An arithmetic-geometric correspondence for character stacks
(based on arXiv:2308.15331, joint work with M. Hablicsek and A. Gonzalez-Prieto)

\[ \Gamma = \text{finitely generated group} \]
\[ G = \text{(linear) algebraic group } \quad (\text{e.g. } \text{GL}_n, \text{SL}_n) \]

\text{Def: } \text{G-representation variety } \mathcal{R}_G(\Gamma) = \text{Hom}(\Gamma, G)
\quad \text{(relations in } \Gamma \text{ yield algebraic equations)}

\text{Ex: } \Gamma = \pi_1(M) \text{ for } M \text{ a compact connected manifold}

\cdot \Gamma = \pi_1(M) = \mathbb{Z}, \quad \mathcal{R}_G(S^1) = G
\quad \text{(shortened notation)}

\cdot \Gamma = \pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{3g} [a_i, b_i] = 1 \rangle \quad (\text{main example})

\mathcal{R}_G(\Sigma_g) = \left\{ (A_1, B_1, \ldots, A_g, B_g) \in G^g \mid \prod_{i=1}^{3g} [A_i, B_i] = 1 \right\} \subseteq G^{3g}
\quad \text{(closed subvariety)}

\text{Why } \Gamma = \pi_1(M) ?

\text{Hom}(\pi_1(M), G) \leftrightarrow \{ \text{G-local systems on } M \} \\
\text{conjugate representations} \leftrightarrow \text{isomorphic local systems}

\text{Def: } \text{G-character stack } \mathcal{X}_G(M) = \left[ \mathcal{R}_G(M)/G \right] \quad (\text{stacky quotient})
\quad (G \text{ acts by conjugation})

This talk
1) Introduce two methods for computing algebraic/cohomological invariants of \( \mathcal{X}_G(\Sigma_g) \)
\quad (arithmetic & geometric method)
2) Show there is a common framework
Arithmetic method (Haussel, Rodriguez-Villegas)

Idea Count \( \mathbb{F}_q \)-points \( \# R_G(\Gamma)(\mathbb{F}_q) = \# R_G(\mathbb{F}_q)(\Gamma) \)

Frobenius formula: If \( G \) is a finite group, then
\[
\# R_G(E_g) = \# G \cdot \sum_{\omega \in \mathcal{O}} (\frac{\# G}{\# \mathcal{O}(\omega)})^2
\]

Ex (\( g=1 \)) \# \{ (A, B) \in G^2 \mid [A, B] = 1 \} = \sum_{\omega \in \mathcal{O}} \frac{\# G}{\# \mathcal{O}(\omega)} (\text{orbit-stabilizer})

= \# G \cdot \# \text{conj classes of } G

= \# G \cdot \hat{G}

Theorem (Katz) If \( X \) is a complex variety, and \( \# X(\mathbb{F}_q) \) is polynomial in \( q \),

then this polynomial is the \( E \)-polynomial of \( X \), with \( q = uv \).

\[
\mathbb{Z}[u, v] \ni e(X) = \sum_{k, p, q} (-1)^k \frac{h_{k, p, q}^\mathcal{O}(X)}{k, p, q} u^k v^q
\]

mixed Hodge numbers of \( X \)

\[
= \sum_{p, q} (-1)^{p+q} \dim_e \mathcal{H}^q(X, \mathcal{O}_X^p) u^p v^q
\]

Ex \( e(A^n) = (uv)^n \)

\( e(P^n) = 1 + uv + \cdots + (uv)^n \) (agrees with the Hodge diamond)

Note Usually, \( e(\mathcal{O}_G(M)) = e(R_G(M)) / e(G) \)
Geometric method (González-Peck, Logares Muñoz, Newstead)

Idea: Compute $E$-polynomial of $R_G(E_g)$ using following properties:

1. (cut-and-paste) $e(X) = e(Z) + e(X \setminus Z)$ for closed subvarieties $Z \subseteq X$

2. (multiplicative) $e(X \times Y) = e(X) \cdot e(Y)$

Note: compatible with Katz’ theorem

Might as well compute invariant in the Grothendieck ring of varieties:

$$K_0(\text{Var}_k) = \bigoplus_{\text{isom classes of varieties }/k} \mathbb{Z} / \left[ x \right] = \left[ Z \right] + \left[ x/\mathbb{Z} \right]$$

Method tries to make smart stratifications to understand:

$$[G^2 \rightarrow G] \in K_0(\text{Var}/G)
\quad \begin{cases} (A,B) \rightarrow [A,B] \end{cases}$$

(Will not go into detail, but there is a way to glue these classes together to get a product of commutators)
Topological Quantum Field Theories

Let's cut $\Sigma_g$ into pieces:

\[ \Sigma_g = \begin{array}{c}
\text{cut}
\end{array} \]

\[ 0 \to \mathcal{X}_0(W) \to \mathcal{X}_0(S') \to \mathcal{X}_0(S') \]

(remark on composition)

\[ \text{2-Bord} \xrightarrow{J} \text{Corr}(\text{Stck}) \]

(field theory)

\[ T \times Z \xrightarrow{g \times T_z} \mathcal{X} \]

\[ f \to K_0(\text{Stck}/Z) \xrightarrow{g!} K_0(\text{Stck}/X) \xrightarrow{g} K_0(\text{Stck}/Y) \]

\[ \text{Corr}(\text{Stck}) \xrightarrow{Q} K_0(\text{Stck}) \text{- Mod} \]

(quantization)

Def The composition $Z = Q \circ J$ is a (law) monoidal functor $n\text{-Bord} \to K_0(\text{Stck}) \text{- Mod}$
that is, a Topological Quantum Field Theory (TQFT)

Note

\[ \emptyset \to \mathcal{M} \to Z \]

\[ \emptyset \to K_0(\text{Stck}/\mathcal{X}_0(M)) \to K_0(\text{Stck}/k) \]

\[ 1 = [k] \to [\mathcal{X}_0(M) \to \mathcal{X}_0(M)] \to [\mathcal{X}_0(M)] \]

We say $Z$ quantizes $[\mathcal{X}_0(M)]$
Arithmetic TQFT

Let's repeat the construction, but with $G$ a finite group

$$0 \xrightarrow{n \text{-Bord}} 0 \xrightarrow{\mathcal{F}^*} \mathcal{X}_G(-) \xrightarrow{\mathcal{X}_G(-)} \text{Corr}(\text{Grpd})$$

$$A \xrightarrow{C} B \xrightarrow{C^*} \text{Corr}(\text{Grpd}) \xrightarrow{\mathbb{Q}^*} \mathbb{C} \text{-Vect}$$

Result: The arithmetic TQFT $\mathbb{Z}^* \rightarrow \mathbb{Q}^* \rightarrow \mathbb{C}^*$ quantizes $\# \mathcal{X}_G(M) = \# R_G(M) / \# G$ (groupoid cardinality)

How is this related to the arithmetic method?

Ex. $(n=2)$ $S' \xrightarrow{S^*} \mathcal{X}_G(S') = G / G \xrightarrow{Q^*} \mathbb{C} \times G \cong R(\mathbb{C}, G)$

$$\mu = \text{convolution} \quad \eta = \text{unit} \quad \delta = \text{comultiplication} \quad E = \text{counit}$$

$\Rightarrow$ Frobenius algebra = algebra + coalgebra + compatibilities

Working out $\mu, \eta, \delta, E$ in terms of characters, we find

$$\mathbb{Z}^*(\Sigma_g) = E \circ (\mu \circ \delta)^g \circ \eta = \sum_{\chi \in \hat{G}} \left( \frac{\# G}{\# \chi} \right)^g \chi(\zeta) = \frac{\# R(\Sigma_g)}{\# G}$$

Note: The arithmetic TQFT $\mathbb{Z}^*$ generalizes the arithmetic method in the sense that it works in any dimension.
Arithmetic-geometric correspondence

Geometric: $n$:Bord $\xrightarrow{F} \text{Corr(Stick)} \xrightarrow{Q} K_0(\text{Stick}) - \text{Mod}$

Arithmetic: $n$:Bord $\xrightarrow{F^*} \text{Corr(Grp)} \xrightarrow{Q^*} \mathbb{C}$-Vect

Counting $\mathbb{F}_q$-points: $K_0(\text{Stick}) \xrightarrow{[X]} \#X(\mathbb{F}_q)$ (ring morphism)

RHS Counting points in fibers: $K_0(\text{Stick}/X) \xrightarrow{C^{X(\mathbb{F}_q)}} [\gamma \mapsto X] \xrightarrow{(x) \mapsto \#f^{-1}(x)}$

Prop This defines a natural transformation "\""

LHS $[R_G(M)/G](\mathbb{F}_q)$ vs. $[R_{G(\mathbb{F}_q)}(M)/G(\mathbb{F}_q)]$

Prop If $G$ is connected, then these groupoids are (naturally) equivalent (Long's theorem)

Theorem If $G$ is connected, then there is a natural transformation $Z \Rightarrow Z^*$
Corollaries

Geometric $\Rightarrow$ Arithmetic

Under the natural transformation,

1. the eigenvalues of $Z(\mathbb{Q} = \mathbb{Q})$ are sent to $\#G(\mathbb{F}_q)/\chi(\cdot)$ for the irreducible characters $\chi$ of $G(\mathbb{F}_q)$,

2. the eigenvectors of $Z(\mathbb{Q} = \mathbb{Q})$ are sent to the sums of equi-dimensional characters of $G(\mathbb{F}_q)$, that is, $\sum_{\chi \in \hat{G}} \chi$

Note From geometric computations, we can deduce (partially) information on the character table of $G(\mathbb{F}_q)$

Arithmetic $\Rightarrow$ Geometric

No formal implication, but it seems one can lift arithmetic eigenvalues/eigenvectors to geometric ones.

Q: Are eigenvalues of $Z(\mathbb{Q} = \mathbb{Q})$ always polynomial in $[A']$?
Q: Is $[R_0(\xi)]$ always polynomial in $[A']$?