The Hodge polynomial of a smooth projective variety $X$ over $\mathbb{C}$ is

$$P(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X) u^p v^q$$

with $h^{p,q}(X) = \dim H^q(X, \Omega^p_X)$ the Hodge numbers of $X$. It satisfies

- **scissor relation**: $P(X) = P(Z) + P(X \setminus Z)$ for $Z \subset X$ a closed subvariety
- **multiplicativity**: $P(X \times Y) = P(X) \cdot P(Y)$

This polynomial extends uniquely to any variety over $\mathbb{C}$ via the Grothendieck ring

$$e : K(\text{Var}_\mathbb{C}) \to \mathbb{Z}[u,v]$$

where $e(X)$ is called the E-polynomial of $X$. Its coefficients are given by the mixed Hodge numbers

$$h^{p,k}_{\text{mixed}}(X) = \dim Gr^k_F Gr^{W} H^p(X, \mathbb{C})$$

of the mixed Hodge structure on the compactly supported cohomology of $X$ [1].

### Examples

- $e(A^1) = e(P^1) - e(pt) = uw = q$, the Lefschetz motive
- $e(P^n) = e(A^n) + e(A^{n-1}) + \cdots + e(A^1) + e(A^0) = q^n + q^{n-1} + \cdots + q + 1$
- To compute $e(SL(2, \mathbb{C})) = e(ad - bc = 1)$, decompose

$$SL(2, \mathbb{C}) = \begin{cases} a = 0, b \neq 0, c = -1/b \\ a \neq 0, d = bc + 1/a \end{cases}$$

and find $e(SL(2, \mathbb{C})) = q(q-1) + q^2(q-1) = q^3 - q$.

### Complete intersections

The Hodge numbers of a smooth hypersurface $X \subset \mathbb{P}^n$ of degree $d$ can be computed recursively from exact sequences

$$0 \to \Omega^p_{\mathbb{P}^n}(-d) \to \Omega^p_{\mathbb{P}^n} \to \Omega^p_X|_X \to 0$$

and cohomology of $\mathbb{P}^n$. This can be generalized as in [2] to compute the Hodge numbers of a smooth complete intersection $X \subset \mathbb{P}^n$ from the degrees $d_i$ of the hypersurfaces.

### Computing E-polynomials

**Setup**: let $X \subset \mathbb{A}^n$ be the variety with ideal $I = (f_1, \ldots, f_k)$. Recursively compute $e(X)$ as follows:

#### Base cases

- if $1 \in I$ then $e(X) = e(\emptyset) = 0$
- if $I = \emptyset$ then $e(X) = e(\mathbb{A}^n) = q^n$

#### Product varieties

- if $F_1 = \{f_1, \ldots, f_m\}$ and $F_2 = \{f_{m+1}, \ldots, f_k\}$ do not share variables, then $X = X_1 \times X_2$, hence $e(X) = e(X_1) \cdot e(X_2)$

#### Factor equations

- if $f_i = gh$ with $g, h$ non-constant, then $e(X) = e(X \cap \{g = 0\}) + e(X \cap \{h = 0\}) - e(X \cap \{g = h = 0\})$

#### Linear equations

- if $f_i = zg + h$ with $g, h$ not containing $z$, then let $Y$ be given by the $f_j$, for $j \neq i$, where $x$ substituted for $-h/g$. Then $e(X) = e(X \cap \{g = 0\}) + e(Y) - e(Y \cap \{g = 0\})$

#### Blowups

- if the singular locus $Z \subset X$ is non-empty, blow up $X$ at $Z$, given by affine patches $U_i$ and exceptional divisor $E$. Then $e(X) = e(Z) + \sum_i e\left( U_i - \cup_{j \neq i} U_j - E \right)$

#### Rehomogenizing

- if $X$ is non-singular, but the projective closure $\overline{X} \subset \mathbb{P}^n$ is singular at another affine patch $Y$ then $e(X) = e(Y) + e(\overline{X} - Y) - e(\overline{X} - X)$

### Smooth projective varieties

- if $X$ defines a smooth projective variety $\overline{X} \subset \mathbb{P}^n$, compute $e(X)$ from the Hodge numbers $h^{p,q}(X) = \dim H^q(X, \Omega^p_X)$

### Application to representation varieties

Used in [3] to automatize the computation of E-polynomials of $G$-representation varieties of closed surfaces

$$X_G(\Sigma_n, G) = \text{Hom}(\pi_1(\Sigma_n), G)$$

using Topological Quantum Field Theory; the E-polynomials can be obtained from the powers of a (large) matrix of E-polynomials of smaller varieties, corresponding to a decomposition of bordisms

![Bordism Diagram](image)

For $G = U_n$ upper triangular matrices of ranks 2, 3 and 4:

$$e(X_U(\Sigma_n)) = q^{2g-1}(q-1)^{2g+1}((q-1)^{2g} + 1),$$

$$e(X_U(\Sigma_4)) = q^{g-3}(q-1)^2(q^2(q-1)^{2g+1} + q^{3g}(q-1)^2 + q^{3g}(q-1)^{2g+1} + q^{3g}(q-1)^{2g+1} + q^{3g}(q-1)^{2g+1} + q^{3g}(q-1)^{2g+1}),$$

and $e(X_U(\Sigma_4)) = q^{g-2}(q-1)^{4g+1} + q^{4g+1}(q-1)^{4g+1} + q^{4g+1}(q-1)^{4g+1} + q^{4g+1}(q-1)^{4g+1} + q^{4g+1}(q-1)^{4g+1} + q^{4g+1}(q-1)^{4g+1} + q^{4g+1}(q-1)^{4g+1}$.

The latter requiring to evaluate $\approx 4000$ E-polynomials.

### What’s next?

- Find more efficient methods for computing the Hodge numbers for non-complete intersections
- Prove the algorithm terminates, e.g. find a numerical invariant that decreases at each step
- Optimize the implementation

### References

[2] SGAI exposé XI, théorème 2.3