Diophantine Equations
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Highlights in the Research Work of T.N. Shorey

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Abstract

We state a number of important results which we owe to Tarlok Shorey.

1 Shorey's Contributions to Linear Form Estimates and Some Applications

One of the first results of Shorey concerns a sharpening of a theorem of Sylvester. Sylvester proved in 1892 that a product of k consecutive positive integers greater than k is divisible by a prime exceeding k. By combining a result of Jutila which depends on estimates for exponential sums and an estimate on linear forms in logarithms, Shorey [45] proved in 1974 that it suffices to take constant times $k(\log\log k)/\log k$ consecutive integers in place of k consecutive integers in the above result of Sylvester. This improved on results of Erdős, Tijdeman, and Ramachandra and Shorey and is still the best known.

The used estimate for the linear form itself is an important contribution of Shorey to the theory on estimating linear forms in logarithms of algebraic numbers which had been developed by Baker in the preceding decade. Since estimates on linear forms play an important role in Shorey's work, we state his result. If a and b are coprime integers then the size of the rational number a/b is defined as $|b| + |a/b|$. All the constants $C_1, C_2, \ldots$ appearing in this article are effectively computable. This means that they can be determined explicitly in terms of the various parameters under consideration. *Let $n > 1$ be an integer. Let*

$$\alpha_1 = \frac{m}{m'},\ \alpha_2 = \frac{p_2}{p_2'},\ \ldots,\ \alpha_n = \frac{p_n}{p_n'}$$

where $p_2, \dots, p_n, p'_2, \dots, p'_n$ *are pairwise distinct prime numbers and none of them is a divisor of the positive integers* m, m'*. Suppose the sizes of* $\alpha_1, \dots, \alpha_n$ *do not exceed* S *and* A *is a constant* > 1 *such that*

$$|\log \alpha_i| \leq \exp(-\frac{1}{A}\log S) \quad for \quad i = 1, \dots, n. \tag{1.1}$$

If $\beta_1, \dots, \beta_{n-1}$ *are rational numbers of size at most* S*, then*

$$|\beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} - \log \alpha_n| > \exp(-(nA)^{C_1 n} \log S) \tag{1.2}$$

where $C_1 > 0$ *is independent of* n, A *and* S*.*

The novelty of Shorey's estimate was two-sided. On the one hand the factor $\log S$ in the exponent of the lower bound of (1.2) is remarkably sharp and in fact the best possible. This was made possible by imposing condition (1.1) which implies that the numbers α_i are quite close to 1. The studies of linear forms in logarithms with α_i's close to 1 were continued by Waldschmidt in 1980 and they led to a remarkable estimate of Laurent, Mignotte and Nesterenko [25] on linear forms in two logarithms in 1995. It has several important applications. For example, it has been applied by Bennett [3] in 2001 to establish the striking theorem that for any positive integer a, the equation

$$(a+1)x^n - ay^n = 1 \quad \text{in integers} \quad x \geq 1, y \geq 1, n \geq 3 \tag{1.3}$$

has no non-trivial solution, i.e. has no solution other than $x = y = 1$. Another application of Shorey's linear form estimate concerns the conjecture of Grimm that if $x, x+1, \dots, x+k-1$ are all composite integers, then the number of distinct prime factors of $x(x+1)\cdots(x+k-1)$ is at least k. Ramachandra, Shorey and Tijdeman [30] confirmed Grimm's conjecture when $(\log x)/(\log k)^2$ exceeds some absolute constant. The assumption that $x, x+1, \dots, x+k-1$ are all composites is not required in this result.

The other novelty in Shorey's estimate (1.2) was that the dependence on n was much better than in previous estimates. Until then there had been a factor n^2 in the exponent. In 1976 Shorey [48] published a linear form estimate with the same dependence on n in the more general case that the numbers α_i and β_i are algebraic numbers of bounded degree and size. Apart from the constant C_1, this estimate was best known with respect to its dependence on n until 2000 when Matveev [26] replaced $n^{C_1 n}$ by $e^{C_2 n}$. The dependence on n has several applications some of which will be mentioned in the next section.

2 Applications of Linear Form Estimates to Values of Polynomials, Recurrence Sequences and Continued Fractions

For an integer ν with $|\nu| > 1$, we denote by $P(\nu)$ the greatest prime factor of ν and by $\omega(\nu)$ the number of distinct prime divisors of ν, respectively. Further we put $P(1) = P(-1) = 1$ and $\omega(1) = \omega(-1) = 0$. Let $f(X)$ be a polynomial with integer coefficients and at least two distinct roots. For a sufficiently large integer x, estimates for linear forms in logarithms yield that $\omega(f(x))$ is at least constant times $\mathrm{loglog}x/\mathrm{logloglog}x$ whenever $\log P(f(x)) \leq (\mathrm{loglog}x)^2$. This implies that $P(f(x))$ at integer x with $|x| \geq C_3$ exceeds $C_4 \log\log|x|$ for some numbers C_3 and $C_4 > 0$ depending only on f. In fact Shorey and Tijdeman [62] obtained lower bounds for

$$\max_{1\leq i\leq y} P(f(x+i))$$

for $\log\, y \leq (\mathrm{loglog}\; x)^{C_5}$ where C_5 is any absolute constant. By applying a p-adic analogue of the above result on linear forms in logarithms, Shorey, van der Poorten, Tijdeman and Schinzel [61] extended the result on a lower bound for $P(f(x))$ to all binary forms with at least three pairwise non-proportional linear factors in their factorizations over $\mathbf{C}$.

For given integers $m > 1$ and $n > 1$ with $mn \geq 6$, a result of Mahler from 1956 states that $P(ax^m - by^n)$ tends to infinity as $\max\,(|x|, |y|) \to \infty$ with $\gcd\,(x, y) = 1$. The proof of Mahler is non-effective but an effective version follows from the theory of linear forms in logarithms. In fact Shorey, van der Poorten, Tijdeman and Schinzel applied this theory to prove that $P(ax^m - by^n)$ tends to infinity with m uniformly in integers x, y with $|x| > 1$ and $\gcd\,(x, y) = 1$. The proof depends on the above mentioned result on the greatest prime factor of a binary form. In 1980 Shorey made the proof independent of this result and it led him to give a quantitative version $P(ax^m - by^n) \geq C_6((\log m)(\log\log m))^{1/2}$ which has been improved by Bugeaud [7] to $P(ax^m - by^n) \geq C_7 \log\, m$ where $C_6 > 0$ and $C_7 > 0$ depend only on a, b and n.

For relatively prime positive integers A and B with $A > B$, it has been conjectured that $P(A^n - B^n)/n$ tends to infinity with n. The first result, from 1904, is due to Birkhoff and Vandiver and states that $P(A^n - B^n) > n$ for $n > 6$. In 1962 this was improved to $P(A^n - B^n) > 2n - 1$ by Schinzel if AB is a square or twice a square unless $n \neq 4, 6, 12$ when $(A, B) = (2, 1)$. In 1975 Stewart confirmed the conjecture for all n with $\omega(n) \leq K \log\log n$ where $0 < K < 1/\log 2$ which is satisfied for almost all n. The year thereafter Erdős and Shorey [15] gave lower bounds for

$P(A^n - B^n)/n$ by applying estimates for linear forms in logarithms. In particular they proved for primes p that

$$P(2^p - 1) > C_8 p \log p$$

where $C_8 > 0$ is an absolute constant. They also combined the theory of linear forms in logarithms with Brun's Sieve to show that

$$P(2^p - 1) > p\ (\log p)^2/(\log\log p)^3$$

for almost all primes p.

The sequence $\{A^n - B^n\}_{n=1}^{\infty}$ is a special case of a binary recursive sequence. Let r and $s \neq 0$ be integers with $r^2 + 4s \neq 0$. Let $u_0, u_1, \ldots$ be integers such that $u_n = ru_{n-1} + su_{n-2}$ for $n = 2, 3, \ldots$. Hence there exist numbers a, b, α, β such that $u_n = a\alpha^n + b\beta^n$ for $n \geq 0$. We assume that $ab \neq 0$ and that α/β is not a root of unity. In 1934 Mahler proved, ineffectively, that $P(u_n)$ tends to infinity with n and an effective version is due to Schinzel in 1967. For $n > m > 0$ with $u_n u_m \neq 0$, Shorey [50] generalized a result of Stewart by proving that

$$P\left(\frac{u_n}{\gcd(u_n, u_m)}\right) \geq C_9 \left(\frac{n}{\log n}\right)^{1/(d_1+1)} \tag{2.1}$$

where $d_1 = [\mathbf{Q}(\alpha, \beta) : \mathbf{Q}]$ and $C_9 > 0$ depends only on α and β. It follows from (2.1) that $u_l \mid u_m$ with $l > m$ implies that l is bounded by a number depending only on the sequence $\{u_n\}$.

Let α be an irrational real number with $[a_0, a_1, \ldots]$ as its simple continued fraction expansion. Let p_n/q_n and $\alpha_n = [a_n, a_{n+1}, \ldots]$ be the n-th convergent and the n-th complete quotient in the simple continued fraction expansion of α, respectively. If α is algebraic of degree ≥ 3 and d_{α_n} denotes the denominator of α_n, then Győry and Shorey [17] showed that $d_{\alpha_n} \geq C_{10} C_{11}^n$ and $P(d_{\alpha_n}) \geq C_{12} \log n$ where $n > 1$ and $C_{10}, C_{11}, C_{12} > 1$ are positive numbers depending only on α. As an application of the estimate on linear forms in logarithms mentioned in the beginning of this article, Shorey [47] derived that $P(p_n q_n) \geq C_{13} \log\log q_n$ if α is algebraic. Here $C_{13} > 0$ depends only on α. This is an improved and effective version of a result of Mahler that $P(p_n q_n)$ tends to infinity with n. In 1939 Erdős and Mahler conjectured that if $P(p_n q_n)$ is bounded for infinitely many n, then α has to be a Liouville number. Shorey [49] showed that if α is a non-Liouville number such that $P(p_{n_k} q_{n_k})$ is bounded for $k \geq 1$ and $n_1 < n_2 < \cdots$, then

$$\lim_{k\to\infty} \frac{\log\log n_k}{\log k} = \infty.$$

3 Some Irrationality Measures and Transcendence Results

Shorey [44] proved a p-adic analogue of a result of Tijdeman on a bound for the number of zeros of a general exponential polynomial in a disk and he applied it to give p-adic analogues of the results of Tijdeman on algebraic independence of certain numbers connected with the exponential function. As an application of much more general theorems he proved that for a prime $p > 2$ at least two of the numbers

$$e^p, e^{pe^p}, e^{pe^{2p}}, e^{pe^{3p}}$$

are algebraically independent. This implies that at least one of the last three numbers is transcendental.

A result of Siegel and Schneider (re-discovered by Lang and Ramachandra) states that

$$|\ 2^{\pi} - \alpha_1\ | + |\ 2^{\pi^2} - \alpha_2\ | + |\ 2^{\pi^3} - \alpha_3\ | \tag{3.1}$$

is positive where α_1, α_2 and α_3 are algebraic numbers. The question whether at least one of the numbers 2^{π} and 2^{π^2} is transcendental remains open and is a special case of the well-known four exponential conjecture. Shorey [46] gave a positive lower bound for (3.1) in terms of the heights and degrees of α_1, α_2 and α_3.

In 2001 the theorem of Baker that a linear form in logarithms of algebraic numbers with algebraic coefficients is either zero or transcendental was applied by Adhikari, Saradha, Shorey and Tijdeman [1] to prove the transcendence of certain infinite series. For example, they showed that $L(1, \chi)$ with χ a non-principal character as well as $\sum_{n=1}^{\infty} \frac{F_n}{n2^n}$ with (F_n) the Fibonacci sequence are transcendental.

4 Results on the Ramanujan τ-function

Consider the Ramanujan τ-function

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{m=1}^{\infty} (1 - q^m)^{24}.$$

Let p be a prime such that $\tau(p) \neq 0$. Shorey [54] applied the theory of linear forms in logarithms to prove that $\tau(p^m) \neq \tau(p^n)$ whenever $m > n$ and $m \geq C_{14}$. In fact he gave an explicit lower bound for the difference of these numbers. Kumar Murty, Ram Murty and Shorey [28] showed that for non-zero odd integer a, the equation

$$\tau(n) = a$$

implies that log $n \leq (2|a|)^{C_{15}}$ where C_{15} is an absolute constant. In particular, the above equation has only finitely many solutions in integers $n \geq 1$.

5 The Ramanujan-Nagell Equation

Ramanujan conjectured and Nagell proved that the equation, now known as the Ramanujan-Nagell equation,

$$x^2 + 7 = 2^n \quad \text{in integers} \quad x \geq 1, n \geq 1$$

has only solutions $(x, n) = (1, 3), (3, 4), (5, 5), (11, 7), (181, 15)$. Let $y \geq 2, D_1$ and D_2 be positive integers such that $\gcd(D_1, D_2) = 1, D = D_1 D_2$ and $\lambda \in \{2^{1/2}, 2\}$. We consider the generalized Ramanujan-Nagell equation

$$D_1 x^2 + D_2 = \lambda^2 y^n \tag{5.1}$$

in integers $x \geq 1$ and $n \geq 1$. We denote by $\mathcal{N}(\lambda, D_1, D_2, y)$ the number of solutions (x, n) of (5.1) and we write p for a prime. Le proved in 1997 and 1999 that $\mathcal{N}(\lambda, D_1, D_2, p) \leq 2$ except for an explicitly given finite set of exceptions. There are three infinite families of triples (D_1, D_2, y) for which $\mathcal{N}(\lambda, D_1, D_2, y) \geq 2$. Bugeaud and Shorey [12] showed that if (D_1, D_2, p) does not belong to any of these three infinite families, then $\mathcal{N}(\lambda, D_1, D_2, p) \leq 1$ except for an explicitly given finite set of possibilities and that if (D_1, D_2, p) belongs to one of these three infinite families, then $\mathcal{N}(\lambda, D_1, D_2, p) = 2$. This settled an old question. The proof depends on a theorem of Bilu, Hanrot and Voutier. The more difficult equation $x^2 + 7 = y^n$ and many similar equations have been completely solved recently by Bugeaud, Mignotte and Siksek, see [10] and [11]. Now all the equations $x^2 + D_2 = y^n$ with $1 \leq D_2 \leq 100$ are completely solved, see [14] and [11].

6 Other Extensions of the Theorem of Sylvester

For positive integers x and $k \geq 2$, we write

$$\triangle_1 = \triangle_1(x, k) = x(x + 1) \cdots (x + k - 1)$$

and give lower bounds for $P(\triangle_1)$ and $\omega(\triangle_1)$. As stated in the first section, Sylvester proved that

$$P(\triangle_1(x,k)) > k \text{ if } x > k.$$

The assumption $x > k$ cannot be removed since $P(\triangle_1(1,k)) \leq k$. Improving on results of Sylvester and Hanson, Laishram and Shorey [22] proved that $P(\triangle_1) > 1.95k$ if $x > k$ except for an explicitly given finite set of possibilities. Here we observe that 1.95 cannot be replaced by 2, since there are arbitrarily long chains of composite positive integers. There is no exception when $k > 270$ or $x > k + 11$.

We turn to lower bounds for $\omega(\triangle_1)$. We see that $k!$ divides $\triangle_1(x,k)$ and therefore Sylvester's theorem can be re-formulated as

$$\omega(\triangle_1) > \pi(k) \text{ if } x > k.$$

A well-known conjecture states that $2^p - 1$ is prime for infinitely many primes p. Thus $\omega(\triangle_1) = 2$ for infinitely many primes p when $x = 2^p - 1, k = 2$ according to the above conjecture. Therefore we assume that $k \geq 3$. Saradha and Shorey [39] improved Sylvester's theorem to

$$\omega(\triangle_1) \geq \pi(k) + [\frac{1}{3}\pi(k)] + 2 \text{ if } x > k$$

except for an explicitly given finite set of possibilities. The above estimate is best known for $k \leq 18$. For $k \geq 19$, Laishram and Shorey [21] sharpened it to

$$\omega(\triangle_1) \geq \pi(k) + [\frac{3}{4}\pi(k)] - 1 \text{ if } x > k$$

except for explicitly given finitely many possibilities. We refer to [39] and [21] for the set of exceptions to the above estimates. These exceptions satisfy $\omega(\triangle_1) \geq \pi(2k) - 1$.

Now we consider Sylvester's theorem and its sharpenings for a product of terms in arithmetic progression. For relatively prime positive integers $x, d \geq 2$ and $k \geq 3$, we put

$$\triangle = \triangle(x,d,k) = x(x+d)\cdots(x+(k-1)d)$$

and we give lower bounds for $P(\triangle)$ and $\omega(\triangle)$. We observe that $P(\triangle(x,d,2)) = 2$ if and only if $x = 1$ and $d+1$ is a power of 2. Therefore we assume that $k \geq 3$. In 1892 Sylvester proved that $P(\triangle) > k$ if $x \geq d + k$. In 1976/77 Langevin replaced the assumption $x \geq d + k$ by $x > k$. Further Shorey and Tijdeman [66] showed that

$$P(\triangle) > k \text{ unless } (x,d,k) = (2,7,3).$$

Laishram and Shorey [23] proved that

$$P(\triangle(x,d,k)) > 2k \quad \text{for} \quad d > 2$$

unless $k = 3, (x, d) = (1, 4), (1, 7), (2, 3), (2, 7), (2, 23), (2, 79), (3, 61), (4, 23), (5, 11), (18, 7); k = 4, (x, d) = (1, 3), (1, 13), (3, 11); k = 10, (x, d) = (1, 3)$. There is no loss of generality in assuming that $d > 2$, since the case $d = 2$ is similar to that of $d = 1$ considered above. A conjecture states that

$$P(\triangle) > ak \quad \text{for} \quad d > a$$

where a is a positive integer. Thus the conjecture has been confirmed for $a = 1, 2$ according to the above inequalities.

Next we consider lower bounds for $\omega(\triangle)$ in terms of $\pi(k)$. Shorey and Tijdeman [64] proved that $\omega(\triangle) \geq \pi(k)$ and Moree showed that $\omega(\triangle) > \pi(k)$ for $k \geq 4$ and $(x, d, k) \neq (1, 2, 5)$. Schinzel's Hypothesis H implies that there are infinitely many d such that $1 + d, 1 + 2d, 1 + 3d, 1 + 4d$ are all primes. Thus Hypothesis H implies that the estimate of Moree is best possible for $k = 4, 5$. For $k \geq 6$, Saradha, Shorey and Tijdeman [43] sharpened and extended the preceding inequality. Their result was further refined by Laishram and Shorey [23] as

$$\omega(\triangle) \geq \pi(2k) - 1 \quad \text{unless} \quad (x, d, k) = (1, 3, 10)$$

confirming a conjecture of Moree. This is best possible when $d = 2$ by considering $\omega(\triangle(k + 1, 2, k)) = \pi(2k) - 1$. The proof of this result depends on explicit estimates for the number of primes in arithmetic progression due to Ramaré and Rumely.

7 Arithmetical Progressions and Perfect Powers

Erdős and Selfridge proved in 1975 that a product of two or more consecutive positive integers is never a power. In 2001 Saradha and Shorey [37] showed that there are no powers other than

$$\frac{6!}{5} = (12)^2, \frac{10!}{7} = (720)^2, 1.2.4 = 2.4 = 2^3$$

which are product of $k-1$ distinct integers out of $k \geq 3$ consecutive positive integers $x, x + 1, \ldots, x + k - 1$. This settled a conjecture of Erdős and Selfridge. The proof depends on combining the elementary method of Erdős and Selfridge with the method of Wiles on the Fermat equation.

Let $m > 2$ be a prime, $k \geq 3$ and $x > k^m$. Erdős and Selfridge showed more generally that a product $x(x+1)\cdots(x+k-1)$ is not of the form by^m with $P(b) < k$. The assumption $P(b) < k$ has been relaxed to $P(b) \leq k$ by Saradha for $k \geq 4$ and by Győry for $k = 3$. The particular case $b = k!$ of the result of Saradha and Győry was already settled by Erdős for $k \geq 4$ and by Győry for $k = 3$. Hanrot, Saradha and Shorey [18] showed that the product in the result of Saradha and Shorey in the preceding paragraph is not of the form by^m with $P(b) < k$ unless $k = 4$ and it is not of the form by^m with $P(b) \leq k$ unless $k \in \{3, 4, 5\}$ which cases were covered by Bennett [4]. The analogous result for $m = 2$ is given in Saradha and Shorey [39] where it has been proved that a product of $k - 1$ distinct integers out of $x, x+1, \ldots, x+k-1$ with $x > k^2$ and $k \geq 4$ is of the form by^2 with $P(b) \leq k$ only when $(x, k) = (24, 4), (47, 4), (48, 4)$. Here the assumption $k \geq 4$ is necessary, since Pell's equations have infinitely many integer solutions.

Some authors have shown that k is bounded if more than $C_m k$ numbers from a block of k consecutive numbers have a product of the form by^m with $P(b) \leq k$, $y > 1, m > 1$ for suitable C_m. Shorey [51], [53] proved that this is true with $C_3 = .84$, $C_4 = .71, C_5 = .65, C_6 = .62$. It follows from the work of Nesternko and Shorey [29] that $C_m = \frac{4}{m}$ suffices for $m \geq 7$.

For relatively prime positive integers x, d and positive integer b with $P(b) \leq k$, we consider the equation

$$x(x+d)\cdots(x+(k-1)d) = by^m \quad \text{in integers } x > 0, y > 0, k \geq 3, m \geq 2. \tag{7.1}$$

We assume that $d \geq 2$ as the case $d = 1$ has already been considered. We always suppose in (7.1) that $(x, d, k) \neq (2, 7, 3)$ so that, as already stated, the left-hand side of (7.1) is divisible by a prime exceeding k. There is no loss of generality in assuming that m is prime in (7.1) which we suppose in this section. We further assume in this paragraph that (7.1) holds and k exceeds a sufficiently large absolute constant. Erdős conjectured that k is bounded by an absolute constant. Marszalek confirmed the conjecture for fixed d. Further Shorey and Tijdeman [65], [67] showed that

$$d \geq k^{C_{16} \log \log k}$$

where $C_{16} > 0$ is an absolute constant and Shorey [59, p.490] applied this inequality to derive the conjecture of Erdős from the *abc*-conjecture if $m > 3$. Further Granville (unpublished) showed that the *abc*-conjecture implies the conjecture of Erdős with $m = 2, 3$. For a proof, see Laishram [20]. Shorey [58], [56] applied linear forms in logarithms with α_i's close to 1 and irrationality measures of Baker obtained by the hypergeometric method to show that $x \geq k^{C_{17}} \log \log k$ for $m \geq 7$ where $C_{17} > 0$ is an absolute constant. Thus k is bounded by a number depending only on x

whenever $m \geq 7$. If $m \geq 3$, Shorey [55] applied the theory of linear forms in logarithms for proving that k is bounded by a number depending only on the greatest prime factor of d. Let d_1 be the maximal divisor of d such that all the prime divisors of d_1 are congruent to 1 mod m. Then Shorey [55] showed that $d_1 > 1$ which implies that we need to verify the preceding assertion for only finitely many m. The proof depends on estimates for the magnitude of solutions of Thue-Mahler equations. Moreover, for a given $m \geq 2$, Shorey and Tijdeman [65] proved that k is bounded by a number depending only on $\omega(d)$.

A stronger version of the conjecture of Erdős, referred as ES, states that if (7.1) holds, then $(k, m) \in (3, 2), (4, 2), (3, 3)$. In each of the above three cases, one can find b such that (7.1) has infinitely many solutions. Let $m > 2$ and $k \geq 4$. Saradha and Shorey [37] showed that Shorey's inequality $d_1 > 1$ for sufficiently large k is valid for all k whenever (7.1) holds. Thus (7.1) implies that d is divisible by a prime congruent to 1 mod m. Consequently (7.1) never holds for d of the form $2^a 3^b 5^c > 1$ where a, b, c are integers. Thus conjecture ES is confirmed for infinitely many d. Saradha and Shorey [40] confirmed conjecture ES for a large number of other values of d. If $\omega(d) = 1$, i.e. d is a prime power, Saradha and Shorey [38] showed that a product of four or more terms in arithmetic progression is never a square. The case $k = 3$ of the preceding result remains open and it is likely that (7.1) with $b = 1, k = 3$ and $\omega(d) = 1$ has infinitely many solutions. Finally Laishram and Shorey [24] confirmed conjecture ES when $b = 1$ and $\omega(d) = 2, 3, 4$.

Now we consider (7.1) with k fixed and without any restriction on d. First we consider the case of squares i.e. $m = 2$. The earliest result is due to Euler that there are no four squares in arithmetic progression. This is also the case when $k = 5$ by Obláth and $6 \leq k \leq 110$ by Hirata-Kohno, Laishram, Shorey and Tijdeman [19]. The cases $6 \leq k \leq 11$ had been covered independently by Bennett, Bruin, Győry and Hajdu [5]. Let $m > 2$ be prime. The result that $n, n + d, n + 2d$ are not all m-th powers is due to Darmon and Merel. Győry showed that (7.1) with $k = 3$ and $P(b) < k$ is not possible. This is also the case when $k = 4, 5, b = 1$ according to Győry, Hajdu, Saradha [16] and when $6 \leq k \leq 11, b = 1$ by Bennett, Bruin, Győry, Hajdu.

8 The Nagell-Ljunggren Equation

Consider the equation

$$y^m = \frac{x^n - 1}{x - 1} \quad \text{in integers } x > 1, y > 1, m > 1, n > 2. \tag{8.1}$$

The equation asks for powers with all the digits equal to 1 in their x-adic expansions. It is called the Nagell-Ljunggren equation as Nagell and Ljunggren made the initial contributions that (8.1) is not possible whenever 4 divides n or $m = 2$, respectively. The equation has solutions given by

$$(x, y, n, m) = (3, 11, 5, 2), (7, 20, 4, 2), (18, 7, 3, 3).$$

It has been conjectured that (8.1) has only finitely many solutions. This is a consequence of the *abc*-conjecture, see [59]. Shorey [51] showed that (8.1) has only finitely many solutions when n is divisible by a prime congruent to 1 mod m. The result of Bennett on (1.3) stated above implies that (8.1) does not hold whenever n is congruent to 1 mod m.

Shorey and Tijdeman [63] showed that (8.1) has only finitely many solutions whenever x is fixed. By using the p-adic analogue of linear forms in logarithms with α_i's close to 1, Bugeaud solved (8.1) completely for several values of x. In particular, Bugeaud and Mignotte [8] settled a problem, due to Inkeri, that there is no m-th power > 1 with digits identically equal to 1 in its decimal expansion. Saradha and Shorey [36] showed that (8.1) is not possible if $x = z^2$ such that z runs through all integers > 31 and $z \in \{2, 3, 4, 8, 9, 16, 27\}$. Further Bugeaud, Mignotte, Roy and Shorey [9] covered the remaining cases. Hence (8.1) is not possible if x is a square. This was also proved, independently, by Bennett [4] who derived it from his general result on (1.3). Further Saradha and Shorey [36] showed that (8.1) implies that x is divisible by a prime congruent to 1 mod m whenever max (x, y, m, n) exceeds a sufficiently large absolute constant.

9 Goormaghtigh's Equation

We turn to an equation of Goormaghtigh:

$$\frac{y^m - 1}{y - 1} = \frac{x^n - 1}{x - 1} \quad \text{in integers } x > 1, y > 1, m > 2, n > 2, m > n. \tag{9.1}$$

We observe that $x > y$ and (9.1) asks for positive integers with all their digits equal to one with respect to two distinct bases. Goormaghtigh observed in 1917 that

$$31 = \frac{2^5 - 1}{2 - 1} = \frac{5^3 - 1}{5 - 1}, \; 8191 = \frac{2^{13} - 1}{2 - 1} = \frac{90^3 - 1}{90 - 1}$$

and it has been conjectured that these are the only solutions of (9.1). It follows from the *abc*- conjecture that (9.1) has only finitely many solutions, see [59, p.473]. In 1961 Davenport, Lewis and Schinzel showed that (8.1) has only finitely many solutions if m and n are fixed. They showed that the underlying polynomial for (9.1)

$$\frac{X^n - 1}{X - 1} - \frac{Y^m - 1}{Y - 1}$$

is irreducible over $\mathbf{C}$ and has positive genus. Then the assertion follows from a well-known theorem of Siegel on integer solutions of polynomial equations in two variables and therefore, is non-effective. On the other hand, they showed that it is effective when $\gcd(m-1, n-1) > 1$. Shorey [57] showed that 31 and 8191 are the only primes N with $\omega(N-1) \leq 5$ such that all the digits of N are equal to one with respect to two distinct bases. For positive integers $A, B, x > 1$ and $y > 1$ with $x \neq y$, Shorey [52] showed that there are at most 24 integers with all the digits equal to A in their $x-$adic expansions and all the digits equal to B in their $y-$adic expansions. If $AB = 1$, Bugeaud and Shorey [13] replaced 24 by 2, and even by 1 if x exceeds 10^{11} or $\gcd(x, y) > 1$. Balasubramanian and Shorey [2] proved that (8.1) implies that max (x, y, m, n) is bounded by a number depending only on the greatest prime factor of x and y.

10 Arithmetical Progressions With Equal Products

It has been conjectured by Erdős and Graham that the equation

$$X(X+1)\cdots(X+K-1)Y(Y+1)\cdots(Y+L-1) = Z^2$$

in integers $K \geq 3, L \geq 3$ and $X \geq Y + L$ has only finitely many solutions in all the integral variables $X > 0, Y > 0, Z > 0, K$ and L. This conjecture implies that

$$x(x+1)\cdots(x+k-1) = y(y+1)\cdots(y+k+l-1)$$

has only finitely many solutions in $x > 0, y > 0, k \geq 3$ and $l \geq 0$ satisfying $x \geq y + k + l$. More generally, for positive integers A and B, Erdős conjectured that there are only finitely many integers $x > 0, y > 0, k \geq 3, l \geq 0$ with $x \geq y + k + l$ satisfying

$$Ax(x+1)\cdots(x+k-1) = By(y+1)\cdots(y+k+l-1). \qquad (10.1)$$

The first result in this direction is due to Mordell that (10.1) with $A = B = 1$ and $k = 2, l = 1$ has no solution in integers $x > 0$ and $y > 0$ and we refer to [6] for more early results. Beukers, Shorey and Tijdeman

[6] applied a well-known theorem of Siegel on integral points on curves to confirm the conjecture if k and l are fixed. The work involves establishing irreducibility and computing genus of the curve under consideration so that the assumptions of the theorem of Siegel are satisfied. Because of the ineffective nature of Siegel's result, we do not know any explicit estimate for the magnitude of the solutions. Saradha and Shorey [31] confirmed the Erdős' conjecture when x and y are composed of fixed primes. The proof depends on several applications of linear forms in logarithms. Further they showed that (10.1) implies that $x - y \geq C_{18}x^{2/3}$ where $C_{18} > 0$ depends only on A and B.

We consider (10.1) with $A = B = 1$ and $k + l$ an integral multiple of k. In this case, for an integer $m \geq 2$,

$$x(x+1)\cdots(x+k-1) = y(y+1)\cdots(y+mk-1) \tag{10.2}$$

in integers $x > 0, y > 0, k \geq 2$.

We refer to [6] for an account of early results. Saradha and Shorey [33], by extending an old effective method of Runge to exponential diophantine equations, proved that (10.2) implies that max (x, y, k) is bounded by a number depending only on m. Saradha and Shorey [32] and Mignotte and Shorey [27] showed that (10.2) with $2 \leq m \leq 6$ implies that $x = 8, y = 1, k = 3, m = 2$. Shorey has conjectured that (10.2) with $m > 6$ has no solution.

For positive integers l, m, d_1 and d_2 with $l < m$ and gcd $(l, m) = 1$, we consider a more general equation than (10.2), namely,

$$x(x+d_1)\cdots(x+(lk-1)d_1) = y(y+d_2)\cdots(y+(mk-1)d_2) \tag{10.3}$$

in integers $x > 0, y > 0, k \geq 2$.

By using Runge's method, Saradha and Shorey [34], [35] and Saradha, Shorey and Tijdeman [41] showed that (10.3) implies that either max (x, y, k) is bounded by a number depending only on m, d_1, d_2 or $m = 2, k = 2, d_1 = 2d_2^2,\ x = y^2 + 3d_2y$. On the other hand, (10.3) with $m = 2$ is satisfied whenever the latter possibilities hold.

Let $l = m = 1$ in (10.3). It is clear that (10.3) with $k = 2$ has infinitely many solutions. Further Gabovich gave an infinite class of solutions of (10.3) with $k = 3, 4$. Some infinite classes of solutions of (10.3) with $k = 5$ were given by Szymiczek and Choudhry where the latter also provided an infinite class of solutions of (10.3) with arbitrary k. Next we take d_1 and d_2 fixed. There is no loss generality in assuming that $x > y$ and gcd $(x, y, d_1, d_2) = 1$. Then $d_1 < d_2$. Saradha, Shorey and Tijdeman [42] proved that either max (x, y, k) is bounded by a number depending only on d_2, or $x = k+1, y = 2, d_1 = 1, d_2 = 4$. The latter possibilities cannot be excluded in view of $(k+1)\cdots(2k) = 2.\cdot 6\cdots(4k-2)$, an observation of Makowski.

Acknowledgements This paper is based on a paper of T.N. Shorey [60] entitled 'Diophantine Approximations, Diophantine equations, Transcendence and Applications" which we have cited without further mention. In that paper the reader can find more details, references and, in particular, a complete list of Shorey's papers until his 60th birthday.

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