Deformation Quantization

Jesse Vogel

based on Chapter 5 of Deformation Quantization and Index Theory by B. Fedosov

Let \((M,\omega)\) be a symplectic manifold, and let \(Z = \mathcal{C}^\infty(M)[[\hbar]]\) be the linear space of formal power series

\[
a = \sum_{k=0}^\infty \hbar^k a_k, \quad \text{with } a_k \in \mathcal{C}^\infty(M).
\]

Definition 1. Deformation quantization of \(\mathcal{C}^\infty(M)\) refers to an associative product \(\star\) on \(Z\), called a star product, satisfying

1. (formal deformation) \(a \star b \mod \hbar = ab\) for all \(a, b \in \mathcal{C}^\infty(M)\).
2. (locality) for any \(a, b \in Z\), we have
\[
a \star b = \sum_{k=0}^\infty \hbar^k c_k,
\]
where \(c_k\) depends on \(\partial^\alpha a_i \partial^\beta b_j\) with \(i + j + |\alpha| + |\beta| \leq k\).
3. (correspondence principle) for all \(a, b \in Z\), we have
\[
[a, b] = a \star b - b \star a = -i\hbar \{a_0, b_0\} + O(\hbar^2),
\]
where \(\{\cdot, \cdot\}\) denotes the Poisson associated to \(\omega\).

Remark 2. Note that deformation quantization differs from Weyl quantization by the fact that the Planck constant \(\hbar\) is no longer a positive number, but a formal parameter.

The Formal Weyl Algebras Bundle

Definition 3. The formal Weyl algebra bundle is the bundle \(W = \hat{\text{Sym}}(T^*M \otimes \mathbb{C})[[\hbar]]\). Locally, its sections are of the form

\[
a = \sum_{k,|\alpha|\geq 0} \hbar^k a_{k,\alpha} y^\alpha,
\]
where \(y^\alpha = (y^1)^{\alpha_1} \cdots (y^{2n})^{\alpha_{2n}}\), with \(y^i\) a basis for \(T^*M\), and \(a_{k,\alpha}\) complex-valued functions on \(M\).

Definition 4. The Weyl product of two sections \(a, b \in \Gamma(W)\) is given (fiberwise) by

\[
a \circ b = \exp \left( -\frac{i\hbar}{2} \omega^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right) a(y)b(z) \bigg|_{z=y} = \sum_{k=0}^\infty \left( -\frac{i\hbar}{2} \right)^k \frac{1}{k!} \omega^{ij_1} \cdots \omega^{ij_k} \frac{\partial^k a}{\partial y^{i_1} \cdots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \cdots \partial y^{j_k}}.
\]

Lemma 5. The center of \(\Gamma(W)\) with respect to the Weyl product is \(Z\).

Proof. Take any \(a\) in the center of \(\Gamma(W)\). If we take \(b = y^k\) for some \(k\), then

\[
a \circ b = ay^k - \frac{i\hbar}{2} \omega^{ij} \frac{\partial a}{\partial y^i} \quad \text{and} \quad b \circ a = ay^k - \frac{i\hbar}{2} \omega^{kj} \frac{\partial a}{\partial y^j}.
\]
so \[ 0 = [a, b] = -i\hbar \omega^{ik} \frac{\partial a}{\partial y^i}. \]

Varying over \( k \), we find that \( \frac{\partial a}{\partial y^i} = 0 \) for all \( i \), so \( a \in Z \). Conversely, it is easy to see that \( Z \) lies in the center of \( W \).

We grade the bundle \( W \) by setting \( \deg y^i = 1 \) and \( \deg \hbar = 2 \). This yields a filtration
\[
\Gamma(W) \supset \Gamma(W_1) \supset \Gamma(W_2) \supset \cdots
\]

Similarly, the bundles of differential forms \( W \otimes \Lambda^q \) are graded, where the degree of any pure \( q \)-form is zero. The Weyl product can be extended to \( W \otimes \Lambda \) using the wedge product \( \wedge \), where the \( y^i \) and \( dx^i \) commute. The commutator of forms \( a \in \Gamma(W \otimes \Lambda^q) \) and \( b \in \Gamma(W \otimes \Lambda^{q_2}) \) is
\[
[a, b] = a \circ b - (-1)^{q_1 q_2} b \circ a.
\]

Similar to Lemma 5, the center of \( \Gamma(W \otimes \Lambda) \) with respect to the Weyl product is \( Z \otimes \Lambda \).

**Notation 6.** For any \( a \in \Gamma(W \otimes \Lambda) \), we write \( a_0 = a|_{y=0} \) and \( a_{00} = a|_{y=0,dx=0} \). Furthermore, for any \( a \in \Gamma(W) \), we write \( \sigma(a) \) for \( a_0 = a|_{y=0} \).

**Definition 7.** Define operations \( \delta \) and \( \delta^* \) on \( \Gamma(W \otimes \Lambda) \) by
\[
\delta : \Gamma(W_p \otimes \Lambda^q) \to \Gamma(W_{p-1} \otimes \Lambda^{q+1}), \quad a \mapsto dx^k \wedge \frac{\partial a}{\partial y^k},
\]
\[
\delta^* : \Gamma(W_p \otimes \Lambda^q) \to \Gamma(W_{p+1} \otimes \Lambda^{q-1}), \quad a \mapsto y^k \iota_{\partial_k} a.
\]

In particular, \( \delta \) lowers the degree by one, while \( \delta^* \) raises the degree by one.

**Lemma 8.** The operations \( \delta \) and \( \delta^* \) do not depend on the choice of local coordinates, and satisfy
\begin{enumerate}[<1>]
\item \( \delta^2 = (\delta^*)^2 = 0 \),
\item \( (\delta \delta^* + \delta^* \delta)(a) = (p + q)a \) for a monomial \( a = y^{i_1} \cdots y^{i_p} dx^{j_1} \cdots dx^{j_q} \).
\item \( \delta(a \circ b) = (\delta a) \circ b + (-1)^q a \circ (\delta b) \) for \( a \in \Gamma(W \otimes \Lambda^q) \) and \( b \in \Gamma(W \otimes \Lambda^{q_2}) \).
\item \( \delta a = -\frac{i}{\hbar} [\omega_{ij} y^i dx^j, a] \).
\end{enumerate}

**Proof.** Straightforward.

**Definition 9.** Let \( a \in \Gamma(W \otimes \Lambda) \), and write \( a_{pq} \) for \( (p,q) \)-homogeneous part. Then define
\[
\delta^{-1} a_{pq} = \begin{cases} \frac{1}{p+q} \delta^* a_{pq} & \text{if } p + q > 0, \\ 0 & \text{otherwise}. \end{cases}
\]

In particular, using Lemma 8(ii), any \( a \in \Gamma(W \otimes \Lambda) \) has a Hodge–De Rham decomposition
\[
a = a_{00} + \delta \delta^{-1} a + \delta^{-1} \delta a. \tag{1}
\]

Recall that there exists a symplectic connection \( \nabla \) on \( M \). Tensorially, there is an induced connection on \( W \otimes \Lambda \), also denoted by \( \nabla \).
Lemma 10.

(i) $\nabla (a \circ b) = \nabla a \circ b + (-1)^q a \circ \nabla b$ for $a \in \Gamma(W \otimes \Lambda^q)$.

(ii) $\nabla (\eta \wedge a) = d\eta \wedge a + (-1)^q \eta \wedge \nabla a$ for $\eta \in \Gamma(\Lambda^q)$.

Proof. Follows from the definition of the Weyl product $\circ$ and the fact that $\nabla$ preserves $\omega$.

Let us work in Darboux local coordinates, with $\Gamma^k_{ij}$ the Christoffel symbols. Recall that for a symplectic connection the numbers $\Gamma^k_{ij} = \omega^k_{\ell j} \Gamma^\ell_{ik}$ are completely symmetric in $ijk$. Although it is cumbersome to write out, it is straightforward to find that

$$\nabla a = da + \frac{i}{\hbar} \left[ \frac{1}{2} \Gamma_{ijk} y^i y^j dx^k, a \right],$$

and we write $\Gamma = \frac{1}{2} \Gamma_{ijk} y^i y^j dx^k$ for the local 1-form with values in $W$.

Now, we want to consider more general (symplectic) connections. Consider connections of the form

$$Da = \nabla a + \frac{i}{\hbar} \left[ \gamma, a \right] = da + \frac{i}{\hbar} \left[ \Gamma + \gamma, a \right],$$

where $\gamma \in \Gamma(W \otimes \Lambda^1)$, a global 1-form. Note that $\gamma$ is determined by $D$ only up to a central one-form, since it appears in a commutator. To enforce uniqueness we impose the Weyl normalization condition, requiring $\gamma_0 = \gamma|_{y=0} = 0$ (like a gauge condition).

Lemma 11. Let $\nabla$ be a symplectic connection on $M$. Then

$$\nabla \delta a + \delta \nabla a = 0$$

and

$$\nabla^2 a = \frac{i}{\hbar} [R, a]$$

where $R = \frac{1}{4} R_{ijkm} y^i y^j dx^k \wedge dx^l$, with $R_{ijkm}$ is the curvature tensor of $\nabla$.

Proof. Follows from the expression of $\nabla$ and $\delta$ as above. Note that the latter equation is a compact form of the Ricci identity.

Definition 12. Let $D$ be a connection on $W$ of the form $D = \nabla + \frac{i}{\hbar} [\gamma, \cdot]$ with $\gamma_0 = 0$. Then the curvature of $D$ is defined as

$$\Omega = R + \nabla \gamma + \frac{i}{\hbar} \gamma^2.$$

Lemma 13. We have

(i) (Bianchi identity) $D \Omega = \nabla \Omega + \frac{i}{\hbar} [\gamma, \Omega] = 0$.

(ii) (Ricci identity) $D^2 a = \frac{i}{\hbar} [\Omega, a]$.

Proof. By definition of $D$ and $\Omega$, we have

$$D \Omega = \nabla R + \nabla^2 \gamma + \frac{i}{\hbar} [\nabla \gamma, \gamma] + \frac{i}{\hbar} [\gamma, R] + \frac{i}{\hbar} [\gamma, \nabla \gamma] + \left( \frac{i}{\hbar} \right)^2 [\gamma, \gamma^2].$$

By the Bianchi identity for $\nabla$, we have $\nabla R = 0$. Furthermore, obviously $[\gamma, \gamma^2] = 0$, and $\nabla^2 \gamma = \frac{i}{\hbar} [R, \gamma]$ as seen earlier. Therefore, $D \Omega = 0$. Part (ii) is straightforward.
Definition 14. A connection $D$ of $W$ is abelian if

$$D^2 a = \frac{i}{\hbar} [\Omega, a] = 0$$

for all $a \in \Gamma(W \otimes \Lambda)$, that is, if the curvature of the connection is a central form.

We will show there exists an abelian connection of the form

$$D = \nabla - \delta - \frac{i}{\hbar} [r, \cdot] = \nabla + i [r, \cdot] + \frac{i}{\hbar} [\omega_{ij} d x^i + r, \cdot],$$

where $\nabla$ is a fixed symplectic connection, and $r \in \Gamma(W_3 \otimes \Lambda^1)$ a globally defined one-form, with Weyl normalization $r_0 = 0$. Computing the curvature of $D$ gives

$$\Omega = -\frac{1}{2} \omega_{ij} d x^i \wedge d x^j + R - \delta r + \nabla r + \frac{i}{\hbar} r^2.$$

It suffices to find an $r$ satisfying

$$\delta r = R + \nabla r + \frac{i}{\hbar} r^2,$$

so that $\Omega = -\omega$ is indeed central.

Theorem 15. The above equation has a unique solution $r$ such that $\deg r \geq 2$ and $\delta^{-1} r = 0$.

Proof. From (1) follows that any such $r$ has $r = \delta^{-1} \delta r$, as $r_0 = 0$ and $\delta \delta^{-1} r = 0$. Applying $\delta^{-1}$ yields

$$r = \delta^{-1} R + \delta^{-1} \left( \nabla r + \frac{i}{\hbar} r^2 \right).$$

Since $\nabla$ preserves the filtration on $W \otimes \Lambda$, and $\delta^{-1}$ raises the degree by 1, one obtains a unique solution by the iteration method. Conversely, one can show that this solution yields an abelian connection (again using the iteration method).

Remark 16. Explicitly, the iterating method yields

$$r = \frac{1}{8} R_{ijkl} y^i y^j y^k d x^l + \frac{1}{20} \nabla_m R_{ijkl} y^i y^j y^k y^m d x^l + \cdots$$

Definition 17. Let $D$ be an abelian connection on $W$. We define $W_D \subset W$ to be the subbundle of flat sections with respect to $D$, that is, $Da = 0$. Note that $\Gamma(W_D)$ is a subalgebra of $\Gamma(W)$ with respect to the Weyl product because of Lemma 10.

Theorem 18. For any $a_0 \in Z$, there exists a unique section $a \in \Gamma(W_D)$ such that $\sigma(a) = a_0$.

Proof. Rewrite the equation $Da = 0$ as

$$\delta a = (D + \delta) a.$$
and note that $D + \delta = \nabla + \frac{i}{\hbar} [r, \cdot]$ does not lower degree since $\deg r \geq 2$. Applying $\delta^{-1}$, we find using the (1) that

$$a = a_0 + \delta^{-1} \delta a = a_0 + \delta^{-1} (D + \delta) a,$$

where we used $\delta \delta^{-1} a = 0$ as $a \in \Gamma(W)$. Since $\delta^{-1}$ raises degree, we can solve this equation (uniquely) via iterations. Conversely, if $a$ is a solution of (1), then $\sigma(a) = a_0$ since $\sigma \circ \delta^{-1} = 0$. Now,

$$\delta^{-1} Da = \delta^{-1} (D + \delta) a - \delta^{-1} \delta a = a - a_0 - \delta^{-1} \delta a = \delta^{-1} a = 0.$$

Since $D$ is abelian, we have $D(Da) = 0$, or $\delta Da = (D + \delta) Da$, and applying $\delta^{-1}$ gives

$$Da = \delta^{-1} (D + \delta) Da.$$

Solve by iterations to get $Da = 0$.

Remark 19. By iterations, we can construct the section $a \in \Gamma(W_D)$ from its symbol $a_0 = \sigma(a)$,

$$a = a_0 + \partial_i a_0 y^i + \frac{1}{2} \partial_i \partial_j a_0 y^i y^j + \frac{1}{6} \partial_i \partial_j \partial_k a_0 y^i y^j y^k - \frac{1}{24} R_{ijkl} \omega^{lm} \partial_m a_0 y^l y^k + \cdots$$

If the curvature tensor $R$ is zero, we have

$$a = \sum_{k=0}^{\infty} \frac{1}{k!} \partial_{i_1} \cdots \partial_{i_k} a_0 y^{i_1} \cdots y^{i_k}.$$

Definition 20. The bijection between $\Gamma(W_D)$ and $Z = C^\infty(M)[[\hbar]]$ allows to define a star product on $Z$, given by

$$a \star b = \sigma(Q(a) \circ Q(b)),$$

where $Q : Z \to W_D$, called the quantization procedure, is the inverse to $\sigma$. One can check that this star product satisfies the properties of Definition 1. The subalgebra $\Gamma(W_D)$ is called the quantum algebra.

Example 21. Let $M = \mathbb{R}^{2n}$ with $\omega = \frac{1}{2} \omega_{ij} dx^i \wedge dx^j$ a constant symplectic form on $M$. The connection

$$D^0 a = da + \frac{i}{\hbar} \omega_{ij} y^i dx^j, a$$

or $D^0 = d - \delta$, is abelian with curvature

$$\Omega = -\omega.$$

Now the corresponding quantum algebra is given by

$$\Gamma(W_{D^0}) = \left\{ a \in \Gamma(W) : \frac{\partial a}{\partial x^i} - \frac{\partial a}{\partial y^i} = 0 \right\}.$$

That is, any $a \in \Gamma(W_{D^0})$ is of the form

$$a = \sum_{|\alpha| \geq 0} \frac{1}{|\alpha|!} \partial_\alpha b y^\alpha,$$

for some $b \in Z = C^\infty(\mathbb{R}^{2n})[[\hbar]]$. Note that the star product now corresponds to the Weyl product.

Remark 22. Later it will be shown that any $W_D$ is locally isomorphic to $W_{D^0}(\mathbb{R}^{2n})$. 5
Theorem 23. The cohomology groups of
\[ \cdots \to \Gamma(W \otimes \Lambda^p) \xrightarrow{D} \Gamma(W \otimes \Lambda^{p+1}) \to \cdots \]
are trivial for \( p > 0 \).

Proof. We can extend the quantization procedure to an isomorphism \( Q : \Gamma(W \otimes \Lambda^p) \xrightarrow{\sim} \Gamma(W \otimes \Lambda^p) \) via
\[ Qa = a + \delta^{-1}(D + \delta)Qa. \]
Indeed, by the iterating method there is a unique solution, and the inverse is given by
\[ Q^{-1}a = a - \delta^{-1}(D + \delta)a. \]

One can show that
\[ Q^{-1}D + \delta Q^{-1} = 0, \]
by substituting for \( Q^{-1} \), and using (1). Then it follows that \( D = -Q\delta Q^{-1} \), so we can replace the complex with \(-\delta\), and then the result follows from the Hodge–De Rham decomposition. Namely, for any \( a \in \Gamma(W \otimes \Lambda^p) \), write \( a = a_{00} + \delta\delta^{-1}a + \delta^{-1}\delta a \). If \( \delta a = 0 \), that is,
\[ \delta a = \delta a_{00} + \delta\delta^{-1}\delta a = \delta a_{00} + (\delta a - \delta^{-1}\delta^2 a) = \delta a_{00} + \delta a = 0, \]
then\( a = a_{00} + \delta\delta^{-1}a + \delta^{-1}\delta a = a_{00} - \delta^{-1}\delta a_{00} + \delta\delta^{-1} a = \delta^{-1} a_{00} + \delta^{-1} a \) lies in the image of \( \delta \), so the sequence is exact for \( p > 0 \). Note that we used that \( p > 0 \) in the line where \( \delta\delta^{-1} a + \delta^{-1} \delta a = a \). \( \square \)

Corollary 24. Any equation \( Da = b \) with \( b \in \Gamma(W \otimes \Lambda^p) \) and \( p > 0 \) has a solution if and only if \( Db = 0 \). The solution may be taken in the form
\[ a = D^{-1}b = -Q\delta^{-1}Q^{-1}b. \]

Generalizations

Note that in the above, the symplectic form \( \omega \) pops up in two places: in the Weyl multiplication rule, and as the curvature of the abelian connection \( D \). In this section we will make them distinct. This is convenient when when we have to vary symplectic structures: we may fix the Weyl multiplication and vary the curvature.

Let \( L \) be a symplectic vector bundle over \( M \) of dimension \( 2n \) with a fixed symplectic structure \( \omega \) and symplectic connection \( \nabla^L \). We assume that \( L \) is isomorphic to \( TM \), but not canonically. Denote by
\[ \theta : TM \to L \]
a bundle isomorphism, and by
\[ \delta : L^* \to T^* M \]
a dual isomorphism. Introducing a local symplectic frame \( (e_1, \ldots, e_{2n}) \) for \( L \) yields a dual frame \( (e^1, \ldots, e^{2n}) \) for \( L^* \), and a frame \( \theta^1 = \delta(e^1), \ldots, \theta^{2n} = \delta(e^{2n}) \) for \( T^* M \), with corresponding vector
fields $X_1, \ldots, X_{2n}$ giving a dual frame for $TM$. The form $\omega$ on $L$ can be transported to $TM$ giving a non-degenerate 2-form on $M$

$$\Omega_0 = -\frac{1}{2} \omega_{ij} \theta^i \wedge \theta^j,$$

but note that it need not be closed. We will use $\theta$ to vary the symplectic structure on $TM$.

**Lemma 25.** Let $\Omega(t)$ be a family of non-degenerate 2-forms on $M$ with $\Omega(0) = \Omega_0 = -\frac{1}{2} \omega_{ij} \theta^i \wedge \theta^j$. Then there exists a family $\theta(t)$ of isomorphisms such that $\Omega(t) = -\frac{1}{2} \omega_{ij} \theta(t)^i \wedge \theta(t)^j$.

**Proof.** Omitted. See [1, Lemma 5.3.1]. □

Analogous to the previous section, we make some definitions.

**Definition 26.** Let $\mathcal{E}$ be a complex vector bundle over $M$ with connection $\nabla^\mathcal{E}$, and let $\mathcal{A} = \text{Hom}(\mathcal{E}, \mathcal{E})$ (the coefficient bundle).

- The formal Weyl bundle with coefficients in $\mathcal{A}$ is the bundle
  $$W(L, \mathcal{A}) = \hat{\text{Sym}}(L^* \otimes \mathfrak{h}) \otimes \mathcal{A}.$$ 

- Using the same rule as in Definition 3, we can define Weyl multiplication on $W(L, \mathcal{A})$, but now the coefficients are taken in $\mathcal{A}$, which means the multiplication may be non-commutative.

- The connections $\nabla^L$ and $\nabla^\mathcal{E}$ induce a connection $\nabla$ on $W(L, \mathcal{A})$.

- In a local symplectic frame of $L$, we can write
  $$\nabla a = da + \frac{i}{\hbar} \left[ \frac{1}{2} \Gamma_{ij} y^i y^j, a \right] + [\Gamma^\mathcal{E}, a]$$
  and
  $$\nabla^2 a = \frac{i}{\hbar} \left[ \frac{1}{2} R_{ij} y^i y^j, a \right] + [R^\mathcal{E}, a],$$
  so we define the curvature of $\nabla$ to be
  $$R = \frac{1}{2} R_{ij} y^i y^j - i\hbar R^\mathcal{E} \in \Gamma(W(L, \mathcal{A}) \otimes \Lambda^2).$$

- Consider more general connections on $W(L, \mathcal{A})$ of the form
  $$D = \nabla + \frac{i}{\hbar} [\gamma, a],$$
  for some globally defined $\gamma \in \Gamma(W(L, \mathcal{A}) \otimes \Lambda^1)$. (Note that there are no unique $\nabla$ and $\gamma$ representing $D$, although we can always choose an arbitrary symplectic connection $\nabla$, and then $\gamma$ is well-defined up to some scalar 1-form $\Delta \gamma \in \Gamma(\Lambda^1[[\hbar]]).$) The curvature of $D$ (with respect to $\nabla$ and $\gamma$) is defined by
  $$\Omega = \nabla \gamma + \frac{i}{\hbar} \gamma^2 + R \in \Gamma(W(L, \mathcal{A}) \otimes \Lambda^2).$$

- Define operators
  $$\delta : \Gamma(W(L, \mathcal{A})_p \otimes \Lambda^q) \to \Gamma(W(L, \mathcal{A})_{p-1} \otimes \Lambda^{q+1}), \quad a \mapsto \theta^k \wedge \frac{\partial a}{\partial y^k},$$
  $$\delta^* : \Gamma(W(L, \mathcal{A})_p \otimes \Lambda^q) \to \Gamma(W(L, \mathcal{A})_{p+1} \otimes \Lambda^{q-1}), \quad a \mapsto y^k \iota_{X_k} a.$$ 

In particular, note that $\delta$ agrees with $L^* \to T^*M$ on linear forms.
The construction of $\delta^{-1}$, the Bianchi identity and Ricci identity (Lemma 13), the Hodge–De Rham decomposition (1), all remain valid.

**Theorem 27.** Let $\Omega = \Omega_0 + \hbar \Omega_1 + h^2 \Omega_2 + \cdots$ be a closed 2-form, and $\theta : TM \to L$ a bundle isomorphism such that $\Omega_0 = -\frac{1}{2} \omega_{ij} \theta^i \wedge \theta^j$. Then for any section $\mu \in \Gamma(W(L, A))$ with $\deg(\mu) \geq 3$ and $\mu|_{y=0} = 0$ there exists a unique section $r \in \Gamma(W(L, A) \otimes \Lambda^1)$ with $\deg(r) \geq 2$ such that $\delta^{-1} r = \mu$, and the corresponding connection $D = \nabla - \delta + \frac{i}{\hbar} [r, \cdot]$ is abelian with curvature $\Omega$.

**Proof.** Omitted. See [1, Theorem 5.3.3].

**Remark 28.** The construction of $D$ as in the theorem depends smoothly on the parameters. That is, if $\Omega(t)$ is a family of closed 2-forms with non-degenerate leading term $\Omega_0(t)$, and a family $\mu(t)$ with $\deg(\mu(t)) \geq 3$ and $\mu(t)|_{y=0} = 0$, there exists a family $r(t)$ satisfying the requirements.

Having constructed the abelian connection $D$, we define a quantum algebra with twisted coefficients $W_D(L, A)$ in the same way as before. Theorems 18 and 23 and Corollary 24 remain valid for the bundle $W(L, A)$. In particular, may define a quantization procedure

$$
\Gamma(\mathcal{A}) \ni \frac{\partial}{\partial a} : \Gamma(W_D(L, A)).
$$

**THE HEISENBERG EQUATION**

Consider the *Heisenberg equation* in $W_D = W_D(L, A)$,

$$
\frac{da}{dt} + \frac{i}{\hbar} [H(t), a] = 0,
$$

with $H(t) \in \Gamma(W_D)$ a given flat section, and $a(t) \in \Gamma(W_D)$ an unknown flat section. If $H(t)$ and $a(t)$ are obtained via quantization, coming from symbols $H_0(t)$ and $a_0(t)$, then the leading term of the equation reads

$$
\frac{d}{dt} a_0(t) + \{H_0, a_0\} = 0,
$$

which corresponds to the Louiville equation in classical mechanics. That is, the Heisenberg equation can be seen as the quantum analogue of the Liouville equation.

Consider a family of abelian connections on $W(L, A)$,

$$
D_t = \nabla + \frac{i}{\hbar} [\gamma_t, \cdot] = \nabla - \delta_t + \frac{i}{\hbar} [r(t), \cdot],
$$

where $\gamma_t = \omega_{ij} y^i \theta(t)^j + r(t)$ with $\deg(r(t)) \geq 2$, and $\theta(t) : TM \to L$ is a family of bundle isomorphisms. Furthermore, let $H(t)$ be a section of $W(L, A)$, called the Hamiltonian, satisfying

1. $\lambda := D_t H(t) - \dot{\gamma}(t)$ lies in $\Lambda^1[[h]]$,
2. there exists a vector field $X_t$ such that $\deg(\iota_{X_t} \Gamma(t) + H(t)) \geq 2$.

Now consider the equation

$$
\frac{da}{dt} + (\iota_{X_t} D_t + D_t \iota_{X_t}) a + \frac{i}{\hbar} [H(t), a] = 0.
$$
Remark 29. When $D$ is time-independent and $a \in \Gamma(W_D)$, the above equation reduces to (2). Namely, in this case $\iota_{X_t}a = 0$ and $Da = 0$, so $(\iota_{X_t}D_t + D_t\iota_{X_t})a = 0$. Furthermore, $\lambda$ is closed since $d\lambda = D\lambda = D^2H = 0$, as $D$ is abelian, so locally we can write $\lambda = -dH_0(t)$ for some scalar function $H_0(t)$. Since $H_0(t)$ is central, we can replace $H(t)$ with $H(t) + H_0(t)$, which is flat as $D(H(t) + H_0(t)) = 0$ by the first property of the Hamiltonian.

Definition 30. Let $W^+ \supset W$ be the bundle whose sections are of the form

$$a = \sum_{2k+|\alpha| \geq 0} \hbar_k a_{k,\alpha} y^\alpha,$$

where $k$ is allowed to be negative, as long as the total degree $2k + |\alpha|$ is non-negative.

Remark 31. Note that the fibers $W_x^+$ are still algebras with respect to the Weyl multiplication, and the connections $\nabla$ and $D$ are well-defined on $W^+$.

Lemma 32. Let $a \in \Gamma(W^+)$ with $Da = 0$, then $a \in \Gamma(W_D)$. That is, $a$ does not contain negative powers of $\hbar$.

Proof. Note that $\sigma(a)$ must only have non-negative powers of $\hbar$, and thus $\sigma(a) \in Z$. By Theorem 18, a flat section is determined by $\sigma(a)$, so it follows that $a \in \Gamma(W_D)$. \qed

Assume that the vector field $X_t$ defines a flow $f_t : M \to M$ for $t \in [0,1]$. (Generally this is only true for small $t$ and $x \in M$ ranging over a compact set.)

Theorem 33. For any initial $a(0) \in \Gamma(W \otimes \Lambda)$, equation (3) has a unique solution $a(t) \in \Gamma(W \otimes \Lambda)$. Moreover, if $a(0) \in \Gamma(W_{D_0})$, then $a(t) \in \Gamma(W_{D_t})$.

Proof. Substituting $D_t = \nabla + \frac{i}{\hbar} [\gamma_t, \cdot]$, we can rewrite (3) as

$$\frac{da}{dt} + (\iota_{X_t} \nabla + \nabla \iota_{X_t})a + \frac{i}{\hbar} [H(t) + \iota_{X_t} \gamma_t, a] = 0.$$

By the second property of the Hamiltonian, we know $\deg (H(t) + \iota_{X_t} \gamma_t) \geq 2$, so we write

$$H(t) + \iota_{X_t} \gamma_t = H_2(t) + H_3(t) = \frac{1}{2} H_{ij}(t) y^i y^j + \hbar \mathcal{H}(t) + H_3(t),$$

where $\mathcal{H}(t)$ is a section of $\mathcal{A}$ and $\deg H_3(t) \geq 3$. We define a pullback $f_t^* : W \otimes \Lambda \to W \otimes \Lambda$ as follows. On differential forms it is the usual pullback, and on sections $a \in \Gamma(W)$ we set

$$(f_t^* a)(x, y) = v_t^{-1} a(f_t(x), \sigma_t(y)) v_t,$$

where $\sigma_t : L_x \to L_{f_t(x)}$ is a linear symplectic lifting of $f_t$, and $v_t : \mathcal{E} \to \mathcal{E}_{f_t(x)}$ an isomorphism of bundles, lifting $f_t$. Now we will see how these lifts are obtained.

Lemma 34. There exist such liftings $\sigma_t$ and $v_t$ such that for any $a \in \Gamma(W \otimes \Lambda)$,

$$\frac{d}{dt} (f_t^* a) = f_t^* \left( (\iota_{X_t} \nabla + \nabla \iota_{X_t})a + \frac{i}{\hbar} [H_2(t), a] \right).$$
Proof. For scalar differential forms, the above equation follows from Cartan’s formula, so it suffices to prove the equation for \( a \in \Gamma(W) \). See [1, Lemma 5.4.4].

Now, for any solution \( a(t) \) of (3), we have that \( b(t) = f_t^* a(t) \) satisfies

\[
\frac{d}{dt} b(t) + \frac{i}{\hbar} [f_t^* H_3, b] = f_t^* \left( \frac{da}{dt} + (\i X, \nabla + \nabla \i X) a + \frac{i}{\hbar} [H_2(t), a] \right) + f_t^* \left( \frac{i}{\hbar} [H_3(t), a] \right)
\]

\[= 0,
\]

and conversely, any such \( b(t) \) gives \( a(t) = (f_t^*)^{-1} b(t) \) a solution of (3). Hence, it suffices to solve for \( b(t) \),

\[ b(t) = b(0) - \frac{i}{\hbar} \int_0^t [f_\tau^* H_3(\tau), b(\tau)] d\tau, \]

which can be done via iterations. Indeed these iterations converge as \( \text{deg}(f_t^* H_3(t)) \geq 3 \).

Remark 35. The solution \( b(t) \) can be expressed in a shortened form as

\[ b(t) = U^{-1}(t) \circ b(0) \circ U(t), \]

where

\[ U(t) = \text{Pexp} \left( \frac{i}{\hbar} \int_0^t f_\tau^* H_3(t) dt \right) \]

is defined by a path-ordered exponential, that is,

\[ U(t) = 1 + \frac{i}{\hbar} \int_0^t (f_\tau^* H_3(\tau) \circ U(\tau)) d\tau. \]

Indeed, such a solution for \( U(t) \) exists as \( \text{deg} \left( \frac{i}{\hbar} f_\tau^* H_3(\tau) \right) \geq 1 \).

It remains to prove the last assertion of the theorem, for which we need the following lemma.

Lemma 36. For any solution \( a(t) \), also \( D_t a(t) \) is a solution.

Proof. We have

\[
\frac{d}{dt} (D_t a) = \frac{d}{dt} \left( \nabla a + \frac{i}{\hbar} [\gamma_t, a] \right) = \nabla \dot{a} + \frac{i}{\hbar} [\gamma_t, \dot{a}] + \frac{i}{\hbar} [\gamma_t, a] = D_t \frac{da}{dt} + \frac{i}{\hbar} [\gamma_t, a].
\]

Since \( a(t) \) is a solution, we can substitute for \( \frac{da}{dt} \) to obtain

\[
\frac{d}{dt} (D_t a) = -D_t \i X, D_t a - \frac{i}{\hbar} [D_t H(t), a] - \frac{i}{\hbar} [H(t), D_t a] + \frac{i}{\hbar} [\gamma_t, a]
\]

\[= - (\i X, D_t + D_t \i X) D_t a - \frac{i}{\hbar} [H(t), D_t a],
\]

using that \( D_t^2 = 0 \) and the fact that \( D_t H - \gamma_t \) is central by the first property of the Hamiltonian.

Finally, if \( a(t) \) is a solution, then so is \( D_t a(t) \) by the above lemma, so whenever \( D_0 a(0) = 0 \) it follows from the uniqueness of the solution that \( D_t a(t) = 0 \) for all \( t \).
Corollary 37. Let $D$ be time-independent, and let $H(t) \in \Gamma(W_D)$ be a flat section with scalar leading term $H_0(t)$. Then for any $a(0) \in \Gamma(W_D)$ there exists a unique solution $a(t) \in \Gamma(W_D)$, and the map $A(t) : a(0) \mapsto a(t)$ is an automorphism of $W_D$.

Proof. The difference $H(t) - H_0(t)$ satisfies the properties of the Hamiltonian, and so the result follows from the above theorem.

References

[1] B. Fedosov, Deformation Quantization and Index Theory