Representation varieties & TQFTs

J.T. Vogel

6 November, 2020
$X$ = connected closed manifold

$\pi_1(X)$ = fundamental group

$G$ = algebraic group over $k$

$\mathcal{X}_G(X) = \text{Hom}(\pi_1(X), G)$  \hspace{1cm} \text{$G$-representation variety of $X$}
When $C$ complex projective curve

\[ \mathcal{M}_B(C, G) \]

Riemann–Hilbert \quad Hitchin–Kobayashi

\[ \mathcal{M}_{DR}(C, G) \quad \mathcal{M}_{Dol}(C, G) \]
E-polynomial

\[ e(X) = \sum_{k,p,q} (-1)^k \ h_c^{k;p,q}(X) \ u^p v^q \in \mathbb{Z}[u, v] \]
Goal: find class of $\mathcal{X}_G(\Sigma_g)$ in $K(\text{Var}_k)$
Goal: find class of $\mathcal{X}_G(\Sigma_g)$ in $K(\text{Var}_k)$

- Counting points over finite fields
Goal: find class of $\mathcal{X}_G(\Sigma_g)$ in $K(\text{Var}_k)$

- Counting points over finite fields

- Geometric method $\Rightarrow$ Topological Quantum Field Theories (TQFTs)
Idea: cut manifold in pieces and ‘compute invariant piecewise’

\[ \sum_{\sigma} = \circ \circ \cdots \circ \]

\( g \) times
More precisely, want a functor $Z : \text{Bdp}_n \to K(\text{Var}_k)\text{-Mod}$
More precisely, want a functor $Z : \text{Bdp}_n \to K(\text{Var}_k)-\text{Mod}$

Bordism category $\text{Bdp}_n$

- Objects $(M, A)$
- Morphisms $(W, A)$
More precisely, want a functor $Z : \text{Bdp}_n \to K(\text{Var}_k)\text{-Mod}$

Bordism category $\text{Bdp}_n$

- Objects $(M, A)$
- Morphisms $(W, A)$

Examples in dimension $n = 2$:

$$D^\dagger : (S^1, \star) \to \emptyset$$

$$L : (S^1, \star) \to (S^1, \star)$$

$$D : \emptyset \to (S^1, \star)$$
\[ \text{Bdp}_n \xrightarrow{\mathcal{F}} \text{Span}(\text{Var}_k) \xrightarrow{\mathcal{Q}} \text{K}(\text{Var}_k)\text{-Mod} \]
\[ \text{Bdp}_n \xrightarrow{\mathcal{F}} \text{Span}(\text{Var}_k) \xrightarrow{Q} \text{K(Var}_k\text{)-Mod} \]

- Define \( \mathcal{F}(M, A) = \mathcal{X}_G(M, A) \)

- Given \((W, A) : (M_1, A_1) \rightarrow (M_2, A_2)\), define

\[
\mathcal{F}(W, A) : \mathcal{X}_G(M_1, A_1) \leftarrow \mathcal{X}_G(W, A) \rightarrow \mathcal{X}_G(M_2, A_2)
\]

- Define \( Q(X) = \text{K(Var}/X) \)

- Define \( Q(X \leftarrow Z \xrightarrow{g} Y) = g! \circ f^* \)
Useful: $\mathbb{Z}$ produces invariants
\[ L : (S^1, \star) \rightarrow (S^1, \star) \]

\[ \mathcal{F}(L) : \quad G \leftrightarrow G^4 \quad \rightarrow \quad G \quad B \leftrightarrow (B, A_1, A_2, C) \quad \rightarrow \quad CB[A_1, A_2]C^{-1} \]
\[ N : (S^1, \star) \rightarrow (S^1, \star) \]

\[ \mathcal{F}(N) : \quad G \leftrightarrow G^3 \rightarrow G \]
\[ B \leftrightarrow (B, A, C) \rightarrow CBA^2C^{-1} \]
So $Z(L), Z(N)$ are maps $K(\text{Var}/G) \to K(\text{Var}/G)$

$$\left[\mathcal{X}_G(\Sigma_g)\right] = \frac{1}{[G]^g} \ Z(D^\dagger) \circ Z(L)^g \circ Z(D)(1)$$
Computed

\[ Z(L) \text{ for } G = U_2, U_3, U_4 \quad \text{and} \quad Z(N) \text{ for } G = U_2, U_3 \]
For $G = U_2$, we have $Z(L) = \begin{bmatrix} q^2(q - 1) & q^2(q - 2) \\ q^2(q - 2)(q - 1) & q^2(q^2 - 3q + 3) \end{bmatrix}$
For $G = \mathbb{U}_2$, we have $Z(L) = \begin{bmatrix} q^2(q - 1) & q^2(q - 2) \\ q^2(q - 2)(q - 1) & q^2(q^2 - 3q + 3) \end{bmatrix}$

For $G = \mathbb{U}_3$, we have $Z(L) = \begin{bmatrix} (q - 1)^2(q^2 + q - 1) & q^2(q - 2)^2 & q^2(q - 2)(q - 1) & q^2(q - 2)(q - 1) & (q - 1)^3(q + 1) \\ q^3(q - 2)^2(q - 1)^2 & q^3(q^2 - 3q + 3)^2 & q^3(q - 2)(q - 1)(q^2 - 3q + 3) & q^3(q - 2)(q - 1)(q^2 - 3q + 3) & q^3(q - 2)^2(q - 1)^2 \\ q^3(q - 2)(q - 1)^2 & q^3(q - 2)(q^2 - 3q + 3) & q^3(q - 1)(q^2 - 3q + 3) & q^3(q - 2)^2(q - 1) & q^3(q - 2)(q - 1)^2 \\ q^3(q - 2)(q - 1)^2 & q^3(q - 2)(q^2 - 3q + 3) & q^3(q - 2)^2(q - 1) & q^3(q - 1)(q^2 - 3q + 3) & q^3(q - 2)(q - 1)^2 \\ (q - 1)^4(q + 1) & q^2(q - 2)^2(q - 1) & q^2(q - 2)(q - 1)^2 & q^2(q - 2)(q - 1)^2 & (q - 1)^2(q^3 - q^2 + 1) \end{bmatrix}$
For $G = \mathbb{U}_2$, we have $Z(L) = \begin{bmatrix} q^2(q - 1) & q^2(q - 2) \\ q^2(q - 2)(q - 1) & q^2(q^2 - 3q + 3) \end{bmatrix}$

For $G = \mathbb{U}_3$, we have $Z(L) = \begin{bmatrix} (q - 1)^2(q^2 + q - 1) & q^2(q - 2)^2 \\ q^3(q - 2)^2(q - 1)^2 & q^3(q^2 - 3q + 3)^2 \\ q^3(q - 2)(q - 1)^2 & q^3(q - 2)(q^2 - 3q + 3) \\ (q - 1)^4(q + 1) & q^2(q - 2)^2(q - 1) \end{bmatrix}$

For $G = \mathbb{U}_4$, we have $Z(L) = \begin{bmatrix} q^2(q - 1) & q^2(q - 2) & q^2(q - 2)(q - 1) & (q - 1)^3(q + 1) \\ q^2(q - 2)(q - 1) & q^2(q^2 - 3q + 3) & q^3(q - 2)(q - 1)(q^2 - 3q + 3) & q^3(q - 2)^2(q - 1)^2 \\ q^3(q - 2)(q - 1)^2 & q^3(q - 1)(q^2 - 3q + 3) & q^3(q - 2)(q - 1)^2 & q^3(q - 2)(q - 1)^2 \\ (q - 1)^4(q + 1) & q^2(q - 2)^2(q - 1) & q^2(q - 2)(q - 1)^2 & (q - 1)^2(q^3 - q^2 + 1) \end{bmatrix}$
Results

\[ \mathcal{X}_{U_2}(\Sigma_g) = q^{2g-1}(q - 1)^{2g+1}((q - 1)^{2g-1} + 1) \]

\[ \mathcal{X}_{U_3}(\Sigma_g) = q^{3g-3}(q - 1)^{2g}(q^2(q - 1)^{2g+1} + q^{3g}(q - 1)^{2g+2} + q^{3g}(q - 1)^{4g} + 2q^{3g}(q - 1)^{2g+1}) \]

\[ \mathcal{X}_{U_4}(\Sigma_g) = \ldots \]

\[ \mathcal{X}_G(N_r) = \ldots \]
Stacky TQFT
Stacky TQFT

Replace

\[ K(\text{Var}_k) \xrightarrow{\cong} K(\text{Stck}_{BG}) \]
\[ \mathcal{X}_G(X) \xrightarrow{\cong} [\mathcal{X}_G(X)/G] \]

to obtain

\[ Z : \text{Bdp}_n \to K(\text{Stck}_{BG})-\text{Mod} \]
For $G = \text{AGL}_1(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\}$

$$Z(L) = \begin{bmatrix} \left[ G_a/G \right] q(q-2) + \left[ G_m/G \right] (q+1) + 1 \\ \left[ G_a/G \right] q(q-2) \end{bmatrix} \begin{bmatrix} \left[ G_a/G \right] [G_m/G] q(q-2) \\ \left[ G_a/G \right] [G_m/G] q(q-2) + q^2 \end{bmatrix}$$
E.g. for $g = 3$, 

$$[\mathcal{X}_G(\Sigma_3)/G] = 1$$

$$+ [\mathbb{G}_a/G] q (q - 2) (q^2 - 3q + 3) (q^2 - q + 1)$$

$$+ [\mathbb{G}_m/G] (q + 1) (q^2 - q + 1) (q^2 + q + 1)$$

$$+ [G/G'] q (q - 2) (q + 1) (q^2 + 1) (q^2 - 3q + 3) (q^2 - q + 1)$$