Generalized geometry and DFT applications to closed string theory

Bachelor Thesis Mathematics and Physics

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Abstract

In this thesis, we introduce and explore the subjects of generalized geometry, string theory and double field theory. Within generalized geometry, a framework is set up leading towards the definition of generalized complex structures. We describe both symplectic and complex structures as particular cases of such structures. From scratch, closed string theory on toroidal backgrounds is developed, and the non-trivial symmetry of T-duality is shown to emerge from this theory. We use double field theory to reformulate the closed string theory in order to turn T-duality into a manifest symmetry. A number of other applications of double field theory are explored, where we find that it is able to unify different concepts. Also we discuss the similarities between double field theory and generalized geometry.

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Chapter 1

Introduction

The motivation for the subjects of this thesis begins with *string theory*, a physical theory of elementary particles. In contrast to usual theories, e.g. the Standard Model, that treat elementary particles as point-like objects, string theory instead describes them as being strings: one-dimensional objects. There are a number of reasons why string theory can be considered interesting. It can be argued that it is mathematically very elegant: all particles can be described by the same type of string, and particle properties, such as mass, charge and spin, are all determined by the vibrational state of the string. Also, string theory has only one adjustable parameter, which makes it quite a unique theory. This parameter, the *string length* ℓ_s, can be imagined as the typical length of a string. Furthermore, string theory is a quantum theory including gravity, so potentially being a unified theory of physics. However, there is also some controversy regarding string theory, mainly coming from the fact that there has still not been any experimental verification of string theory. Nevertheless, string theory is a very interesting theory, and a better understanding of it might yield predictions that can be tested in experiment.

Surprisingly, a Lorentz invariant quantum string theory requires that spacetime is 26-dimensionsal. In order to make this compatible with the 4-dimensional spacetime we observe, string theory allows for backgrounds, i.e. shapes of spacetime, different from the usual Euclidean space $\mathbb{R}^D$, where D denotes the amount of spacetime dimensions. In particular, in this thesis we will consider *toroidal* backgrounds, in which a number of dimensions are 'curled up'. This is done by the identification of points $x \sim x + L$, where L denotes the length of the dimension. Spacetime may then e.g. look like $\mathbb{R}^4 \times \mathbb{T}^d$, where $\mathbb{R}^4$ is the usual Minkowski spacetime and $\mathbb{T}^d$ a $d = D - 4$ dimensional torus. It turns out that a string in the presence of such compact dimensions can be equivalently described as a string living on compact dimensions of some different length. This is a non-trivial duality of string theory that appears on toroidal backgrounds and is referred to as *T-duality*, where the T stands for toroidal.

Double field theory (DFT) is a framework that was proposed [1] in order to reformulate the string theory that we consider such that T-duality becomes a manifest symmetry of the theory. Besides this original motivation, it turns out that in DFT a number of different concepts can be seen as the same. The unification of different concepts is a recurring theme in DFT, which is interesting to study, as it may point to some deeper connection between such concepts. However, DFT is still relatively new and being developed. In this thesis, we will discuss a number of results and proposals from various papers. This will be done in Chapter 4.

DFT shows great similarities with *generalized geometry*, a mathematical subject in differential geometry. Generalized geometry was first introduced by Hitchin [7] and developed further by his students, among which Gualtieri [6]. In this geometry, the tangent bundle T of a manifold is replaced by $T \oplus T^*$, a sum of the tangent and cotangent bundle, allowing for new interesting structures to be defined. One of the main themes in generalized geometry is that certain classical geometrical structures can be seen as special cases of these new structures, e.g. complex and symplectic structures that can be seen as particular cases of generalized complex structures. We will later see what exactly these are, and what is meant by this.

Besides its relation to DFT, generalized geometry is also quite interesting on its own. So, before going into the physics, we will first discuss the mathematics of generalized geometry in Chapter 2. We will introduce a number of concepts, some of which will reappear when discussing DFT, be it in a slightly different form. We will work towards the definition of the aforementioned generalized complex structures. We finish the chapter by showing how Riemannian geometry is incorporated into generalized geometry.

In Chapter 3 we will give an introduction to string theory. Starting from scratch, we will develop the physics of strings, in particular of closed strings. In analogue to the physics of point particles, we will construct the physics of classical relativistic strings which will then be quantized. We will discuss the implications of compact dimensions, and how T-duality is found in the theory.

This thesis is mostly based on the works of [6, 14, 1, 11] and it aims to give an understanding of the different subjects and to show how they are connected to each other, on the level an advanced undergraduate.

Chapter 2

Generalized geometry

The main difference between generalized geometry and usual geometry is that we replace the tangent bundle T of a manifold M by

$$T \oplus T^* = \Big\{(p, X, \xi) : p \in M,\ X \in T_p,\ \xi \in T_p^*\Big\},$$

where T^* denotes the cotangent bundle of M. In generalized geometry, instead of considering sections of T, i.e. vector fields, we consider sections of $T \oplus T^*$: elements of the form $X + \xi$, where X is a vector field and ξ is a one-form. The set of smooth sections of a bundle E we denote by $\Gamma(E)$, e.g. the set of smooth vector fields is denoted by $\Gamma(T)$ and the set of smooth one-forms by $\Gamma(T^*)$.

We start in the first section by looking at the linear algebra on fibers of $T \oplus T^*$, those can be thought of as a generalization of the tangent spaces. An inner product is defined on these spaces and we look at the orthogonal group and its corresponding Lie algebra. Then, some particular transformations are discussed that will appear and be useful throughout the chapter, and also later in Chapter 4.

In the following sections, some more definitions are made, and the linear theory is transported onto the manifold. The Courant bracket will be defined, which can be seen as an analogue of the Lie bracket for sections of $T \oplus T^*$.

Throughout this chapter, we will work towards the definition of generalized complex structures, which are discussed in Section 2.5. In this section, we first introduce complex structures and symplectic structures, and then we will see how generalized complex structures turn out to be a generalization of them both.

We end the chapter by showing how Riemannian geometry is incorporated into generalized geometry. Here, we will see how the Courant bracket can also be used to compute covariant derivatives.

2.1 Inner product on $V \oplus V^*$

Let V be a n-dimensional vector space. We consider the vector space $V \oplus V^*$, where V^* denotes the dual space of V, with indefinite inner product given by

$$\langle X+\xi, Y+\eta\rangle = \tfrac{1}{2}\left(\xi(X)+\eta(Y)\right),$$

which has signature (n,n). This can quickly be verified by taking a basis $\{\mathbf{e}_i : i=1,\dots,n\}$ for V with corresponding dual basis $\{\mathbf{e}^i : i=1,\dots,n\}$ for V^*. Then $\{\mathbf{e}_i \pm \mathbf{e}^i : i=1,\dots,n\}$ is a basis for $V \oplus V^*$ for which the inner product admits a diagonal form with ± 1 corresponding to $\mathbf{e}_i \pm \mathbf{e}^i$.

We consider the orthogonal group of transformations, consisting of the linear transformations leaving the inner product invariant,

$$O(V\oplus V^*) = \Big\{A \in \mathrm{GL}(V\oplus V^*) : \langle Av, Aw\rangle = \langle v, w\rangle \text{ for all } v,w \in V\oplus V^*\Big\}.$$

The corresponding Lie algebra, which is the algebra of infinitesimal transformations, is denoted by $\mathfrak{so}(V\oplus V^*)$. Elements $R \in \mathfrak{so}(V\oplus V^*)$ are those that satisfy $\langle Rv, w\rangle + \langle v, Rw\rangle = 0$. Using the splitting $V \oplus V^*$, we write R in block form

$$R = \begin{pmatrix} A & \beta \\ B & D \end{pmatrix},$$

with $A : V \to V, \beta : V^* \to V, B : V \to V^*$ and $D : V^* \to V^*$, all linear maps. Then explicitly, the condition on R is

$$\langle (AX+\beta\xi)+(BX+D\xi), Y+\eta\rangle + \langle X+\xi, (AY+\beta\eta)+(BY+D\eta)\rangle = 0. \tag{2.1}$$

In particular, for $X = Y = 0$ this reduces to $\eta(\beta\xi)+\xi(\beta\eta) = \eta(\beta\xi)+\eta(\beta^*\xi) = 0$, which must hold for all η, ξ, hence $\beta^* = -\beta$. Note here that β^* denotes the adjoint of β, also known as the transpose or pullback. Similarly, by setting $\xi = \eta = 0$ we find $B^* = -B$ and for $Y = 0, \xi = 0$ we find $D = -A^*$. Also, these three resulting conditions are sufficient to satisfy the above condition. Therefore, all elements $R \in \mathfrak{so}(V\oplus V^*)$ are of the form

$$R = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix},$$

where $A \in \mathrm{End}(V), B : V \to V^*$ and $\beta : V^* \to V$, with $B^* = -B$ and $\beta^* = -\beta$. We say B and β are *skew*. We may view B as a 2-form in $\wedge^2 V^*$ and β as an element of $\wedge^2 V$ via

$$B(X,Y) = (B(X))(Y), \qquad \beta(\xi,\eta) = \eta(\beta(\xi)), \qquad X,Y \in V,\ \xi,\eta \in V^*,$$

where the skewness implies the alternativity. We conclude

$$\mathfrak{so}(V\oplus V^*) = \mathrm{End}(V) \oplus \wedge^2 V^* \oplus \wedge^2 V.$$

We will discuss the following three types of orthogonal transformations in particular.

- A *B-transform* is a transformations that is generated by an element of the form $R = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$. Note that we usually just write $R = B$. The corresponding orthogonal transformation is obtained by exponentiation:

  $$\exp(B) = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix},$$

  This transformation acts on an element $X + \xi$ as $X + \xi \mapsto X + \xi + i_X B$, and can be called a *shear transformation* as it shifts V^* but keeps V invariant. This type of transformation will be of more interest to us later.

- A *β-transform* is a transformation that is generated by an element of the form $R = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$. The corresponding orthogonal transformation is

  $$\exp(\beta) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix},$$

  which acts on an element $X + \xi$ as $X + \xi \mapsto X + \xi + i_\xi \beta$. This type of transformation shifts V and leaves V^* invariant.

- Exponentiation of elements of the form $R = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$ results in the transformation

  $$\exp(A) = \begin{pmatrix} \exp A & 0 \\ 0 & (\exp A^*)^{-1} \end{pmatrix},$$

  yielding an embedding of $GL^+(V)$ into $SO(V \oplus V^*)$. The obvious extension

  $$S \mapsto \begin{pmatrix} S & 0 \\ 0 & (S^*)^{-1} \end{pmatrix}$$

  embeds the full $GL(V)$ into $O(V \oplus V^*)$.

2.2 Maximal isotropic subspaces

Definition 2.1. A subspace $L \subset V \oplus V^*$ is called *isotropic* if $\langle v, w \rangle = 0$ for all $v, w \in V \oplus V^*$. As the inner product has signature (n, n), the maximal dimension such a subspace can have is n. If this is the case, it is called a *maximal isotropic* subspace. These subspaces are also known as *linear Dirac structures.*

To see exactly why the dimension of these isotropic subspaces cannot exceed n, we first show that $V \subset V \oplus V^*$, which is easily seen to be an isotropic subspace of dimension n, cannot be extended to a higher dimensional isotropic subspace. Suppose $L \supset V$ is such an extension, then it should contain some $X + \xi \in L$ with $\xi \neq 0$. However, this means $\langle Y, \xi \rangle = \frac{1}{2}\xi(Y) \neq 0$ for some $Y \in V \subset L$, contradicting that L is isotropic. Now, let $N \subset V \oplus V^*$ be any isotropic subspace of dimension n and $\varphi : N \to V$ be any linear bijection. By Witt's theorem [5], φ can be extended to an orthogonal endomorphism of $V \oplus V^*$. This endomorphism will map any isotropic extension of N to an isotropic extension of V, which was just shown to be impossible. Hence, the maximal dimension for isotropic subspaces is n.

We are interested in maximal isotropic subspaces as we will need them later on. Right now, we want to describe all maximal isotropic subspaces. For this, we start with a simple family of maximal isotropic subspaces. Let E be a subspace of V, and define the *annihilator* of E as $\mathrm{Ann}(E) = \{\xi \in V^* : \xi(E) = 0\}$. The subspace $E \oplus \mathrm{Ann}(E) \subset V \oplus V^*$ is then trivially a maximal isotropic subspace.

Note that if we apply any orthogonal transformation to some maximal isotropic subspace L, we obtain a new maximal isotropic subspace, as orthogonal transformation respect the inner product. In particular this is true for B-transforms. Our claim is that any maximal isotropic subspace can be obtained as a B-transform of a maximal isotropic subspace of the form $E \oplus \mathrm{Ann}(E)$. Consider the following proposition.

Proposition 2.1. *All maximal isotropic subspaces* $L \subset V \oplus V^*$ *are of the form* $L(E, \varepsilon) = \{X + \xi \in E \oplus V^* : \xi|_E = \varepsilon(X)\}$*, where* $E \subset V$ *and* $\varepsilon \in \wedge^2 E^*$.

Proof. Let L be a maximal isotropic subspace and define $E = \pi_V L$, where π_V is the canonical projection to V. Take $X \in E$, and $\xi \in V^*$ such that $X + \xi \in L$. Note that $X + \eta \in L$ exactly when $(X + \eta) - (X + \xi) = \eta - \xi \in L$, which is the case if and only if $\langle \eta - \xi, \tilde{X} + \tilde{\xi} \rangle = (\eta - \xi)(\tilde{X}) = 0$ for all $\tilde{X} + \tilde{\xi} \in L$, i.e. $\eta - \xi|_E = 0$. As L is maximal, also all elements $X + \eta \in L$ whenever $\eta - \xi|_E = 0$. Define $\varepsilon : E \to E^*$ by $\varepsilon(X) = \xi|_E$, which is well-defined by this argument. Now clearly $L = L(E, \varepsilon)$. ∎

In the particular case that $\varepsilon = 0$, we obtain $L(E, 0) = E \oplus \mathrm{Ann}(E)$. As mentioned, B-transforms leave V invariant and we see that

$$\begin{aligned} \exp(B)\, L(E, \varepsilon) &= \{X + \xi + i_X B \in E \oplus V^* : \xi|_E = \varepsilon(X)\} \\ &= \{X + \xi + i_X B \in E \oplus V^* : (\xi + i_X B)|_E = (\varepsilon + i^* B)(X)\} \\ &= L(E, \varepsilon + i^* B), \end{aligned}$$

with $i : E \to V$ the inclusion map. This means we can obtain any maximal isotropic as a B-transform of some $L(E, 0)$, i.e. any maximal isotropic subspace is the B-transform of some $E \oplus \mathrm{Ann}(E)$.

Definition 2.2. The *type* of a maximal isotropic subspace L is defined by $k := n - \dim \pi_V(L)$.

Note that since B-transforms are shear transformations, they leave projections to V invariant, and thus the type of a maximal isotropic subspace is also invariant under B-transforms.

The last thing we want to discuss regarding maximal isotropic subspaces is complexification. The inner product $\langle \cdot, \cdot \rangle$ extends by complexification to $(V \oplus V^*) \otimes \mathbb{C}$:

$$\langle u \otimes z, v \otimes w \rangle = \langle u, v \rangle \, zw, \qquad u, v \in V \oplus V^*, \; z, w \in \mathbb{C}.$$

So we can talk about maximal isotropic subspaces of $(V \oplus V^*) \otimes \mathbb{C}$. A *maximal isotropic complex subspace* $L \subset (V \oplus V^*) \otimes \mathbb{C}$ is an isotropic subspace of complex dimension n. By replacing V with $V \otimes \mathbb{C}$, the above theory yields similar results for maximal isotropic complex subspaces. All maximal isotropic complex subspace are of the form $L(E, \varepsilon)$ with a subspace $E \subset V \otimes \mathbb{C}$ and a complex 2-form $\varepsilon \in \wedge^2 E^*$. Similar to the above, we define the type of such spaces is $k := n - \dim_{\mathbb{C}} \pi_{V \otimes \mathbb{C}}(L)$.

We denote complex conjugation by a bar, e.g. the complex conjugate of L is denoted by $\bar{L}$. The *real index* of a maximal isotropic complex subspace L is defined as $r = \dim_{\mathbb{C}} L \cap \bar{L}$.

2.3 The Courant bracket

We want to transport the concepts we introduced onto the manifold, that is to make the step from $V \oplus V^*$ to $T \oplus T^*$. Before we do so, in analogue to the Lie bracket which is defined on sections of the tangent bundle T, we will define a bracket on sections of $T \oplus T^*$.

Definition 2.3. The *Courant bracket* is a skew-symmetric bracket defined for smooth sections $X + \xi, Y + \eta$ of $T \oplus T^*$:

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \tfrac{1}{2} \mathrm{d}(i_X \eta - i_Y \xi),$$

where $[X, Y]$ is the usual Lie bracket. Note that we use the same notation for the Courant bracket as for the Lie bracket, but there should be no confusion as the Courant bracket for vector fields reduces to the Lie bracket.

From its definition, it is easy to see that the Courant bracket satisfies bilinearity and skewness. However it does not define a Lie bracket on $T \oplus T^*$ as it does not satisfy the Jacobi identity. We define the *Jacobiator*

$$\mathrm{Jac}(A, B, C) = [[A, B], C] + [[B, C], A] + [[C, A], B],$$

for $A, B, C \in \Gamma(T \oplus T^*)$. This operator tells us in what way the Courant bracket fails to satisfy the Jacobi identity. The following proposition will show the Jacobiator is the differential of the so-called *Nijenhuis operator*. This will imply the Courant bracket satisfies the Jacobi identity up to an exact term.

Proposition 2.2.

$$Jac(A, B, C) = \mathrm{d}(Nij(A, B, C)),$$

where Nij is the Nijenhuis operator

$$Nij(A, B, C) = \tfrac{1}{3}(\langle [A, B], C \rangle + \langle [B, C], A \rangle + \langle [C, A], B \rangle).$$

Proof. For convenience, we introduce the *Dorfman bracket* $(X+\xi)\circ(Y+\eta) = [X,Y] + \mathcal{L}_X\eta - i_Y d\xi$, whose skew-symmetrization is the Courant bracket, i.e.

$$[A,B] = \tfrac{1}{2}(A\circ B - B\circ A).$$

This follows directly from the skewness of the Lie bracket, and Cartan's formula [8] $\mathcal{L}_X\eta = \mathrm{d}(i_X\eta) + i_Y\mathrm{d}\eta$. This formula also shows that the difference between the brackets is given by

$$[A,B] = A\circ B - \mathrm{d}\langle A,B\rangle.$$

The advantage to using the Dorfman bracket is that it satisfies the following rule:

$$A\circ(B\circ C) = (A\circ B)\circ C + B\circ(A\circ C).$$

This is proven as follows. Set $A = X+\xi, B = Y+\eta$ and $C = Z+\zeta$. Then,

$$\begin{aligned}
&(A\circ B)\circ C + B\circ(A\circ C)\\
&\qquad = [[X,Y],Z] + [Y,[Z,X]] + \mathcal{L}_{[X,Y]}\zeta - i_Z\mathrm{d}(\mathcal{L}_X\eta - i_Y\mathrm{d}\xi) + \mathcal{L}_Y(\mathcal{L}_X\zeta - i_Z\mathrm{d}\xi) - i_{[X,Z]}\mathrm{d}\eta\\
&\qquad = [X,[Y,Z]] + \mathcal{L}_X\mathcal{L}_Y\zeta - \mathcal{L}_X i_Z\mathrm{d}\eta - \mathcal{L}_Y i_Z\mathrm{d}\xi + i_Z d i_Y\mathrm{d}\xi\\
&\qquad = [X,[Y,Z]] + \mathcal{L}_X(\mathcal{L}_Y\zeta - i_Z\mathrm{d}\eta) - i_{[Y,Z]}\mathrm{d}\xi\\
&\qquad = A\circ(B\circ C),
\end{aligned}$$

using that $i_{[X,Y]} = [\mathcal{L}_X, i_Y]$ and $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X,\mathcal{L}_Y]$. Now,

$$\begin{aligned}
[[A,B],C] &= [A,B]\circ C - \mathrm{d}\langle[A,B],C\rangle\\
&= (A\circ B - \mathrm{d}\langle A,B\rangle)\circ C - \mathrm{d}\langle[A,B],C\rangle\\
&= (A\circ B)\circ C - \mathrm{d}\langle[A,B],C\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
4\,[[A,B],C] &= (A\circ B)\circ C - C\circ(A\circ B) - (B\circ A)\circ C + C\circ(B\circ A)\\
&= A\circ(B\circ C) - B\circ(A\circ C) - C\circ(A\circ B) - B\circ(A\circ C) + A\circ(B\circ C) + C\circ(B\circ A)\\
&= A\circ(B\circ C) - B\circ(A\circ C)\\
&= (A\circ B)\circ C\\
&= [[A,B],C] + \mathrm{d}\langle[A,B],C\rangle.
\end{aligned}$$

Adding all cyclic permutations of the above equality results in

$$4\,\mathrm{Jac}(A,B,C) = \mathrm{Jac}(A,B,C) + 3\,\mathrm{d}(\mathrm{Nij}(A,B,C)).$$

Therefore $\mathrm{Jac}(A,B,C) = \mathrm{d}(\mathrm{Nij}(A,B,C))$. ☐

Before continuing, we first prove a property of the Courant bracket that we will need later on.

Proposition 2.3. *Let $A,B\in\Gamma(T\oplus T^*)$ and $f\in C^\infty(M)$. Then the Courant bracket satisfies*

$$[A,fB] = f[A,B] + (\pi(A)f)B - \langle A,B\rangle\,\mathrm{d}f,$$

where $\pi: T\oplus T^*\to T$ is the natural projection.

Proof. Let $A = X + \xi, B = Y + \eta$. Then,

$$\begin{aligned}[X+\xi, f(Y+\eta)] &= [X, fY] + \mathcal{L}_X(f\eta) - \mathcal{L}_{fY}\xi - \tfrac{1}{2}\mathrm{d}(i_X(f\eta) - i_{fY}\xi) \\ &= f[X+\xi, Y+\eta] + (Xf)Y + (Xf)\eta - (i_Y\xi)\mathrm{d}f - \tfrac{1}{2}(i_X\eta - i_Y\xi)\,\mathrm{d}f \\ &= f[X+\xi, Y+\eta] + (Xf)(Y+\eta) - \langle X+\xi, Y+\eta\rangle\,\mathrm{d}f,\end{aligned}$$

using that the Lie bracket and derivative satisfy $[X, fY] = f[X,Y] + (Xf)Y$ and $\mathcal{L}_X(f\eta) = (Xf)\eta + f\mathcal{L}_X\eta$ and $\mathcal{L}_{fY}\xi = f\mathcal{L}_Y\xi + i_Y\xi\,\mathrm{d}f$ [8], and Cartan's formula. □

Next, we are interested in symmetries of the Courant bracket. The Courant bracket is clearly invariant under diffeomorphisms, as it is defined by a coordinate-free expression, i.e. it is *generally covariant*. The Lie bracket has no more symmetries [6], but the Courant bracket has additional symmetry. Namely, it is also preserved under *B-field transformations*, which are B-transforms with B closed. This is proven by the following proposition.

Proposition 2.4. *The transformation* $\exp(B)$ *is an automorphism of the Courant bracket if and only if* B *is closed, i.e.* $\mathrm{d}B = 0$.

Proof. Let $X + \xi, Y + \eta$ be smooth sections of $T \oplus T^*$, then

$$\begin{aligned}\left[e^B(X+\xi), e^B(Y+\eta)\right] &= [X+\xi+i_XB, Y+\eta+i_YB] \\ &= [X+\xi, Y+\eta] + [X, i_YB] + [i_XB, Y] \\ &= [X+\xi, Y+\eta] + \mathcal{L}_X i_Y B - \tfrac{1}{2}\mathrm{d}i_X i_Y B - \mathcal{L}_Y i_X B + \tfrac{1}{2}\mathrm{d}i_Y i_X B \\ &= [X+\xi, Y+\eta] + \mathcal{L}_X i_Y B - i_Y\mathcal{L}_X B + i_Y i_X \mathrm{d}B \\ &= [X+\xi, Y+\eta] + i_{[X,Y]}B + i_Y i_X \mathrm{d}B \\ &= e^B\,[X+\xi, Y+\eta] + i_Y i_X \mathrm{d}B.\end{aligned}$$

Hence, e^B is an automorphism of the Courant bracket if and only if $i_Y i_X \mathrm{d}B$ is zero for all X, Y, i.e. $\mathrm{d}B = 0$. □

An *orthogonal Courant automorphism* is a pair (f, F) consisting of diffeomorphisms f of M and F of $T \oplus T^*$ such that F is an orthogonal linear map on each fiber of $T \oplus T^*$, satisfying $F([A, B]) = [F(A), F(B)]$ for all $A, B \in \Gamma(T \oplus T^*)$. Together with the operation of composition this defines the group of orthogonal Courant automorphisms of $T \oplus T^*$.

As mentioned, the Courant bracket is generally covariant, so invariant under diffeomorphisms. Under a diffeomorphism f of M, smooth sections of $T \oplus T^*$ transform according to

$$F_f = \begin{pmatrix} f_* & 0 \\ 0 & (f^*)^{-1} \end{pmatrix},$$

where f_* and f^* denote the pushforward and pullback of f, respectively. Hence the pair (f, F_f) is an orthogonal Courant automorphism. We obtain a subgroup of orthogonal Courant automorphisms

$$\mathrm{Diff}(M) = \Big\{(f, F_f) : f \text{ is a diffeomorphism of } M\Big\}.$$

From Proposition 2.4 we also obtain the subgroup

$$\Omega^2_{closed}(M) = \Big\{(id, e^B) : B \text{ is a closed 2-form}\Big\}.$$

We finish this section by showing that every orthogonal Courant automorphism can be made from a diffeomorphism and a B-field transformation.

Proposition 2.5. *Every orthogonal Courant automorphism of $T \oplus T^*$ can be uniquely written as the composition of an element of Diff(M) with one from $\Omega^2_{closed}(M)$.*

Proof. Let (f, F) be an orthogonal Courant automorphism. Set $G = F_f^{-1} \circ F$, then the pair (id, G) is also an orthogonal Courant automorphism. In particular, for any $A, B \in \Gamma(T \oplus T^*)$ and $h \in C^\infty(M)$, we have $G([hA, B]) = [G(hA), G(B)]$. On the one hand, by Proposition 2.3

$$\begin{aligned} G([hA, B]) &= G\Big(h[A, B] - (\pi(B)h)A - \langle A, B\rangle \, \mathrm{d}h\Big) \\ &= hG([A, B]) - (\pi(B)h)G(A) - \langle A, B\rangle G(\mathrm{d}h), \end{aligned}$$

where $\pi : T \oplus T^* \to T$ is the natural projection. By the same proposition

$$\begin{aligned} [G(hA), G(B)] &= h[G(A), G(B)] - (\pi(G(B))h)\, G(A) - \langle G(A), G(B)\rangle \, \mathrm{d}h \\ &= hG([A, B]) - (\pi(G(B))h)\, G(A) - \langle A, B\rangle \, \mathrm{d}h. \end{aligned}$$

Equality between the two yields

$$(\pi(B)h)\, G(A) + \langle A, B\rangle \, G(\mathrm{d}h) = (\pi(G(B))h)\, G(A) + \langle A, B\rangle \, \mathrm{d}h.$$

Choose $A = X, B = Y$ to be smooth sections of T. Then $\langle A, B\rangle = 0$, so we obtain $(Yh)G(X) = (\pi(G(Y))h)\, G(X)$ for all X, Y, h. This can only hold when $\pi(G(Y)) = Y$ for all Y, so G admits the form $\begin{pmatrix} 1 & * \\ * & * \end{pmatrix}$. The previous equation now reduces to

$$\langle A, B\rangle \, G(\mathrm{d}h) = \langle A, B\rangle \, \mathrm{d}h,$$

hence $G = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$. As G is orthogonal, it must be that $G = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} = e^B$ with B skew. Proposition 2.4 says B must be closed. Now $F = F_f \circ e^B$, which proves the claim. □

2.4 Dirac structures

Definition 2.4. A *Lie algebroid* is a vector bundle L on a smooth manifold M equipped with a Lie bracket $[\cdot, \cdot]$ on its smooth sections, and a smooth bundle map $a : L \to T$ called the *anchor*, satisfying

$$\begin{aligned} a([X, Y]) &= [a(X), a(Y)], \\ [X, fY] &= f[X, Y] + (a(X)f)Y, \end{aligned} \tag{2.2}$$

for $X, Y \in \Gamma(L), f \in C^\infty(M)$.

Intuitively, a Lie algebroid can be seen as a generalization of the tangent bundle T, satisfying the properties of a Lie algebra when projected (or anchored) to the tangent bundle. As a candidate Lie algebroid, the bundle $T \oplus T^*$ equipped with the Courant bracket has a natural choice for anchor, namely the projection $\pi : T \oplus T^* \to T$. This makes it automatically satisfy the first condition of (2.2) as the Courant bracket is equal to the Lie bracket when restricted to sections of T. However, the second condition is not automatically satisfied, and we still have that the Courant bracket is not even a Lie bracket as the Jacobiator is non-zero. This makes that $(T \oplus T^*, [\cdot, \cdot])$ is not a Lie algebroid. Both of these problems are solved if we instead restrict to a subbundle $L \subset T \oplus T^*$ that is both *involutive*, i.e. closed under the Courant bracket, and isotropic. Then the inner products in the Nijenhuis operator vanish, so that $(L, [\cdot, \cdot], \pi)$ would be a Lie algebroid. Also the second condition is satisfied by Proposition 2.3. This motivates the following definition.

Definition 2.5. A maximal isotropic subbundle $L \subset T \oplus T^*$ is called an *almost Dirac structure*. If L is involutive, i.e. closed under the Courant bracket, then it is said to be *integrable*, or simply a *Dirac structure*. Similarly, a maximal isotropic and involutive complex subbundle $L \subset (T \oplus T^*) \otimes \mathbb{C}$ is called a *complex Dirac structure*, and is an instance of a complex Lie algebroid.

All Lie algebroids we consider are Dirac structures. Note that we can use B-field transformations to map Dirac structures to new Dirac structures. This is because B-field transformations preserve the Courant bracket, ensuring the involutivity, and also are orthogonal transformations, ensuring the isotropicness. This property will be used in the next section.

2.5 Generalized complex structures

Generalized complex structures are one of the main subjects of Gualtieri and Hitchin [6, 7]. Roughly speaking, generalized complex structures are structures on $T \oplus T^*$ that are generalizations of complex structures and symplectic structures. We start by first discussing what complex and symplectic structures are, and discussing some of their properties.

A *complex structure* on V is a linear map $J : V \to V$ such that $J^2 = -1$. If we extend J to the complexification $V \otimes \mathbb{C}$, then we see J has complex eigenvalues $\pm i$, leading to the decomposition $V \otimes \mathbb{C} = V_{1,0} \oplus V_{0,1}$, where $V_{1,0}$ and $V_{0,1}$ denote the $\pm i$-eigenspaces of J, respectively. It can easily be seen that these are related by complex conjugation, i.e. $\bar{V}_{1,0} = V_{0,1}$, which shows the dimensions of $V_{1,0}$ and $V_{0,1}$ are the same. Since $\det(J)^2 = (-1)^n$, we find that complex structures can only exist on even-dimensional spaces $n = 2m$. Conversely, also every even-dimensional space V admits a complex structure J. Namely, let $\{\mathbf{e}_i : i = 1, \ldots, 2m\}$ be a basis for V, then J defined by $\mathbf{e}_{2k} \mapsto \mathbf{e}_{2k+1}$ and $\mathbf{e}_{2k+1} \mapsto -\mathbf{e}_{2k}$ for $k = 1, \ldots, m$, is a complex structure. If $J : T \to T$ is a complex structure on each fiber of T of a manifold, we say J is an *almost complex structure*. We say J is integrable to a *complex structure* if in addition it satisfies the following integrability condition: that the manifold everywhere admits local coordinates $x^1, y^1, \ldots, x^m, y^m$, such that

$$J\frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k} \qquad \text{and} \qquad J\frac{\partial}{\partial y^k} = -\frac{\partial}{\partial x^k}, \qquad k = 1, \ldots, m,$$

with holomorphic transition maps between the charts. Frobenius theorem [8] states that this integrability condition is equivalent to saying that $T_{1,0}$ is involutive, i.e. the space of its sections is closed under the Lie bracket $[\cdot, \cdot]$.

A *symplectic structure* on V is a non-degenerate 2-form $\omega \in \wedge^2 V^*$. As before, we can view ω as a map $V \to V^*$ that is skew, i.e. $\omega^* = -\omega$. Since $\det(\omega) = \det(-\omega^*) = (-1)^n \det(\omega)$, it follows that symplectic structures can only exist on even-dimensional subspaces as well. Also every even-dimensional space V admits a symplectic structure, e.g. $\omega = \mathbf{e}^1 \wedge \mathbf{e}^2 + \cdots + \mathbf{e}^{2m-1} \wedge \mathbf{e}^{2m}$. If $\omega \in \wedge^2 T$ is a symplectic structure on each fiber of T, we call it an *almost symplectic structure*. We say ω is integrable to a *symplectic structure* if in addition it satsifies the following integrability condition: that the manifold admits everywhere local coordinates $x^1, y^1, \ldots, x^m, y^m$, such that

$$\omega = \mathrm{d}x^1 \wedge \mathrm{d}y^1 + \cdots + \mathrm{d}x^m \wedge \mathrm{d}y^m,$$

with symplectic transition maps between the charts. By Darboux theorem [2] this is the case exactly when $\mathrm{d}\omega = 0$, i.e. ω is closed. Hence, we refer to $\mathrm{d}\omega = 0$ as the integrability condition for symplectic structures.

In analogue to the complex and symplectic structures, we will now define a generalized complex structure on V. This definition will be extended to generalized complex structures on T, and we will specify the integrability condition for these structures.

Definition 2.6. A *generalized complex structure* on V is an endomorphism $\mathcal{J}$ of $V \oplus V^*$ which is both complex, i.e. $\mathcal{J}^2 = -1$, and symplectic, i.e. $\mathcal{J}^* = -\mathcal{J}$.

Note that in this definition, technically the adjoint $\mathcal{J}^*$ is an endomorphism of the dual space $(V \oplus V^*)^*$. However, we identify $V \oplus V^*$ with its dual space using the inner product $\langle \cdot, \cdot \rangle$. Furthermore, note that the latter condition in this definition could also be replaced by requiring that $\mathcal{J}$ is an orthogonal transformation. Namely, as $\mathcal{J}$ is invertible, $\mathcal{J}^* = -\mathcal{J} \iff \mathcal{J}^*\mathcal{J} = -\mathcal{J}^2 = 1$. We now wish to understand the space of all generalized complex structures.

Proposition 2.6. *A generalized complex structure on V is equivalent to the specification of a maximal isotropic complex subspace $L \subset (V \oplus V^*) \otimes \mathbb{C}$ of real index zero, i.e. such that $L \cap \bar{L} = \{0\}$.*

Proof. Let $\mathcal{J}$ be a generalized complex structure, and let $L \subset (V \oplus V^*) \otimes \mathbb{C}$ be its $+i$-eigenspace. For $x, y \in L$, we have using the orthogonality of $\mathcal{J}$ that $\langle x, y \rangle = \langle \mathcal{J}x, \mathcal{J}y \rangle = \langle ix, iy \rangle = -\langle x, y \rangle$, implying $\langle x, y \rangle = 0$. Together with the fact that L has complex dimension n as it is the $+i$-eigenspace of $\mathcal{J}$, it follows that it is maximal isotropic. As $\bar{L}$ is the $-i$-eigenspace of $\mathcal{J}$, we also have $L \cap \bar{L} = \{0\}$.

Conversely, let L be maximal isotropic complex subspace of real index zero. Then $(V \oplus V^*) \otimes \mathbb{C} = L \oplus \bar{L}$, and we can define $\mathcal{J}$ to be multiplication by i on L and multiplication by $-i$ on $\bar{L}$. Restricting this map to $V \oplus V^*$ and taking the real part gives us a generalized complex structure on V. ☐

As done for the complex and symplectic structures, we will now define generalized (almost) complex structures on the manifold.

Definition 2.7. If an endomorphism $\mathcal{J}$ of $T \oplus T^*$ defines a generalized almost complex structure $\mathcal{J}$ on each tangent space, it is called a *generalized almost complex structure*. It is said to be *integrable* to a generalized complex structure when its $+i$ eigenbundle $T_{1,0} \subset (T \oplus T^*) \otimes \mathbb{C}$ is Courant involutive.

Hence, looking at Definition 2.5 and Proposition 2.6, we see that an integrable generalized complex structure is equivalent to the specification of a complex Dirac structure of real index zero.

Now we will show how symplectic and complex structures are particular cases of generalized complex structures, and how their integrability conditions are equivalent to the integrability condition for generalized complex structures.

Consider a symplectic structure ω on V. It can be described by the generalized complex structure

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

The corresponding maximal isotropic subspace, i.e. the $+i$-eigenspace of $\mathcal{J}_\omega$, is given by

$$L_\omega = \Big\{ X - i\omega(X) : X \in V \otimes \mathbb{C} \Big\}.$$

As $\pi_{V\otimes\mathbb{C}}(L_\omega) = V \otimes \mathbb{C}$, it follows that this structure has type $k = 0$. Actually, it can be proven that any type $k = 0$ generalized complex structure is a B-field transformation of a symplectic structure [6]. Now suppose ω is an almost symplectic structure on T. It can be described by the generalized almost complex structure

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix},$$

which has corresponding maximal isotropic subbundle

$$L_\omega = \Big\{ X - i\omega(X) : X \in \Gamma(T \otimes \mathbb{C}) \Big\}.$$

Note that $L_\omega = e^{-i\omega}T$, i.e. it can be viewed as a B-transform of T. Since T is Courant involutive, Proposition 2.4 implies that L is Courant involutive if and only if $\mathrm{d}\omega = 0$, which is precisely the integrability condition for symplectic structures. In this case, L_ω is indeed a Dirac structure of real index zero, as the non-degeneracy of ω implies L_ω has real index zero.

Consider a complex structure J on V. It can be described by the generalized complex structure

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}.$$

The corresponding maximal isotropic subspace, i.e. the $+i$-eigenspace of $\mathcal{J}_J$, is given by

$$L_J = V_{0,1} \oplus V^*_{1,0}.$$

We see $\pi_{V\otimes\mathbb{C}}(L_J) = V_{0,1}$ and hence the type is $k = n/2$. In fact, in [6] it is proven that any generalized complex structure of type $k = n/2$ is the B-field transformation of a complex structure. Now suppose J is an almost complex structure on T. It can be described by the generalized almost complex structure

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix},$$

with corresponding maximal isotropic subbundle

$$L_J = T_{0,1} \oplus T^*_{1,0}.$$

Suppose that $\mathcal{J}_J$ is integrable, i.e. L_J is Courant involutive. Since the Courant bracket reduces to the Lie bracket on T, it follows that $T_{0,1}$ is Lie involutive, so J is integrable. Conversely, suppose that J is integrable. Let $X+\xi, Y+\eta$ be sections of L_J. Then,

$$\begin{aligned}[X+\xi, Y+\eta] &= [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + \tfrac{1}{2}\mathrm{d}(i_X\eta - i_Y\xi) \\ &= [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi.\end{aligned}$$

As J is integrable, $[X,Y]$ is a section of $T_{0,1}$. Using Cartan's formula, $\mathcal{L}_X\eta = i_X(\mathrm{d}\eta) + d(i_X\eta) = i_X(\mathrm{d}\eta) \in \Gamma(T^*_{1,0})$ and similarly for $\mathcal{L}_Y\xi$, it follows that $\mathcal{L}_X\eta - \mathcal{L}_Y\xi \in \Gamma(T^*_{1,0})$ and thus is L_J Courant involutive. This shows that the integrability conditions for $\mathcal{J}_J$ and J are the same. And indeed, in this case L_J becomes a Dirac structure of real index zero.

We finish this section by mentioning the generalized Darboux theorem. As said, all generalized complex structures of type $k = 0$ are B-field transforms of symplectic structures, and those of type $k = n/2$ are B-field transforms of complex structures. In fact, the type k of a generalized complex structure turns out to be a measure for the degree to which it is symplectic or complex. This is described by the following theorem. Its proof of goes beyond the theory discussed here, so for a proof we refer the reader to [6].

Theorem 2.1 (Generalized Darboux Theorem)**.** *Consider a generalized complex structure $\mathcal{J}$ on a manifold. Any regular point, i.e. a point where the type k of $\mathcal{J}$ is locally constant, has a neighborhood that admits coordinates $x^1, y^1, \dots x^m, y^m$ such that $\mathcal{J}$ is a B-field transformation of*

$$\mathcal{J}_0 = \begin{pmatrix} -J_0 & -\omega_0^{-1} \\ \omega_0 & J_0^* \end{pmatrix},$$

with J_0 defined by

$$J_0\,\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j}, \qquad J_0\,\frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j}, \qquad \textit{for } j = 1,\dots,k,$$

and

$$\omega_0 = \mathrm{d}x^{k+1} \wedge \mathrm{d}y^{k+1} + \dots + \mathrm{d}x^m \wedge \mathrm{d}y^m.$$

2.6 Riemannian geometry

We want to finish this chapter by a short description of how Riemannian geometry is incorporated into the framework of generalized geometry. Riemannian geometry is given by a manifold M equipped with a symmetric positive definite $(0,2)$-tensor called the metric g. Using the interior product, we can consider the metric as a map $g : T \to T^*$:

$$g : X \mapsto i_X g.$$

The non-degeneracy of g makes this map an isomorphism between T and T^*.

For any tangent vector X, we use the notation $X^+ = X + gX$ and $X^- = X - gX$. Consider the following subbundles of $T \oplus T^*$:

$$C_+ = \Big\{ X^+ = X + gX : X \in T \Big\},$$
$$C_- = \Big\{ X^- = X - gX : X \in T \Big\}.$$

These subbundles can also be seen as the graphs of $\pm g : T \to T^*$. On C_+, the inner product reduces to

$$\langle X + gX, Y + gY \rangle = \tfrac{1}{2} \left(g(X,Y) + g(Y,X) \right) = g(X,Y),$$

i.e. a positive definite inner product. Similarly, on C_- the inner product $\langle X - gX, Y - gY \rangle = -g(X,Y)$ becomes negative definite. Also we see that C_+ and C_- are orthogonal. Since g is an isomorphism, we have $C_+ \cap C_- = \{0\}$. Both have half the rank of $T \oplus T^*$, hence

$$T \oplus T^* = C_+ \oplus C_-.$$

Denote the projections to C_+ and C_- by π_{C_+} and π_{C_-}, respectively. We see that

$$\pi_{C_+}(X) = \pi_{C_+} \left(\tfrac{1}{2}(X + gX + X - gX) \right) = \tfrac{1}{2} X^+,$$

and similarly $\pi_{C_-}(X) = \frac{1}{2} X^-$.

Now we want to focus on the covariant derivative. As usual in Riemannian geometry, the covariant derivative allows us to take derivatives of vector fields along other vector fields. In coordinates, this covariant derivative is given by

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k,$$

where we use the notation $\partial_i \equiv \frac{\partial}{\partial x^i}$, and

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right)$$

denote the *Christoffel symbols*. More generally, we can define a covariant derivative along a vector field on any vector bundle E.

Definition 2.8. If X is a vector field on M, a *covariant derivative along* X on a vector bundle E is a map

$$\nabla_X : \Gamma(E) \to \Gamma(E),$$

satisfying

$$\begin{aligned}\nabla_X(v+w) &= \nabla_X v + \nabla_X w,\\ \nabla_{fX+hY} v &= f\nabla_X v + h\nabla_Y v,\\ \nabla_X(fv) &= f\nabla_X v + (Xf)v,\end{aligned}$$

for all $v, w \in \Gamma(E)$ and $f \in C^\infty(M)$.

One can easily check that the usual covariant derivative in Riemannian geometry satisfies this definition. The following proposition gives us a covariant derivative on C_+.

Proposition 2.7. *A covariant derivative on C_+ is given by*

$$\nabla_X Y^+ = \pi_{C_+}[X^-, Y^+], \qquad X \in \Gamma(T),\ Y \in \Gamma(C_+).$$

Proof. From the definition, additivity in both X and Y^+ are clear. Now see that, by Proposition 2.3

$$\begin{aligned}\nabla_{fX} Y^+ &= \pi_{C_+}[fX^-, Y^+]\\ &= \pi_{C+}\Big(f[X^-, Y^+] - (Yf)X^- + \langle X^-, Y^+\rangle \,\mathrm{d}f\Big)\\ &= f\pi_{C+}[X^-, Y^+]\\ &= f\nabla_X Y^+,\end{aligned}$$

using the orthogonality of C_+ and C_-. This shows linearity in X. Similarly,

$$\begin{aligned}\nabla_X(fY^+) &= \pi_{C_+}[X^-, fY^+]\\ &= \pi_{C_+}\left(f[X^-, Y^+] + (Xf)Y^+ - \langle X^-, Y^+\rangle \mathrm{d}f\right)\\ &= f\pi_{C_+}[X^-, Y^+] + (Xf)Y^+\\ &= f\nabla_X Y^+ + (Xf)Y^+,\end{aligned}$$

which shows the product rule. This proves that ∇ is a covariant derivative on C_+. □

In terms of local coordinates x^i on M, the vector fields ∂_i^+ form a basis for C_+. Now,

$$\begin{aligned}\nabla_{\partial_i}\partial_j^+ &= \pi_V\left[\partial_i - g_{ik}\mathrm{d}x^k, \partial_j + g_{jl}\mathrm{d}x^l\right]\\ &= \pi_V\left([\partial_i, \partial_j] + \mathcal{L}_{\partial_i} g_{jl}\mathrm{d}x^l + \mathcal{L}_{\partial_j} g_{ik}\mathrm{d}x^k - \tfrac{1}{2}\mathrm{d}\left(g_{jl}\delta_i^l + g_{ik}\delta_j^k\right)\right)\\ &= \pi_V\left(\partial_i g_{jk}\mathrm{d}x^k + \partial_j g_{ik}\mathrm{d}x^k - \tfrac{1}{2}(g_{ji} + g_{ij})\right)\\ &= \pi_V\left(\partial_i g_{jk}\mathrm{d}x^k + \partial_j g_{ik}\mathrm{d}x^k - \partial_k g_{ij}\mathrm{d}x^k\right)\\ &= \tfrac{1}{2}\left(\mathrm{d}x^k + g^{kl}\partial_l\right)\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)\\ &= \tfrac{1}{2}g^{kl}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)\partial_l^+\\ &= \Gamma_{ij}^k \partial_k^+,\end{aligned}$$

with usual formula for the Christoffel symbols Γ^k_{ij}. By the identification of C_+ with T using the isomorphism g, this results in the usual covariant derivative on T. The interesting conclusion we can make here is that the Courant bracket in generalized geometry can be used to compute covariant derivatives in ordinary Riemannian geometry, an surprising other application of the Courant bracket.

Chapter 3

String theory

As mentioned in the introduction, in string theory we replace the concept of point particles by string-like particles. Strings are one-dimensional objects, so they can be parametrized by a coordinate σ. Strings have finite length, and we usually let σ run from 0 to σ_1. There are two types of strings: *open strings*, which have two endpoints at $\sigma = 0$ and $\sigma = \sigma_1$, and *closed strings*, for which the endpoints coincide.

Similar to point particles, strings have position and momentum. However, they can also vibrate in certain ways. The way in which a string vibrates, we refer to as the *vibrational state* of the string. Roughly speaking, in string theory all particles are of the same type of string, either open or closed, but properties of the particle, such as mass, charge or spin, are determined by their vibrational state.

The contents of this chapter is as follows. We start by making a description for a free classical relativistic string. We construct an action for strings, leading to the equations of motion. We will then restrict ourselves to closed strings, as these will be the subject of the later sections. Choosing a suitable parametrization for the string we are able to solve these. Using canonical quantization, we will obtain a quantum mechanical closed string. While doing the quantization, we assume the reader is familiar with the basics of quantum field theory for point particles. If not, we refer to [13]. In the last two sections, we will see what the implications of compact dimensions are to closed strings, show how T-duality emerges as a duality of the theory.

3.1 String action

As is familiar, point particles trace out a world-line through spacetime. Usually, to determine the trajectory of a particle, we come up with an action, and choose the world-line that minimizes the action. This is known as the principle of least action. The simplest Lorentz invariant action for a point particle is given by its proper time, i.e.

$$S = -mc \int \mathrm{d}s. \tag{3.1}$$

This action can be interpreted as the length of the world-line, so then the principle of least action implies the particle always takes shortest path through spacetime.

For strings, we start completely analogously. The only difference is that instead of particles as points, we consider particles as strings. Therefore, particles no longer trace out a one-dimensional world-line in spacetime, but rather a two-dimensional world-sheet. We parametrize this world-sheet by $X^\mu(\tau, \sigma)$. We call X^μ the *string coordinates*, and τ and σ are the coordinates of the world-sheet. Roughly speaking, τ will have the interpretation of a time-like coordinate, and σ that of a space-like coordinate or that of the length along the string.

We induce the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ of spacetime onto the world-sheet,

$$\gamma_{\alpha\beta} \equiv \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta},$$

where $(\xi^0, \xi^1) = (\tau, \sigma)$, and α and β are world-sheet indices, running from 0 to 1. For notational convenience, we also introduce the notation

$$\dot{X} \equiv \frac{\partial X}{\partial \tau}, \qquad X' \equiv \frac{\partial X}{\partial \sigma},$$

also stressing the time- and space-like interpretation of τ and σ, respectively. It follows that

$$\gamma_{\alpha\beta} = \eta_{\mu\nu} \dot{X}^\mu X'^\nu = \begin{pmatrix} (\dot{X})^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & (X')^2 \end{pmatrix}.$$

As mentioned, we want to come up with an action analogously to (3.1), but instead of minimizing the length of a world-line through spacetime, we want to minimize the area of the world-sheet. An area element of the world-sheet is expressed as

$$\mathrm{d}A = \mathrm{d}\tau \, \mathrm{d}\sigma \, \sqrt{|\gamma|}.$$

From this, we construct the *Nambu–Goto action*,

$$S = \frac{-1}{2\pi\alpha'\hbar c^2} \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \sqrt{|\gamma|}, \tag{3.2}$$

where the constant in front is to make the units match. The constant α' is called the *slope parameter* and has units of inverse energy squared. It is related to the string length by $\ell_s = \hbar c \sqrt{\alpha'}$. Alternatively, some express the action in terms of the *string tension* $T_0 = (2\pi\alpha\hbar c)^{-1}$. For notational convenience however, we will use natural units, i.e. $\hbar = c = 1$. Plugging in the expression for $\sqrt{|\gamma|}$,

$$S = \frac{-1}{2\pi\alpha'} \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}.$$

The corresponding Lagrangian density is

$$\mathcal{L}(\dot{X}^\mu, X'^\mu) = \frac{-1}{2\pi\alpha'} \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}. \tag{3.3}$$

We introduce the canonical momenta

$$\mathcal{P}^\tau_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}, \qquad \mathcal{P}^\sigma_\mu \equiv \frac{\partial \mathcal{L}}{\partial X'^\mu}.$$

Explicitly, we find

$$\begin{aligned}
\mathcal{P}^\tau_\mu &= \frac{-1}{2\pi\alpha'} \frac{(\dot{X} \cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}, \\
\mathcal{P}^\sigma_\mu &= \frac{-1}{2\pi\alpha'} \frac{(\dot{X} \cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}.
\end{aligned} \tag{3.4}$$

In terms of the canonical momenta, the Euler–Lagrange equations become

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0. \tag{3.5}$$

Despite the simple form, looking at (3.4), this equation is rather difficult to solve for X^μ. Our strategy will be to use the freedom in parametrization to simplify the equation of motion in terms of X^μ enormously.

In addition, in the case of open strings, we need boundary conditions. For each endpoint σ^* and dimension μ, we could consider boundary conditions of two types. Either *Dirichlet boundary conditions*

$$\frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_*) = 0,$$

or *free endpoint boundary conditions*

$$\mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0.$$

We will give a remark on the interpretation of these boundary conditions. Each endpoint σ_* satisfies the Dirichlet boundary condition in some dimensions μ, implying the endpoints are fixed in those dimensions. For all other μ it satisfies the free endpoint boundary condition, so that it is free to move along those dimensions. More specifically, some endpoint σ_* has p values of μ for which it has free endpoint boundary conditions, and the other $(D-p)$ are Dirichlet boundary conditions. Then the string endpoint moves freely on a p-dimensional object. These objects are called *D-branes*, in particular *Dp-branes*, where the D stands for Dirichlet.

For closed strings, we do not have boundary conditions, they do not have endpoints connected to a D-brane. However, they do have a periodicity condition.

$$X^\mu(\tau, \sigma + \sigma_1) = X^\mu(\tau, \sigma).$$

Note that this periodicity also gives rise to an ambiguity on the specification of the $\sigma = 0$ point: as there are no endpoints, we can start anywhere. This ambiguity will be taken into account later.

3.2 String momentum and parametrization freedom

As mentioned, the equation of motion (3.5) is rather difficult to solve for X^μ, so we want to use the freedom in parametrization of the world-sheet to simplify it. Before we do so, we first need to define some property of the string.

Consider the transformation

$$X^\mu(\tau,\sigma) \to X^\mu(\tau,\sigma) + \epsilon^\mu,$$

for some constant ϵ^μ, i.e. a spacetime translation. As the Lagrangian (3.3) only depends on derivatives of X^μ, this transformation is a symmetry of the Lagrangian. By Noether's theorem [14], this leads to the conserved current

$$j^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)},$$

where the world-sheet index α runs from 0 to 1. We have

$$(j^0_\mu, j^1_\mu) = (\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu).$$

Note that the conservation equation $\partial_\alpha j^\alpha_\mu = 0$ is the same as the equation of motion (3.5). The corresponding conserved charges are normally obtained by integrating j^0_μ over the spatial coordinates. In this case, this gives

$$p_\mu(\tau) = \int_0^{\sigma_1} \mathcal{P}^\tau_\mu \mathrm{d}\sigma.$$

Note that this charge gets the name and interpretation of momentum, as is the usual conserved charge when Noether's theorem is applied to spacetime translations. We see that

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = \int_0^{\sigma_1} \frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau}\mathrm{d}\sigma = -\int_0^{\sigma_1} \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma}\mathrm{d}\sigma = -\left.\mathcal{P}^\sigma_\mu\right|_0^{\sigma_1}. \tag{3.6}$$

For closed strings, the points $\sigma = 0$ and $\sigma = \sigma_1$ are identified, so that the above expression evaluates to zero, i.e. p_μ is conserved. For open string, p_μ will be constant for all μ where the endpoints satisfy the free endpoint boundary conditions. However, in the case of endpoints satisfying Dirichlet boundary conditions instead, p_μ may fail to be conserved. In fact, this corresponds to current flowing from the string in and out of the D-branes that they are connected to [14]. From the definition of p_μ, we can interpret $\mathcal{P}^\tau_\mu$ as the σ-density of spacetime momentum carried by the string.

It can be proven [14] that we can express

$$p_\mu = \int_\gamma \mathcal{P}^\tau_\mu \mathrm{d}\sigma - \mathcal{P}^\sigma_\mu \mathrm{d}\tau,$$

where for open strings, γ is any curve across the world-sheet, and for closed strings, γ is any curve around the world-sheet tube. In fact, this shows p_μ is independent of the chosen parametrization for the world-sheet.

Now we are at the point that we introduce a restriction on the parametrization of the world-sheet that will simplify the equation of motion. For this, we first restrict ourselves to the types of parametrizations given by

$$n \cdot X(\tau,\sigma) = \lambda\tau.$$

This type of parametrization depends on some vector n_μ and scalar λ, and fixes the τ-parametrization. The string with value of τ is the intersection of the world-sheet with the plane given by the above equation. We rewrite the condition as

$$n \cdot X(\tau,\sigma) = (n \cdot p)\tilde{\lambda}\tau,$$

with $\lambda = (n \cdot p)\tilde{\lambda}$. Here we require $n \cdot p$ to be constant. This naturally holds for closed strings and open strings with free endpoint boundary conditions, as p_μ is constant for those. For open strings with Dirichlet boundary conditions in some dimensions, we will assume $n \cdot \mathcal{P}^\sigma = 0$, so that (3.6) dotted with n implies $n \cdot p = 0$. Furthermore,

$$\tilde{\lambda} = \beta\alpha',$$

where we use $\beta = 1$ for closed strings and $\beta = 2$ for open strings.

Next, we will choose our σ-parametrization by demanding that $n \cdot \mathcal{P}^\tau$ increases constantly as σ increases. This is done as follows. Suppose, we are given some parametrizations σ and $\tilde{\sigma}$, then

$$\frac{\partial X^\mu}{\partial \sigma} = \frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\sigma}\frac{\partial X^\mu}{\partial \tilde{\sigma}}.$$

From (3.4) we see that $\mathcal{P}^{\tau\mu}$ scales linearly with X'^μ, hence

$$\mathcal{P}^{\tau\mu}(\tau, \sigma) = \frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\sigma}\mathcal{P}^{\tau\mu}(\tau, \tilde{\sigma}),$$

and thus

$$n \cdot \mathcal{P}^\tau(\tau, \sigma) = \frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\sigma} n \cdot \mathcal{P}^\tau(\tau, \tilde{\sigma}).$$

This implies we can choose a σ-parametrization where $n \cdot \mathcal{P}^\tau$ is independent of σ. Note that this property is invariant under a rescaling of the parameter σ, which allows us to choose the range of σ. For open strings we will choose $\sigma \in [0, \pi]$ and for closed strings $\sigma \in [0, 2\pi]$, as this we be convenient later on. Independence of σ means $n \cdot \mathcal{P}^\tau = a(\tau)$ for some function $a(\tau)$. We see

$$a(\tau) = \frac{1}{\pi}\int_0^\pi a(\tau)\mathrm{d}\sigma = \frac{1}{\pi}\int_0^\pi n \cdot \mathcal{P}^\tau \mathrm{d}\sigma = \frac{n \cdot p}{\pi},$$

which is constant. The same for closed strings, with π replaced by 2π. Hence, the σ value assigned to a point is proportional to the amount of $n \cdot p$ momentum carried by the portion of the string from the endpoint $\sigma = 0$ to that point.

Dotting the equation of motion (3.5) with n, and using that $n \cdot \mathcal{P}^\tau$ is constant results in

$$\partial_\sigma \left(n \cdot \mathcal{P}^\sigma\right) = 0,$$

i.e. $n \cdot \mathcal{P}^\tau$ is independent of σ. For open strings we assumed $n \cdot \mathcal{P}^\sigma = 0$ at the string endpoints, so it follows that $n \cdot \mathcal{P}^\sigma = 0$ everywhere on the world-sheet. We would like to have the same result for closed strings. Here we recall that the $\sigma = 0$ point for closed strings can be arbitrarily defined. We deal with these two matters simultaneously. We select the $\sigma = 0$ point arbitrarily on some string. Then we select the $\sigma = 0$ point on all other strings by requiring that $n \cdot \mathcal{P}^\sigma = 0$. We compute

$$n \cdot \mathcal{P}^\sigma = -\frac{1}{2\pi\alpha'}\frac{(\dot{X} \cdot X')\partial_\tau(n \cdot X) - (\dot{X})^2\partial_\sigma(n \cdot X)}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2(X')^2}}.$$

Since $\partial_\sigma(n \cdot X) = \partial_\sigma((n \cdot p)\beta\alpha'\tau) = 0$, this reduces to

$$n \cdot \mathcal{P}^\sigma = -\frac{1}{2\pi\alpha'}\frac{(\dot{X} \cdot X')\partial_\tau(n \cdot X)}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2(X')^2}}. \tag{3.7}$$

Since $\partial_\tau(n \cdot X) = \alpha'(n \cdot p)$ is non-zero constant, we must have $\dot{X} \cdot X' = 0$ at some point on each string. We choose the σ-parametrization such that $X(\tau, \sigma)$ becomes an orthogonal parametrization. This fixes the $\sigma = 0$ point everywhere else on the world-sheet, and this implies that $n \cdot \mathcal{P}^\sigma = 0$ everywhere. Summarizing, for both open and closed strings we can restrict ourselves to a family of parametrizations such that

$$\begin{aligned} n \cdot X(\tau, \sigma) &= \beta\alpha'(n \cdot p)\tau, \\ n \cdot p &= \frac{2\pi}{\beta} n \cdot \mathcal{P}^\tau, \\ n \cdot \mathcal{P}^\sigma &= 0, \end{aligned} \tag{3.8}$$

where $\beta = 1$ for open strings, and $\beta = 2$ for closed strings.

By restricting ourselves to this type of parametrization, constraints are put on the solutions X^μ. The vanishing of $n \cdot \mathcal{P}^\sigma$ together with (3.7) yields

$$\dot{X} \cdot X' = 0. \tag{3.9}$$

Now by (3.8) and (3.4),

$$n \cdot p = \frac{2\pi}{\beta} n \cdot \mathcal{P}^\tau = \frac{1}{\beta\alpha'} \frac{(X')^2 (n \cdot \dot{X})}{\sqrt{-(\dot{X})^2 (X')^2}} = n \cdot p \frac{(X')^2}{\sqrt{-(\dot{X})^2 (X')^2}},$$

as $n \cdot \dot{X} = \partial_\tau(n \cdot X) = \beta\alpha'(n \cdot p)$. Hence, $(\dot{X})^2 + (X')^2 = 0$. Together with (3.9), we can compactly write these constraints as

$$(\dot{X} \pm X')^2 = 0. \tag{3.10}$$

These constraints hold for open strings as well as closed strings. We can simplify the expression (3.4) for $\mathcal{P}^{\tau\mu}$ now considerably using these constraints. In particular note that $\sqrt{-(\dot{X})^2(X')^2} = \sqrt{(X')^2(X')^2} = (X')^2$. We find

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu. \tag{3.11}$$

Similarly, the expression for $\mathcal{P}^{\sigma\mu}$ simplifies to

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X'^\mu. \tag{3.12}$$

And the equation of motion (3.5) now takes the form

$$\ddot{X}^\mu - X''^\mu = 0, \tag{3.13}$$

which is simply a wave equation in all components X^μ.

We have arrived at the equation of motion for both open and closed strings, having some restrictions set on the parametrization. From now on, we will restrict ourselves to closed strings, as the symmetry of T-duality will apply to them. The physics of open strings can be done for a large part very similar to what we will do in the next sections. For more in-depth derivations, we refer the reader to [14].

3.3 Relativistic closed strings

The most general solution to the wave equation (3.13) is

$$X^\mu(\tau,\sigma) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma),$$

for some functions X_L^μ and X_R^μ denoting the left- and right-moving waves, respectively. For closed strings, the periodicity in σ, i.e. $\sigma \sim \sigma + 2\pi$, gives

$$X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma). \tag{3.14}$$

In terms of a change of variables $u \equiv \tau+\sigma, v \equiv \tau-\sigma$, this yields

$$X_L^\mu(u+2\pi) + X_R^\mu(v-2\pi) = X_L^\mu(u) + X_R^\mu(v). \tag{3.15}$$

Differentiating w.r.t. u and v yields that $X_L'^\mu$ and $X_R'^\mu$ are 2π-periodic, respectively. Hence, we expand X_L and X_R in Fourier series

$$\begin{aligned} X_L'^\mu(u) &= \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \bar{\alpha}_n^\mu e^{-inu}, \\ X_R'^\mu(v) &= \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-inv}, \end{aligned} \tag{3.16}$$

where we put the constants of $\sqrt{\frac{\alpha'}{2}}$ in front to make the coefficients α_n^μ and $\bar{\alpha}_n^\mu$ dimensionless. Note that the bar should not be confused with complex conjugation, α_n^μ and $\bar{\alpha}_n^\mu$ are really different variables. Integration leads to

$$\begin{aligned} X_L^\mu(u) &= \frac{1}{2}x_{L0}^\mu + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^\mu u + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\bar{\alpha}_n^\mu}{n} e^{-inu}, \\ X_R^\mu(v) &= \frac{1}{2}x_{R0}^\mu + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu v + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-inv}, \end{aligned} \tag{3.17}$$

with constants of integration x_{L0}^μ and x_{R0}^μ. From (3.15) now follows that

$$\alpha_0^\mu = \bar{\alpha}_0^\mu.$$

Putting together these expressions, we obtain the full solution

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n}\left(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}\right),$$

with $x_0^\mu \equiv \frac{1}{2}\left(x_{L0}^\mu + x_{R0}^\mu\right)$. From (3.11) we now find

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu = \frac{1}{2\pi\alpha'}\sqrt{2\alpha'}\alpha_0^\mu + \cdots,$$

where the dots denote oscillating terms in σ. Hence,

$$p^\mu = \int_0^{2\pi} \mathcal{P}^{\tau\mu}\mathrm{d}\sigma = 2\pi\frac{1}{2\pi\alpha'}\sqrt{2\alpha'}\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu \quad \Longrightarrow \quad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu. \tag{3.18}$$

Plugging this expression for α_0^μ into the expression we have for X^μ, we obtain

$$X^\mu(\tau,\sigma) = x_0^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \left(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}\right).$$

Here we can clearly identify the components that make up X^μ. We have a zero-mode x_0^μ, a term corresponding to the momentum, and the oscillators of the string. Note that if all oscillators were to vanish, i.e. if we drop the σ-dependence, we are left with the motion of a point particle. The oscillators determine the vibrational state of the string, which in turn determine the properties of particle.

It is convenient to record the τ and σ derivatives,

$$\begin{aligned} \dot{X}^\mu &= X_L'^\mu(\tau+\sigma) + X_R'^\mu(\tau-\sigma), \\ X'^\mu &= X_L'^\mu(\tau+\sigma) - X_R'^\mu(\tau-\sigma). \end{aligned} \tag{3.19}$$

These lead to the following linear combinations:

$$\begin{aligned} \dot{X}^\mu + X'^\mu &= 2X_L'^\mu(\tau+\sigma) = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \bar{\alpha}_n^\mu e^{-in(\tau+\sigma)}, \\ \dot{X}^\mu - X'^\mu &= 2X_R'^\mu(\tau-\sigma) = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-in(\tau-\sigma)}. \end{aligned} \tag{3.20}$$

Note however that not all α_n^μ will give valid solutions. Namely, they must satisfy the parametrization constraints (3.10). In order to handle these constraints well, we introduce a change of coordinates, called *light-cone coordinates*:

$$x^+ = \frac{x^0 + x^1}{\sqrt{2}}, \qquad x^- = \frac{x^0 - x^1}{\sqrt{2}}, \qquad x^I = (x^2, \dots, x^{D-1}).$$

Here I denotes a *transverse index*, running over the *transverse dimensions*, from 2 to $D-1$. These coordinates are merely introduced so that it is easier to solve the equations together with the constraints. They are called light-cone coordinates because the x^+ and x^- axes are world-lines of beams of light in the $\pm x^+$ directions. Using these coordinates, the inner product looks like

$$\begin{aligned} v \cdot w &= -v^0 w^0 + v^1 w^1 + \cdots + v^{D-1} w^{D-1} \\ &= -v^- w^+ - v^+ w^- + v^2 w^2 + \cdots + v^{D-1} w^{D-1}. \end{aligned}$$

In addition to this change of coordinates, we also impose the *light-cone gauge*, by which we mean that we take the particular choice of parametrization corresponding to $n_\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0\right)$, so that $n \cdot X = X^+$ and $n \cdot p = p^+$. By (3.8), this yields

$$X^+(\tau,\sigma) = \alpha' p^+ \tau, \qquad p^+ = 2\pi\, \mathcal{P}^{\tau +}.$$

The strategy behind this is to show that there is no dynamics in X^-, up to a zero mode, and that all the dynamics is in the transverse string coordinates $X^2, \dots, X^{D-1}$.

In terms of light-cone coordinates, the constraints (3.10) can be expressed as

$$-2(\dot{X}^+ \pm X'^+)(\dot{X}^- \pm X'^-) + (\dot{X}^I \pm X'^I)^2 = 0.$$

By the light-cone gauge condition, we have $X'^{+} = 0$ and $\dot{X}^{+} = \beta\alpha' p^{+}$, so we find

$$\dot{X}^{-} \pm X'^{-} = \frac{1}{\beta\alpha'}\frac{1}{2p^{+}}(\dot{X}^{I} \pm X'^{I})^{2}. \tag{3.21}$$

Note that this step relies on the assumption that $p^{+} \neq 0$. We will assume this and say that our analysis fails to hold otherwise. Now, from this and the fact that

$$\mathrm{d}X^{-} = \dot{X}^{-}\,\mathrm{d}\tau + X'^{-}\,\mathrm{d}\sigma,$$

by integration X^{-} can be completely determined from X^{I} up to a constant of integration. However, we also require that $\mathrm{d}X^{-}$ integrated around the string gives zero, in order for X^{-} to be well-defined. Choosing loops of constant τ, this reduces to the constraint that $\int_{0}^{2\pi}\partial_\sigma X^{-}\mathrm{d}\sigma = 0$.

Now, the full solution is determined by the objects

$$X^{I}(\tau,\sigma),\;\; p^{+},\;\; x_{0}^{-}. \tag{3.22}$$

Using (3.20) and (3.21), we can solve for X^{-}. For the plus sign we obtain

$$\begin{aligned}
\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\bar{\alpha}_{n}^{-}e^{-in(\tau+\sigma)} &= \frac{1}{\alpha'}\frac{1}{2p^{+}}\left(\dot{X}^{I} + X'^{I}\right)^{2} \\
&= \frac{1}{p^{+}}\sum_{p,q\in\mathbb{Z}}\bar{\alpha}_{p}^{I}\bar{\alpha}_{q}^{I}e^{-i(p+q)(\tau+\sigma)} \\
&= \frac{1}{p^{+}}\sum_{n,p\in\mathbb{Z}}\bar{\alpha}_{p}^{I}\bar{\alpha}_{n-p}^{I}e^{-in(\tau+\sigma)} \\
&= \frac{1}{p^{+}}\sum_{n\in\mathbb{Z}}\left(\sum_{p\in\mathbb{Z}}\bar{\alpha}_{p}^{I}\bar{\alpha}_{n-p}^{I}\right)e^{-in(\tau+\sigma)}.
\end{aligned}$$

So we can identify

$$\sqrt{2\alpha'}\bar{\alpha}_{n}^{-} = \frac{1}{p^{+}}\sum_{p\in\mathbb{Z}}\bar{\alpha}_{p}^{I}\bar{\alpha}_{n-p}^{I}.$$

A similar expression follows when taking the minus sign, then with unbarred α's. We introduce the *transverse Virasoro modes*, as they will be very useful in our later analysis:

$$\bar{L}_{n}^{\perp} \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}}\bar{\alpha}_{p}^{I}\bar{\alpha}_{n-p}^{I}, \qquad L_{n}^{\perp} \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_{p}^{I}\alpha_{n-p}^{I}, \tag{3.23}$$

so that

$$\sqrt{2\alpha'}\bar{\alpha}_{n}^{-} = \frac{2}{p^{+}}\bar{L}_{n}^{\perp}, \qquad \sqrt{2\alpha'}\alpha_{n}^{-} = \frac{2}{p^{+}}L_{n}^{\perp}.$$

As mentioned earlier, for $n = 0$ we have the constraint that $\bar{\alpha}_{0}^{-} = \alpha_{0}^{-}$, from which follows that

$$\bar{L}_{0}^{\perp} = L_{0}^{\perp}.$$

We refer to this constraint as the *level-matching constraint*. Note that we can write

$$X^{-}(\tau,\sigma) = x_{0}^{-} + \frac{1}{p^{+}}L_{0}^{\perp}\tau + \frac{i}{p^{+}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(L_{n}^{\perp}e^{in\sigma} + \bar{L}_{n}^{\perp}e^{-in\sigma}\right),$$

which shows the transverse Virasoro modes are the expansion modes of $X^{-}(\tau,\sigma)$.

3.4 Quantization of closed strings

Now we want to quantize the closed string. By means of canonical quantization, we impose the following commutation relations on our set of independent objects (3.22):

$$\begin{aligned}\left[X^I(\tau,\sigma),\mathcal{P}^{\tau J}(\tau,\sigma')\right] &= i\eta^{IJ}\delta(\sigma-\sigma'),\\ \left[x_0^-,p^+\right] &= -i,\end{aligned} \tag{3.24}$$

and all other commutators zero. This declares $(X^I,\mathcal{P}^{\tau I})$ and (x_0^-,p^+) to be conjugate position and momentum pairs. Now we will derive all other commutation relations from these.

Differentiating the first of (3.24) w.r.t. σ, and using (3.11), we obtain

$$\left[X'^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\right] = 2\pi\alpha' i\eta^{IJ}\tfrac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma').$$

Take the derivative of $[X^I,X^J]=0$ w.r.t. both σ and σ' to obtain $[X'^I,X'^J]=0$. From $[\mathcal{P}^{\tau I},\mathcal{P}^{\tau J}]=0$ follows that $[\dot{X}^I,\dot{X}^J]=0$. Together, these relations imply

$$\begin{aligned}\left[(\dot{X}^I\pm X'^I)(\tau,\sigma),(\dot{X}^J\pm X'^J)(\tau,\sigma')\right] &= \pm\left[\dot{X}^I(\tau,\sigma),X'^J(\tau,\sigma')\right]\pm\left[X'^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\right]\\ &= \pm 2\pi\alpha' i\eta^{IJ}\tfrac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma')\mp 2\pi\alpha' i\eta^{IJ}\tfrac{\mathrm{d}}{\mathrm{d}\sigma'}\delta(\sigma'-\sigma)\\ &= \pm 4\pi\alpha' i\eta^{IJ}\tfrac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma'),\end{aligned}$$

where we have used that $\frac{\mathrm{d}}{\mathrm{d}\sigma'}\delta(\sigma'-\sigma)=-\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma')$. Similarly, we find that

$$\begin{aligned}\left[(\dot{X}^I\pm X'^I)(\tau,\sigma),(\dot{X}^J\mp X'^J)(\tau,\sigma')\right] &= \mp\left[\dot{X}^I(\tau,\sigma),X'^J(\tau,\sigma')\right]\pm\left[X'^I(\tau,\sigma),\dot{X}^J(\tau,\sigma')\right]\\ &= 0.\end{aligned}$$

Applying this to the expressions (3.20), for the plus-signs we find

$$2\alpha'\sum_{n,m\in\mathbb{Z}} e^{-in(\tau+\sigma)}e^{-im(\tau+\sigma')}\left[\bar{\alpha}_m^I,\bar{\alpha}_n^J\right] = 4\pi\alpha' i\eta^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma').$$

An inverse Fourier transform yields

$$\begin{aligned}e^{-i(m+n)\tau}\left[\bar{\alpha}_{m'}^I,\bar{\alpha}_{n'}^J\right] &= i\eta^{IJ}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma e^{im\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}\int_0^{2\pi}\mathrm{d}\sigma' e^{in\sigma'}\delta(\sigma-\sigma')\\ &= i\eta^{IJ}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma e^{im\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}e^{in\sigma}\\ &= -n\eta^{IJ}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma e^{i(m+n)\sigma}\\ &= -n\eta^{IJ}\delta_{m+n,0}\\ &= m\eta^{IJ}\delta_{m+n,0},\end{aligned}$$

from which we can conclude that

$$\left[\bar{\alpha}_m^I,\bar{\alpha}_n^J\right] = m\eta^{IJ}\delta_{m+n,0}.$$

A very similar computation which follows from taking the minus-sign gives $\left[\alpha_m^I, \alpha_n^J\right] = m\eta^{IJ}\delta_{m+n,0}$ and by taking opposite signs, we find $\left[\alpha_m^I, \bar{\alpha}_n^J\right] = 0$. Therefore, the commutation relations among the oscillators are

$$\left[\bar{\alpha}_m^I, \bar{\alpha}_n^J\right] = \left[\alpha_m^I, \alpha_n^J\right] = m\eta^{IJ}\delta_{m+n,0}, \qquad \left[\alpha_m^I, \bar{\alpha}_n^J\right] = 0. \tag{3.25}$$

Now, we define canonical creation and annihilation operators

$$\alpha_n^I = a_n^I\sqrt{n}, \qquad \alpha_{-n}^I = a_n^{I\dagger}\sqrt{n}, \qquad n \geq 1,$$
$$\bar{\alpha}_n^I = \bar{a}_n^I\sqrt{n}, \qquad \bar{\alpha}_{-n}^I = \bar{a}_n^{I\dagger}\sqrt{n}, \qquad n \geq 1,$$

As expected for such operators, we obtain the following commutation relations:

$$\left[\bar{a}_m^I, \bar{a}_n^{J\dagger}\right] = \delta_{m,n}\eta^{IJ}, \qquad \left[a_m^I, a_n^{J\dagger}\right] = \delta_{m,n}\eta^{IJ}.$$

Usually in quantum field theory, we would encounter creation and annihilation operators when quantizing a field. The creation and annihilation operators applied to the state of the field would then correspond to the creation and annihilation of particles. However, now we are not quantizing a field, but a string. The interpretation of these operators is as follows: when a creation operator is applied to the state of the string, it adds a vibrational mode to the string. When annihilation operators are applied, these vibrational modes are removed again. We will discuss this in more detail in Section 3.5.

Lastly, we want to find the commutation relations involving x_0^I. We start by integrating the first of (3.24) over $\sigma \in [0, 2\pi]$, then together with (3.11) this gives

$$[x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau, \dot{X}^J] = \alpha' i\eta^{IJ}.$$

As can be seen from (3.20), $\dot{X}^J$ is a sum of α_n and $\bar{\alpha}_n$ operators, all of which commute with α_0^I by (3.25). This leaves us, after expanding by (3.20), with

$$\sum_{n\in\mathbb{Z}}[x_0^I, \bar{\alpha}_n^I]e^{-in(\tau+\sigma)} + [x_0^I, \alpha_n^I]e^{-in(\tau-\sigma)} = \sqrt{2\alpha'}i\eta^{IJ}.$$

An inverse Fourier transform yields

$$\left[x_0^I, \bar{\alpha}_0^I\right] + \left[x_0^I, \alpha_0^I\right] = \sqrt{2\alpha'}i\eta^{IJ} \qquad \text{and} \qquad \left[x_0^I, \bar{\alpha}_n^I\right]e^{-in\tau} + \left[x_0^I, \alpha_n^I\right]e^{in\tau} = 0, \text{ for } n \neq 0.$$

Since the latter of these holds for all τ, we find $[x_0^I, \bar{\alpha}_n^J] = [x_0^I, \alpha_n^J] = 0$ for $n \neq 0$. The first, together with the fact that $\bar{\alpha}_0^I = \alpha_0^I$ implies $[x_0^I, \alpha_0^J] = \sqrt{\frac{\alpha'}{2}}i\eta^{IJ}$. In particular, using (3.18), we see that

$$[x_0^I, p^J] = i\eta^{IJ}.$$

Therefore, p^I is the momentum conjugate to x_0^I, as expected.

In analogue to the classical case (3.23), we define the *transverse Virasoro operators*

$$\bar{L}_n^\perp \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}}\bar{\alpha}_{n-p}^I\bar{\alpha}_p^I, \qquad L_n^\perp \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_{n-p}^I\alpha_p^I.$$

Note that we did not pay attention to the ordering of the α's. In the classical case this was fine, but now that the α's are operators, they may fail to commute, so that the ordering might matter. As seen from (3.25), this is only the case for $n = 0$. We want to make the operators *normal ordered*, meaning the all creation operators are left of the annihilation operators. We will not go through the trouble, but it turns out [14] that when trying to make $L_0^\perp$ normal ordered, it introduces a constant diverging term. The way we deal with this is to redefine the transverse Virasoro operators for $n = 0$ to be the normal ordered term without this diverging term:

$$L_0^\perp \equiv \frac{1}{2}\alpha_0^I\alpha_0^I + N^\perp, \qquad \bar{L}_0^\perp \equiv \frac{1}{2}\bar{\alpha}_0^I\bar{\alpha}_0^I + \bar{N}^\perp,$$

with the usual number operators associated with the barred and unbarred creation and annihilation operators

$$N^\perp \equiv \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I, \qquad \bar{N}^\perp \equiv \sum_{n=1}^{\infty} n \bar{a}_n^{I\dagger} \bar{a}_n^I.$$

To add to the discussion above, these operators do not count the number of particles, but they count in some sense the amount of vibrations of the string.

It is shown in [14] that in order for the quantum theory to be Lorentz invariant, we need the requirement that $D = 26$, and that we then have

$$\sqrt{2\alpha'}\alpha_0^- = \frac{2}{p^+}(L_0^\perp - 1), \qquad \sqrt{2\alpha'}\bar{\alpha}_0^- = \frac{2}{p^+}(\bar{L}_0^\perp - 1).$$

The level-matching constraint that emerged from $\bar{\alpha}_0 = \alpha_0$ remains. In terms of the number operators, we can also write this constraint as

$$N^\perp - \bar{N}^\perp = 0.$$

Finally, we want to find expressions for the mass of the string and the Hamiltonian. Using the above, write

$$\alpha' p^- = \sqrt{2\alpha'}\alpha_0^- = \frac{1}{p^+}(L_0^\perp + \bar{L}_0^\perp - 2).$$

The mass of the string, defined as $M \equiv -p^2$, can then be written as

$$\begin{aligned} M^2 &= 2p^+p^- - p^I p^I \\ &= \frac{2}{\alpha'}(L_0^\perp + \bar{L}_0^\perp - 2) - p^I p^I \\ &= \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2). \end{aligned} \tag{3.26}$$

Regarding the Hamiltonian, we know that p^- generates translation in X^+ since p^- is the momentum conjugate to X^+ as postulated by (3.24). Also, from the light-cone gauge condition, we have $X^+ = \alpha' p^+ \tau$, which results in $\partial_\tau = \alpha' p^+ \partial_{X^+}$. Combining these with the fact that the Hamiltonian is the generator for time evolution, it must be that

$$H = \alpha' p^+ p^-.$$

Similarly as for the expression for the mass, we can write the Hamiltonian as

$$\begin{aligned} H &= L_0^{\perp} + \bar{L}_0^{\perp} - 2 \\ &= \frac{\alpha'}{2} p^I p^I + N^{\perp} + \bar{N}^{\perp} - 2. \end{aligned} \tag{3.27}$$

This completes our analysis of the quantum operators of the closed string.

3.5 State space for closed strings

The quantum states can be labelled by the eigenvalues of a maximal commuting set of operators. We have the canonical pairs (x_0^-, p^+) and (x_0^I, p^I). As usual, it is convenient to work in momentum space, so we take p^+, p^I as our maximally commuting subset. We denote the eigenvalues by p^+ and $\vec{p}_T$, respectively. We introduce states,

$$|p^+, \vec{p}_T\rangle,$$

which are called *ground states* for all values of $p^+, \vec{p}_T$. Also they are declared to be vacuum states for all oscillators, i.e. they annihilate by all the a_n^I,

$$a_n^I |p^+, \vec{p}_T\rangle = 0, \qquad n \geq 1, \quad I = 2, \ldots, D-1.$$

States are build from these ground states by acting on them with creation operators. The states we obtain in this fashion have the form

$$\left[\prod_{n=1}^{\infty} \prod_{I=2}^{D-1} \left(a_n^{I\dagger}\right)^{\lambda_{n,I}}\right] \left[\prod_{m=1}^{\infty} \prod_{J=2}^{D-1} \left(\bar{a}_m^{J\dagger}\right)^{\bar{\lambda}_{m,J}}\right] |p^+, \vec{p}_T\rangle,$$

where $\lambda_{n,I}$ and $\bar{\lambda}_{m,J}$ are non-negative integers. As mentioned in the previous section, each of these states corresponds to a one-particle state of a certain vibrational state, depending on the λ-values. The vibrational state of the string then determines what type the particle is. The eigenvalues of $N^{\perp}$ and $\bar{N}^{\perp}$ for these states are

$$N^{\perp} = \sum_{n=1}^{\infty} \sum_{I=2}^{D-1} n \lambda_{n,I}, \qquad \bar{N}^{\perp} = \sum_{m=1}^{\infty} \sum_{J=2}^{D-1} n \bar{\lambda}_{m,J}.$$

However, note that not all such states are valid states. Only states satisfying the level-matching condition $N^{\perp} = \bar{N}^{\perp}$ are valid. The space spanned by these states then form the *Hilbert space* of the string. This is not an equation that we can solve for, we must check the states on a case by case basis. We give the first few basis states which have the lowest mass and energy, as these are physically most relevant.

The first type of states are the ground states, corresponding to all $\lambda_{n,I} = \bar{\lambda}_{m,J} = 0$, and has $N^{\perp} = \bar{N}^{\perp} = 0$. It follows from (3.26) that $M^2 = -\frac{4}{\alpha'} < 0$. These closed string states are called *tachyons*: their mass-squared is negative, and they have no association with any known particle. A lot can be said about tachyons, but we will not discuss them any further.

The next case is for $N^{\perp} = \bar{N}^{\perp} = 1$. These states must be built out of two oscillators, one barred and one unbarred, both $n = 1$. They are of the form

$$a_1^{I\dagger}\bar{a}_1^{J\dagger}\ |p^+, \vec{p}_T\rangle,$$

of which there are $(D-2)^2$, due to possibilities of I and J. By (3.26) these states are massless, and thus of great interest to us. The general superposition of such states is written as

$$\sum_{I,J} R_{IJ}\ a_1^{I\dagger}\bar{a}_1^{J\dagger}\ |p^+, \vec{p}_T\rangle.$$

We decompose the matrix R_{IJ} into a symmetric traceless part, an asymmetric part, and a trace part, i.e.

$$R_{IJ} = S_{IJ} + A_{IJ} + S'\delta_{IJ},$$

We can split the states into three groups, one for each part. Our claim is that these correspond to one-particle graviton states, one-particle states of the so-called *Kalb–Ramond field* (an antisymmetric field with two indices), and one-particle states of the *dilaton field* (a scalar field). These fields are denoted by $g_{\mu\nu}, b_{\mu\nu}$ and ϕ, respectively.

The motivation for the interpretation of these types of particles goes together with together with the step from one-particle states to multi-particle states. We will first explain the concepts of first and second quantization. Usually, when trying to describe a system of particles, we start with a classical one-particle system. When quantizing this system, this is referred to as *first quantization*. The one-particle states that arise satisfy a certain Schrödinger equation. Then, we can reinterpret this equation as a classical field equation. *Second quantization* refers to the quantization of this classical field. From there emerge multi-particle states, together forming the *Fock space*. This is exactly what we will do now. In Section 3.3 we derived the physics of a classical closed string, and in Section 3.4 we quantized this system, so that we arrived at one-particle states. What we will do now, is determine the Schrödinger equation that these states satisfy and reinterpret this equation as a classical field equation.

Wavefunctions $\psi_{IJ}(\tau, p^+, \vec{p}_T)$ describe the time-dependent states at the massless level:

$$|\psi, \tau\rangle = \int \mathrm{d}p^+\ \mathrm{d}\vec{p}_T\ \psi_{IJ}(\tau, p^+, \vec{p}_T)\ a_1^{I\dagger}\bar{a}_1^{J\dagger}\ |p^+, \vec{p}_T\rangle.$$

They satisfy the following Schrödinger equation

$$i\partial_\tau|\psi, \tau\rangle = H|\psi, \tau\rangle.$$

Using the expression for the Hamiltonian (3.27), and noting that these states satisfy $N^{\perp} = \bar{N}^{\perp} = 1$, we obtain

$$i\frac{\partial\psi_{IJ}}{\partial\tau} = \frac{\alpha'}{2}p^K p^K \psi_{IJ}.$$

The wavefunctions become the classical fields, and the Schrödinger equation becomes the classical field equation.

It was shown in [14] that, in light-cone coordinates, the graviton field, the Kalb–Ramond field and the dilaton field all satisfy the equation

$$\left(i\frac{\partial}{\partial x^+} - \frac{1}{2p^+}p^K p^K\right)\phi^\bullet(x^+, p^+, \vec{p}_T) = 0,$$

where $\bullet$ refers to any relevant light-cone indices for these fields. Substituting $x^+ = \alpha' p^+ \tau$, we find

$$\left(i\frac{\partial}{\partial \tau} - \frac{\alpha'}{2}p^K p^K\right)\phi^\bullet(\tau, p^+, \vec{p}_T) = 0,$$

corresponding to the Schrödinger equation above. Therefore, these groups of states can correctly be identified with the different types of particles as claimed. The next step would be to quantize these fields, i.e. do second quantization, to obtain the Fock space consisting of multi-particle states. However, besides the interpretation of these particles, we are not interested in the multi-particle systems living on these fields.

3.6 Closed strings in the presence of compact dimensions

As mentioned in the introduction, in string theory we allow for different backgrounds, i.e. shapes or topologies of spacetime different from the usual Euclidean one. In particular, we will consider backgrounds consisting of *compact dimensions*. These are dimensions on which points with coordinate x are identified by the relation $x \sim x + 2\pi R$. Here R is called the *radius* of the compact dimension, stressing on the intuition that such a dimension can be seen as a circular dimension. When multiple dimensions are simultaneously compactified in this way, topologically we obtain a higher dimensional torus.

In this section, we consider closed strings in the presence of d compact dimensions. We use string coordinates

$$X^+, X^-, X^i, X^a, \text{ with } i = 2, \ldots, D-d-1, \quad a = D-d, \ldots, D-1.$$

Here i is an index for the non-compact dimensions, and a is an index for the compact dimensions. The periodicity condition of the string (3.14) is adjusted to incorporate for the periodicity of the dimension:

$$X^a(\tau, \sigma + 2\pi) = X^a(\tau, \sigma) + m^a\, 2\pi R^a, \qquad m^a \in \mathbb{Z}, \tag{3.28}$$

where R^a is the radius of the compact dimension a, and the numbers m^a are called the *winding numbers*. We define the *winding* as

$$w^a \equiv \frac{m^a R^a}{\alpha'}. \tag{3.29}$$

The expansions (3.16) of $X_L^a(u)$ and $X_R^a(v)$ still hold, except for that now $\bar{\alpha}_0^a$ is not necessarily equal to α_0^a. Instead (3.28) implies,

$$\bar{\alpha}_0^a - \alpha_0^a = \sqrt{2\alpha'} w^a.$$

Note that the momentum is given by

$$p^a = \frac{1}{2\pi\alpha'}\int_0^{2\pi}(\dot{X}^a_L + \dot{X}^a_R)\mathrm{d}\sigma = \frac{1}{\sqrt{2\alpha'}}(\bar{\alpha}^a_0 + \alpha^a_0),$$

yielding the symmetric expressions

$$p^a = \frac{1}{\sqrt{2\alpha'}}(\bar{\alpha}^a_0 + \alpha^a_0), \qquad w^a = \frac{1}{\sqrt{2\alpha'}}(\bar{\alpha}^a_0 - \alpha^a_0). \tag{3.30}$$

In total, we have

$$X^a(\tau,\sigma) = x^a_0 + \alpha' p^a \tau + \alpha' w^a \sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\bar{\alpha}^a_n e^{-in\sigma} + \alpha^a_n e^{in\sigma}). \tag{3.31}$$

Again, we quantize this system. We impose the same commutation relations as (3.25). Actually, the derivation as described in Section 3.4 can be applied here as well, and we obtain

$$\left[\bar{\alpha}^a_m, \bar{\alpha}^b_n\right] = \left[\alpha^a_m, \alpha^b_n\right] = m\eta^{ab}\delta_{m+n,0}, \qquad \left[\bar{\alpha}^a_m, \alpha^b_n\right] = 0. \tag{3.32}$$

Because of (3.30) and the fact that α^a_0 and $\bar{\alpha}^a_0$ commute with all oscillators, it follows that p^a and w^a also commute with all oscillators. Also, similarly as in Section 3.4 we can derive

$$\left[x^a_0, \bar{\alpha}^b_0\right] = \left[x^a_0, \alpha^b_0\right] = i\sqrt{\frac{\alpha'}{2}}\eta^{ab}.$$

In combination with (3.30) this yields

$$[x^a_0, p^b] = i\delta^{ab}, \qquad [x^a_0, w^b] = 0.$$

Hence, and as expected, p^a is conjugate to x^a, therefore p^a generates x^a translation, i.e. $\exp(-ip^a s)\,\psi(x^a) = \psi(x^a + s)$. Because of the identification $x^a \sim x^a + 2\pi R^a$ for compact dimensions, it must be that $\exp(-i2\pi R^a p^a)$ behaves like the unit operator, so

$$p^a = \frac{n^a}{R^a}, \qquad n^a \in \mathbb{Z}.$$

Also looking at (3.29), we see that both p^a and w^a have discrete spectra. Note that the spectra also make intuitive sense: for large R it is hard for the string to wind around the dimension, corresponding to a very largely spaced winding spectrum. Also, for large R the compact dimension looks more like a normal dimension, corresponding to the momentum spectra becoming almost continuous. For small R it is exactly the other way around.

The level-matching constraint $L^\perp_0 = \bar{L}^\perp_0$ remains, as the x^- dimension is not curled up. However, we now have

$$\begin{aligned}\bar{L}^\perp_0 &= \frac{1}{2}\bar{\alpha}^I_0\bar{\alpha}^I_0 + \bar{N}^\perp = \frac{\alpha'}{4}p^i p^i + \frac{1}{2}\bar{\alpha}^a_0\bar{\alpha}^a_0 + \bar{N}^\perp,\\ L^\perp_0 &= \frac{1}{2}\alpha^I_0\alpha^I_0 + N^\perp = \frac{\alpha'}{4}p^i p^i + \frac{1}{2}\alpha^a_0\alpha^a_0 + N^\perp.\end{aligned} \tag{3.33}$$

Hence, we find by means of (3.30) that

$$\begin{aligned}L^\perp_0 - \bar{L}^\perp_0 &= \frac{1}{2}\left(\alpha^a_0\alpha^a_0 - \bar{\alpha}^a_0\bar{\alpha}^a_0\right) + N^\perp - \bar{N}^\perp\\ &= -\alpha' p^a w^a + N^\perp - \bar{N}^\perp.\end{aligned} \tag{3.34}$$

So the level-matching constraint can be written as

$$N^{\perp} - \bar{N}^{\perp} = \alpha' p^a w^a = n^a m^a.$$

Now the ground states are labelled like $|p^+, \vec{p}_T; \vec{n}, \vec{m}\rangle$, even though not all of them are allowed by our theory due to the above level-matching constraint. The general basis candidate states are of the form

$$\left[\prod_{r=1}^{\infty}\prod_{i=2}^{D-d-1}\left(a_r^{i\dagger}\right)^{\lambda_{i,r}}\right]\left[\prod_{s=1}^{\infty}\prod_{j=2}^{D-d-1}\left(\bar{a}_s^{j\dagger}\right)^{\bar{\lambda}_{j,s}}\right]\left[\prod_{k=1}^{\infty}\prod_{a=D-d}^{D-1}\left(a_k^{a\dagger}\right)^{\lambda_{a,k}}\right]\left[\prod_{l=1}^{\infty}\prod_{b=D-d}^{D-1}\left(\bar{a}_l^{b\dagger}\right)^{\bar{\lambda}_{b,l}}\right]|p^+, \vec{p}_T; \vec{n}, \vec{m}\rangle,$$

The operators $N^{\perp}$ and $\bar{N}^{\perp}$ give eigenvalues

$$N^{\perp} = \sum_{r=1}^{\infty}\sum_{i=2}^{D-d-1} r\lambda_{i,r} + \sum_{k=1}^{\infty}\sum_{a=D-d}^{D-1} k\lambda_{a,k}, \qquad \bar{N}^{\perp} = \sum_{s=1}^{\infty}\sum_{j=2}^{D-d-1} s\bar{\lambda}_{j,s} + \sum_{l=1}^{\infty}\sum_{b=D-d}^{D-1} l\bar{\lambda}_{b,l}.$$

To give an expression for the mass-squared, we consider an observer in the $(D-d)$-dimensional Minkowski spacetime that does not take the compact dimensions into account.

$$\begin{aligned} M^2 &= 2p^+p^- - p^ip^i \\ &= \frac{2}{\alpha'}\left(\bar{L}_0^{\perp} + L_0^{\perp} - 2\right) - p^ip^i \\ &= \frac{1}{\alpha'}\left(\bar{\alpha}_0^a\bar{\alpha}_0^a + \alpha_0^a\alpha_0^a\right) + \frac{2}{\alpha'}\left(N^{\perp} + \bar{N}^{\perp} - 2\right) \\ &= p^ap^a + w^aw^a + \frac{2}{\alpha'}\left(N^{\perp} + \bar{N}^{\perp} - 2\right) \end{aligned} \tag{3.35}$$

Alternatively, using the expressions for p^a and w^a, we could write

$$M^2 = \left(\frac{n^a}{R^a}\right)^2 + \left(\frac{m^aR^a}{\alpha'}\right)^2 + \frac{2}{\alpha'}\left(N^{\perp} + \bar{N}^{\perp} - 2\right). \tag{3.36}$$

Using (3.33), the Hamiltonian is given by

$$\begin{aligned} H &= L_0^{\perp} + \bar{L}_0^{\perp} - 2 \\ &= \frac{\alpha'}{2}p^ip^i + \frac{\alpha'}{2}(p^ap^a + w^aw^a) + N^{\perp} + \bar{N}^{\perp} - 2. \end{aligned} \tag{3.37}$$

We will finish this section by discussing some of the low-energy states particles. Starting with the case $\vec{n} = \vec{m} = 0$, corresponding to a state with no momentum nor winding, the level-matching constraint becomes $N^{\perp} - \bar{N}^{\perp} = 0$. Similarly as before, when $\bar{N}^{\perp} = N^{\perp} = 0$, we obtain closed string tachyons with $M^2 = -\frac{4}{\alpha'}$. When $N^{\perp} - \bar{N}^{\perp} = 1$, we obtain the four types of massless states

$$\begin{aligned} &a_1^{i\dagger}\bar{a}_1^{j\dagger}\;|p^+, \vec{p}_T; 0, 0\rangle, \\ &a_1^{i\dagger}\bar{a}_1^{a\dagger}\;|p^+, \vec{p}_T; 0, 0\rangle, \\ &a_1^{a\dagger}\bar{a}_1^{i\dagger}\;|p^+, \vec{p}_T; 0, 0\rangle, \\ &a_1^{a\dagger}\bar{a}_1^{b\dagger}\;|p^+, \vec{p}_T; 0, 0\rangle. \end{aligned}$$

Basically, what happened is that the massless states were reorganized upon compactification. The first line corresponds to states that will form a gravity field, Kalb–Ramond field and dilaton field on the $(D-d)$ non-compact dimensions.

Looking at the mass spectrum (3.36), it is possible to obtain tachyonic, massless and massive if $N^\perp - \bar{N}^\perp = 0$ but with $\vec{n}$ or $\vec{m}$ non-zero, depending on R^a. However, we will leave the discussion of states here.

3.7 T-duality for closed strings

In this section, we will prove a duality that appears in the theory of closed strings on compact dimensions. A duality of a theory is a bit different than a symmetry. Whereas symmetries often refer to the system being invariant under certain (coordinate) transformations, duality rather refers to a system being equivalently described by a theory that uses different parameters. In particular, and as we shall see, closed strings on a compact dimension of radius R are equivalently described by closed strings on a compact dimension of α'/R.

Take a look at the mass spectrum (3.35). We see that it treats the momentum p^a and the winding w^a on the same footing, i.e. it is invariant under an interchange of them. Alternatively, in terms of (3.36), we can describe this invariance as the interchange of n^a and m^a together with the replacement of R^a by α'/R^a. This invariance holds for each compact dimension a separately. Our claim is that this not merely a symmetry of the mass spectrum, but rather gives us a symmetry of the whole closed string theory. This is what we call *T-duality.*

For compact dimensions a, we introduce *dual coordinate operators*,

$$\tilde{X}^a(\tau,\sigma) \equiv X^a_L(\tau+\sigma) - X^a_R(\tau-\sigma).$$

Expanding using (3.17) gives

$$\tilde{X}^a(\tau,\sigma) = q_0^a + \alpha' w^a \tau + \alpha' p^a \sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \left(\bar{\alpha}^a_n e^{-in\sigma} - \alpha^a_n e^{in\sigma} \right),$$

with $q_0^a \equiv \frac{1}{2}(x^a_{L0} - x^a_{R0})$. The momentum associated with the coordinate $\tilde{x}^a_0$ is therefore w^a, as it appears with τ, and the winding associated with $\tilde{x}^a_0$ is p^a, as it appears with σ. This motivates us to write

$$\tilde{x}^a_0 = q^a_0, \qquad \tilde{p}^a = w^a, \qquad \tilde{w}^a = p^a, \qquad \tilde{\bar{\alpha}}^a_n = \bar{\alpha}^a_n, \qquad \tilde{\alpha}^a_n = -\alpha^a_n,$$

so that

$$\tilde{X}^a(\tau,\sigma) = \tilde{x}^a_0 + \alpha' \tilde{p}^a \tau + \alpha' \tilde{w}^a \sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \left(\tilde{\bar{\alpha}}^a_n e^{-in\sigma} + \tilde{\alpha}^a_n e^{in\sigma} \right),$$

having the exact same form as the string coordinates (3.31). Likewise, we introduce dual momenta $\tilde{\mathcal{P}}^\tau$ operators,

$$\tilde{\mathcal{P}}^{\tau a} \equiv \frac{1}{2\pi\alpha'} \partial_\tau \tilde{X}^a = \frac{1}{2\pi\alpha'} (\dot{X}^a_L - \dot{X}^a_R).$$

We postulate the commutator

$$[\tilde{X}^a(\tau,\sigma), \tilde{\mathcal{P}}^{\tau b}(\tau,\sigma')] = i\eta^{ab}\delta(\sigma-\sigma'),$$

and demand that commutators between two coordinates or two momenta vanish.

Since the dual oscillators are the same as the oscillators (up to a minus sign for the unbarred oscillators), we see that the commutation relations (3.32) also hold for the dual oscillator operators. The pair $(\tilde{x}_0^a, \tilde{p}^a) = (q_0^a, w^a)$ appears in $\tilde{X}^a$ just as (x_0^a, p^a) appears in X^a. Therefore, the derivation in Section 3.4 can be done analogously to conclude that $[\tilde{x}_0^a, \tilde{p}^a] = [q_0, w] = i$, meaning $\tilde{p}^a = w^a$ is the momentum associated to the coordinate $\tilde{x}_0^a = q_0^a$. From the quantized spectra of this dual momentum $\tilde{p}^a = w^a = m^a R^a/\alpha'$, we infer that the coordinate $\tilde{x}_0 = q_0^a$ must live on a compact dimension of radius $\tilde{R}^a \equiv \alpha'/R^a$. Hence, the dual string coordinate $\tilde{X}^a$ lives on a dimension of radius $\tilde{R}^a$ as well.

Looking at the expressions (3.33), we can see the Hamiltonian $H = L_0^\perp + \bar{L}_0^\perp - 2$ can equivalently be written as

$$H = \tilde{L}_0^\perp + \tilde{\bar{L}}_0^\perp - 2.$$

Hence, this yields an interpretive ambiguity. The closed string can be described in two equivalent ways, either by the string coordinate X^μ or the dual string coordinate $\tilde{X}^\mu$, both giving the same Hamiltonian. T-duality corresponds to the replacement of every operator by its dual operator. The physical system is equally well described by this new or dual theory, even though it describes a string on a compact dimension of radius $\tilde{R}^a = \alpha'/R^a$ instead of R^a.

For the record, T-duality corresponds to the map sending all objects to their dual:

$$\begin{aligned} x_0^a &\longrightarrow \tilde{q}_0^a, \qquad & p^a &\longrightarrow \tilde{w}^a, \qquad & \alpha_n^a &\longrightarrow -\tilde{\alpha}_n^a, \\ q_0^a &\longrightarrow \tilde{x}_0^a, \qquad & w^a &\longrightarrow \tilde{p}^a, \qquad & \bar{\alpha}_n^a &\longrightarrow \tilde{\bar{\alpha}}_n^a. \end{aligned} \tag{3.38}$$

Note that T-duality is actually very non-trivial. At no point in the derivation of the theory did we assume any relation between the momentum and winding of the closed strings. Still, they seem to be somehow related to each other. This is the motivation towards the next chapter on double field theory, where the momentum and winding of the string will be unified into a single object, as well as any other object with its dual.

Chapter 4

Double field theory

As mentioned in the introduction, DFT is still a relatively new subject. Hence, it is not built from a set of rigorous axioms, but rather it consists of a collection of ideas and proposals that is still being developed. The idea to keep in mind through this chapter is that DFT is all about symmetry and that it tries to unify different concepts by geometrical means. As originally proposed, DFT allows for a reformulation of closed string theory on compact dimensions (see Section 3.6) such that T-duality, a non-trivial duality of the theory, becomes a manifest symmetry, in particular a coordinate transformation. This idea we will discuss in Section 4.1. Later on, we will see how the metric g and Kalb–Ramond field b can be combined into a single generalized metric, and how diffeomorphisms and b-field gauge transformations can be unified into generalized coordinate transformations.

DFT is not so much a physical theory by itself, but rather a framework in which physical theories can be described. In this framework we double the manifold M that describes our spacetime. The copy is denoted by $\tilde{M}$ and the new space is given by

$$\hat{M} = M \times \tilde{M}.$$

We have coordinates x^i on M, and *dual coordinates* $\tilde{x}_i$ on $\tilde{M}$, $i = 1, \ldots, D$. In terms of Section 3.7, this dual space $\tilde{M}$ is where the dual string coordinates $\tilde{X}^\mu$ live. Note that we denote the dual coordinates as being contravariant. In fact, this agrees with the idea that if a coordinate x^i lives on a dimension with radius R, then the dual coordinate $\tilde{x}_i$ lives on a dimension with radius α'/R. Together, they form the *generalized coordinate*

$$X^I = (x^i, \tilde{x}_i),$$

where $^I = (\ ^i,\ _i)$ denotes a *doubled index* ranging over all (both normal and dual) coordinates. We remark that some papers reverse the order of the coordinates and dual coordinates, but we choose this convention to make any connection with generalized geometry more clear. Doubled indices are raised and lowered with the constant metric

$$\eta_{IJ} = \begin{pmatrix} 0 & \delta_i{}^j \\ \delta^i{}_j & 0 \end{pmatrix}.$$

Note that whenever we represent a rank-2 tensor with doubled indices by a matrix, the first and second index are associated with the row and column index, respectively. Also, the entries are sometimes denoted by $\eta_{ij}, \eta_i{}^j$, etc.

This setup might remind one of generalized geometry as discussed in Chapter 2. Indeed, if we compare η_{IJ} and the inner product $\langle \cdot, \cdot \rangle$ defined in Section 2.1, by identifying tangent vectors along the dual coordinates with cotangent vectors in generalized geometry (note: both are contravariant), the inner products on the tangent spaces agree (up to a factor $\frac{1}{2}$). In particular, both have signature (D, D) and the same group of orthogonal transformations. The difference between DFT and generalized geometry lies in the underlying space. In generalized geometry, merely the tangent bundle is doubled, whereas in DFT the manifold M is doubled. Hence, DFT can be seen as a generalization of generalized geometry.

In DFT, although the amount of coordinates is doubled, we do impose constraints on the fields that live on the doubled space: they cannot depend on all coordinates. In DFT, the *weak constraint* refers to the condition that $\partial^M \partial_M A = 0$ for all fields A. Here derivatives are denoted by $\partial_M = (\partial_i, \tilde{\partial}^i)$. The *strong constraint*, also known as the *section condition*, requires in addition that $\partial^M A \partial_M B = 0$ for all fields A and B. By the product rule, this is equivalent to the condition that $\partial^M \partial_M (AB) = 0$ for all products of fields A and B. In Section 4.3 we will see how the weak constraint follows from the level-matching condition in string theory, and we will see how, when the strong constraint is imposed, DFT becomes equivalent to generalized geometry.

At last a small remark. As said, in DFT the coordinates of spacetime are doubled. However, it is also possible to double only a subset of the coordinates, e.g. only the coordinates of the compact dimensions. This can be seen as doubling all coordinates and setting $\tilde{\partial}^i = 0$ for all coordinates i that you do not intend to double, rendering them inactive while solving the strong constraint for the coordinates ${}^I = ({}^i, {}_i)$. In particular, throughout this chapter we will not double the time coordinate, which is the reason that we will sometimes say that δ_{ij} denotes a flat spacetime metric, instead of the Minkowski metric.

4.1 T-duality as DFT symmetry

In this section we will formulate the physics of sections 3.6 and 3.7 in a DFT-fashion. We aim to make T-duality manifest symmetry. Looking at (3.38), we see certain pairs of objects that are dual to each other. This motivates to define

$$P_M = (p_m, w^m), \qquad x_0^M = (x_0^m, q_{0,m}), \qquad \bar{\alpha}_n^M = (\bar{\alpha}_n^m, \bar{\alpha}_{n,m}), \qquad \alpha_n^M = (\alpha_n^m, -\alpha_{n,m}),$$

i.e. generalized momenta, coordinates and oscillators, respectively. Here we have $\bar{\alpha}_{n,i} \equiv \mathcal{E}_{ij}\ \bar{\alpha}_n^j$ and $\alpha_{n,i} \equiv \mathcal{E}_{ji}\ \alpha_n^j$, with for now $\mathcal{E}_{ij} = \delta_{ij}$ so that oscillators with indices downstairs have the same value as with indices upstairs. This is because the original Nambu–Goto action (3.2) assumes a flat background, i.e. $g_{ij} = \delta_{ij}$ and $b_{ij} = 0$. Later, in Section 4.2, we will see how $\mathcal{E}$ is changed when non-flat backgrounds are incorporated.

Now, we can write the generalized string coordinates $X^M = (X^m, \tilde{X}_m)$ as

$$X^M(\tau, \sigma) = x_0^M + \alpha' \mathcal{H}^{MN} P_N \tau + \alpha' P^M \sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{e^{-in\tau}}{n} \left(\bar{\alpha}_n^M e^{-in\sigma} + \alpha_n^M e^{in\sigma} \right), \tag{4.1}$$

where for now

$$\mathcal{H}^{MN} = \begin{pmatrix} \delta^{mn} & 0 \\ 0 & \delta_{mn} \end{pmatrix},$$

which denotes a flat metric on the doubled space. It is flat for the same reasons as described earlier, and in Section 4.2 we will see how it changes for non-flat backgrounds.

In these terms, T-duality can now simply be described by the coordinate transformation

$$X^I \to X'^I = T^I{}_J X^J,$$

where

$$T^I{}_J = \begin{pmatrix} \delta^i{}_j - t^i{}_j & t^{ij} \\ t_{ij} & \delta_i{}^j - t_i{}^j \end{pmatrix},$$

with t being a diagonal matrix containing a 1 for each coordinate i over which we want to perform T-duality, and 0 otherwise. From the form of this matrix, a rotation or permutation matrix, it is clear that this transformation swaps the coordinates X^i and $\tilde{X}_i$ for which $t_{ii} = 1$. This transformation is easily checked to be an orthogonal transformation, i.e. preserving the metric

$$\eta^{IJ} = T^I{}_K T^J{}_L \eta^{KL},$$

hence in the framework of DFT, T-duality has simply become an orthogonal coordinate transformation!

As mentioned before, the metric η_{IJ} is very similar to the inner product $\langle \cdot, \cdot \rangle$ from Section 2.1. The linear algebra on tangent space of $\hat{M}$ and fibers of $T \oplus T^*$ in generalized geometry is therefore the same. In particular, we can copy the analysis of the group of orthogonal transformations. Also in DFT, we have B-transforms

$$h^I{}_J = \begin{pmatrix} \delta^i{}_j & 0 \\ B^{ij} & \delta_i{}^j \end{pmatrix},$$

with $B^{ij} = -B^{ji}$. Also, we have $GL(D)$ transformations

$$h^I{}_J = \begin{pmatrix} A^i{}_j & 0 \\ 0 & (A^{-1})_i{}^j \end{pmatrix}.$$

We refer to these transformations as $O(D, D)$-transformations. Any $O(D, D)$-transformation can be written as a composition of T-duality transforms, B-transforms and $GL(D)$-transforms [3].

4.2 The generalized metric

As seen in Section 3.5, the low-energy sector of closed strings gives rise to three massless fields: a gravitational field, a Kalb–Ramond field and a dilaton field, denoted by g, b and ϕ, respectively. In DFT, we combine the first two of these into a single object, receiving the name and interpretation of a *generalized metric*

$$\mathcal{H}_{IJ} = \begin{pmatrix} g_{ij} - b_{ik}g^{kl}b_{lj} & b_{ik}g^{kj} \\ -g^{ik}b_{kj} & g^{ij} \end{pmatrix}. \tag{4.2}$$

The dilaton is replaced by $e^{-2d} = \sqrt{|g|}e^{-2\phi}$. The reason we do this is because these objects turn out to be $O(D,D)$-covariant, and the field theory action for these massless fields can be lifted to an $O(D,D)$-covariant form [11]:

$$S_{DFT} = \int d^{2D}X\, e^{-2d}\, \mathcal{R}(\mathcal{H}, d), \tag{4.3}$$

where $\mathcal{R}$ is a scalar depending $O(D,D)$-covariantly on $\mathcal{H}$ and d, and such that when we set $\tilde{\partial}^i = 0$, it reduces to the original field theory action. The generalized metric also allows for more $O(D,D)$-covariant expressions for, e.g. the mass and Hamiltonian of the closed string [3, 15].

We will give a short derivation for how $\mathcal{H}$ can be constructed. We define

$$\mathcal{E}_{ij} \equiv g_{ij} + b_{ij},$$

in particular a flat spacetime corresponds to $\mathcal{E}_{ij} = \delta_{ij}$. The action of an $O(D,D)$-transformation $h_I{}^J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\mathcal{E}$ is given by [15]

$$\mathcal{E}' = (a\mathcal{E} + b)(c\mathcal{E} + d)^{-1}. \tag{4.4}$$

We are looking for a transformation that takes δ_{ij} to any $\mathcal{E}_{ij}$. Since g is symmetric, we can write $g_{ij} = e_i{}^k e_j{}^k$ for some invertible $e_i{}^j$. Then consider the transformation

$$(h_\mathcal{E})_I{}^J = \begin{pmatrix} e_i{}^j & b_{ik}(e^{-1})_j{}^k \\ 0 & (e^{-1})_j{}^i \end{pmatrix},$$

which is orthogonal as it is the composition of a $GL(D)$-transform (by e) and a B-transform (by b). This transformation transforms to any $\mathcal{E}_{ij}$ from the flat δ_{ij}. Namely,

$$\begin{aligned} h_\mathcal{E}\, \delta &= (e\, \delta + b(e^t)^{-1})(0\, \delta + (e^t)^{-1})^{-1} \\ &= e\delta e^t + b \\ &= g + b \\ &= \mathcal{E}. \end{aligned}$$

Note that from (4.4) it follows that $h_\mathcal{E}$ is unique up to an element of $O(D) \times O(D)$. Now define,

$$\mathcal{H}_{IJ} = (h_\mathcal{E})_I{}^K (h_\mathcal{E})_J{}^K,$$

for which this ambiguity cancels out. Note that under a transformation $h : \mathcal{E} \mapsto \mathcal{E}'$ we have $h_{\mathcal{E}'} = h \circ h_{\mathcal{E}}$, so that $\mathcal{H}$ transforms as $\mathcal{H}'^{IJ} = h^I{}_K h^J{}_L \mathcal{H}^{KL}$, implying it is $O(D,D)$-covariant. Explicitly, we find

$$\mathcal{H} = \begin{pmatrix} e & b(e^t)^{-1} \\ 0 & (e^t)^{-1} \end{pmatrix} \begin{pmatrix} e^t & 0 \\ -e^{-1}b & e^{-1} \end{pmatrix} = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix},$$

agreeing with (4.2).

Coming back to the remarks made in Section 4.1, as shown in [12], when adjusting the string action to incorporate for non-flat fields g and b, the string coordinates and the dual string coordinates take the form as in (4.1), but now with $\mathcal{E}$ and $\mathcal{H}$ as introduced in this section.

4.3 The weak and strong constraint

As mentioned, in DFT we have a constraint on the fields that we allow for on our enlarged space $\hat{M}$. The weak version of the constraint says that $\partial^M \partial_M A = 0$ for all fields A, and the strong version additionally requires $\partial_M A \partial^M B = 0$ for all fields A and B. Equivalently, the strong constraint says that $\partial^M \partial_M (AB) = 0$ for all products of fields A and B. We will show how the weak version of the constraint is implied in low-energy closed string theory.

Consider a field A in terms of its Fourier modes, labelled by generalized momenta $P_M = (p_m, w^m)$:

$$A = \int A_P \, e^{iP_M X^M} \mathrm{d}P,$$

where we integrate over the momentum space $\mathbb{R}^{D+D}$. Now note that

$$\partial^M \partial_M A = -\int A_P P_N P^N e^{iP_M X^M} \mathrm{d}P,$$

which implies that $\partial^M \partial_M A = 0$ if and only if $P_M P^M = 0$ whenever $A_P \neq 0$. Hence, the weak constraint on fields is equivalent to the condition that all fields only have null momentum modes. In terms of the components $P_M = (p_m, w^m)$, this condition becomes $P_M P^M = 2p_m w^m = 0$, which is equivalent to the level-matching condition (3.34) as obtained in string theory, for the massless states $N = \bar{N} = 1$,

$$L_0 - \bar{L}_0 = -\alpha' p_i w^i = 0.$$

Now we consider the strong constraint. Use the same expansion as above for fields A and B. Then

$$\begin{aligned} \partial_M A \partial^M B &= \left(\int A_P \; iP^N e^{iP_M X^M} \mathrm{d}P \right) \left(\int B_{P'} \; iP'_N e^{iP'_K X^K} \mathrm{d}P' \right) \\ &= -\int \int A_P B_{P'} \; P^N P'_N \; e^{iP_M X^M} e^{iP'{}_K X^K} \mathrm{d}P \, \mathrm{d}P'. \end{aligned}$$

Therefore, $\partial_M A \partial^M B = 0$ if and only if $P_N P'^N = 0$ whenever $A_P \neq 0$ and $B_{P'} \neq 0$. Hence, the strong constraint on fields is equivalent to the condition that all fields have momentum modes that lie in an isotropic subspace of the momentum space.

We will show that the strong constraint reduces DFT equivalent to generalized geometry. Assuming the strong constraint, all fields have momentum modes that lie in a totally isotropic subspace, denote it by $L^k \subset \mathbb{R}^{D+D}$, where k denotes its dimension. As shown in Chapter 2, isotropic subspaces must have $k \leq D$. Let $\{\mathbf{f}_i : i = 1, \dots, k\}$ be a basis for L^k and $\{\mathbf{e}_i : i = 1, \dots, D\}$ be the standard basis for $\mathbb{R}^D$. We construct a linear map by

$$\varphi : L^k \to \mathbb{R}^{D+D} : \mathbf{f}_i \mapsto \begin{pmatrix} \mathbf{e}_i \\ 0 \end{pmatrix},$$

which is clearly an isometry. Witt's theorem [5] says that this map can be extended to an isometry of $\mathbb{R}^{D+D}$, i.e. an $O(D, D)$-transformation of $\mathbb{R}^{D+D}$. This implies we can make a coordinate transformation such that all fields have no dependence anymore on any winding mode. This implies $\tilde{\partial}^i = 0$, i.e. the fields have no dependence on their dual coordinates. Then the components of a generalized vector field $V^M = (V^i, V_i)$ can be interpreted as vector fields and 1-forms on M, and hence DFT has been reduced to generalized geometry.

The immediate question arises: if the strong constraint removes dependence on any dual coordinate, what is the value of introducing them? Note that beyond the massless sector of strings we can have $L_0^\perp - \bar{L}_0^\perp = N^\perp - \bar{N}^\perp - p_i w^i = 0$ with $N^\perp - \bar{N}^\perp \neq 0$, so that $\partial^M \partial_M$ takes on integer values when acting on fields. Therefore, for the full string theory, the dual coordinates are definitely real.

4.4 Generalized coordinate transformations

Lastly, we want to look at at how gauge transformations of the g- and b-fields can be considered in the setting of DFT. The b-field is an antisymmetric field, it transforms as a two-form, and appears in the action only via its field strength $H = \mathrm{d}b$ [11]. Therefore, adding exact terms to b, i.e. $\delta_{\tilde{\xi}} b_{ij} = \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i$, does not change the physics. These transformations are gauge transformations of b, parametrized by a one-form $\tilde{\xi}_i$. Also we have diffeomorphism invariance, meaning that the physics does not change under a change of coordinates. Under a diffeomorphism $x \to x'$, the g- and b-field transform according to

$$g'_{ij} = \Lambda_i{}^k \Lambda_j{}^l g_{kl}, \qquad b'^{ij} = (\Lambda^{-1})_k{}^i (\Lambda^{-1})_l{}^j b^{kl},$$

where $\Lambda^i{}_j \equiv \frac{\partial x'_i}{\partial x_j}$. Diffeomorphisms can be parametrized by vector fields ξ^i as the flow of a vector field yields a diffeomorphism. We consider these as gauge transformations as well. It can be shown [10] that these two types of gauge transformations can be combined and lifted into a single $O(D, D)$-gauge transformation parametrized by $\xi^M = (\xi^m, \tilde{\xi}_m)$:

$$\begin{aligned} \delta_\xi \mathcal{H}_{IJ} &= \xi^K \partial_K \mathcal{H}_{IJ} + (\partial_I \xi^K - \partial^K \xi_I)\mathcal{H}_{KJ} + (\partial_J \xi^K - \partial^K \xi_J)\mathcal{H}_{IK}, \\ \delta_\xi \left(e^{-2d}\right) &= \partial_M \left(\xi^M e^{-2d}\right). \end{aligned} \tag{4.5}$$

Using the definition of $\mathcal{H}$ (4.2) it can be shown this transformation will reduce to the usual gauge transformations for $\tilde{\partial}_i = 0$. The way in which generalized rank-2 tensors and scalars transform in this way is referred to as the *generalized Lie derivative*.

In this section, we will discuss a proposal from [9], that finds that these gauge transformations can be described as *generalized coordinate transformations* $X^I \to X'^I$. As proposed in this paper, a generalized vector V_I transforms under such transformation $X^I \to X'^I$ as

$$V'_I = \mathcal{F}_I{}^J V_J,$$

where

$$\mathcal{F}_I{}^J = \frac{1}{2}\left(\frac{\partial X^K}{\partial X'^I}\frac{\partial X'_K}{\partial X_J} + \frac{\partial X'_I}{\partial X_K}\frac{\partial X^J}{\partial X'^K}\right). \tag{4.6}$$

Here we use the notation

$$\frac{\partial X^I}{\partial X'^J} = \begin{pmatrix} \frac{\partial x^i}{\partial x'^j} & \frac{\partial x^i}{\partial \tilde{x}'_j} \\ \frac{\partial \tilde{x}_i}{\partial x'^j} & \frac{\partial \tilde{x}_i}{\partial \tilde{x}'_j} \end{pmatrix}, \qquad \frac{\partial X'_I}{\partial X_J} = \begin{pmatrix} \frac{\partial \tilde{x}'_i}{\partial \tilde{x}_j} & \frac{\partial \tilde{x}'_i}{\partial x^j} \\ \frac{\partial x'^i}{\partial \tilde{x}_j} & \frac{\partial x'^i}{\partial x^j} \end{pmatrix}.$$

On tensors of higher rank, each doubled index transforms by a factor of $\mathcal{F}_I{}^J$, e.g. the generalized metric will transform as

$$\mathcal{H}'_{IJ} = \mathcal{F}_I{}^K \mathcal{F}_J{}^L \mathcal{H}_{KL}. \tag{4.7}$$

First we will see how the gauge transformations of diffeomorphisms are obtained. Consider the following transformation that leaves the coordinates $\tilde{x}_i$ invariant,

$$x^i \to x'^i = x'^i(x), \qquad \tilde{x}_i \to \tilde{x}'_i = \tilde{x}_i. \tag{4.8}$$

We compute

$$\frac{\partial X^I}{\partial X'^J} = \begin{pmatrix} (\Lambda^{-1})^i{}_j & 0 \\ 0 & \delta_i{}^j \end{pmatrix}, \qquad \frac{\partial X'_I}{\partial X_J} = \begin{pmatrix} \delta_i{}^j & 0 \\ 0 & \Lambda^i{}_j \end{pmatrix},$$

where as before we use the notation $\Lambda^i{}_j \equiv \frac{\partial x'_i}{\partial x_j}$. Using (4.6) we obtain

$$\mathcal{F}_I{}^J = \frac{1}{2}\left[\begin{pmatrix} (\Lambda^{-1})^j{}_i & 0 \\ 0 & \Lambda^i{}_j \end{pmatrix} + \begin{pmatrix} (\Lambda^{-1})^j{}_i & 0 \\ 0 & \Lambda^i{}_j \end{pmatrix}\right] = \begin{pmatrix} (\Lambda^{-1})^j{}_i & 0 \\ 0 & \Lambda^i{}_j \end{pmatrix}.$$

Note that $\mathcal{F}_I{}^J$ here corresponds to a $GL(D)$-transform, which is what one might have expected from a diffeomorphism transformation. We consider the transformation of $\mathcal{H}_{IJ}$ (4.7). In particular the $\mathcal{H}^{ij}$-component yields

$$g'^{ij} = \Lambda^i{}_k \Lambda^j{}_l g^{kl},$$

i.e. g transforms as usual under diffeomorphisms. The $\mathcal{H}_i{}^j$-component of this transformation yields

$$b'_{ik} g'^{kj} = (\Lambda^{-1})^k{}_i \Lambda^j{}_l b_{km} g^{ml},$$

which, using the transformation rule for g we just obtained, reduces to

$$b'_{ij} = (\Lambda^{-1})^k{}_i (\Lambda^{-1})^l{}_j b_{kl},$$

i.e. the usual transformation rule for the b-field under diffeomorphisms. This shows that the transformation (4.8) correctly reproduces the usual gauge transformations associated to diffeomorphisms. Furthermore, we want to note that regarding the transformation of η_{IJ}, we have

$$\eta'_{IJ} = \begin{pmatrix} 0 & (\Lambda^{-1})^k{}_i \Lambda^j{}_l \delta_k{}^l \\ \Lambda^i{}_k (\Lambda^{-1})^l{}_j \delta^k{}_l & 0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_i{}^j \\ \delta^i{}_j & 0 \end{pmatrix} = \eta_{IJ}.$$

This is as expected, since ${\mathcal{F}_I}^J$ was just noted to be an orthogonal transformation: a $GL(D)$-transform. This shows that these general coordinate transformation leave η_{IJ} invariant.

Next we will consider a type of generalized coordinate transformation that will reproduce the b-field gauge transformations,

$$x^i \to x'^i = x^i, \qquad \tilde{x}_i \to \tilde{x}'_i = \tilde{x}_i - \tilde{\xi}_i(x).$$

For this coordinate transformation we compute

$$\frac{\partial X^I}{\partial X'^J} = \begin{pmatrix} {\delta^i}_j & 0 \\ \partial_j \zeta_i & {\delta_i}^j \end{pmatrix}, \qquad \frac{\partial X'_I}{\partial X_J} = \begin{pmatrix} {\delta_i}^j & -\partial_j \zeta_i \\ 0 & {\delta^i}_j \end{pmatrix},$$

so that

$${\mathcal{F}_I}^J = \begin{pmatrix} {\delta_i}^j & \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i \\ 0 & {\delta^i}_j \end{pmatrix}.$$

Here we note that ${\mathcal{F}_I}^J$ corresponds to a B-transform, even a B-field transform, which is again what one might have expected from these type of transformations. Therefore also this ${\mathcal{F}_I}^J$ is orthogonal, leaving η_{IJ} invariant. Looking at the $\mathcal{H}^{ij}$- and ${\mathcal{H}_i}^j$-component of the transformation of $\mathcal{H}_{IJ}$ (4.7), we see that

$$g'^{ij} = g^{ij}, \qquad b'_{ik} g'^{kj} = (b_{ik} + \partial_i \tilde{\xi}_k - \partial_k \tilde{\xi}_i) g'^{kj},$$

implying that g^{ij} stays invariant and $b'_{ij} = b_{ij} + \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i$ under this transformation. So indeed, this type of coordinate transformation corresponds to a b-field gauge transformation.

Lastly, we want to consider simultaneous diffeomorphisms and b-field gauge transformations, given by

$$x^i \to x'^i = x'^i(x), \qquad \tilde{x}_i \to \tilde{x}'_i = \tilde{x}_i - \zeta_i(x).$$

We compute

$$\frac{\partial X^I}{\partial X'^J} = \begin{pmatrix} {(\Lambda^{-1})^i}_j & 0 \\ {(\Lambda^{-1})^k}_j \partial_k \zeta_i & {\delta_i}^j \end{pmatrix}, \qquad \frac{\partial X'_I}{\partial X_J} = \begin{pmatrix} {\delta_i}^j & -\partial_j \zeta_i \\ 0 & {\Lambda^i}_j \end{pmatrix}.$$

and, using (4.6), we obtain

$${\mathcal{F}_I}^J = \begin{pmatrix} {(\Lambda^{-1})^j}_i & \mathcal{F}_{ij} \\ 0 & {\Lambda^i}_j \end{pmatrix},$$

with

$$\begin{aligned} \mathcal{F}_{ij} &= \tfrac{1}{2} \left({(\Lambda^{-1})^l}_i \partial_l \zeta_j - \partial_j \zeta_i - {(\Lambda^{-1})^k}_i \partial_j \zeta_k + {(\Lambda^{-1})^l}_i {\Lambda^k}_j \partial_l \zeta_k \right) \\ &= \tfrac{1}{2} \left({(\Lambda^{-1})^l}_i {\Lambda^k}_j \partial_l \zeta_k - \partial_j \zeta_i + {(\Lambda^{-1})^l}_i (\partial_l \zeta_j - \partial_j \zeta_l) \right). \end{aligned}$$

Note that previous two types of generalized coordinate transformations are special cases of this one. They can be obtained by either setting $\zeta = 0$ or $\Lambda = \delta$. Looking at $\mathcal{H}^{ij}$-component, we still find

$$g'^{ij} = {\Lambda^i}_k {\Lambda^j}_l g^{kl},$$

i.e. g still transforms as usual under diffeomorphisms. Inspecting the ${\mathcal{H}^i}_j$-component, we can obtain that the b-field transforms as

$$b'_{ij} = {(\Lambda^{-1})^k}_i {(\Lambda^{-1})^l}_j (b_{kl} + \tfrac{1}{2}(\partial_k \zeta_l - \partial_l \zeta_k)) + \tfrac{1}{2} \left(\partial'_i \zeta_j - \partial'_j \zeta_i \right),$$

where $\partial'_i \equiv (\Lambda^{-1})^j{}_i \partial_j$. This transformation can be seen as a b-field gauge transformation, followed by a diffeomorphism, followed by another b-field gauge transformation. From this interpretation we see that the metric η_{IJ} is also invariant under such transformations. What this means, is that in DFT we can allow for a constant metric η_{IJ} without having many restrictions on the allowed transformations.

Here we shall leave the discussion on generalized coordinate transformations. However, we do want to mention an interesting topic following up on this discussion: the composition of generalized coordinate transformations. It turns out the commutator between the gauge parameters $\xi^M = (\xi^m, \tilde{\xi}_m)$ is the *C-bracket*, an $O(D,D)$-covariant version of the Courant bracket as known from Chapter 2, that reduces to the Courant bracket for $\tilde{\partial}^i = 0$. As for the Courant bracket, the Jacobiator of the C-bracket is the differential of the $O(D,D)$-covariant version of the Nijenhuis operator. As a consequence, generalized coordinate transformations are not associative, except when applied to fields. Namely, the transformation by some exact $\tilde{\xi}$ yields a special case of the b-field gauge transformations discussed above, where $\mathcal{F}_I{}^J$ reduces to the identity, i.e. all fields are left invariant. For the interested reader, we will refer to [9, 11] where more can be read about this.

Chapter 5

Conclusion

In this thesis, we have given introductions to generalized geometry, string theory and double field theory, and for each subject a number of interesting concepts and results were discussed. Regarding string theory, we have developed the physics of closed strings and in particular we have seen how one-particle states of the gravitational, Kalb–Ramond and dilaton field arise from certain vibrational states of closed strings. When imposed on toroidal backgrounds, we found T-duality as a non-trivial symmetry of the theory. DFT was applied to this theory, and we have seen how it turned T-duality into an orthogonal coordinate transformation.

Although they are not equivalent, we have seen great similarities between generalized geometry and DFT. Not only did the mathematical structures turn out to be very similar, e.g. we were we able to use some results from generalized geometry in DFT, but also did we see that both subjects aim at the unification of different concepts. In Chapter 2 we have seen generalized geometry allows for a description of complex and symplectic structures where they become special cases of the same type of structure: generalized complex structures. In Chapter 4 we have seen DFT is able to combine the g- and b-field into a single object, the generalized metric $\mathcal{H}$, as well as their respective gauge transformations: diffeomorphisms and b-field gauge transformations were unified in generalized coordinate transformations. It might be interesting to see what more concepts can be unified in both subjects.

There is still a lot to be investigated in DFT. One can go into more depth regarding the constraints put on the fields, e.g. look for ways in which the strong constraint can be relaxed. Also, it might be interesting to see if and how DFT can be formulated with more mathematical rigor, possibly built as an extension upon generalized geometry. Also we recommend looking into *generalized Kähler geometry*, which is also described by Gualtieri [6], in which the generalized metric $\mathcal{H}$ as seen in DFT appears as a metric on $T \oplus T^*$. This might give more insight into how the generalized metric in DFT arises from the g- and b-field.

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