\sum_{n=1}^{\infty} \frac{p_n}{2^n}$ either.

Theorem 6.1 *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a monotonic sequence of positive integers such that $\lim_{n \to \infty} \frac{a_n}{\log n} = \infty$. Then $S = \sum_{n=1}^{\infty} \frac{p_n}{a_1 \dots a_n}$ is rational if and only if $\frac{p_n}{a_n - 1}$ is constant for $n \geq n_0$.*

Proof. Suppose $S = \frac{r}{q}$ for some $r \in \mathbb{Z}$, $q \in \mathbb{N}$. Since $p_n \leq 2n \log n$ and $p_{n+1} \leq 1.1 p_n$ for large n, we have

$$\sum_{i=K_n}^{\infty} \frac{p_{n+i}}{a_n \dots a_{n+i}} \leq \frac{p_{n+K_n}}{a_n \dots a_{n+K_n}} \sum_{i=K_n}^{\infty} (0.55)^{i-K_n}$$

$$\leq \frac{6(n+K_n)\log(n+K_n)}{2^{K_n+1}} < \frac{7n \log n}{n^{2 \log 2}} < \frac{1}{8q}$$

for $K_n = 2[\log n] + 1$ when n is large enough.

In view of Theorem 5.1 we may assume without loss of generality that there are infinitely many integers N such that $a_{2N} < N^{0.6}$. We choose such an N so large that all the inequalities in the rest of the proof are satisfied. By $a_{2N} < N^{0.6}$ there are at most $N^{0.6}$ integers $n \in [N, 2N)$ such that $a_{n+1} > a_n$. Put $K = K_{2N}$. Hence $K \leq 2.1 \log N$. Since $p_{2N} < 4N \log N$, we have $p_{n+1} - p_n > 20(\log N)^2$ for at most $\frac{N}{5 \log N}$ integers $n \in [N, 2N)$. Hence there is a set A of at least $N - 2KN^{0.6} - \frac{KN}{5 \log N} > \frac{N}{2}$ integers $n \in [N, 2N)$ such that $a_n = \dots = a_{n+K}$ and $p_{n+i+1} - p_{n+i} \leq 20(\log N)^2$ for $i = 1, \dots, K$. For $n \in A$ we have, assuming that $a_N > 20q \log N$,

$$\left|\sum_{i=1}^{K-1} \left(\frac{p_{n+i}}{a_n \dots a_{n+i}} - \frac{p_{n+i+1}}{a_{n+1} \dots a_{n+i+1}}\right)\right| \leq \sum_{i=1}^{K-1} \frac{|p_{n+i} - p_{n+i+1}|}{a_n^{i+1}} \leq$$

$$\sum_{i=1}^{K-1} \frac{20(\log N)^2}{(20q\log N)^{i+1}} < \frac{1}{4q}.$$

Hence, for $n \in A$,

$$|(S_n - S_{n+1}) - (\frac{p_n}{a_n} - \frac{p_{n+1}}{a_{n+1}})| = |\sum_{i=1}^{\infty}(\frac{p_{n+i}}{a_n \dots a_{n+i}} - \frac{p_{n+i+1}}{a_{n+1} \dots a_{n+i+1}})| \le$$

$$\sum_{i=1}^{K-1} \frac{|p_{n+i} - p_{n+i+1}|}{a_n^{i+1}} + \sum_{i=K}^{\infty} \frac{p_{n+i}}{a_n \dots a_{n+i}} + \sum_{i=K}^{\infty} \frac{p_{n+i+1}}{a_{n+1} \dots a_{n+i+1}} <$$

$$\frac{1}{4q} + 2\frac{1}{8q} = \frac{1}{2q}. \tag{6}$$

Let $n \in A$. If $S_n = S_{n+1}$, then $(a_n - 1)qS_n = qp_n$. Since $a_n \to \infty$, we infer $p_n \mid (a_n - 1)$ which contradicts $a_n < n^{0.6}$. Thus $S_n \ne S_{n+1}$. If $S_{n+1} < S_n$, then, by $q(S_n - S_{n+1}) \in \mathbb{Z}$ and (6), we have $\frac{p_n - p_{n+1}}{a_n} = \frac{p_n}{a_n} - \frac{p_{n+1}}{a_{n+1}} \ge \frac{1}{2q} > 0$ which is impossible. Thus $S_{n+1} > S_n$ for every $n \in A$. For such n we obtain from (6) and $q(S_n - S_{n+1}) \in \mathbb{Z}$ that $\frac{p_{n+1}}{a_{n+1}} - \frac{p_n}{a_n} \ge \frac{1}{2q}$, whence $p_{n+1} - p_n \ge \frac{1}{2q}a_N > 10\log N$. Since the cardinality of A is at least $\frac{N}{2}$, this implies $p_{2N} > 5N\log N$ which is in contradiction with $p_{2N} < 4N\log N$. □

The primality of b_n in Theorem 6.1 is used, but not crucial. By the same method the following result can be proved.

Theorem 6.2 *Let $\{a_n\}_{n=1}^{\infty}$ be an unbounded monotonic sequence of positive integers. Then $S = \sum_{n=1}^{\infty} \frac{n}{a_1 \dots a_n}$ is rational if and only if $\frac{n}{a_n - 1}$ is constant for $n \ge n_0$.*

Proof. We only work out the new features of the proof. For N with $a_{2N} < N^{0.7}$ we apply an argument similar to the derivation of (6) in the proof of Theorem 6.1. Hence, in that case, for a set A of at least $N - 4N^{0.7}\log N$ integers $n \in [N, 2N)$ we have $|(S_n - S_{n+1}) - (\frac{n}{a_n} - \frac{n+1}{a_{n+1}})| < \frac{1}{2q}$. We assume that N is so large that $a_N > 10q$. If $S_{n+1} > S_n$, then $1 > \frac{1}{2q}a_N > 5$. If $S_n > S_{n+1}$, then we obtain $n - (n+1) > 0$, which is also impossible. Suppose it happens that $S_n = S_{n+1} = S_{n+2}$ for $n, n+1 \in A$. Then $(a_n - 1)S_n = n$, $(a_{n+1} - 1)S_n = n + 1$. Since $S_n > 0$, this implies $a_{n+1} > a_n$ in contradiction with $n \in A$. Note that $a_n \le n^{0.6}$ implies $a_{2n} \le (2n)^{0.6} < n^{0.7}$ when n is large.

It remains to consider the case that $a_n > n^{0.6}$ for every n larger than some n_0. By (3) we have $|S_n - \frac{n}{a_n}| < \frac{1}{4q}$. If $S_{n+1} > S_n$, then we obtain $\frac{n+1}{a_{n+1}} - \frac{n}{a_n} \ge \frac{1}{2q}$ whence $a_n \le 2q$. It follows that such n are bounded. Then there are only finitely many n for which $S_{n+1} < S_n$. Thus $S_n = S_{n+1}$ for n larger than some n_1 whence $\frac{n}{a_n - 1} = S_n$ has the same value for all $n > n_1$. □

Note that if $\{a_n\}_{n=1}^{\infty}$ is a bounded monotonic sequence of positive integers, then it is constant for n larger than some n_0 so that $\sum_{n=1}^{\infty} \frac{n}{a_1 \dots a_n}$ is rational.

References

[1] Erdős P.: On arithmetical properties of Lambert series, J. Indian. Math. Soc. 12 (1948), 63–66.

[2] Erdős P.: On the irrationality of certain series, Indagationes Math. 60 (1957), 212–219.

[3] Erdős P.: Sur certaines series à valeur irrationnelle, Enseignement Math. 4 (1958), 93–100.

[4] Erdős P. and Graham R. L.: Old and New Problems and Results in Combinatorial Number Theory, In: Monographies Math. 38 de L'Enseignement Math. Genève, (1980), 60–66.

[5] Erdős P. and Kac M.: Problem 4518, Amer. Math. Monthly 61 (1954), 264.

[6] Erdős P. and Straus E. G.: Some number theoretic problems, Pacific J. Math. 36 (1971), 635–646.

[7] Erdős P. and Straus E. G. : On the irrationality of certain series, Pacific J. Math. 55 (1974), 85-92.

[8] Hančl J.: A note to the rationality of infinite series I, Acta Math. Inf. Univ. Ostr. 5 (1997), 5–11.

[9] Hančl J.: A note to a paper of Oppenheim and Šalát concerning the series of Cantor type, (to appear).

[10] Mozzochi C. J.: On the difference between consecutive primes, J. Number Th. 24 (1986), 181–187.

[11] Oppenheim A.: Criteria for irrationality of certain classes of numbers, Amer. Math. Monthly 61 (1954), 235–241.

[12] Westzynthius E.: Über die Verteilung der Zahlen, die zu den n ersten Primzahlen teilerfremd sind, Comm. Phys.-Math. Soc. Sci. Fenn.5, Nr. 25 (1931), 1–37.

Department of Mathematics, University of Ostrava, Dvořákova 7, 701 03 Ostrava 1, Czech Republic, e-mail: hancl@osu.cz

Mathematisch Instituut, Universiteit Leiden, Niels Bohrweg 1, Postbus 9512, 2300 RA Leiden, the Netherlands, e-mail: tijdeman@math.leidenuniv.nl