Computing Virtual Classes of Representation Varieties using TQFTs

Master Thesis Mathematics

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supervised by

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Abstract

In this thesis, we study a method for computing virtual classes of representation varieties of closed orientable surfaces using Topological Quantum Field Theories (TQFTs). Along the way, we introduce the Grothendieck ring of varieties and discuss its general properties. Also we look at monoidal categories and the category of bordisms, which we use to define TQFTs. In dimension 1 and 2 a classification of TQFTs is given. As an application of the method, we compute the classes of the representation varieties for the upper triangular matrices of rank $2, 3$ and 4. Finally we discuss and give expressions for the corresponding moduli spaces of representations.

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1 Introduction

Let X be a path-connected topological space with finitely generated fundamental group $\pi_1(X)$, and G an algebraic group over a field k. The set of representations $\rho : \pi_1(X) \to G$, denoted

$$\mathfrak{X}_G(X) = \mathrm{Hom}(\pi_1(X), G),$$

has the structure of an algebraic variety, and is called the G-representation variety of X. The group G acts on it by conjugation, and one can consider the moduli space of representations

$$\mathcal{M}_G(X) = \mathfrak{X}_G(X) /\!\!/ G,$$

where the quotient denotes the GIT quotient. This space is known as the Betti moduli space of X, and for reductive groups G it is also known as the G-character variety of X.

When X is the underlying space of a smooth complex variety, the Betti moduli space is one of the moduli spaces studied in non-abelian Hodge theory. In the case of a smooth complex projective curve C and the algebraic group $G = \mathrm{GL}_n(\mathbb{C})$, the Betti moduli space parametrizes vector bundles over C of rank n and degree zero equipped with a flat connection. With this identification, the Riemann–Hilbert correspondence [28] provides a real analytic isomorphism between the character variety $\mathcal{M}_G(C)$ and the moduli space of G-flat connections on the curve C. Moreover, the Hitchin–Kobayashi correspondence [27] gives a real analytic isomorphism between $\mathcal{M}_G(C)$ and the moduli space of semistable G-Higgs bundles of rank n and degree zero on C. These correspondences were used by Hitchin [16] to compute the Poincaré polynomial of character varieties for $G = \mathrm{GL}_2(\mathbb{C})$.

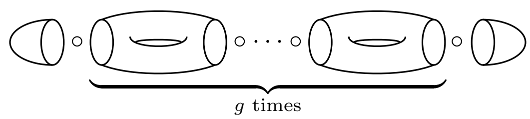
These correspondences are far from being algebraic. As a result, the mixed Hodge structures of the above-mentioned moduli spaces have been extensively studied via the Deligne–Hodge polynomial, or E-polynomial. For a complex variety X it is given by

$$e(X) = \sum_{k,p,q} (-1)^k h_c^{k;p,q}(X) u^p v^q$$

with $h_c^{k;p,q}(X)$ the mixed Hodge numbers of the compactly supported cohomology of X. Inspired by the Weil conjectures, Hausel and Rodríguez-Villegas [15] developed an arithmetic approach to compute the E-polynomial of the $\mathrm{GL}_n(\mathbb{C})$-character variety by counting its number of points over finite fields using the character table of $\mathrm{GL}_n(\mathbb{F}_q)$. However, the expressions were not explicit, but given in terms of generating functions. Mereb [21] did the same for $G = \mathrm{SL}_n(\mathbb{C})$ and gave an explicit expression when $n = 2$. Baraglia and Hekmati [2] gave explicit expressions for the cases $G = \mathrm{GL}_3(\mathbb{C}), \mathrm{SL}_3(\mathbb{C})$. A disadvantage of this method is that it does not give much algebraic or geometric insight.

A geometric approach was introduced by Logares, Muñoz and Newstead [18]. Their idea was to divide the representation variety into pieces, and compute the E-polynomial piecewise. Then they studied what identifications are made when passing to the GIT quotient. Based on this technique, Martínez and Muñoz [20] gave an explicit expression for the E-polynomial of the $\mathrm{SL}_2(\mathbb{C})$-representation variety.

The recursive patterns in these computations lead González-Prieto, Logares and Muñoz to a new method [13], which uses Topological Quantum Field Theories (TQFTs) to compute the class of the representation variety in the Grothendieck ring of varieties $\mathrm{K}(\mathbf{Var}_{\mathbb{C}})$. TQFTs, coming from physics, were first introduced by Witten [30] and axiomatized by Atiyah [1]: a TQFT is given by a monoidal functor $Z : \mathbf{Bd}_n \to R\text{-}\mathbf{Mod}$ from the category of bordisms to the category of R-modules. In particular, a closed manifold X can be seen as a bordism $X : \varnothing \to \varnothing$, so we have $Z(X) : R \to R$, since $Z(\varnothing) = R$ by monoidality, and thus X has an associated invariant $Z(X)(1) \in R$. This new method uses a TQFT with $R = \mathrm{K}(\mathbf{Var}_{\mathbb{C}})$ and where the invariants are the classes of $\mathfrak{X}_G(X)$. Now, a closed surface $X = \Sigma_g$ of genus g can be considered as a composition of bordisms



so computing the TQFT for these smaller bordisms will yield the class of $\mathfrak{X}_G(\Sigma_g)$ for all g. This method was used in [11] and [12] to compute expressions for the (parabolic) $\mathrm{SL}_2(\mathbb{C})$-character variety. An advantage of this method is that it not only computes the E-polynomial, but it keeps track of the virtual classes in the Grothendieck ring.

In this thesis, we lay out the theory needed to construct the TQFT used in this method. In Section 2 we define the Grothendieck ring of varieties, and discuss some basic constructions and properties. We will give an algorithm that computes the virtual class of certain affine varieties over $\mathbb{C}$ in this ring, in terms of the class of the affine line. Monoidal categories and TQFTs will be discussed in Section 3, and in particular, following [17], we will look at the equivalence of categories that arises in dimension 2 between TQFTs and Frobenius algebras. The actual TQFT used for this method, which is only lax monoidal, will be constructed in Section 4. Also we describe a criterion for when a TQFT can be modified or reduced to simplify the computations. Finally, in Section 5 we will apply this theory to the groups of upper triangular matrices $G = \mathbb{U}_n$ of rank $n = 2, 3, 4$, to compute the virtual class of the G-representation variety. These computations have not been done before, and the results can be summarized as follows.

Theorem 1.1. *Let $q = [\mathbb{A}^1_{\mathbb{C}}]$ be the class of the affine line in the Grothendieck ring of varieties. Then*

- *the virtual class of the $\mathbb{U}_2$-representation variety $\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)$ is*

$$[\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)] = q^{2g-1}(q-1)^{2g+1}((q-1)^{2g-1}+1),$$

- *the virtual class of the $\mathbb{U}_3$-representation variety $\mathfrak{X}_{\mathbb{U}_3}(\Sigma_g)$ is*

$$[\mathfrak{X}_{\mathbb{U}_3}(\Sigma_g)] = q^{3g-3}(q-1)^{2g}\left(q^2(q-1)^{2g+1} + q^{3g}(q-1)^2 + q^{3g}(q-1)^{4g} + 2q^{3g}(q-1)^{2g+1}\right),$$

- *the virtual class of the $\mathbb{U}_4$-representation variety $\mathfrak{X}_{\mathbb{U}_4}(\Sigma_g)$ is*

$$\begin{aligned}[\mathfrak{X}_{\mathbb{U}_4}(\Sigma_g)] = {} & q^{8g-2}(q-1)^{4g+2} + q^{8g-2}(q-1)^{6g+1} + q^{10g-4}(q-1)^{2g+3} \\ & + q^{10g-4}(q-1)^{4g+1}\left(2q^2-6q+5\right)^g + 3q^{10g-4}(q-1)^{4g+2} + q^{10g-4}(q-1)^{6g+1} \\ & + q^{12g-6}(q-1)^{8g} + q^{12g-6}(q-1)^{2g+3} + 3q^{12g-6}(q-1)^{4g+2} + 3q^{12g-6}(q-1)^{6g+1}.\end{aligned}$$

The result for $G = \mathbb{U}_2$ can be compared to a result from a very recent paper [14] where the computations were done for $G = \mathrm{AGL}_1(\mathbb{C})$, the group of affine transformations of the line. The resemblance comes from the fact that $\mathbb{U}_2 \simeq \mathbb{C}^* \times \mathrm{AGL}_1(\mathbb{C})$ as affine group varieties.

Finally, in Section 5.2 we describe the moduli space of $\mathbb{U}_n$-representations and discuss its relation to the $\mathbb{U}_n$-character variety.

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2 The Grothendieck ring of varieties

Let k be a field.

Definition 2.1. Let S be a variety over k (i.e. a reduced separated scheme of finite type over k). The *Grothendieck ring of varieties* over S, denoted $\mathrm{K}(\mathbf{Var}/S)$, is defined as the quotient of the free abelian group on the set of isomorphism classes of varieties over S, by relations of the form

$$[X] = [X \backslash Z] + [Z]$$

where Z is a closed subvariety of X. Multiplication is distributively induced by

$$[X] \cdot [Y] = [(X \times_S Y)_{\mathrm{red}}].$$

By Lemma 2.3 below, this operation is well-defined. From the definition we see it is associative and commutative as well. It follows that $[\varnothing] = 0$ and $[S] = 1$ in $\mathrm{K}(\mathbf{Var}/S)$.

Remark 2.2. Note that we could include isomorphism classes of non-reduced X over S in the definition above. However, since we always have a closed immersion $X_{\mathrm{red}} \subset X$, it follows that we would have $[X] = [\varnothing] + [X_{\mathrm{red}}]$, and since $[\varnothing] = 0$, we have $[X] = [X_{\mathrm{red}}]$. Hence, these extra classes would not contain any additional information.

Lemma 2.3. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of varieties over k. Let Z be a closed subvariety of X, and $U = X \backslash Z$ its open complement. Then $Z \times_S Y$ is closed in $X \times_S Y$, and its open complement is $U \times_S Y$. In particular, $([U] + [Z]) \cdot [Y] = [U] \cdot [Y] + [Z] \cdot [Y]$.*

Proof. This follows from the fact that closed and open immersions are stable under pullback. □

When $S = \operatorname{Spec} k$, we write $\mathrm{K}(\mathbf{Var}_k)$ for the corresponding Grothendieck ring. To distinguish between the classes of different rings, we will write $[X]_S$ for the class of X in $\mathrm{K}(\mathbf{Var}/S)$ and for the class of X in $\mathrm{K}(\mathbf{Var}_k)$ we will simply write $[X]$. We will also write $\times_k$ or $\times$ instead of $\times_{\operatorname{Spec} k}$.

In this section we will consider some of the basic properties and constructions related to the Grothendieck ring, that will be needed in later sections. For more information on this topic, see [4, 5, 23].

Example 2.4. Writing q for the class $[\mathbb{A}^1_k]$, we find that $[\mathbb{A}^n_k] = q^n$. Also it follows that $[\mathbb{P}^n_k] = q^n + q^{n-1} + \ldots + 1$, using inductively that $\mathbb{P}^n_k \backslash \mathbb{P}^{n-1}_k = \mathbb{A}^n_k$.

Example 2.5. Consider the *special linear group* $\mathrm{SL}_2(\mathbb{C}) = \operatorname{Spec} \mathbb{C}[a, b, c, d]/(ad - bc - 1)$ over $k = \mathbb{C}$. This is an affine algebraic group, i.e. an affine variety with a group structure given by morphisms of varieties: the multiplication map $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ is given by the ring morphism

$$\begin{aligned} A \to A \otimes_{\mathbb{C}} A : \quad & a \mapsto a \otimes a + b \otimes c, \qquad && b \mapsto a \otimes b + b \otimes d, \\ & c \mapsto c \otimes a + d \otimes c, \qquad && d \mapsto c \otimes b + d \otimes d, \end{aligned}$$

and the inversion map by

$$A \to A: \quad a \mapsto d, \qquad b \mapsto -b,$$
$$c \mapsto -c, \qquad d \mapsto a,$$

where $A = \mathbb{C}[a,b,c,d]/(ad-bc-1)$. (Indeed this simply describes multiplication and inversion of matrices). We compute the class $[\mathrm{SL}_2(\mathbb{C})] \in \mathrm{K}(\mathbf{Var}_\mathbb{C})$ as follows:

$$[\mathrm{SL}_2(\mathbb{C}) \cap \{a=0\}] = [\{(b,c,d) \in \mathbb{C}^3 : bc-1\}] = q(q-1)$$

using that $\mathrm{Spec}\,\mathbb{C}[b,c]/(bc-1) \simeq \mathbb{C}^* \simeq \mathbb{A}^1_\mathbb{C}\backslash\{0\}$ has class $q-1$. Furthermore,

$$[\mathrm{SL}_2(\mathbb{C}) \cap \{a \neq 0\}] = [\{(a,b,c,d) \in \mathbb{C}^* \times \mathbb{C}^3 : d = (bc+1)/a\}] = (q-1)q^2.$$

Hence $[\mathrm{SL}_2(\mathbb{C})] = q(q-1) + (q-1)q^2 = q(q-1)(q+1)$.

Example 2.6. Consider the *general linear group* $\mathrm{GL}_2(\mathbb{C}) = \mathrm{Spec}\,\mathbb{C}[a,b,c,d,(ad-bc)^{-1}]$ over $k=\mathbb{C}$, an affine algebraic group as well. We find that

$$[\mathrm{GL}_2(\mathbb{C})] = [\mathbb{C}^4] - [\{(a,b,c,d) \in \mathbb{C}^4 : ad-bc=0\}]$$

with

$$[\{(a,b,c,d) \in \mathbb{C}^4 : ad-bc=0\} \cap \{a=0\}] = [\{(b,c,d) \in \mathbb{C}^3 : bc=0\}] = q(2q-1)$$

and

$$[\{(a,b,c,d) \in \mathbb{C}^4 : ad-bc=0\} \cap \{a \neq 0\}] = [\{(a,b,c,d) \in \mathbb{C}^* \times \mathbb{C}^3 : d = bc/a\}] = (q-1)q^2.$$

Hence, $[\mathrm{GL}_2(\mathbb{C})] = q^4 - q(2q-1) - (q-1)q^2 = q(q-1)^2(q+1)$. Comparing to the previous example, it is no coincidence that $[\mathrm{GL}_2(\mathbb{C})] = (q-1)[\mathrm{SL}_2(\mathbb{C})]$, since in general we have an isomorphism of varieties

$$\mathbb{C}^* \times \mathrm{SL}_n(\mathbb{C}) \xrightarrow{\sim} \mathrm{GL}_n(\mathbb{C}),$$

sending (x, A) to the matrix A with its first column multiplied by x. An inverse map is given by $A \mapsto (\det(A), A')$, where A' is A with its first column divided by $\det(A)$. Note that this is not an isomorphism of algebraic groups.

Example 2.7. The main point of the thesis is to compute the class of *representation varieties* w.r.t. groups of upper triangular matrices $\mathbb{U}_n \subset \mathrm{GL}_n(\mathbb{C})$. As a variety, $\mathbb{U}_n$ can be identified with $(\mathbb{C}^*)^n \times \mathbb{C}^{n(n-1)/2}$ since it consists of all upper triangular matrices with non-zero elements on the diagonal. Therefore, its class is $[\mathbb{U}_n] = q^{n(n-1)/2}(q-1)^n$.

The Grothendieck ring of varieties is of particular interest when studying *additive invariants* of varieties. An additive invariant is a map $\lambda : \mathrm{Ob}(\mathbf{Var}_k) \to R$ with R a ring, such that (i) $\lambda(X) = \lambda(Y)$ for isomorphic varieties X and Y, (ii) $\lambda(X) = \lambda(X\backslash Y) + \lambda(Y)$ when $Y \subset X$ is a closed subscheme, and (iii) $\lambda(X \times_k Y) = \lambda(X) \cdot \lambda(Y)$. From the definition of the Grothendieck ring we see that such maps will factor as

$$\mathrm{Ob}(\mathbf{Var}_k) \to \mathrm{K}(\mathbf{Var}_k) \to R.$$

For example, when $k = \mathbb{F}_q$ is a finite field, the map $N : \mathrm{Ob}(\mathbf{Var}_k) \to \mathbb{Z} : X \mapsto \#X(k)$ which yields the number of k-points is an additive invariant. For $k = \mathbb{C}$, an important example is the *E-polynomial*, also known as the *Deligne-Hodge polynomial* [15, 18, 23]. When X is a smooth projective variety, its E-polynomial is given by its Hodge polynomial

$$e(X) = \sum_{p,q=0}^{\dim(X)} (-1)^{p+q} h^{p,q}(X) u^p v^q \in \mathbb{Z}[u, v],$$

where $h^{p,q}(X) = \dim_{\mathbb{C}} H^q(X, \Omega_X^p)$ are the *hodge numbers* of X. It can be shown that the classes of smooth projective varieties generate the Grothendieck ring [4], and that e extends uniquely in this way to an additive invariant defined for all complex varieties [23]. Alternatively, one can define the E-polynomial of a complex variety X to be

$$e(X) = \sum_{k,p,q} (-1)^k h_c^{k;p,q}(X) u^p v^q,$$

where the coefficients $h_c^{k;p,q}(X)$ are the mixed Hodge numbers of the compactly supported cohomology of X. When X is a smooth projective variety, its Euler characteristic

$$\chi(X) = \sum_{p,q=0}^{\dim(X)} (-1)^{p+q} h^{p,q}(X)$$

can be obtained by evaluating $e(X)$ in $u = v = 1$. The *virtual Euler characteristic* of any complex variety X is obtained by evaluating $e(X)$ in $u = v = 1$.

Remark 2.8. Although the classes in the examples treated so far can all be expressed in terms of $q = [\mathbb{A}_k^1]$, this is not true for every class in $\mathrm{K}(\mathbf{Var}_k)$, and we can show this using E-polynomials. From the cohomologies of $\mathbb{P}_{\mathbb{C}}^1$ and the point $\star$ one can compute $e(\mathbb{P}_{\mathbb{C}}^1) = 1 + uv$ and $e(\star) = 1$, hence $e(\mathbb{A}_{\mathbb{C}}^1) = e(\mathbb{P}_{\mathbb{C}}^1) - e(\star) = uv$. But from the cohomology of an elliptic curve E over $\mathbb{C}$, one can show $e(E) = 1 - u - v + uv$, which is not generated by $e(\mathbb{A}_{\mathbb{C}}^1) = uv$, so it follows that $[E]$ is not generated by $[\mathbb{A}_{\mathbb{C}}^1]$.

Definition 2.9. A *stratification* for a variety X is a collection of disjoint locally closed subsets $X_i \subset X$ covering X.

Lemma 2.10. *Let X be a variety stratified by subvarieties $X_i \subset X$. Then only finitely many of the X_i are non-empty, and $[X] = \sum_{i=1}^n [X_i]$.*

Proof. We will prove this by induction on the dimension of X. If this dimension is zero, then X is a finite set of points, and the result is clear. Now we can assume that the result holds for all varieties of dimension less than $\dim X$.

First we consider the case where X is irreducible. Note that some $U = X_i$ contains the generic point of X and is therefore open. The complement $Z = X \backslash U$ must be of smaller dimension than X, and is stratified by the other X_i. We have $[X] = [U] + [Z]$, so the result follows from the induction hypothesis.

Now consider the case where X is reducible. Take an irreducible component and remove the intersections with the other irreducible components, which gives an irreducible open subset $U \subset X$.

The complement $Z = X \backslash U$ is a closed subvariety with fewer irreducible components than X. Since $\{Z \cap X_i\}$ is a stratification of Z and $\{U \cap X_i\}$ is one for U, apply induction on the number of irreducible components of X to find that only finitely many are non-empty, and we have

$$[U] = \sum_i [U \cap X_i], \qquad \text{and} \qquad [Z] = \sum_i [Z \cap X_i].$$

Since $[X_i] = [U \cap X_i] + [Z \cap X_i]$, it now follows that $[X] = [U] + [Z] = \sum_i [X_i]$. $\square$

Lemma 2.11. *Let $f : X \to Y$ be an algebraic bundle with fiber F, i.e. there exists a Zariski-open cover $Y = \cup_{i \in I} U_i$ and isomorphisms $g_i : f^{-1}(U_i) \to U \times F$ such that each*

$$\begin{tikzcd} f^{-1}(U_i) \arrow[rr, "g_i"] \arrow[dr, "f"'] & & U_i \times F \arrow[dl, "\pi_i"] \\ & U_i & \end{tikzcd}$$

commutes. Then $[X] = [Y] \cdot [F]$ in $K(\boldsymbol{Var}_k)$.

Proof. From the given open cover, one can construct a stratification for Y. Let $Z_0 = Y$ and inductively define Z_{j+1} for $j \geq 0$ as follows: if $Z_j \neq \varnothing$, then there exists some $i \in I$ such that $Z_j \cap U_i \neq \varnothing$, and we set $Z_{j+1} = Z_j \backslash (Z_j \cap U_i)$. Since Y is noetherian, this results in a finite descending chain of closed sets

$$Y = Z_0 \supsetneq Z_1 \supsetneq \ldots \supsetneq Z_n \supsetneq Z_{n+1} = \varnothing.$$

Now the locally closed sets $Y_j = Z_j \backslash Z_{j+1}$ for $j = 0, 1, \ldots, n$ form a stratification for Y. Moreover, as $Y_j \subset U_i$ for some i by construction, we have that f is a trivial fibration over each Y_i, i.e. $f^{-1}(Y_i) \simeq Y_i \times F$. Using Lemma 2.10 we conclude

$$[X] = \sum_{j=0}^{n} \left[f^{-1}(Y_j) \right] = \sum_{i=0}^{n} [Y_j] \cdot [F] = [Y] \cdot [F].$$

$\square$

Example 2.12. An alternative way to compute class of $\mathrm{SL}_2(\mathbb{C})$ would be to note that the map $\mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}^2 - \{(0,0)\}$ which sends a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $A \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$ is an algebraic bundle with fiber $\mathbb{C}$. Indeed, $U_1 = \{(a,c) \in \mathbb{C}^2 : a \neq 0\}$ and $U_2 = \{(a,c) \in \mathbb{C}^2 : c \neq 0\}$ define an open cover on which the map is a trivial fibration. By the above lemma we find $[\mathrm{SL}_2(\mathbb{C})] = [\mathbb{C}^2 - \{(0,0)\}] \cdot [\mathbb{C}] = (q^2 - 1)q = q^3 - q$.

2.1 Modules over the Grothendieck ring

Let X be a variety over k. We can give $\mathrm{K}(\mathbf{Var}/X)$ a $\mathrm{K}(\mathbf{Var}_k)$-module structure induced by

$$[V] \cdot [Y]_X = [V \times_k Y]_X$$

for V a variety over k and Y a variety over X. Again using Lemma 2.3 we see that this gives a well-defined module-structure.

Let $f : X \to Y$ be a morphism of varieties over k. Composition with f yields a functor

$$\begin{aligned} f_! : \mathbf{Var}/X &\to \mathbf{Var}/Y \\ (V \xrightarrow{g} X) &\mapsto (V \xrightarrow{fg} Y). \end{aligned}$$

As $f_!$ sends isomorphisms to isomorphisms, and also $S \times_k f_! V = f_!(S \times_k V)$ for any variety S, we have that $f_!$ induces a $\mathrm{K}(\mathbf{Var}_k)$-module morphism

$$f_! : \mathrm{K}(\mathbf{Var}/X) \to \mathrm{K}(\mathbf{Var}/Y).$$

Note that this map will in general not be a ring morphism. E.g. the unit $[X]_X \in \mathrm{K}(\mathbf{Var}/X)$ need not be sent to the unit $[Y]_Y \in \mathrm{K}(\mathbf{Var}/Y)$.

Similarly, pulling back along f yields a functor

$$f^* : \mathbf{Var}/Y \to \mathbf{Var}/X$$

sending $W \xrightarrow{h} Y$ to $W \times_Y X \xrightarrow{f^*h} X$. Again, f^* sends isomorphisms to isomorphisms, and in combination with Lemma 2.3, we obtain the induced map

$$f^* : \mathrm{K}(\mathbf{Var}/Y) \to \mathrm{K}(\mathbf{Var}/X)$$

which is also a $\mathrm{K}(\mathbf{Var}_k)$-module morphism as $S \times_k f^*(V) = S \times_k (V \times_Y X) = (S \times_k V) \times_Y X = f^*(S \times_k V)$. In contrast to $f_!$, the map f^* is a ring morphism as $(V \times_Y W) \times_Y X = (V \times_Y X) \times_X (W \times_Y X)$ for any V, W over Y.

Remark 2.13. The functors f^* and $f_!$ are adjoint, as for any varieties $V \xrightarrow{v} X$ and $W \xrightarrow{w} Y$ there is a bijection

$$\mathrm{Hom}_{\mathbf{Var}/Y}(f_! V, W) \simeq \mathrm{Hom}_{\mathbf{Var}/X}(V, W \times_Y X)$$

natural in V and W. Namely, by the universal property of the fiber product, to give a morphism $\varphi : V \to W \times_Y X$ is to give morphisms $V \xrightarrow{r} W$ and $V \xrightarrow{s} X$ such that $w \circ r = f \circ s$, and requiring φ to be over X means to have $s = v$. Hence, to give φ over X is to give $V \xrightarrow{r} W$ such that $w \circ r = v$, i.e. a morphism $V \xrightarrow{r} W$ over Y. The naturality of this bijection is easily seen.

Finally, we note that there is also the $\mathrm{K}(\mathbf{Var}_k)$-module morphism

$$\begin{array}{ccc} \mathrm{K}(\mathbf{Var}/X) \otimes_{\mathrm{K}(\mathbf{Var}_k)} \mathrm{K}(\mathbf{Var}/Y) & \to & \mathrm{K}(\mathbf{Var}/(X \times_k Y)) \\ [V] \otimes [W] & \mapsto & [V \times_k W]. \end{array} \tag{2.1}$$

This map need not be an isomorphism.

When studying varieties over X, it can happen to be more convenient to look at their restrictions to open or closed subsets of X. The following lemma shows how one can decompose the module $\mathrm{K}(\mathbf{Var}/X)$.

Lemma 2.14. *Let X be a variety over k, $Z \subset X$ a closed subvariety, and denote $U = X \backslash Z$. Write $j : Z \to X$ and $i : U \to X$ for the corresponding closed/open immersions. Then there is a*

$K(\textbf{\textit{Var}}_k)$*-module isomorphism induced by*

$$\begin{aligned} K(\textbf{\textit{Var}}/X) &\simeq K(\textbf{\textit{Var}}/Z) \oplus K(\textbf{\textit{Var}}/U) \\ x &\mapsto (j^*(x), i^*(x)) \\ j_!(z) + i_!(u) &\leftarrow\!\shortmid (z, u). \end{aligned}$$

Proof. Indeed both maps are $\mathrm{K}(\mathbf{Var}_k)$-module morphisms. For any variety $V \xrightarrow{f} X$, we have $[V] = [f^{-1}(U)] + [f^{-1}(Z)]$, so $x = j_!(j^*(x)) + i_!(i^*(x))$ for all $x \in \mathrm{K}(\mathbf{Var}/X)$. On the other hand, $j^*(j_!(z)+i_!(u)) = z$ and $i^*(j_!(z)+i_!(u)) = u$ for all $(z, u) \in \mathrm{K}(\mathbf{Var}/Z) \oplus \mathrm{K}(\mathbf{Var}/U)$ since $Z \cap U = \varnothing$ and both $j^* j_!$ and $i^* i_!$ are identity maps. ☐

Corollary 2.15. *Let X be a variety over k, and $\{X_1, \dots, X_n\}$ a stratification for X. Then we have a* $K(\textbf{\textit{Var}}_k)$*-module isomorphism*

$$\begin{aligned} K(\textbf{\textit{Var}}/X) &\simeq K(\textbf{\textit{Var}}/X_1) \oplus \cdots \oplus K(\textbf{\textit{Var}}/X_n) \\ x &\mapsto (j_1^*(x), \dots, j_n^*(x)) \\ (j_1)_!(x_1) + \cdots + (j_n)_!(x_n) &\leftarrow\!\shortmid (x_1, \dots, x_n). \end{aligned}$$

☐

2.2 Algorithmically computing classes

In later sections, a lot of classes in $\mathrm{K}(\mathbf{Var}_{\mathbb{C}})$ will need to be computed of affine varieties over $\mathbb{C}$ in terms of $q = [\mathbb{A}^1_{\mathbb{C}}]$. These computations can partly be automated. For this section, we will use the following notation. Let $S = \{x_1, \dots, x_n\}$ be a finite set (of variables), and F, G be finite subsets of $\mathbb{C}[S]$. Then we write $X(S, F, G)$ for the (reduced) subvariety of $\mathbb{C}^n$ given by $f = 0$ for all $f \in F$ and $g \neq 0$ for all $g \in G$. For example,

$$\mathbb{C}^n = X(\{x_1, \dots, x_n\}, \varnothing, \varnothing) \quad \text{and} \quad \mathrm{GL}_2(\mathbb{C}) = X(\{a, b, c, d\}, \varnothing, \{ad - bc\}).$$

For convenience we will write $\mathrm{ev}_x(f, u)$ with $f \in \mathbb{C}[S]$ for the polynomial where $x \in S$ in f is substituted for $u \in \mathbb{C}[S]$. Then we write

$$\mathrm{ev}_x(F, u) = \{\mathrm{ev}_x(f, u) : f \in F\}$$

and

$$\mathrm{ev}_x(F, u, v) = \{v^{\deg_x(f)} \cdot \mathrm{ev}_x(f, u/v) : f \in F\}$$

for a set of polynomials F and $x \in S$ and $u, v \in \mathbb{C}[S]$. Note that for $\mathrm{ev}_x(F, u, v)$ the substituted polynomials are multiplied by a suitable number of factors v, in order to clear denominators. Now consider the following recursive algorithm.

Algorithm 2.16. Let $X = X(S, F, G)$ for some S, F and G as above.

1. If F contains a non-zero constant or if $0 \in G$, then $X = \varnothing$, so $[X] = 0$.

2. If $F = \varnothing$ and $G = \varnothing$, then $X = \mathbb{C}^{\#S}$, so $[X] = q^{\#S}$.

3. If some $x \in S$ 'does not appear' in any $f \in F$ and any $g \in G$, then we can factor $X \simeq \mathbb{C} \times X'$ with $X' = X(S - \{x\}, F, G)$. We have $[X] = q[X']$.

4. If $f = u^n$ (with $n > 1$) for some $f \in F$ and $u \in \mathbb{C}[S]$, then we can replace f with u, not changing X. That is, $X = X(S, F - \{f\} + \{u\}, G)$. Similarly, if $g = u^n$ (with $n > 1$) for some $g \in G$ and $u \in \mathbb{C}[S]$, then $X = X(S, F, G - \{g\} + \{u\})$.

5. If some $f \in F$ is univariate in $x \in S$, we write $f = (x - \alpha_1) \cdots (x - \alpha_m)$, and we have $[X] = \sum_{i=1}^{m} [X_i]$ with $X_i = X(S - \{x\}, \mathrm{ev}_x(F - \{f\}, \alpha_i), \mathrm{ev}_x(G, \alpha_i))$.

6. If $f = uv$ for some $f \in F$ and $u, v \in \mathbb{C}[S]$ (both not constant), then $X_1 = X(S, F - \{f\} + \{u\}, G) = X \cap \{u = 0\}$ and $X_2 = X(S, F - \{f\} + \{v\}, G + \{u\}) = X \cap \{u \neq 0, v = 0\}$ define a stratification for X, and thus $[X] = [X_1] + [X_2]$.

7. If $f = xu + v$ for some $f \in F$, $x \in S$ and $u, v \in \mathbb{C}[S]$ with x not appearing in u and v, then we consider the following cases. For any point p of X, either $u(p) = 0$, implying $v(p) = 0$ as well, or $u(p) \neq 0$, implying $x(p) = -v(p)/u(p)$. Therefore $[X] = [X_1] + [X_2]$ with $X_1 = X(S, F - \{f\} + \{u, v\}, G)$ and $X_2 = X(S, \mathrm{ev}_x(F - \{f\}, -v, u)), \mathrm{ev}_x(G, -v, u) + \{u\})$.

8. Suppose $f = x^2 u + xv + w$ for some $f \in F$, $x \in S$ and $u, v, w \in \mathbb{C}[S]$ with x not appearing in u, v and w. Moreover, suppose that the discriminant $D = v^2 - 4uw$ is a square, i.e. we can write $D = h^2$ for some $h \in \mathbb{C}[S]$. Then for any point p of X, we consider the following cases. Either $u(p) = 0$, in which case $(xv + w)(p) = 0$. If $u(p) \neq 0$, we distinguish between $D(p) = 0$ and $D(p) \neq 0$. In the first case we find that $x(p) = \left(\frac{-v}{2u}\right)(p)$, and in the latter case we have the two possibilities $x(p) = \left(\frac{-v \pm h}{2u}\right)(p)$. Hence $[X] = [X_1] + [X_2] + [X_3] + [X_4]$, with

$$\begin{aligned}
X_1 &= X(S, F - \{f\} + \{u, xv + w\}, G), \\
X_2 &= X(S, \mathrm{ev}_x(F - \{f\}, -v, 2u) + \{D\}, \mathrm{ev}_x(G, -v, 2u) + \{u\}), \\
X_3 &= X(S, \mathrm{ev}_x(F - \{f\}, -v - h, 2u), \mathrm{ev}_x(G, -v - h, 2u) + \{u, D\}), \\
X_4 &= X(S, \mathrm{ev}_x(F - \{f\}, -v + h, 2u), \mathrm{ev}_x(G, -v + h, 2u) + \{u, D\}).
\end{aligned}$$

9. If $G \neq \varnothing$, pick any $g \in G$. We have $[X] = [X_1] - [X_2]$ where $X_1 = X(S, F, G - \{g\})$ and $X_2 = X(S, F + \{g\}, G)$.

An implementation of this algorithm in Python can be found at [29], together with the code for the computations done in the later sections. Note that this algorithm may not be optimally efficient, and as pointed out in Remark 2.8 not every variety's class is generated by q. However, the varieties we will deal with do have a class generated by q, so this algorithm is sufficient for our purposes.

Example 2.17. The computations in examples 2.5 and 2.6 for $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{GL}_2(\mathbb{C})$ were instances of this algorithm done by hand.

We have $\mathrm{SL}_3(\mathbb{C}) = X(\{a, b, c, d, e, f, g, h, i\}, \{aei - afh - bdi + bfg + cdh - ceg - 1\}, \varnothing)$.

$$[\mathrm{SL}_3(\mathbb{C})] = q^3(q-1)^2(q+1)(q^2+q+1)$$

We have $\mathrm{GL}_3(\mathbb{C}) = X(\{a, b, c, d, e, f, g, h, i\}, \varnothing, \{aei - afh - bdi + bfg + cdh - ceg\})$.

$$[\mathrm{GL}_3(\mathbb{C})] = q^3(q-1)^3(q+1)(q^2+q+1)$$

The last of these can be checked against [5, Lemma 2.6] for $d = 3$. As pointed out before, we indeed have $[\mathrm{GL}_n(\mathbb{C})] = (q-1)[\mathrm{SL}_n(\mathbb{C})]$.

3 Topological Quantum Field Theories

As mentioned, the first axiomatic description of TQFTs was given by Atiyah [1]. Roughly speaking, an n-dimensional TQFT over a commutative ring R, assigns an R-module to every $(n-1)$-dimensional closed manifold, and R-linear maps between them for every n-dimensional manifold that connects two boundaries. This assignment should satisfy certain functorial and multiplicative properties, which are best described in terms of monoidal categories and monoidal functors.

3.1 Monoidal categories

Definition 3.1. A *monoidal category* is a category $\mathcal{C}$ with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (the tensor product), an object 1 in $\mathcal{C}$ (the unital object) and natural isomorphisms

$$\begin{array}{ccc} \alpha : - \otimes (- \otimes -) \Rightarrow (- \otimes -) \otimes -, & \lambda : 1 \otimes - \Rightarrow \mathrm{id}_{\mathcal{C}}, & \rho : - \otimes 1 \Rightarrow \mathrm{id}_{\mathcal{C}} \\ \text{(associator)} & \text{(left unitor)} & \text{(right unitor)} \end{array}$$

such that the triangle

$$\begin{tikzcd} (A \otimes 1) \otimes B \arrow[rr, "\alpha_{A,1,B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"'] & & A \otimes (1 \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_B"] \\ & A \otimes B & \end{tikzcd}$$

and the pentagon

$$\begin{tikzcd} & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\ ((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] & & A \otimes (B \otimes (C \otimes D)) \\ (A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "\mathrm{id}_A \otimes \alpha_{B,C,D}"'] \end{tikzcd}$$

commute for all objects A, B, C and D in $\mathcal{C}$. In case the natural isomorphisms α, λ and ρ are all equalities, we say that such a category is *strict*. The above triangle and pentagon then commute automatically.

Definition 3.2. A *symmetric* monoidal category is a monoidal category $\mathcal{C}$ equipped with natural isomorphisms

$$\tau_{A,B} : A \otimes B \to B \otimes A$$

such that

$$\tau_{B,A} \circ \tau_{A,B} = \mathrm{id}_{A \otimes B}$$

and the diagrams

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\tau_{A,B} \otimes \mathrm{id}_C"'] & A \otimes (B \otimes C) \arrow[r, "\tau_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\ (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[r, "\mathrm{id}_B \otimes \tau_{A,C}"] & B \otimes (C \otimes A) \end{tikzcd}$$

and

$$\begin{array}{ccccc} A\otimes(B\otimes C) & \xrightarrow{\alpha^{-1}_{A,B,C}} & (A\otimes B)\otimes C & \xrightarrow{\tau_{A\otimes B,C}} & C\otimes(A\otimes B) \\ \Big\downarrow{\scriptstyle \mathrm{id}_A\otimes\tau_{B,C}} & & & & \Big\downarrow{\scriptstyle \alpha^{-1}_{C,A,B}} \\ A\otimes(B\otimes C) & \xrightarrow{\alpha^{-1}_{A,C,B}} & (A\otimes C)\otimes B & \xrightarrow{\tau_{A,C}\otimes\mathrm{id}_B} & (C\otimes A)\otimes B \end{array}$$

commute for all A, B and C in $\mathcal{C}$.

Example 3.3.
- The category **Set** with the disjoint union operator $\sqcup$ as tensor product, and the empty set $\varnothing$ as unital object is a symmetric monoidal category. Also **Set** with the Cartesian product $\times$ as tensor product and a singleton set as unital object is a symmetric monoidal category.
- Equivalent to **Set** is the category of cardinal numbers, which is monoidal with addition as tensor product and 0 as unital object, and also with multiplication and 1. Moreover, in both cases this is a strict monoidal category.
- Let R be a commutative ring, then the category R-**Mod** (or R-**Alg**) with the tensor product $\otimes_R$ and R as unital object is a symmetric monoidal category. Also R-**Mod** with $\oplus$ as tensor product and 0 as unital object is a symmetric monoidal category.
- The category R-**Bimod** of R-bimodules is monoidal as well (with $\otimes_R$ and R), but not necessarily symmetric. For example, take $R = k$ a field with two non-commuting automorphisms σ, τ. Let $M = {}_1k_\sigma$ be the abelian group k with k-bimodule structure given by $a \cdot x \cdot b = ax\sigma(b)$, and let $N = {}_1k_\tau$ similarly. Then we have a k-bimodule isomorphism $\varphi : M \otimes_k N \xrightarrow{\sim} {}_1k_{\sigma\tau}$ given by $x \otimes y \mapsto x\sigma(y)$, and similarly $N \otimes_k M = {}_1k_{\tau\sigma}$. If there were to exist some k-bimodule isomorphism $\psi : {}_1k_{\sigma\tau} \xrightarrow{\sim} {}_1k_{\tau\sigma}$ it would be given by $\psi(x) = x\psi(1)$ (since ψ is a left k-module isomorphism). However, as ψ is a right k-module isomorphism as well, we must have $\psi(x) = \psi(1)\tau\sigma\tau^{-1}\sigma^{-1}(x)$, which implies that $x = \tau\sigma\tau^{-1}\sigma^{-1}(x)$ for all $x \in k$, but we assumed τ and σ did not commute. Hence such ψ does not exist, and k-**Bimod** cannot be symmetric.
- Geometrically, the category **Sch**/S of schemes over S, with the fiber product $\times_S$ as tensor product, and S as unital object is also a symmetric monoidal category.
- Let $\mathbf{GrVect}_k$ be the category whose objects are graded linear vector spaces $V = \oplus_{n\in\mathbb{Z}} V_n$ over k, and whose morphisms are linear maps that respect the grading. The tensor product of two graded vector spaces V and W is again graded, with grading $(V \otimes W)_n = \oplus_{p+q=n}(V_p \otimes W_q)$, making $\mathbf{GrVect}_k$ into a monoidal category, where the unital object is the ground field k concentrated in degree zero.

  There is more than one way to make $\mathbf{GrVect}_k$ symmetric. As the map $V \otimes W \to W \otimes V$ we could take the usual $\tau : v \otimes w \to w \otimes v$. However, we could also take the map $\kappa : v \otimes w \mapsto (-1)^{pq} w \otimes v$ with $p = \deg(v)$ and $q = \deg(w)$. One can check that κ indeed satisfies the axioms.

Definition 3.4. A *lax monoidal functor* is a functor $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories together with a natural transformation

$$\mu : F(-) \otimes_{\mathcal{D}} F(-) \Rightarrow F(- \otimes_{\mathcal{C}} -)$$

between $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$ functors, and a morphism $\varepsilon : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$, such that the diagrams

$$\begin{tikzcd}
(F(A) \otimes_{\mathcal{D}} F(B)) \otimes_{\mathcal{D}} F(C) \arrow[r, "\alpha^{\mathcal{D}}_{F(A),F(B),F(C)}"] \arrow[d, "\mu_{A,B} \otimes \mathrm{id}_{F(C)}"'] & F(A) \otimes_{\mathcal{D}} (F(B) \otimes_{\mathcal{D}} F(C)) \arrow[d, "\mathrm{id}_{F(A)} \otimes \mu_{B,C}"] \\
F(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{D}} F(C) \arrow[d, "\mu_{A \otimes_{\mathcal{C}} B, C}"'] & F(A) \otimes_{\mathcal{D}} F(B \otimes_{\mathcal{C}} C) \arrow[d, "\mu_{A, B \otimes_{\mathcal{C}} C}"] \\
F((A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C) \arrow[r, "F(\alpha^{\mathcal{C}}_{A,B,C})"] & F(A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C))
\end{tikzcd} \qquad \text{(associativity)}$$

and

$$\begin{tikzcd}
1_{\mathcal{D}} \otimes_{\mathcal{D}} F(A) \arrow[r, "\varepsilon \otimes \mathrm{id}_{F(A)}"] \arrow[d, "\lambda^{\mathcal{D}}_{F(A)}"'] & F(1_{\mathcal{C}}) \otimes_{\mathcal{D}} F(A) \arrow[d, "\mu_{1_{\mathcal{C}},A}"] \\
F(A) & F(1_{\mathcal{C}} \otimes_{\mathcal{C}} A) \arrow[l, "F(\lambda^{\mathcal{C}}_A)"]
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
F(A) \otimes_{\mathcal{D}} 1_{\mathcal{D}} \arrow[r, "\mathrm{id}_{F(A)} \otimes \varepsilon"] \arrow[d, "\rho^{\mathcal{D}}_{F(A)}"'] & F(A) \otimes_{\mathcal{D}} F(1_{\mathcal{C}}) \arrow[d, "\mu_{A,1_{\mathcal{C}}}"] \\
F(A) & F(A \otimes_{\mathcal{C}} 1_{\mathcal{C}}) \arrow[l, "F(\rho^{\mathcal{C}}_A)"]
\end{tikzcd} \qquad \text{(unitality)}$$

commute for all objects A, B and C in $\mathcal{C}$. Such a functor is said to be a *strong monoidal functor* or simply a *monoidal functor* if all $\mu_{A,B}$ and ε are isomorphisms.

A monoidal functor between symmetric monoidal categories is said to be *symmetric* if it respects the symmetric structure, i.e. the diagram

$$\begin{tikzcd}
F(A) \otimes_{\mathcal{D}} F(B) \arrow[r, "\tau^{\mathcal{D}}_{F(A),F(B)}"] \arrow[d, "\mu_{A,B}"'] & F(B) \otimes_{\mathcal{D}} F(A) \arrow[d, "\mu_{B,A}"] \\
F(A \otimes_{\mathcal{C}} B) \arrow[r, "F(\tau^{\mathcal{C}}_{A,B})"] & F(B \otimes_{\mathcal{C}} A)
\end{tikzcd}$$

commutes for all A and B in $\mathcal{C}$.

Example 3.5.

- Let $R \to R'$ be a morphism of commutative rings. Changing base $M \mapsto M \otimes_R R'$ yields a monoidal functor R-**Mod** $\to R'$-**Mod** which is symmetric.
- Analogously, geometrically the base change $X \mapsto X \times_S S'$ yields a monoidal functor $\mathbf{Sch}/S \to \mathbf{Sch}/S'$. For a morphism $S' \to S$.
- Not every monoidal functor between symmetric monoidal categories is automatically symmetric as well. The identity functor $\mathrm{id}_{\mathbf{GrVect}_k}$ is clearly monoidal, but it is not symmetric when seen as a functor $(\mathbf{GrVect}_k, \tau) \Rightarrow (\mathbf{GrVect}_k, \kappa)$ as it does not respect the symmetry.

For more information on monoidal categories, see [19].

3.2 The category of bordisms

Let $i : M \to \partial W$ be an inclusion, where W is an n-dimensional oriented manifold with boundary, and M an $(n-1)$-dimensional closed oriented manifold. (All manifolds we consider are assumed to be smooth.) Take a point $x \in i(M)$, let $\{v_1, \ldots, v_{n-1}\}$ be a positively oriented basis for $T_x i(M)$ w.r.t. the orientation induced by M, and pick some $w \in T_x i(M)$ that points inwards compared to

W. Then if $\{v_1, \ldots, v_{n-1}, w\}$ is a positively oriented basis for T_xW, we say x is an *in-boundary point*, and an *out-boundary point* otherwise. Note that this is independent of the chosen vectors v_i and w. If all $x \in i(M)$ are in-boundary (resp. out-boundary) points, we say i is an *in-boundary* (resp. *out-boundary*).

Definition 3.6. Given two $(n-1)$-dimensional closed oriented manifolds M and M', a *bordism* from M to M' is an n-dimensional oriented manifold W (with boundary) with maps

$$M' \xrightarrow{i'} W \xleftarrow{i} M$$

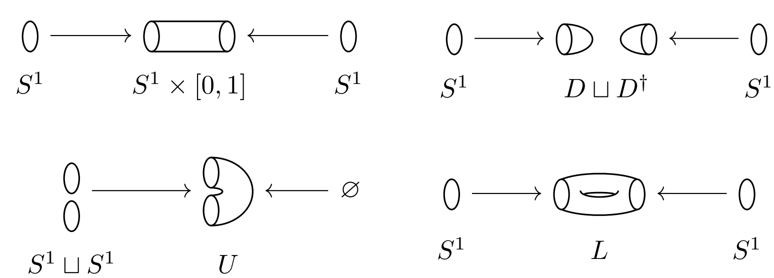
where i is an in-boundary, i' an out-boundary and $\partial W = i(M) \sqcup i'(M')$. (In the literature people also use the notation $\partial W = M \sqcup \overline{M'}$.) Two such bordisms W, W' are said to be *equivalent* if there exists an orientation-preserving diffeomorphism $W \xrightarrow{\sim} W'$ such that

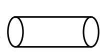
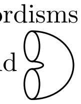
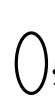
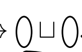
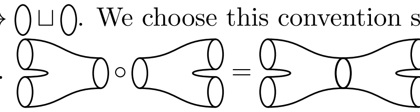
$$\begin{tikzcd} & W \arrow[dd, "\wr"] & \\ M' \arrow[ur] \arrow[dr] & & M \arrow[ul] \arrow[dl] \\ & W' & \end{tikzcd}$$

commutes.

For a more precise definition of bordisms, see [22] or [17].

Example 3.7. We provide some pictorial examples for $n = 2$ (without explicitly choosing orientations).



Remark 3.8. For simplicity, we will use a notation where we do not write down the source and target of the bordisms, but implicitly the bordisms goes from the 'boundary on the right' to the 'boundary on the left'. E.g. although  and  are diffeomorphic as manifolds, we use the first to denote the bordism  $\to$ , while the latter denotes the bordism $\varnothing \to$ . We choose this convention so that compositions of bordisms can be written down nicely, e.g. .

Suppose we have bordisms $W : M \to M'$ and $W' : M' \to M''$. One can glue W and W' as topological spaces by identifying the images of M', which we denote by $W \sqcup_{M'} W'$. By [22, Theorem 1.4], there exists a smooth manifold structure on $W \sqcup_{M'} W'$ such that the inclusions $W \to W \sqcup_{M'} W'$ and $W' \to W \sqcup_{M'} W'$ are diffeomorphisms onto their images, which is unique up to (non-unique) diffeomorphism. Hence $W \sqcup_{M'} W'$ belongs to a well-defined equivalence class, and moreover this class only depends on the classes of W and W'. Namely, if $\tilde{W} : M \to M'$ and $\tilde{W}' : M' \to M''$

are equivalent to W and W', respectively, then any such manifold structure on $W \sqcup_{M'} W'$ induces such a manifold structure on $\tilde{W} \sqcup_{M'} \tilde{W}'$ via the homeomorphism $W \sqcup_{M'} W' \to \tilde{W} \sqcup_{M'} \tilde{W}'$, showing $W \sqcup_{M'} W'$ and $\tilde{W} \sqcup_{M'} \tilde{W}'$ are equivalent. This implies that equivalence classes of bordisms can be composed to obtain an equivalence class of bordisms $M \to M''$.

Definition 3.9. The *category of n-bordisms*, denoted $\mathbf{Bd}_n$, is defined as follows. Its objects are $(n-1)$-dimensional closed oriented manifolds, and morphisms $M \to M'$ are equivalence classes of bordisms from M to M'. Composition is given by gluing along the common boundary: if $W : M \to M'$ and $W' : M' \to M''$, then $W' \circ W = W \sqcup_{M'} W' : M \to M''$.

It is easy to see that for any object M, the identity bordism is given by (the class of) the cylinder $M \times [0,1]$, with orientation such that the inclusion $M \to M \times \{0\}$ is an in-boundary and $M \to M \times \{1\}$ is an out-boundary.

Note that $\mathbf{Bd}_n$ is a symmetric monoidal category, the tensor product being given by taking disjoint unions, and $\varnothing$ being the unital object.

Lemma 3.10. *For an $(n-1)$-dimensional closed oriented manifold M, let $F(M)$ be M as an object in $\boldsymbol{Bd}_n$. For a diffeomorphism $\varphi : M \to M'$ between such manifolds, let $F(\varphi)$ be the bordism $M \to M'$ given by*

$$\begin{array}{ccccc} M' & \longrightarrow & M' \times [0,1] & \longleftarrow & M \\ x' & \mapsto & (x',1), \quad (\varphi(x),0) & \leftarrow\!\shortmid & x. \end{array}$$

Then F defines a functor from the category of $(n-1)$-dimensional closed oriented manifolds with diffeomorphisms to $\boldsymbol{Bd}_n$.

Proof. Indeed $F(\mathrm{id}_M)$ is the identity bordism $M \to M$. Given diffeomorphisms $\varphi : M \to M'$ and $\varphi' : M' \to M''$, note that $F(\varphi)$ can be seen as the bordism given by

$$\begin{array}{ccccc} M' & \longrightarrow & M'' \times [0,1] & \longleftarrow & M \\ x' & \mapsto & (\varphi'(x'),1), \quad (\varphi' \circ \varphi(x),0) & \leftarrow\!\shortmid & x, \end{array}$$

as φ' induces a diffeomorphism $M' \times [0,1] \to M'' \times [0,1]$. Now it is clear that $F(\varphi' \circ \varphi) = F(\varphi') \circ F(\varphi)$. $\square$

We can now give the definition of a TQFT.

Definition 3.11. Let R be a commutative ring with unit. A *Topological Quantum Field Theory* (TQFT) in dimension n, or n-TQFT, is a monoidal functor $Z : \mathbf{Bd}_n \to R$-$\mathbf{Mod}$, where the monoidal structure on R-$\mathbf{Mod}$ is given by the tensor product $\otimes_R$ and the unital object R. We say the TQFT is *lax monoidal* or *symmetric* if it is so as a monoidal functor.

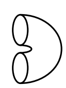
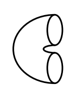
It is possible to change the definition of bordisms in many ways to obtain different categories of bordisms, and hence different types of TQFTs. For example, one may omit the conditions on orientability. A common addition is to equip the manifolds M and W with some extra structure, such as a metric or physical fields.

Remark 3.12. In physics one most commonly takes $R = \mathbb{C}$, and the interpretation of a TQFT is as follows. The objects in $\mathbf{Bd}_n$ can be interpreted as physical space and the bordisms as spacetime. The vector space $Z(M)$ resembles the quantum mechanical state space/Hilbert space associated to M, and the k-linear maps $Z(W) : Z(M) \to Z(M')$ are interpreted as time-evolution operators. The monoidality condition $Z(M \sqcup M') = Z(M) \otimes Z(M')$ makes sense from a quantum mechanical point of view: the Hilbert space associated to two disjoint systems is the tensor product of the individual Hilbert spaces. If one were to take $\oplus$ as the monoidal tensor product on R-**Mod**, one would obtain a classical theory. Lastly, the diffeomorphism invariance reflects the relativistic aspect of the theory. For more information on the physical side of TQFTs, see [9].

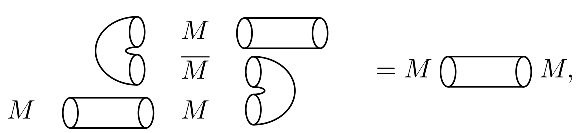
3.3 Some properties of TQFTs

One of the main properties of TQFTs that we will make use of in the later sections, is that they associate invariants to closed oriented manifolds. Namely, if X is a closed oriented manifold of dimension n, it can be seen as a bordism $X : \varnothing \to \varnothing$. If $Z : \mathbf{Bd}_n \to R\text{-}\mathbf{Mod}$ is a TQFT, then $Z(X) : R \to R$ is completely specified by the value $Z(X)(1) \in R$, which we refer to as the *invariant associated to* X *by* Z. Indeed $Z(X)(1)$ only depends on the isomorphism class of X. Moreover we have $Z(X \sqcup Y)(1) = Z(X)(1) \cdot Z(Y)(1)$ for any two such closed oriented manifolds by monoidality of Z.

Consider the case where $R = k$ is a field, and let $Z : \mathbf{Bd}_n \to \mathbf{Vect}_k$ be a TQFT. Let M be an object in $\mathbf{Bd}_n$, i.e. an $(n-1)$-dimensional closed oriented manifold. Denote by $\overline{M}$ the manifold M with opposite orientation. Let $U_M : \varnothing \to M \sqcup \overline{M}$ and $U_M^\dagger : M \sqcup \overline{M} \to \varnothing$ be the bordisms given by $M \times [0, 1]$, with inclusions to the endpoints. Picturing both M and $\overline{M}$ by circles, we denote these bordisms as

$U_M = \begin{matrix} M \\ \overline{M} \end{matrix}$  and $U_M^\dagger =$  $\begin{matrix} M \\ \overline{M} \end{matrix}$.

The map $Z(U_M) : k \to Z(M) \otimes Z(\overline{M})$ is uniquely determined by the element $Z(U_M)(1) \in Z(M) \otimes Z(\overline{M})$. Note that $Z(U_M)(1) = \sum_{i=1}^m v_i \otimes \overline{v}_i$ for some $v_i \in Z(M)$ and $\overline{v}_i \in Z(\overline{M})$, and moreover we can pick such v_i linearly independent and $\overline{v}_i$ linearly independent. Namely, if $v_m = \sum_{i=1}^{m-1} \alpha_i v_i$ for some $\alpha_i \in k$, then $Z(U_M)(1) = \sum_{i=1}^{m-1} v_i \otimes (\overline{v}_i + \alpha_i \overline{v}_m)$, and one can make a similar substitution if $\overline{v}_m = \sum_{i=1}^{m-1} \overline{\alpha}_i \overline{v}_i$ for some $\overline{\alpha}_i \in k$, so the claim follows by induction on m. Now, since

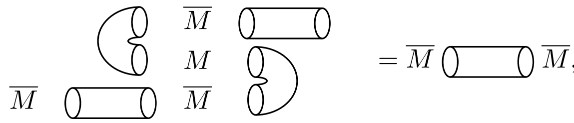


it follows that

$$v = \sum_{i=1}^{m} Z(U_M^\dagger)(v \otimes \overline{v}_i) v_i \qquad \text{for all } v \in Z(M).$$

In particular this implies $Z(M)$ is finite-dimensional and that $\{v_1, \ldots, v_m\}$ is a basis for $Z(M)$.

Completely analogous, switching the roles of M and $\overline{M}$ we find

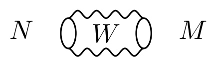


implying

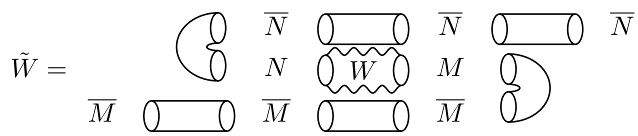
$$\overline{v} = \sum_{i=1}^{m} Z(U_M^\dagger)(v_i \otimes \overline{v})\overline{v}_i \qquad \text{for all } \overline{v} \in Z(\overline{M}),$$

so $Z(\overline{M})$ is finite-dimensional as well, and $\{\overline{v}_1, \ldots, \overline{v}_m\}$ is a basis for $Z(\overline{M})$. Moreover, this shows that the vector space $Z(\overline{M})$ can be identified as the dual space of $Z(M)$, with $\{\overline{v}_1, \ldots, \overline{v}_m\}$ as the dual basis of $\{v_1, \ldots, v_m\}$ w.r.t. the non-degenerate pairing $Z(M) \otimes Z(\overline{M}) \to k : v \otimes \overline{v} \mapsto Z(U_M^\dagger)(v \otimes \overline{v})$.

Consider a bordism $W : M \to N$, giving the linear map $Z(W) : Z(M) \to Z(N)$.



Note that the manifold of W can also be considered as a bordism $\tilde{W} : \overline{N} \to \overline{M}$ by swapping the source and target, yielding a map $Z(\tilde{W}) : Z(\overline{N}) \to Z(\overline{M})$. Indeed this is the dual map of $Z(W)$, which can be seen from the expression

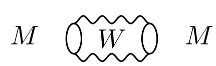


where $Z(U_N)(1) = \sum_{i=1}^{n} w_i \otimes \overline{w}_i$. This expression shows that $Z(\tilde{W})$ is given by

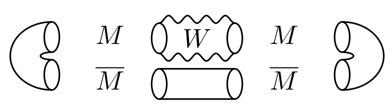
$$\overline{w} \mapsto \sum_{i=1}^{n} Z(U_N^\dagger)(Z(W)(v_i) \otimes \overline{w})\overline{v}_i.$$

Hence, this is the dual map to $Z(W)$.

Now consider a bordism $W : M \to M$, which gives the endomorphism $Z(W) : Z(M) \to Z(M)$.



One can glue the copies of M at the boundary together to obtain a closed manifold. One way to do this is to consider the composition $U_M^\dagger \circ (W \sqcup \mathrm{id}_{\overline{M}}) \circ U_M$. Pictorially,



We find that

$$Z(U_M^\dagger \circ (W \sqcup \mathrm{id}_{\overline{M}}) \circ U_M)(1) = Z(U_M)\left(\sum_{i=1}^{n} Z(W)(v_i) \otimes \overline{v}_i\right) = \mathrm{tr}(Z(W)).$$

Hence we can obtain the trace of W by gluing its ends. Note that this construction nicely reflects the cyclic property of the trace. If k is of characteristic zero, we obtain in particular for $W = \mathrm{id}_M$ the identity

$$\dim_k(Z(M)) = \mathrm{tr}(\mathrm{id}_{Z(M)}) = Z(M \times S^1)(1).$$

We encounter the simplest types of TQFT in dimension 1.

Proposition 3.13. *Let k be a field. There is an equivalence of categories*

$$\textbf{\textit{1TQFT}}_k \simeq \textbf{\textit{FinVect}}_k,$$

between the category of TQFTs over k in dimension 1 *(its arrows are natural transformations) and the category of finite dimensional k-vector spaces.*

Proof. The objects of $\mathbf{Bd}_1$ are finite sets of disjoint points, each with some orientation ± 1. Write $\star$ and $\overline{\star}$ for points with orientation $+1$ and -1, respectively. As seen above, $Z(\overline{\star})$ is the dual vector space of $Z(\star)$, so using monoidality, specifying $Z(\star)$ determines Z for all objects. The only (connected) bordisms are given by $S^1 : \varnothing \to \varnothing$ or $[0,1]$, which depending on orientation can be seen as a bordism

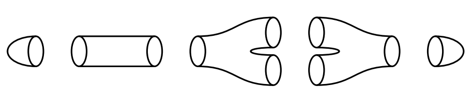
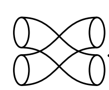
$$\mathrm{id}_\star : \star \to \star, \qquad U_\star : \varnothing \to \star \sqcup \overline{\star}, \qquad U_\star^\dagger : \star \sqcup \overline{\star} \to \varnothing, \quad \text{or} \quad \mathrm{id}_{\overline{\star}} : \overline{\star} \to \overline{\star},$$

all of whose associated k-linear maps are canonical. Write Z_V for the TQFT specified by $Z(\star) = V$ in this way for a k-vector space V. A k-linear map $A : V \to V'$ naturally induces a natural transformation $Z_A : Z_V \Rightarrow Z_{V'}$ with $Z_A(\star) = A$. This yields a functor $Z_{(-)} : \mathbf{FinVect}_k \to \mathbf{1TQFT}_k$.

Conversely, a TQFT Z yields a vector space $Z(\star)$, which was already shown to be finite-dimensional, and a natural transformation $Z \Rightarrow Z'$ between TQFTs yields the linear map $Z(\star) \to Z'(\star)$. This way we obtain a functor $\mathbf{1TQFT}_k \to \mathbf{FinVect}_k$, which is clearly a pseudo-inverse of $Z_{(-)}$. This shows the equivalence of categories. $\square$

3.4 TQFTs in dimension 2 and Frobenius algebras

For dimension $n = 2$, every object in $\mathbf{Bd}_2$, that is every oriented closed 1-dimensional manifold, is diffeomorphic to a finite number of disjoint circles. So using Lemma 3.10, in order to specify a 2-TQFT Z it is sufficient to give $Z(S^1)$ (which by monoidality determines $Z\left(\sqcup_{i=1}^m S^1\right) = \bigotimes_{i=1}^m Z(S^1)$) and $Z(W)$ for all bordisms W between disjoint unions of circles. As shown in [17], these bordisms of $\mathbf{Bd}_2$ are generated, that is under taking disjoint unions and composition, by the six bordisms

 and . (3.1)

Since 2-dimensional oriented surfaces are (topologically) well understood, we can try to understand all TQFTs in dimension 2. It turns out there is a relation in the spirit of Proposition 3.13 between these TQFTs and so-called *Frobenius algebras*, first observed by Dijkgraaf [8]. Following the notes of [17], we will discuss this relation.

Let $R = k$ be a field, so that R-$\mathbf{Mod} = \mathbf{Vect}_k$, the category of vector spaces over k.

Definition 3.14. A *Frobenius algebra* over a field k is a k-algebra A equipped with an associative nondegenerate bilinear form $\beta : A \otimes A \to k$, called the *Frobenius pairing.* Associativity of β means that $\beta(ab \otimes c) = \beta(a \otimes bc)$ for all $a, b, c \in A$. Nondegeneracy of β means that there exists a k-linear map $\gamma : k \to A \otimes A$ such that $(\beta \otimes \mathrm{id}_A)(a \otimes \gamma(1)) = a = (\mathrm{id}_A \otimes \beta)(\gamma(1) \otimes a)$ for all $a \in A$. We say a Frobenius algebra is *commutative* if it is commutative as k-algebra.

We write $\eta : k \to A$ for the unit and $\mu : A \otimes A \to A$ for the multiplication on A, i.e. $\eta(1) = 1 \in A$ and $ab = \mu(a \otimes b)$.

There are multiple equivalent definitions for a Frobenius algebra, see [17] for a more elaborate discussion on this.

Lemma 3.15. *Any Frobenius algebra is of finite dimension.*

Proof. Write $\gamma(1) = \sum_{i=1}^n a_i \otimes b_i$ for some $a_i, b_i \in A$, and note for any $a \in A$ we have

$$a = (\mathrm{id}_A \otimes \beta) \circ (\gamma \otimes \mathrm{id}_A)(1 \otimes a) = (\mathrm{id}_A \otimes \beta)\left(\sum_{i=1}^n a_i \otimes b_i \otimes a\right) = \sum_{i=1}^n a_i \cdot \beta(b_i \otimes a), \tag{3.2}$$

so in particular $a \in \langle a_1, \dots, a_n \rangle$, implying that A is finite-dimensional. $\square$

Remark 3.16. This definition of non-degeneracy of β implies the usual notion of non-degeneracy. Namely, if $a \in A$ is such that $\beta(b \otimes a) = 0$ for all $b \in A$, then (3.2) implies $a = 0$. To show non-degeneracy in the other argument, a completely similar argument shows that $(\beta \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \gamma(1)) = \mathrm{id}_A$, which can be used to show it.

Note that the map γ is unique (if it exists). Namely, suppose γ and γ' both satisfy the conditions, and write $\gamma(1) - \gamma'(1) = \sum_{i=1}^n a_i \otimes b_i$ with $a_i, b_i \in A$ such that the a_i are linearly independent. Then by linearity, $0 = (\mathrm{id}_A \otimes \beta)(\sum_{i=1}^n a_i \otimes b_i \otimes a) = \sum_{i=1}^n a_i \cdot \beta(b_i \otimes a)$ for all $a \in A$. In particular, all $b_i = 0$, so $\gamma(1) = \gamma'(1)$.

A Frobenius algebra A also naturally carries a k-coalgebra structure. As counit, we take the map

$$\begin{aligned} \varepsilon : A &\to k \\ a &\mapsto \beta(1 \otimes a) = \beta(a \otimes 1), \end{aligned} \tag{3.3}$$

which is well-defined by associativity of β. As comultiplication, we take

$$\begin{aligned} \delta : A &\to A \otimes A \\ a &\mapsto (\mu \otimes \mathrm{id}_A)(a \otimes \gamma(1)) = (\mathrm{id}_A \otimes \mu)(\gamma(1) \otimes a). \end{aligned} \tag{3.4}$$

To see why this is well-defined, write $\gamma(1) = \sum a_i \otimes b_i$ as before, then the associativity and non-degeneracy of β imply that

$$\begin{aligned} (\mu \otimes \mathrm{id}_A)(a \otimes \gamma(1)) = \sum_{i=1}^n a a_i \otimes b_i &= \sum_{i,j=1}^n \beta(b_j \otimes a a_i) a_j \otimes b_i \\ &= \sum_{i,j=1}^n \beta(b_i a \otimes a_j) a_i \otimes b_j = \sum_{i=1}^n a_i \otimes b_i a = (\mathrm{id}_A \otimes \mu)(\gamma(1) \otimes a) \end{aligned}$$

for all $a \in A$.

Lemma 3.17. *The k-linear maps $\delta : A \to A \otimes A$ and $\varepsilon : A \to k$ make A into a coalgebra. That is, $(id_A \otimes \delta) \circ \delta = (\delta \otimes id_A) \circ \delta$ (i.e. δ is coassociative), and $(id_A \otimes \varepsilon) \circ \delta = id_A = (\varepsilon \otimes id_A) \circ \delta$ (i.e. ε is a counit for the comultiplication).*

Proof. Write $\gamma(1) = \sum_{i=1}^{n} a_i \otimes b_i$ as before. For any $a \in A$, we have

$$(\mathrm{id}_A \otimes \delta) \circ \delta(a) = \sum_{i,j=1}^{n} aa_i \otimes b_i a_j \otimes b_j = \sum_{i,j=1}^{n} aa_i \otimes b_i a_j \otimes b_j = \sum_{i,j=1}^{n} aa_j a_i \otimes b_i \otimes b_j = (\delta \otimes \mathrm{id}_A) \circ \delta(a).$$

Furthermore,

$$(\mathrm{id}_A \otimes \varepsilon)(\delta(a)) = \sum_{i=1}^{n} aa_i \cdot \beta(b_i \otimes 1) \overset{(3.2)}{=} a \cdot 1 = a$$

and similarly we have $(\varepsilon \otimes \mathrm{id}_A)(\delta(a)) = a$. □

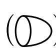
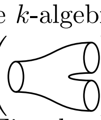
Definition 3.18. A *Frobenius algebra morphism* $A \to B$ is an k-algebra morphism which is also a coalgebra morphism.

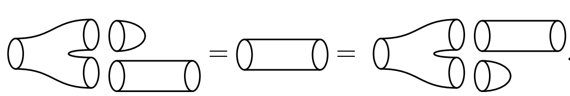
Remark 3.19. By associativity of β, we have $\beta(a \otimes b) = \beta(ab \otimes 1)$ in particular, so $\beta = \varepsilon \circ \mu$. Hence, Frobenius algebra morphisms preserve the Frobenius pairing. Conversely, a map of k-algebras preserving the pairing, also preserves ε and δ by their definitions. (Although, as a step in between one must show that γ is preserved as well, but this is clear from the uniqueness of γ). Hence, one could equivalently define a Frobenius algebra morphism to be a k-algebra morphism preserving the Frobenius pairing.

Definition 3.20. The *category of Frobenius algebras over k*, denoted $\mathbf{FA}_k$, has Frobenius algebras over k as objects, and Frobenius algebra morphisms as arrows. We write $\mathbf{CFA}_k$ for the full subcategory of $\mathbf{FA}_k$ of commutative Frobenius algebras.

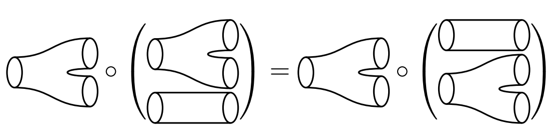
Example 3.21. Some examples of Frobenius algebras include the following.

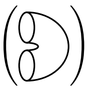
- Let A be a finite field extension of k, and $\varepsilon : A \to k$ any non-zero k-linear map. Then A with the pairing $\beta : a \otimes b \mapsto \varepsilon(ab)$ is a Frobenius algebra.
- The ring $\mathrm{Mat}_n(k)$ of $n \times n$ matrices over k, with pairing $\beta : A \otimes B \mapsto \mathrm{tr}(AB)$.

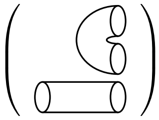
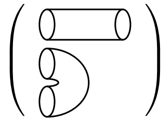
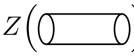
We will now see how a (commutative) Frobenius algebra arises from a symmetric 2-TQFT Z. Let $A = Z\left(S^1\right)$, whose k-algebra structure is given by the unit $\eta = Z$ () $: k \to A$ and the multiplication map $\mu = Z$ () $: A \otimes A \to A$. Furthermore, we take the bilinear map $\beta = Z$ () $:$ $A \otimes A \to k$. First let us check that A with η and μ define an k-algebra. Indeed η is a unit for the multiplication μ as $\mu \circ (\eta \otimes \mathrm{id}_A) = \mathrm{id}_A = \mu \circ (\mathrm{id}_A \otimes \eta)$ because



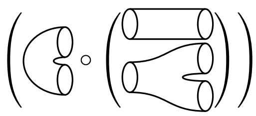
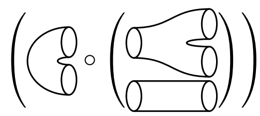
Note that the multiplication given by μ is associative, i.e. $\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \mu)$, since



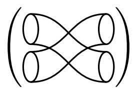
Let $\gamma = Z$  $: k \to A \otimes A$. Then

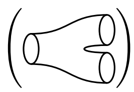
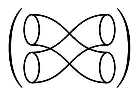
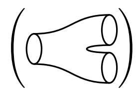
$$(\beta \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \gamma) = Z$$  $$\circ\, Z$$  $$= Z$$  $$= \mathrm{id}_A$$

and similarly $(\mathrm{id}_A \otimes \beta) \circ (\gamma \otimes \mathrm{id}_A) = \mathrm{id}_A$, which shows that β is non-degenerate. Note here the similarity with the arguments used in Section 3.3 and (3.2). To see why β is associative, note that

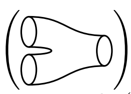
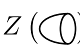
$$\beta \circ (\mathrm{id}_A \otimes \mu) = Z$$  $$= Z$$  $$= \beta \circ (\mu \otimes \mathrm{id}_A).$$

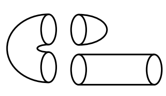
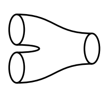
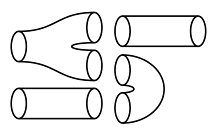
This shows that A is a Frobenius algebra.

That Z is symmetric implies that Z  $= \tau : A \otimes A \to A \otimes A$ is the twist map $a \otimes b \mapsto b \otimes a$. Hence we find that

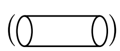
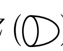
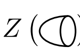
$$\mu \circ \tau = Z$$  $$\circ\, Z$$  $$= Z$$  $$= \mu,$$

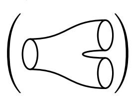
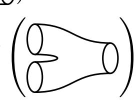
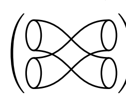
i.e. $\mu(a \otimes b) = \mu(b \otimes a)$. Hence multiplication in A is commutative and we conclude that A with β is a commutative Frobenius algebra.

Note that $\delta = Z$  $: A \to A \otimes A$ and $\varepsilon = Z$  $: A \to k$ give the coalgebra structure on A. Indeed this agrees with (3.3) and (3.4) by the relations

 $=$  and  $=$  .

Now we would like to do the converse. Given a commutative Frobenius algebra A with maps η, μ, β, we define a symmetric 2-TQFT Z over k by putting

$Z(S^1) = A$, Z  $= \mathrm{id}_A$, Z  $= \eta$, Z  $= \varepsilon$,

Z  $= \mu$, Z  $= \delta$, Z  $= \tau$,

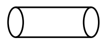
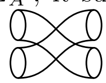
which determines Z completely using functoriality and monoidality. However, in order to check compatibility with relations between these bordisms in $\mathbf{Bd}_2$, one must check that a finite number of relations are satisfied, which can be found in 1.4.24 – 1.4.28 of [17]. We will not go into detail, but all of these relations will follow directly from the axioms of a commutative Frobenius algebra.

These constructions are (up to isomorphism) inverse to each other, which leads to the following theorem.

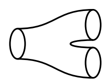
Theorem 3.22. *There is an equivalence of categories*

$$\boldsymbol{2SymTQFT}_k \simeq \boldsymbol{CFA}_k$$

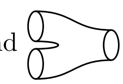
between the category of symmetric 2-TQFTs over k (its arrows are natural transformations) and the category of commutative Frobenius algebras over k.

Proof. Assign to any commutative Frobenius algebra A the symmetric 2-TQFT as described above. A Frobenius algebra morphism $f : A \to A'$ induces maps $A^{\otimes n} \to A'^{\otimes n}$ for any $n \geq 1$. To see that this defines a natural transformation $Z_A \Rightarrow Z_{A'}$, it suffices to check the naturality condition for generators (3.1) only. That is, for  and  we must have commuting diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"'] & A' \arrow[d, "\mathrm{id}_{A'}"] \\ A \arrow[r, "f"] & A' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes A \arrow[r, "f\otimes f"] \arrow[d, "\tau"'] & A' \otimes A' \arrow[d, "\tau'"] \\ A \otimes A \arrow[r, "f\otimes f"] & A' \otimes A' \end{tikzcd}$$

which is trivially the case. For  and  we must have commuting diagrams

$$\begin{tikzcd} k \arrow[r, "\mathrm{id}"] \arrow[d, "\eta"'] & k \arrow[d, "\eta'"] \\ A \arrow[r, "f"] & A' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes A \arrow[r, "f\otimes f"] \arrow[d, "\mu"'] & A' \otimes A' \arrow[d, "\mu'"] \\ A \arrow[r, "f"] & A' \end{tikzcd}$$

which is the case since f is a k-algebra morphism. For  and  we must have commuting diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varepsilon"'] & A' \arrow[d, "\varepsilon'"] \\ k \arrow[r, "\mathrm{id}"] & k \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\delta"'] & A' \arrow[d, "\delta'"] \\ A \otimes A \arrow[r, "f\otimes f"] & A' \otimes A' \end{tikzcd}$$

which is the case since f is a k-coalgebra morphism as well.

Conversely, to any symmetric 2-TQFT Z, assign the commutative Frobenius algebra $A_Z = Z(S^1)$ as described above. Given a natural transformation of TQFTs $F : Z \Rightarrow Z'$, we find using the same six diagrams that $f = F\left(S^1\right) : A_Z \to A_{Z'}$ is a Frobenius morphism.

Finally, the constructions are clearly pseudo-inverse to each other, showing the equivalence of categories. □

3.5 Bordism category variations

As remarked before, one can put extra structure on the manifolds to obtain different types bordisms. We will consider the following two variations, which will be used in the later sections.

Definition 3.23. The *category of n-bordisms of pairs*, denoted $\mathbf{Bdp}_n$, is the 2-category consisting of:

- Objects: pairs (M, A) with M an $(n-1)$-dimensional closed oriented manifold, and $A \subset M$ a finite set of points intersecting each connected component of M.

- 1-morphisms: a map $(M_1, A_1) \to (M_2, A_2)$ is given by a class of pairs (W, A) with $W : M_1 \to M_2$ a bordism, and $A \subset W$ a finite set intersecting each connected component of W such that $A \cap M_1 = A_1$ and $A \cap M_2 = A_2$. Two such pairs (W, A) and (W', A') are in the same class if there is a diffeomorphism $F : W \to W'$ such that $F(A) = A'$ and such that the diagram

$$\begin{tikzcd} & W \arrow[dd, "\wr"] & \\ M_2 \arrow[ur] \arrow[dr] & & M_1 \arrow[ul] \arrow[dl] \\ & W' & \end{tikzcd} \tag{3.5}$$

  commutes.

  The composition of $(W, A) : (M_1, A_1) \to (M_2, A_2)$ and $(W', A') : (M_2, A_2) \to (M_3, A_3)$ is $(W \sqcup_{M_2} W', A \cup A') : (M_1, A_1) \to (M_3, A_3)$.

- 2-morphisms: a map $(W, A) \to (W', A')$ is given by a diffeomorphism $F : W \to W'$ such that $F(A) \subset A'$ and (3.5) commutes.

Note that so far, no identity morphism exists for (M, A), unless $M = A = \varnothing$. For this reason, we loosen the definition of a bordism a bit, and allow M itself to be seen as a bordism $M \to M$, so that (M, A) will be the identity morphism for (M, A).

The last type of bordism category we consider is one in which the manifolds carry a so-called *parabolic structure*. Fix a set Λ and call it the *parabolic data*. We say a *parabolic structure* on a manifold M is a finite set $Q = \{(S_1, \mathcal{E}_1), \dots, (S_s, \mathcal{E}_s)\}$ with $\mathcal{E}_i \in \Lambda$ and the S_i are pairwise disjoint compact submanifolds of M of codimension 2 with a co-orientation (i.e. an orientation of its normal bundle) such that $S_i \cap \partial M = \partial S_i$ transversally.

Definition 3.24. Let Λ be a set. The *category of n-bordisms of pairs with parabolic structures over* Λ, denoted $\mathbf{Bdp}_n(\Lambda)$, is the 2-category consisting of:

- Objects: triples (M, A, Q) with M an $(n-1)$-dimensional closed oriented manifold, Q a parabolic structure on M, and $A \subset M$ a finite set of points not intersecting any of the S_i of Q.

- 1-morphisms: a map $(M_1, A_1, Q_1) \to (M_2, A_2, Q_2)$ is given by a class of triples (W, A, Q) where $W : M_1 \to M_2$ is a bordism, Q a parabolic structure on W, and $A \subset W$ a finite set intersecting each connected component of W but not intersecting any of the S_i of Q, such that $A \cap M_1 = A_1$, $A \cap M_2 = A_2$, $Q|_{M_1} = Q_1$ and $Q|_{M_2} = Q_2$. Here we use the notation $Q|_{M_i} = \{(S_j \cap M_i, \mathcal{E}_j) : (S_j, \mathcal{E}_j) \in \Lambda\}$. Two such triples (W, A, Q) and (W', A', Q') are in the same class if there is a diffeomorphism $F : W \to W'$ such that $F(A) = A'$ and $(S, \mathcal{E}) \in Q$ if and only if $(F(S), \mathcal{E}) \in Q'$ and such that the diagram

$$\begin{tikzcd} & W \arrow[dd, "\wr"] & \\ M_2 \arrow[ur] \arrow[dr] & & M_1 \arrow[ul] \arrow[dl] \\ & W' & \end{tikzcd} \tag{3.6}$$

commutes.

The composition of bordisms $(W, A, Q) : (M_1, A_1, Q_1) \to (M_2, A_2, Q_2)$ and $(W', A', Q') : (M_2, A_2, Q_2) \to (M_3, A_3, Q_3)$ is given by $(W, A, Q) \circ (W', A', Q') = (W \sqcup_{M_2} W', A \cup A', Q \sqcup_{M_2} Q')$, where $Q \sqcup_{M_2} Q'$ denotes the union of Q and Q' but where we glue pairs $(S, \mathcal{E}) \in Q$ and $(S', \mathcal{E}) \in Q'$ that have a common boundary (in M_2).

- 2-morphisms: a map $(W, A, Q) \to (W', A', Q')$ is given by a diffeomorphism $F : W \to W'$ such that $F(A) \subset A'$ and $(F(S), \mathcal{E}) \in Q'$ for each $(S, \mathcal{E}) \in Q$ and such that (3.6) commutes.

Actually, $\mathbf{Bdp}_n$ can be seen as a particular case of $\mathbf{Bdp}_n(\Lambda)$ for $\Lambda = \varnothing$. The category $\mathbf{Bdp}_n(\Lambda)$ (and thus $\mathbf{Bdp}_n$ as well) is a monoidal category. The tensor product is given by taking disjoint unions:

$$(M, A, Q) \sqcup (M', A', Q') = (M \sqcup M', A \cup A', Q \cup Q')$$

for objects, and similarly for bordisms. The unital object is $(\varnothing, \varnothing, \varnothing)$, which we also denote simpy by $\varnothing$.

Note that (non-empty) parabolic structures can only exist on manifolds of dimension ≥ 2. In particular for $\mathbf{Bdp}_2(\Lambda)$, its 1-dimensional objects have $Q = \varnothing$ and the parabolic structures of its 2-dimensional bordisms are of the form $\{(p_1, \mathcal{E}_1), \ldots, (p_s, \mathcal{E}_s)\}$ with p_i points on the interior of the bordism that have a preferred orientation of small loops around them.

4 Field theory and quantization functor

Let X be a closed path-connected manifold X with finitely generated fundamental group $\pi_1(X)$, and G an algebraic group over a field k. One can consider the set of group representations $\rho : \pi_1(X) \to G$:

$$\mathfrak{X}_G(X) = \mathrm{Hom}(\pi_1(X), G),$$

which we will call the *G-representation variety* of X. Indeed this set naturally carries the structure of a variety: let $\gamma_1, \ldots, \gamma_n$ be a set of generators for $\pi_1(X)$, then the morphism

$$\begin{array}{rcl} \mathfrak{X}_G(X) & \to & G^n \\ \rho & \mapsto & (\rho(\gamma_1), \ldots, \rho(\gamma_n)) \end{array}$$

identifies $\mathfrak{X}_G(X)$ with a subvariety of G^n (the relations between the generators define an algebraic set) giving it a k-variety structure. This structure can be shown to be independent of the chosen generators. In particular, when $X = \Sigma_g$ is a closed surface of genus g, its fundamental group is given by

$$\pi_1(\Sigma_g) = \left\langle a_1, b_1, \ldots, a_g, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = \mathrm{id} \right\rangle,$$

where $\prod_{i=1}^{g} [a_i, b_i]$ denotes the product $[a_1, b_1] \cdots [a_g, b_g]$. This gives

$$\mathfrak{X}_G(\Sigma_g) = \left\{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] = \mathrm{id} \right\}, \tag{4.1}$$

a closed subvariety of G^{2g}.

Our goal is to compute the class of $\mathfrak{X}_G(\Sigma_g)$ in $\mathrm{K}(\mathbf{Var}_k)$, which we will do using a technique involving TQFTs based on [13, 10]. We will construct a lax monoidal TQFT Z over the ring $\mathrm{K}(\mathbf{Var}_k)$ such that the invariant associated to a closed manifold X is precisely $Z(X)(1) = [\mathfrak{X}_G(X)]$. This construction will allow to solve a more general problem: if Λ is the set of conjugacy-closed subsets of G, one can put a parabolic structure $Q = \{(\star, \mathcal{E}_1), \ldots, (\star, \mathcal{E}_s)\}$ with data in Λ on Σ_g, such that the invariant associated to (Σ_g, Q) will be the class of the variety

$$\mathfrak{X}_G(\Sigma_g, Q) = \left\{ (A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_s) \in G^{2g+s} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{s} C_i = \mathrm{id} \text{ and } C_i \in \mathcal{E}_i \right\}.$$

4.1 Span categories

Definition 4.1. Given a category $\mathcal{C}$ with pullbacks, we define the 2-category $\mathrm{Span}(\mathcal{C})$ as follows:

- Its objects are the objects of $\mathcal{C}$.
- An arrow from C to C' is given by a diagram $C \leftarrow D \to C'$ in $\mathcal{C}$, called a *span*. Composition of the spans $C \leftarrow D \to C'$ and $C' \leftarrow D' \to C''$ is given by the span $C \leftarrow E \to C''$ such that the square in

$$\begin{tikzcd} & & E \arrow[dl] \arrow[dr] & & \\ & D \arrow[dl] \arrow[dr] & & D' \arrow[dl] \arrow[dr] & \\ C & & C' & & C'' \end{tikzcd}$$

is a pullback square.

- A 2-morphism from $C \leftarrow D \rightarrow C'$ to $C \leftarrow D' \rightarrow C'$ is given by an arrow $D \rightarrow D'$ such that the following diagram commutes:

$$\begin{tikzcd} & D \arrow[dl] \arrow[d] \arrow[dr] & \\ C & & C' \\ & D' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The category $\mathrm{Span}^{\mathrm{op}}(\mathcal{C})$ is defined analogously on categories with pushouts, where we reverse all arrows.

If $\mathcal{C}$ is a monoidal category, then $\mathrm{Span}(\mathcal{C})$ naturally has the structure of a monoidal category as well. The tensor product and unital object naturally carry over, the associator will be given by the span

$$A \otimes (B \otimes C) \xleftarrow{\mathrm{id}} A \otimes (B \otimes C) \xrightarrow{\alpha_{A,B,C}} (A \otimes B) \otimes C$$

and the left and right unitor by

$$I \otimes A \xleftarrow{\mathrm{id}} I \otimes A \xrightarrow{\lambda_A} A \qquad \text{and} \qquad A \otimes I \xleftarrow{\mathrm{id}} A \otimes I \xrightarrow{\rho_A} A \ .$$

4.2 Constructing the TQFT

Definition 4.2. Let (X, A) be a pair of topological spaces. The *fundamental groupoid of X w.r.t. A* denoted $\Pi(X, A)$ is the groupoid category whose objects are elements of A, and an arrow $a \to b$ for each homotopy class of paths from a to b. Composition of morphisms is given by concatenation of paths. Indeed this construction only depends on the homotopy type of (X, A). In particular, if $A = \{x_0\}$ is a single point, we obtain the fundamental group $\pi_1(X, x_0)$.

Note that if $f : (X, A) \to (X', A')$ is a map of pairs, there is an induced functor $\Pi(X, A) \to \Pi(X', A')$ between groupoids, mapping an object $a \in A$ to $f(a) \in A'$ and an arrow $\gamma : a \to b$ to $f \circ \gamma$. This allows us to construct the following functor.

Definition 4.3. The *geometrization functor* is a 2-functor $\Pi : \mathbf{Bdp}_n \to \mathrm{Span}^{\mathrm{op}}(\mathbf{Grpd})$, with $\mathbf{Grpd}$ the category of groupoids, defined as follows:

- To each object (X, A) we assign the fundamental groupoid $\Pi(X, A)$.
- For each 1-morphism $(W, A) : (X_1, A_1) \to (X_2, A_2)$ we assign the cospan

$$\Pi(X_1, A_1) \xrightarrow{i_1} \Pi(W, A) \xleftarrow{i_2} \Pi(X_2, A_2)$$

  with i_1 and i_2 induced by inclusions.

- For each 2-morphism $(W, A) \to (W, A')$ given by diffeomorphism $F : W \to W'$ with $F(A) \subset A'$, we obtain a groupoid morphism ΠF yielding the commutative diagram

$$\begin{tikzcd}
& \Pi(W,A) \arrow[dd, "\Pi F"] & \\
\Pi(X_1,A_1) \arrow[ur, "i_1"] \arrow[dr, "i_1'"'] & & \Pi(X_2,A_2) \arrow[ul, "i_2"'] \arrow[dl, "i_2'"] \\
& \Pi(W',A') &
\end{tikzcd}$$

which is a 2-morphism in $\mathrm{Span}^{\mathrm{op}}(\mathbf{Grpd})$.

- To show functoriality, consider two bordisms $(W,A) : (X_1,A_1) \to (X_2,A_2)$ and $(W',A') : (X_2,A_2) \to (X_3,A_3)$, and write $(W'',A'') = (W \sqcup_{X_2} W', A \cup A')$ for their composition. Let $U \subset W''$ be a sufficiently small open neighborhood of X_2 such that $U \cap A'' = A_2$, and set $V = W \cup U$ and $V' = W' \cup U$. Now $\{V, V'\}$ is an open cover of W'' such that $(V, A'' \cap V), (V', W'' \cap V')$ and $(V \cap V', A'' \cap V \cap V')$ are homotopically equivalent to $(W,A), (W',A')$ and (X_2,A_2), respectively. Therefore, by the Seifert-van Kampen theorem for fundamental groupoids [6], we obtain a pushout diagram for $\Pi(W'',A'')$ induced by inclusions, making

$$\begin{tikzcd}
& & \Pi(W'',A'') & & \\
& \Pi(W,A) \arrow[ur] & & \Pi(W',A') \arrow[ul] & \\
\Pi(X_1,A_1) \arrow[ur] & & \Pi(X_2,A_2) \arrow[ul] \arrow[ur] & & \Pi(X_3,A_3) \arrow[ul]
\end{tikzcd}$$

precisely $\Pi(W',A') \circ \Pi(W,A)$.

Suppose X is a compact connected manifold (possibly with boundary), $A \subset X$ a finite set of points, and denote $\mathcal{G} = \Pi(X,A)$. We write $\mathcal{G}_a = \mathrm{Hom}_{\mathcal{G}}(a,a)$ for a in $\mathcal{G}$. Since a compact connected manifold has the homotopy type of a finite CW-complex, every $\mathcal{G}_a = \pi_1(X,a)$ is a finitely generated group.

The groupoid $\mathcal{G}$ has finitely many connected components, where we say objects $a, b \in \mathcal{G}$ are *connected* if $\mathrm{Hom}_{\mathcal{G}}(a,b)$ is non-empty. Pick a subset $S = \{a_1, \ldots, a_s\} \subset A$ such that each connected component of $\mathcal{G}$ contains exactly one of the a_i. Also pick an arrow $f_a : a_i \to a$ for each $a \in A$ (with $a_i \in S$ in the connected component of a) such that $f_{a_i} = \mathrm{id}_{\mathcal{G}_{a_i}}$ for each $a_i \in S$. Now if G is a group, then a morphism of groupoids $\rho : \mathcal{G} \to G$ is uniquely determined by the group morphisms $\rho_i : \mathcal{G}_{a_i} \to G$ and a choice of $\rho(f_a) \in G$. Namely, any $\gamma : a \to b$ in $\mathcal{G}$ can be written as $\gamma = f_b \circ \gamma' \circ (f_a)^{-1}$ for some $\gamma' \in \mathcal{G}_{a_i}$ (with $a_i \in S$ in the connected component of a and b). The elements $\rho(f_a)$ can take any value for $a \notin A \backslash S$ (and $\rho(f_{a_i}) = \mathrm{id} \in G$ for $a_i \in S$), so if $\mathcal{G}$ has n objects and s connected components, we have

$$\mathrm{Hom}_{\mathbf{Grpd}}(\mathcal{G}, G) \simeq \mathrm{Hom}(\mathcal{G}_{a_1}, G) \times \cdots \times \mathrm{Hom}(\mathcal{G}_{a_s}, G) \times G^{n-s}. \tag{4.2}$$

If G is an algebraic group, each of these factors naturally carries the structure of an algebraic variety. Namely, each $\mathcal{G}_{a_i}$ is finitely generated, so $\mathrm{Hom}(\mathcal{G}_{a_i}, G)$ can be identified with a subvariety of G^m for some $m > 0$ as in the beginning of Section 4. This gives $\mathrm{Hom}(\mathcal{G}, G)$ the structure of an algebraic variety, and this structure can be shown not to depend on the choices [10].

Definition 4.4. Let X be a compact connected manifold (possibly with boundary) and $A \subset X$ a finite set. Then we define the *G-representation variety* of (X,A) to be

$$\mathfrak{X}_G(X,A) = \mathrm{Hom}_{\mathbf{Grpd}}(\Pi(X,A), G).$$

Note that the functor $\mathrm{Hom}_{\mathbf{Grpd}}(-,G)$ sends pushouts to pullbacks, so we obtain an induced functor

$$\mathcal{F} : \mathbf{Bdp}_n \to \mathrm{Span}(\mathbf{Var}_k)$$

which we refer to as the *field theory*. This functor sends an object (M, A) to $\mathfrak{X}_G(M, A)$ and a bordism $(W, A) : (M_1, A_1) \to (M_2, A_2)$ to the span

$$\mathfrak{X}_G(M_1, A_1) \longleftarrow \mathfrak{X}_G(W, A) \longrightarrow \mathfrak{X}_G(M_2, A_2).$$

Recall that $\mathbf{Var}_k$ is a monoidal category, the tensor product being $\times_k$ and the unital object $\operatorname{Spec} k$, so the category $\mathrm{Span}(\mathbf{Var}_k)$ is monoidal as well. Following the above construction, one can see that $\mathcal{F}$ is a monoidal functor. Indeed, $\mathfrak{X}_G(\varnothing) = \mathrm{Hom}_{\mathbf{Grpd}}(\varnothing, G)$ is a point, and $\mathfrak{X}_G(X \sqcup X', A \cup A') = \mathfrak{X}_G(X, A) \times_k \mathfrak{X}_G(X', A')$ as can be easily shown from (4.2). Also $\mathcal{F}$ is seen to be symmetric.

The last step in constructing the TQFT is the *quantization functor*

$$\mathcal{Q} : \mathrm{Span}(\mathbf{Var}_k) \to K(\mathbf{Var}_k)\text{-}\mathbf{Mod}$$

which assigns to an object X the module $\mathrm{K}(\mathbf{Var}/X)$, and to a span $X_1 \xleftarrow{f} Y \xrightarrow{g} X_2$ the module morphism $g_! \circ f^* : \mathrm{K}(\mathbf{Var}/X_1) \to \mathrm{K}(\mathbf{Var}/X_2)$.

Lemma 4.5. *$\mathcal{Q}$ is a functor.*

Proof. It is clear that $\mathcal{Q}(\mathrm{id}_X) = \mathrm{id}_X$ for any object X. Consider a composition of spans

$$\begin{tikzcd} & & Z \arrow[dl, "h"'] \arrow[dr, "j"] & & \\ & Y_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & Y_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ X_1 & & X_2 & & X_3, \end{tikzcd}$$

so the square is a pullback square. For $\mathcal{Q}$ to be a functor, we want to prove that $\mathcal{Q}(X_1 \leftarrow Z \to X_3) = (g_2 \circ j)_! \circ (f_1 \circ h)^*$ equals $\mathcal{Q}(X_1 \leftarrow Y_1 \to X_2) \circ \mathcal{Q}(X_2 \leftarrow Y_2 \to X_3) = g_{2!} \circ f_2^* \circ g_{1!} \circ f_1^*$. It suffices to prove that $j_! \circ h^* = f_2^* \circ g_{1!}$ as maps $\mathbf{Var}/Y_1 \to \mathbf{Var}/Y_2$. Indeed this holds, for if $W \to Y_1$ then

$$\begin{tikzcd} W \times_{Y_1} Z \arrow[r] \arrow[d] & Z \arrow[r, "j"] \arrow[d, "h"] & Y_2 \arrow[d, "f_2"] \\ W \arrow[r] & Y_1 \arrow[r, "g_1"] & X_2 \end{tikzcd}$$

is a pullback rectangle as both squares are pullback squares. Hence $f_2^*(g_{1!}(W)) = W \times_{Y_1} Z = j_!(h^*(W))$. ☐

Note that $\mathcal{Q}$ is a lax (symmetric) monoidal functor. Indeed there is a map $K(\mathbf{Var}/X) \otimes K(\mathbf{Var}/Y) \to K(\mathbf{Var}/(X \times Y))$, see (2.1), which is functorial in X and Y: given spans

$$X \leftarrow Z \to X' \quad \text{and} \quad Y \leftarrow W \to Y'$$

and varieties $U \to X$ and $V \to Y$, we have an isomorphism

$$(U \times_X Z) \times (V \times_Y W) \simeq (U \times V) \times_{(X \times Y)} (Z \times W)$$

over $X' \times Y'$, so the square

$$\begin{array}{ccc} \mathrm{K}(\mathbf{Var}/X) \otimes \mathrm{K}(\mathbf{Var}/Y) & \longrightarrow & \mathrm{K}(\mathbf{Var}/(X \times Y)) \\ {\scriptstyle \mathcal{Q}(X \leftarrow Z \rightarrow X') \otimes \mathcal{Q}(Y \leftarrow W \rightarrow Y')} \downarrow & & \downarrow {\scriptstyle \mathcal{Q}(X \times Y \leftarrow Z \times W \rightarrow X' \times Y')} \\ \mathrm{K}(\mathbf{Var}/X') \otimes \mathrm{K}(\mathbf{Var}/Y') & \longrightarrow & \mathrm{K}(\mathbf{Var}/(X' \times Y')) \end{array}$$

commutes. However, since the map $K(\mathbf{Var}/X) \otimes K(\mathbf{Var}/Y) \to K(\mathbf{Var}/(X \times Y))$ need not be an isomorphism, $\mathcal{Q}$ is only lax monoidal. It is clear to see that $\mathcal{Q}$ is symmetric as well.

At last, we define the symmetric lax monoidal TQFT as the composition of the field theory and the quantization functor

$$Z = \mathcal{Q} \circ \mathcal{F} : \mathbf{Bdp}_n \to \mathrm{K}(\mathbf{Var}_k)\text{-}\mathbf{Mod}.$$

Now, when X is a closed connected oriented manifold of dimension n, we can choose a point $\star$ on X, and view it as a bordism $(X, \star) : \varnothing \to \varnothing$. Then,

$$\mathcal{F}(X, \star) : \star \xleftarrow{t} \mathfrak{X}_G(X, \star) = \mathfrak{X}_G(X) \xrightarrow{t} \star$$

and hence $Z(X, \star)(1) = t_! t^*(\star) = t_! \left([\mathfrak{X}_G(X)]_{\mathfrak{X}_G(X)} \right) = [\mathfrak{X}_G(X)]$ as desired.

4.3 Parabolic structures

Let Λ be a set of conjugacy-closed subsets of G. We will slightly modify the construction above to obtain a (lax symmetric) TQFT $Z_\Lambda : \mathbf{Bdp}_n(\Lambda) \to \mathrm{K}(\mathbf{Var}_k)\text{-}\mathbf{Mod}$. This will be an extension of Z in the sense that it yields the same modules and morphisms as Z in the absence of parabolic structures, i.e. $Z_\Lambda(X, A, \varnothing) = Z(X, A)$.

Let X be a compact manifold (possibly with boundary) with a parabolic structure Q given by

$$Q = \{(S_1, \mathcal{E}_1), \ldots, (S_s, \mathcal{E}_s)\},$$

and $A \subset X$ a finite set intersecting each connected component of X, but not intersecting $S = \cup S_i$. Then the representation variety of (X, A, Q) is defined as

$$\mathfrak{X}_G(X, A, Q) = \left\{ \rho : \Pi(X - S, A) \to G \;\middle|\; \begin{array}{c} \scriptstyle \rho(\gamma) \in \mathcal{E}_i \text{ for all loops } \gamma \text{ around } S_i \\ \scriptstyle \text{positive w.r.t. the co-orientation,} \\ \scriptstyle \text{for all } (S_i, \mathcal{E}_i) \in Q \end{array} \right\}, \tag{4.3}$$

where 'γ around S_i' means a non-zero loop γ in $\Pi(X - S, A)$ which is zero in $\Pi(X - (S - S_i), A)$. Since the $\mathcal{E}_i$ are conjugacy-closed, the condition on the loops γ around S_i is independent on the chosen base point. Indeed this definition of $\mathfrak{X}_G(X, A, Q)$ agrees with Definition 4.4 for $Q = \varnothing$. When X is connected, we write

$$\mathfrak{X}_G(X, Q) = \mathfrak{X}_G(X, \star, Q).$$

In the particular case of $X = \Sigma_g$ with parabolic structure $Q = \{(\star, \mathcal{E}_1), \ldots, (\star, \mathcal{E}_s)\}$ (co-orientation induced from orientation on Σ_g), we find

$$\mathfrak{X}_G(\Sigma_g, Q) = \left\{ (A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_s) \in G^{2g+s} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{s} C_i = \mathrm{id} \text{ and } C_i \in \mathcal{E}_i \right\}. \tag{4.4}$$

Consider the modified field theory $\mathcal{F}_\Lambda : \mathbf{Bdp}_n(\Lambda) \to \mathrm{Span}(\mathbf{Var}_k)$ that maps an object (M, A, Q) to $\mathfrak{X}_G(M, A, Q)$ and a bordism $(W, A, Q) : (M_1, A_1, Q_1) \to (M_2, A_2, Q_2)$ to the span

$$\mathfrak{X}_G(M_1, A_1, Q_1) \longleftarrow \mathfrak{X}_G(W, A, Q) \longrightarrow \mathfrak{X}_G(M_2, A_2, Q_2)$$

induced by the inclusions. By a similar argument as in Definition 4.3, one can show [10] that for bordisms $(W, A, Q) : (M_1, A_1, Q_1) \to (M_2, A_2, Q_2)$ and $(W', A', Q') : (M_2, A_2, Q_3) \to (M_3, A_3, Q_3)$ the diagram induced by the inclusions

$$\begin{tikzcd}
 & \mathfrak{X}_G(W'', A'', Q'') \arrow[dl] \arrow[dr] & \\
\mathfrak{X}_G(W, A, Q) \arrow[dr] & & \mathfrak{X}_G(W', A', Q') \arrow[dl] \\
 & \mathfrak{X}_G(M_2, A_2, Q_2) &
\end{tikzcd}$$

where $(W'', A'', Q'') = (W \sqcup_{M_2} W', A \cup A', Q \sqcup_{M_2} Q')$ denotes the composition, is a pullback diagram, making $\mathcal{F}_\Lambda$ indeed a functor. Also it is easy to see that this functor is still monoidal. We obtain the resulting (lax symmetric) TQFT

$$Z_\Lambda = \mathcal{Q} \circ \mathcal{F}_\Lambda : \mathbf{Bdp}_n(\Lambda) \to \mathrm{K}(\mathbf{Var}_k)\text{-}\mathbf{Mod}.$$

To a closed connected oriented manifold X with parabolic structure Q is now associated the invariant

$$Z_\Lambda(X, A, Q)(1) = [\mathfrak{X}_G(X, A, Q)].$$

Since Z_Λ is understood to be an extension of the earlier TQFT $Z : \mathbf{Bd}_n \to \mathrm{K}(\mathbf{Var}_k)\text{-}\mathbf{Mod}$, and since it is clear what set Λ we consider, we will just write Z for Z_Λ.

4.4 Field theory in dimension 2

We focus on the case of dimension $n = 2$. Let $X = \Sigma_g$ be a closed oriented 2-dimensional surface of genus g, possibly with a parabolic structure Q. Now Σ_g can be considered as a bordism $\varnothing \to \varnothing$, and after taking a suitable finite set $A \subset \Sigma_g$, be written as a composition of the following bordisms:

$D^\dagger : (S^1, \star) \to \varnothing$ $\quad$ $L : (S^1, \star) \to (S^1, \star)$ $\quad$ $L_\mathcal{E} : (S^1, \star) \to (S^1, \star)$ $\quad$ $D : \varnothing \to (S^1, \star)$ (4.5)

Here $L_\mathcal{E}$ denotes the cylinder with parabolic structure $\{(\star, \mathcal{E})\}$. Now indeed, if we write $Q = \{(\star, \mathcal{E}_1), \ldots, (\star, \mathcal{E}_s)\}$ for the parabolic structure on Σ_g, we have

$$(\Sigma_g, A, Q) = D^\dagger \circ L^g \circ L_{\mathcal{E}_1} \circ \cdots \circ L_{\mathcal{E}_s} \circ D. \tag{4.6}$$

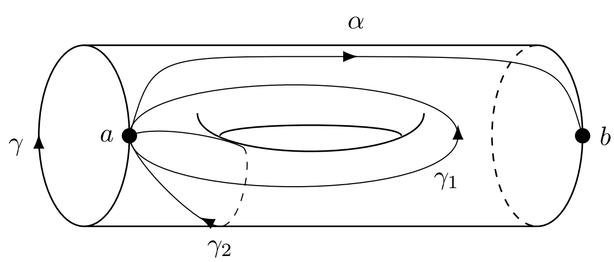
Of course, the category $\mathbf{Bdp}_2(\Lambda)$ consists of more objects and morphisms than just the ones mentioned in (4.5). However, as we are only interested in closed connected surfaces (possibly with a parabolic structure), we will restrict our attention to subcategory of $\mathbf{Bdp}_2(\Lambda)$: we say a *strict tube* is any composition of the bordisms in (4.5), and let $\mathbf{Tb}_2(\Lambda)$ be the subcategory of $\mathbf{Bdp}_2(\Lambda)$ whose objects are disjoint copies of $(S^1, \star)$ and bordisms are disjoint unions of strict tubes. Note that $\mathbf{Tb}_2(\Lambda)$ is still monoidal (with the same monoidal structure as $\mathbf{Bdp}_2(\Lambda)$). We refer to $\mathbf{Tb}_2(\Lambda)$ as the *category of tubes*.

We restrict Z to a functor $\mathbf{Tb}_2(\Lambda) \to \mathrm{K}(\mathbf{Var}_k)\text{-}\mathbf{Mod}$, and explicitly describe what the TQFT does to our objects and bordisms in (4.5).

The fundamental groups $\pi_1(D)$ and $\pi_1(D^\dagger)$ are trivial, implying $\mathfrak{X}_G(D) = \mathfrak{X}_G(D^\dagger) = \star$. Since $\pi_1(S^1, \star) = \mathbb{Z}$, we have $\mathfrak{X}_G(S^1, \star) = \mathrm{Hom}(\mathbb{Z}, G) = G$ and since $\Pi(\varnothing)$ is the empty groupoid, we have $\mathfrak{X}_G(\varnothing) = \star$. Hence the field theory for D and $D^\dagger$ is given by

$$\begin{array}{rccccc} \mathcal{F}(D): & \star & \longleftarrow & \star & \longrightarrow & G \\ & \star & \leftarrow\!\shortmid & \star & \mapsto & \mathrm{id} \end{array} \quad \text{and} \quad \begin{array}{rccccc} \mathcal{F}(D^\dagger): & G & \longleftarrow & \star & \longrightarrow & \star \\ & \mathrm{id} & \leftarrow\!\shortmid & \star & \mapsto & \star. \end{array}$$

For the bordism L, call its two basepoints a and b. The surface of L is homotopic to a torus with two punctures, so its fundamental group (w.r.t. a) is the free group F_3. We pick generators $\gamma, \gamma_1, \gamma_2$ as depicted in the following image, and a path α connecting a and b.



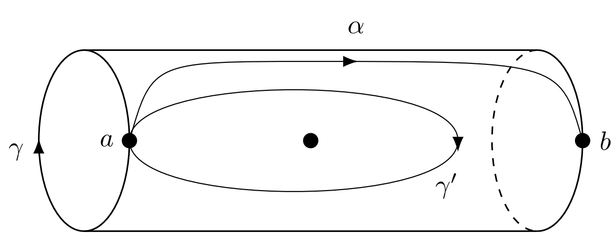
According to (4.2) we can now identify

$$\begin{aligned} \mathfrak{X}_G(L) &\simeq \mathrm{Hom}(F_3, G) \times G \simeq G^4 \\ \rho &\mapsto (\rho(\gamma), \rho(\gamma_1), \rho(\gamma_2), \rho(\alpha)). \end{aligned}$$

A generator for $\pi_1(S^1, b)$ is given by $\alpha\gamma[\gamma_1, \gamma_2]\alpha^{-1}$, and so the field theory for L is found to be

$$\begin{array}{rccccc} \mathcal{F}(L): & G & \longleftarrow & G^4 & \longrightarrow & G \\ & g & \leftarrow\!\shortmid & (g, g_1, g_2, h) & \mapsto & hg[g_1, g_2]h^{-1}. \end{array} \tag{4.7}$$

Finally for the bordism $L_{\mathcal{E}}$, call its two basepoints a and b. The fundamental group (w.r.t. a) of the cylinder with a puncture is the free group F_2. We pick generators γ, γ' as depicted in the following image, and a path α connecting a and b.



Using (4.3) we can now identify

$$\begin{aligned}\mathfrak{X}_G(L_{\mathcal{E}}) &\simeq G^2 \times \mathcal{E} \\ \rho &\mapsto (\rho(\gamma), \rho(\alpha), \rho(\gamma')).\end{aligned}$$

A generator for $\pi_1(S^1, b)$ is given by $\alpha\gamma\gamma'\alpha^{-1}$, and so the field theory for $L_{\mathcal{E}}$ is found to be

$$\begin{array}{rccccc} \mathcal{F}(L_\lambda): & G & \longleftarrow & G^2 \times \mathcal{E} & \longrightarrow & G \\ & g & \leftarrow\!\shortmid & (g, h, \xi) & \mapsto & hg\xi h^{-1}. \end{array}$$

Finally, using (4.2) and (4.6) we can express the class of the character variety $\mathfrak{X}_G(\Sigma_g, Q)$ in terms of the TQFT:

$$\begin{aligned}[\mathfrak{X}_G(\Sigma_g, Q)] &= \frac{1}{[G]^{g+s}}[\mathfrak{X}_G(\Sigma_g, \{g+s+1 \text{ points}\}, Q)] \\ &= \frac{1}{[G]^{g+s}} Z(D^\dagger \circ L^g \circ L_{\mathcal{E}_1} \circ \ldots \circ L_{\mathcal{E}_s} \circ D)(1) \end{aligned} \tag{4.8}$$

Note that $[G]$ might not be invertible, but one can consider a suitable localization of $\mathrm{K}(\mathbf{Var}_k)$ in which this is the case. As a consequence, the class $[\mathfrak{X}_G(\Sigma_g, Q)]$ is only defined in this localization.

4.5 Case G abelian

In this section, suppose that G is an abelian group. We write $+$ for the group operation and 0 for the unit of G. Quite some simplifications occur for the field theory that is described above. First of all, note that the conjugacy classes of G are all singletons $\mathcal{E} = \{\xi\}$. The field theory for L and $L_{\{\xi\}}$ as described in the previous section now simplify to

$$\begin{array}{rccccc} \mathcal{F}(L): & G & \xleftarrow{p} & G^4 & \xrightarrow{q} & G \\ & g & \leftarrow\!\shortmid & (g, g_1, g_2, h) & \mapsto & g \end{array}$$

and

$$\begin{array}{rccccc} \mathcal{F}(L_{\{\xi\}}): & G & \xleftarrow{r} & G^2 \times \{\xi\} & \xrightarrow{s} & G \\ & g & \leftarrow\!\shortmid & (g, h, \xi) & \mapsto & g + \xi. \end{array}$$

Suppose that X is a variety over G that factors as $X \to \{g\} \to G$ for some $g \in G$. Then we find that $Z(L)([X]_G) = q_! p^* [X]_G = [G]^3 \cdot [X]_G$. Also we see that $Z(L_{\{\xi\}})([X]_G) = s_! r^* [X]_G = [G] \cdot [X']_G$ with X' being isomorphic to X, lying over $g + \xi \in G$.

With these observations the computations become very easy. Let $X = (\Sigma_g, A, Q)$, with $Q = \{(\star, \{\xi_1\}), \ldots, (\star, \{\xi_s\})\}$ some parabolic structure and $A \subset X$ a finite set of $g+s+1$ points, viewed as a bordism $\varnothing \to \varnothing$. We have

$$\begin{aligned} Z(X)(1) &= Z(D^\dagger \circ L^g \circ L_{\{\xi_1\}} \circ \cdots \circ L_{\{\xi_s\}} \circ D)(1) \\ &= Z(D^\dagger \circ L^g \circ L_{\{\xi_1\}} \circ \cdots \circ L_{\{\xi_s\}})([\{0\}]_G) \\ &= Z(D^\dagger \circ L^g)([G]^s \cdot [\{\textstyle\sum_i \xi_i\}]_G) \\ &= Z(D^\dagger)([G]^{3g+s} \cdot [\{\textstyle\sum_i \xi_i\}]_G) \\ &= \begin{cases} [G]^{3g+s} & \text{if } \sum_i \xi_i = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By (4.8) we conclude that

$$[\mathfrak{X}_G(\Sigma_g, Q)] = \begin{cases} [G]^{2g} & \text{if } \sum_i \xi_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

When looking back at expression (4.4), this result makes sense. Namely, in an abelian group all elements commute so we have

$$\left\{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \;\middle|\; \sum_{i=1}^{g} [A_i, B_i] \sum_{j=1}^{s} \xi_j = 0 \right\} = \begin{cases} G^{2g} & \text{if } \sum_i \xi_i = 0, \\ \varnothing & \text{otherwise,} \end{cases}$$

from which the same result could also be derived.

4.6 Reduction of the TQFT

Let $Z : \mathcal{B} \to R\text{-}\mathbf{Mod}$ be a (lax monoidal) TQFT, with $\mathcal{B}$ some kind of bordism category (e.g. $\mathbf{Bd}_n$, $\mathbf{Bdp}_n$ or $\mathbf{Bdp}_n(\Lambda)$). In some cases there is a symmetry, like a group acting on the R-modules, which can be used to 'reduce' the TQFT. We represent this as follows.

For each object $M \in \mathcal{B}$ let

$$Z(M) \underset{\beta_M}{\overset{\alpha_M}{\rightleftarrows}} N_M$$

be R-module morphisms, with N_M some R-module. Assume $N_\varnothing = Z(\varnothing)$ with $\alpha_\varnothing$ and $\beta_\varnothing$ the identity maps. Let $V_M \subset N_M$ be R-submodules such that $(\alpha_{M'} \circ Z(W) \circ \beta_M)(V_M) \subset V_{M'}$ for all bordisms $W : M \to M'$ in $\mathcal{B}$. In particular $(\alpha_M \circ \beta_M)(V_M) \subset V_M$ for any M.

Lemma 4.6. *Suppose that $\alpha_M \circ \beta_M : V_M \to V_M$ is invertible for all $M \in \mathcal{B}$. Then for every $W : M \to M'$ there exists a unique morphism $\tilde{Z}(W) : V_M \to V_{M'}$ such that the following diagram commutes.*

$$\begin{tikzcd} \beta_M(V_M) \arrow[r, "Z(W)"] \arrow[d, "\alpha_M"'] & Z(W)(\beta_M(V_M)) \arrow[d, "\alpha_{M'}"] \\ V_M \arrow[r, dashed, "\tilde{Z}(W)"'] & V_{M'} \end{tikzcd}$$

Proof. Indeed the above diagram is well-defined by the assumption that $(\alpha_{M'} \circ Z(W) \circ \beta_M)(V_M) \subset V_{M'}$. We are looking for a map $\tilde{Z}(W) : V_M \to V_{M'}$ such that $\tilde{Z}(W) \circ \alpha_M = \alpha_{M'} \circ Z(W)$. Precomposing this equality with $\beta_M \circ (\alpha_M \circ \beta_M)^{-1}$ gives

$$\tilde{Z}(W) = \alpha_{M'} \circ Z(W) \circ \beta_M \circ (\alpha_M \circ \beta_M)^{-1}.$$

This shows there is a unique choice of $\tilde{Z}(W)$, and it is easy to see that this choice makes the diagram commute: any $x \in \beta_M(V_M)$ can be written as $x = \beta_M(y)$ for some $y \in V_M$, so

$$\tilde{Z}(W) \circ \alpha_M(x) = \tilde{Z}(W) \circ (\alpha_M \circ \beta_M)(y) = \alpha_{M'} \circ Z(W) \circ \beta_M(y) = \alpha_{M'} \circ Z(W)(x).$$

$\square$

Our goal is to construct a functor $\tilde{Z} : \mathcal{B} \to R\text{-}\mathbf{Mod}$ that assigns $\tilde{Z}(M) = V_M$ and $\tilde{Z}(W) : V_M \to V_{M'}$ for any $W : M \to M'$ as above. Note that functoriality follows from the additional assumption that

$$Z(W)(\beta_M(V_M)) \subset \beta_{M'}(V_{M'}) \qquad \text{for any bordism } W : M \to M'.$$

Namely if so, let $W : M \to M'$ and $W' : M' \to M''$ be bordisms. Then

$$\begin{tikzcd}
\beta_M(V_M) \arrow[r, "Z(W)"] \arrow[d, "\alpha_M"'] & Z(W)(\beta_M(V_M)) \arrow[r, "Z(W')"] \arrow[d, "\alpha_{M'}"] & Z(W' \circ W)(\beta_M(V_M)) \arrow[d, "\alpha_{M''}"] \\
V_M \arrow[r, "\tilde{Z}(W)"] & V_{M'} \arrow[r, "\tilde{Z}(W')"] & V_{M''}
\end{tikzcd}$$

is a commutative diagram by the previous lemma. We have $\tilde{Z}(W') \circ \tilde{Z}(W) \circ \alpha_M = \alpha_{M''} \circ Z(W') \circ Z(W) = \alpha_{M''} \circ Z(W' \circ W)$ so uniqueness implies that $\tilde{Z}(W' \circ W) = \tilde{Z}(W') \circ \tilde{Z}(W)$, and hence $\tilde{Z}$ is a functor. Summarizing, we obtain the following definition.

Definition 4.7. For each object M in $\mathcal{B}$, let $Z(M) \underset{\beta_M}{\overset{\alpha_M}{\rightleftarrows}} N_M$ be R-module morphisms with N_M an R-module, and $V_M \subset N_M$ a submodule. If

(i) $N_\varnothing = Z(\varnothing)$ and $\alpha_\varnothing, \beta_\varnothing$ are identity maps,

(ii) $(\alpha_{M'} \circ Z(W) \circ \beta_M)(V_M) \subset V_{M'}$ for all bordisms $W : M \to M'$,

(iii) the restriction $\alpha_M \circ \beta_M : V_M \to V_M$ is invertible for all M,

(iv) $Z(W)(\beta_M(V_M)) \subset \beta_{M'}(V_{M'})$ for all bordisms $W : M \to M'$,

then we speak of a *reduction* of the TQFT and call $\tilde{Z}$ the *reduced TQFT*.

The useful fact about the reduced TQFT $\tilde{Z}$ is that it computes the same invariants as Z for closed manifolds. Namely, if $W : \varnothing \to \varnothing$ is a bordism, then

$$\tilde{Z}(W)(1) = \tilde{Z}(W) \circ \alpha_\varnothing(1) = \alpha_\varnothing \circ Z(W)(1) = Z(W)(1).$$

We will apply this in Section 5 to the category $\mathcal{B} = \mathbf{Tb}_2(\Lambda)$ as follows. We have $Z(S^1, \star) = K(\mathbf{Var}/G)$, and there is an action of G on itself by conjugation. Suppose there are conjugacy-closed strata $\mathcal{C}_1, \ldots, \mathcal{C}_n$ for G, with maps $\pi_i : \mathcal{C}_i \to C_i$ whose fibers are precisely the orbits of G. Then by Lemma 2.14 we have $K(\mathbf{Var}/G) = K(\mathbf{Var}/\mathcal{C}_1) \oplus \cdots \oplus K(\mathbf{Var}/\mathcal{C}_n)$, and the maps $(\pi_i)_!$ and $(\pi_i)^*$ induce maps

$$\mathrm{K}(\mathbf{Var}/\mathcal{C}_1) \oplus \cdots \oplus \mathrm{K}(\mathbf{Var}/\mathcal{C}_n) \underset{\pi^*}{\overset{\pi_!}{\rightleftarrows}} \mathrm{K}(\mathbf{Var}/C_1) \oplus \cdots \oplus \mathrm{K}(\mathbf{Var}/C_n) \ , \tag{4.9}$$

which by slight abuse of notation we have denoted π^* and $\pi_!$. For bordisms $W : (S^1, \star) \to (S^1, \star)$, we write $Z_\pi(W) = \pi_! \circ Z(W) \circ \pi^*$ as a shorthand. Let $V \subset \oplus_i \mathrm{K}(\mathbf{Var}/C_i)$ be a submodule on which $\eta = \pi_! \pi^*$ is invertible. Now, not any such stratification will satisfy the assumptions for a reduction. To make this precise, consider $\Delta = \{(g_1, g_2) \in G^2 : g_1 \sim g_2\}$, where '$\sim$' means 'is conjugate to', and $p_1, p_2 : \Delta \to G$ the projections of the components. There is the conjugation map

$$c : G^2 \to \Delta : (g, h) \mapsto (g, hgh^{-1}).$$

Note that the stratification of G naturally induces a stratification of Δ, whose pieces we denote by $\Delta_i = \{(g_1, g_2) \in \mathcal{C}_i^2 : g_1 \sim g_2\}$. Now the map $c_! c^*$ can be restricted to a map $\mathrm{K}(\mathbf{Var}/\Delta_i) \to \mathrm{K}(\mathbf{Var}/\Delta_i)$ for each i.

Lemma 4.8. *Let G be an algebraic group, stratified by conjugacy-closed strata $\mathcal{C}_i$, with maps $\pi_i : \mathcal{C}_i \to C_i$ whose fibers are precisely the orbits of G, and let $V \subset \oplus_i K(\mathbf{Var}/C_i)$ be a submodule. Assume that*

(i) $Z_\pi(W)(V) \subset V$ for all bordisms $W : (S^1, \star) \to (S^1, \star)$,

(ii) the map $\eta = \pi_! \pi^* : V \to V$ is invertible,

(iii) $[\{id\}] \in \pi^* V$ and whenever $X \in V$ then $X_i = X|_{C_i} \in V$ as well,

(iv) the restrictions $c_! c^* : K(\mathbf{Var}/\Delta_i) \to K(\mathbf{Var}/\Delta_i)$ are given by multiplication by a constant α_i.

Then the maps in (4.9) and the submodule V yield a reduction of the TQFT.

Proof. The only remaining condition to show is (iv) of Definition 4.7, and it suffices to show this holds for the bordisms $D, D^\dagger, L$ and $L_\mathcal{E}$. This holds for D by assumption (iii), and for $D^\dagger$ trivially because $V_\varnothing = \mathrm{K}(\mathbf{Var}_k)$. To L is associated the span

$$\begin{array}{ccccc} G & \xleftarrow{p} & G^4 & \xrightarrow{q} & G \\ g & \mapsfrom & (g, g_1, g_2, h) & \mapsto & hg[g_1, g_2]h^{-1} \end{array}$$

and we also consider the modified span

$$\begin{array}{ccccc} G & \xleftarrow{\tilde{p}} & G^3 & \xrightarrow{\tilde{q}} & G \\ g & \mapsfrom & (g, g_1, g_2) & \mapsto & g[g_1, g_2]. \end{array}$$

It is not hard to see that

$$q_! p^* = (p_2)_! c_! c^* (p_1)^* \tilde{q}_! \tilde{p}^* \tag{4.10}$$

as both sides of the equality map $X \xrightarrow{f} G$ to

$$\begin{array}{ccc} \{(x,g,g_1,g_2,h) \in X \times G^4 : g = f(x)\} & \longrightarrow & G \\ (x,g,g_1,g_2,h) & \mapsto & hg[g_1,g_2]h^{-1}. \end{array}$$

Now take any $X \in V$, write $Y = \tilde{q}_! \tilde{p}^* \pi^* X \in \mathrm{K}(\mathbf{Var}/G)$, and decompose $Y = \sum_i Y_i$ with each $Y_i \in \mathrm{K}(\mathbf{Var}/\mathcal{C}_i)$ according to the stratification of G. We have $\pi_! Y = \pi_! \tilde{q}_! \tilde{p}^* \pi^* X = \frac{1}{[G]} Z_\pi(L)(X)$, which lies in V by (i). Hence $\pi_! Y_i = (\pi_! Y)_i \in V$ by (iii), and using (iv) we see that

$$(p_2)_! c_! c^* (p_1)^* Y_i = \alpha_i (p_2)_! (p_1)^* Y_i = \alpha_i \pi^* \pi_! Y_i \in \pi^* V,$$

so it follows using (4.10) that

$$Z(L)(\pi^* X) = \sum_i \pi^* \pi_! Y_i \in \pi^* V.$$

A completely similar argument shows that $Z(L_\mathcal{E})(\pi^* V) \subset \pi^* V$ as well. $\square$

Remark 4.9. For the groups G and strata $\mathcal{C}_i$ we consider in the next sections, it might be that the map $\eta = \pi_! \pi^*$ is not invertible as a $\mathrm{K}(\mathbf{Var}_k)$-module morphism. However, one can replace the ring $\mathrm{K}(\mathbf{Var}_k)$ by a suitable localization (often it suffices to invert $[G]$), to make η is invertible. As a consequence, the resulting classes $[\mathfrak{X}_G(X,Q)]$ will only be defined in that localization. This is not unreasonable, since $[G]$ needs to be invertible anyway in order to apply (4.8). Also in many cases we can still extract algebraic data from the localized class: given a multiplicative system $S \subset \mathrm{K}(\mathbf{Var}_k)$ and an element $\overline{x} \in S^{-1}\mathrm{K}(\mathbf{Var}_k)$ that admits a lift $x \in \mathrm{K}(\mathbf{Var}_k)$, this lift is defined up to a sum of annihilators of elements of S. If $\varphi : \mathrm{K}(\mathbf{Var}_k) \to R$ is a ring morphism with R a domain such that $\varphi(s) \neq 0$ for all $s \in S$, then $\varphi(a) = 0$ for any annihilator a of any $s \in S$. Hence $\varphi(x)$ is independent on the choice of lift.

The example to have in mind here is the E-polynomial $e : \mathrm{K}(\mathbf{Var}_k) \to \mathbb{Z}[u,v]$. Since $e(q) = uv \neq 0$, to compute the E-polynomial of some variety X over k it is sufficient to know its class in the localized ring $S^{-1}\mathrm{K}(\mathbf{Var}_k)$ for $S = \{1, q, q^2, \ldots\}$. (Similarly we could invert $q-1$ or $q+1$.)

5 Applications

In this section we apply the technique developed in Section 4 to compute the class of the G-representation varieties $\mathfrak{X}_G(\Sigma_g)$ in the Grothendieck ring $\mathrm{K}(\mathbf{Var}_{\mathbb{C}})$ with G the groups of upper triangular $n \times n$ matrices, for $n = 2, 3, 4$. This section consists of original work and the main results are given by theorems 5.2, 5.8 and 5.10.

5.1 Upper triangular 2×2 matrices

Let $\mathbb{U}_2$ be the group of 2×2 upper triangular matrices over $\mathbb{C}$, i.e.

$$\mathbb{U}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, c \neq 0 \right\}.$$

We are going to apply the theory from the previous sections to compute the class of the $\mathbb{U}_2$-representation variety of Σ_g. First we discuss some generalities about this group. Its class is seen to be $[\mathbb{U}_2] = q(q-1)^2$. The group contains the following three types of conjugacy classes. All scalar matrices $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ have a singleton orbit. All matrices of the form $\left(\begin{smallmatrix} \lambda & b \\ 0 & \lambda \end{smallmatrix}\right)$ with $b \neq 0$ are conjugate to the Jordan block $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$, and thus have an orbit isomorphic to $\mathbb{C}^*$. All remaining matrices in $\mathbb{U}_2$ are of the form $\left(\begin{smallmatrix} \lambda & b \\ 0 & \mu \end{smallmatrix}\right)$, with $\lambda, \mu \neq 0$, which are conjugate if and only if they have the same diagonal. Hence, these matrices have an orbit isomorphic to $\mathbb{C}$. Let

$$\mathcal{S} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \neq 0 \right\}, \quad \mathcal{J} = \left\{ \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} : \lambda, b \neq 0 \right\}, \quad \mathcal{M} = \left\{ \begin{pmatrix} \lambda & b \\ 0 & \mu \end{pmatrix} : \lambda, \mu \neq 0, \lambda \neq \mu, b \in \mathbb{C} \right\}.$$

The orbits we denote by

$$\mathcal{S}_\lambda = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\}, \qquad \mathcal{J}_\lambda = \left\{ \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} : b \neq 0 \right\}, \qquad \mathcal{M}_{\lambda,\mu} = \left\{ \begin{pmatrix} \lambda & b \\ 0 & \mu \end{pmatrix} : b \in \mathbb{C} \right\}$$

for any $\lambda, \mu \neq 0$ with $\lambda \neq \mu$. It is easy to see that

$$\begin{array}{ccc} [\mathcal{S}] = q - 1, & [\mathcal{J}] = (q-1)^2, & [\mathcal{M}] = q(q-1)(q-2), \\ [\mathcal{S}_\lambda] = 1, & [\mathcal{J}_\lambda] = q - 1, & [\mathcal{M}_{\lambda,\mu}] = q. \end{array}$$

We denote the orbit spaces by $S = \mathbb{C}^*$, $J = \mathbb{C}^*$ and $M = \{(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^* : \lambda \neq \mu\}$. The quotient maps that identify the orbits are

$$\pi_{\mathcal{S}} : \mathcal{S} \to S : \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mapsto \lambda, \qquad \pi_{\mathcal{J}} : \mathcal{J} \to J : \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} \mapsto \lambda, \qquad \pi_{\mathcal{M}} : \mathcal{M} \to M : \begin{pmatrix} \lambda & b \\ 0 & \mu \end{pmatrix} \mapsto (\lambda, \mu).$$

These maps induce the morphisms

$$K(\mathbf{Var}/\mathcal{S}) \oplus K(\mathbf{Var}/\mathcal{J}) \oplus K(\mathbf{Var}/\mathcal{M}) \underset{\pi^*}{\overset{\pi_!}{\rightleftarrows}} K(\mathbf{Var}/S) \oplus K(\mathbf{Var}/J) \oplus K(\mathbf{Var}/M) \ .$$

In order to show condition (iv) of Lemma 4.8 is satisfied for this stratification, it is convenient to consider representatives of the conjugacy classes

$$\xi^{\mathcal{S}}_\lambda = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right), \qquad \xi^{\mathcal{J}}_\lambda = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right), \qquad \xi^{\mathcal{M}}_{\lambda,\mu} = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right).$$

The stabilizers

$$\mathrm{Stab}(\xi_\lambda^{\mathcal{S}}) = \mathbb{U}_2, \qquad \mathrm{Stab}(\xi_\lambda^{\mathcal{J}}) = \left\{ \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right) : \alpha \neq 0 \right\}, \qquad \mathrm{Stab}(\xi_{\lambda,\mu}^{\mathcal{M}}) = \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) : \alpha, \beta \neq 0 \right\}$$

are independent of λ, μ so denote them by $\mathrm{Stab}_{\mathcal{S}}, \mathrm{Stab}_{\mathcal{J}}$ and $\mathrm{Stab}_{\mathcal{M}}$. For all $\mathcal{E} = \mathcal{S}, \mathcal{J}, \mathcal{M}$ it is straightforward to come up with a map $\sigma : \mathcal{E} \to G$ such that $g = \sigma(g) \xi^{\mathcal{E}}_{\pi_{\mathcal{E}}(g)} \sigma(g)^{-1}$ for any $g \in \mathcal{E}$. E.g. for $\mathcal{E} = \mathcal{J}$ we can take $\sigma \left(\begin{smallmatrix} \lambda & b \\ 0 & \lambda \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/b \end{smallmatrix}\right)$ as

$$\begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix},$$

and for $\mathcal{E} = \mathcal{M}$ one can take $\sigma \left(\begin{smallmatrix} \lambda & b \\ 0 & \mu \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & b/(\mu-\lambda) \\ 0 & 1 \end{smallmatrix}\right)$ as

$$\begin{pmatrix} 1 & b/(\mu-\lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & -b/(\mu-\lambda) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & b \\ 0 & \mu \end{pmatrix}.$$

Now for any variety $X \xrightarrow{(f_1, f_2)} \Delta_{\mathcal{E}}$ we have an isomorphism

$$\begin{array}{ccc} X \times \mathrm{Stab}_{\mathcal{E}} & \xrightarrow{\sim} & c_! c^* X = \{(x, h) \in X \times G : f_2(x) = h f_1(x) h^{-1}\} \\ (x, s) & \mapsto & (x, \sigma(f_2(x)) s \sigma(f_1(x))^{-1}) \end{array}$$

which shows that $c_! c^*$ restricted to $\Delta_{\mathcal{E}}$ is just multiplication by $\mathrm{Stab}_{\mathcal{E}}$.

Write $T_{S_\lambda} \in K(\mathbf{Var}/S), T_{J_\lambda} \in K(\mathbf{Var}/J)$ and $T_{M_{\lambda,\mu}} \in K(\mathbf{Var}/M)$ for the classes of the points $\{\lambda\} \to S, \{\lambda\} \to J$ and $\{(\lambda, \mu)\} \to M$. We consider the submodule $V = \langle T_{S_\lambda}, T_{J_\lambda}, T_{M_{\lambda,\mu}} \rangle$. From the computations that follow, it will be clear that V is invariant under $\eta = \pi_! \circ \pi^*$ and Z_π. Hence all conditions from Lemma 4.8 are satisfied, so we have a reduction of the TQFT.

Since all fibrations $\mathcal{S} \to S, \mathcal{J} \to J$ and $\mathcal{M} \to M$ are trivial, we immediately find that

$$\eta(T_{S_\lambda}) = [\mathcal{S}_\lambda] T_{S_\lambda} = T_{S_\lambda}, \quad \eta(T_{J_\lambda}) = [\mathcal{J}_\lambda] T_{J_\lambda} = (q-1) T_{J_\lambda}, \quad \eta(T_{M_{\lambda,\mu}}) = [\mathcal{M}_{\lambda,\mu}] T_{M_{\lambda,\mu}} = q T_{M_{\lambda,\mu}},$$

that is,

$$\eta = \left(\begin{array}{c|ccc} & T_{S_\lambda} & T_{J_\lambda} & T_{M_{\lambda,\mu}} \\ \hline T_{S_\lambda} & 1 & 0 & 0 \\ T_{J_\lambda} & 0 & q-1 & 0 \\ T_{M_{\lambda,\mu}} & 0 & 0 & q \end{array} \right).$$

For computing $Z_\pi(L)$, recall that $\mathcal{F}(L)$ is given by

$$\begin{array}{ccccc} \mathbb{U}_2 & \xleftarrow{p} & \mathbb{U}_2^4 & \xrightarrow{q} & \mathbb{U}_2 \\ g & \longleftarrow & (g, g_1, g_2, h) & \longmapsto & h g [g_1, g_2] h^{-1}. \end{array}$$

- First we compute $Z_\pi(L)(T_{S_\lambda})$. We have $\pi^*(T_{S_\lambda}) = [\mathcal{S}_\lambda]_{\mathbb{U}_2}$. Note that for any $g_1, g_2 \in \mathbb{U}_2$, the commutator $[g_1, g_2] = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ with $x = (a_1 b_2 - a_2 b_1 + b_1 c_2 - b_2 c_1)/(c_1 c_2)$, writing $g_i = \left(\begin{smallmatrix} a_i & b_i \\ 0 & c_i \end{smallmatrix}\right)$. Hence $q(p^*(\mathcal{S}_\lambda)) \subset \mathcal{S}_\lambda \cup \mathcal{J}_\lambda$, and thus $Z_\pi(L)(T_{S_\lambda})$ is generated by T_{S_λ} and T_{J_λ}. Then we have that

$$\begin{aligned} Z_\pi(L)(T_{S_\lambda})|_{T_{S_\lambda}} &= [\mathcal{S}_\lambda \times \mathbb{U}_2^3 \cap q^{-1}(\mathcal{S}_\lambda)] \\ &= [\{g_1, g_2 \in \mathbb{U}_2 : [g_1, g_2] = \mathrm{id}\}] \cdot [\mathbb{U}_2] \\ &= [\{a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{C} : a_1 b_2 - a_2 b_1 + b_1 c_2 - b_2 c_1 = 0 \text{ and } a_1 c_1, a_2 c_2 \neq 0\}] \cdot [\mathbb{U}_2], \end{aligned}$$

which is $q^2(q-1)^3 \cdot [\mathbb{U}_2] = q^3(q-1)^5$. Now it follows that $Z_\pi(L)(T_{S_\lambda})|_{T_{J_\lambda}}$ is simply $[\mathcal{S}_\lambda \times \mathbb{U}_2^3] - q^3(q-1)^5 = q^3(q-1)^5(q-2)$.

- Next we compute $Z_\pi(L)(T_{J_\lambda})$. We have $\pi^*(T_{J_\lambda}) = [\mathcal{J}_\lambda]_{\mathbb{U}_2}$. By the same observation as above about the commutator, we see that $Z_\pi(L)(T_{J_\lambda})$ is also generated by T_{S_λ} and T_{J_λ}. Note that

$$\left(\begin{smallmatrix} \lambda & b \\ 0 & \lambda \end{smallmatrix}\right)[g_1, g_2] = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) \iff [g_1, g_2] = \left(\begin{smallmatrix} 1 & -b/\lambda \\ 0 & 1 \end{smallmatrix}\right),$$

which implies that $Z_\pi(L)(T_{J_\lambda})|_{T_{S_\lambda}} = Z_\pi(L)(T_{S_\lambda})|_{T_{J_\lambda}} = q^3(q-1)^5(q-2)$. Now it follows that $Z_\pi(L)(T_{J_\lambda})|_{T_{J_\lambda}} = [\mathcal{J}_\lambda \times \mathbb{U}_2^3] - q^3(q-1)^5(q-2) = q^3(q-1)^5(q^2-3q+3)$.

- Lastly we compute $Z_\pi(L)(T_{M_{\lambda,\mu}})$. We have $\pi^*(T_{M_{\lambda,\mu}}) = [\mathcal{M}_{\lambda,\mu}]_{\mathbb{U}_2}$. By the observation about the commutator, we immediately see that $Z_\pi(L)(T_{M_{\lambda,\mu}})$ must be generated by $T_{M_{\lambda,\mu}}$. Therefore, $Z_\pi(L)(T_{M_{\lambda,\mu}}) = [\mathcal{M}_{\lambda,\mu} \times \mathbb{U}_2^3] T_{M_{\lambda,\mu}} = q^4(q-1)^6 T_{M_{\lambda,\mu}}$.

In summary,

$$Z_\pi(L) = q^3(q-1)^5 \left(\begin{array}{c|ccc} & T_{S_\lambda} & T_{J_\lambda} & T_{M_{\lambda,\mu}} \\ \hline T_{S_\lambda} & 1 & q-2 & 0 \\ T_{J_\lambda} & q-2 & q^2-3q+3 & 0 \\ T_{M_{\lambda,\mu}} & 0 & 0 & q(q-1) \end{array}\right).$$

Finally, we can also compute $Z_\pi(L_{\mathcal{S}_\lambda}), Z_\pi(L_{\mathcal{J}_\lambda})$ and $Z_\pi(L_{\mathcal{M}_{\lambda,\mu}})$. Recall that $\mathcal{F}(L_\mathcal{E})$ is given by

$$\begin{array}{ccccc} \mathbb{U}_2 & \xleftarrow{\;r\;} & \mathbb{U}_2^2 \times \mathcal{E} & \xrightarrow{\;s\;} & \mathbb{U}_2 \\ g & \longleftarrow & (g, h, \xi) & \longmapsto & hg\xi h^{-1}. \end{array}$$

- Let $g \in \mathcal{S}_\lambda$, and note that if $\xi \in \mathcal{S}_\sigma$ then $g\xi \in \mathcal{S}_{\lambda\sigma}$, if $\xi \in \mathcal{J}_\sigma$ then $g\xi \in \mathcal{J}_{\lambda\sigma}$, and if $\xi \in \mathcal{M}_{\sigma,\rho}$ then $g\xi \in \mathcal{M}_{\lambda\sigma,\lambda\rho}$. Hence we have

$$Z_\pi(L_{\mathcal{S}_\lambda}) = \left(\begin{array}{c|ccc} & T_{S_\sigma} & T_{J_\sigma} & T_{M_{\sigma,\rho}} \\ \hline T_{S_{\lambda\sigma}} & [\mathcal{S}_\sigma \times \mathbb{U}_2] & 0 & 0 \\ T_{J_{\lambda\sigma}} & 0 & [\mathcal{J}_\sigma \times \mathbb{U}_2] & 0 \\ T_{M_{\lambda\sigma,\lambda\rho}} & 0 & 0 & [\mathcal{M}_{\sigma,\rho} \times \mathbb{U}_2] \end{array}\right) = q(q-1)^2 \left(\begin{array}{c|ccc} & T_{S_\sigma} & T_{J_\sigma} & T_{M_{\sigma,\rho}} \\ \hline T_{S_{\lambda\sigma}} & 1 & 0 & 0 \\ T_{J_{\lambda\sigma}} & 0 & q-1 & 0 \\ T_{M_{\lambda\sigma,\lambda\rho}} & 0 & 0 & q \end{array}\right).$$

- Now let $g \in \mathcal{J}_\lambda$. We see that if $\xi \in \mathcal{S}_\sigma$ then $g\xi \in \mathcal{J}_{\lambda\sigma}$, and if $\xi \in \mathcal{M}_{\sigma,\rho}$ then $g\xi \in \mathcal{M}_{\lambda\sigma,\lambda\rho}$. If $\xi \in \mathcal{J}_\sigma$, then $g\xi \in \mathcal{S}_{\lambda\sigma}$ precisely if $g = \lambda\sigma\xi^{-1}$ and otherwise $g\xi \in \mathcal{J}_{\lambda\sigma}$. Hence we have

$$Z_\pi(L_{\mathcal{J}_\lambda}) = \left(\begin{array}{c|ccc} & T_{S_\sigma} & T_{J_\sigma} & T_{M_{\sigma,\rho}} \\ \hline T_{S_{\lambda\sigma}} & 0 & [\mathcal{J}_\sigma \times \mathbb{U}_2] & 0 \\ T_{J_{\lambda\sigma}} & [\mathcal{S}_\sigma \times \mathbb{U}_2][\mathcal{J}_\lambda] & [\mathcal{J}_\sigma \times \mathbb{U}_2]([\mathcal{J}_\lambda]-1) & 0 \\ T_{M_{\lambda\sigma,\lambda\rho}} & 0 & 0 & [\mathcal{M}_{\sigma,\rho} \times \mathbb{U}_2][\mathcal{J}_\lambda] \end{array}\right)$$

$$= q(q-1)^2 \left(\begin{array}{c|ccc} & T_{S_\sigma} & T_{J_\sigma} & T_{M_{\sigma,\rho}} \\ \hline T_{S_{\lambda\sigma}} & 0 & q-1 & 0 \\ T_{J_{\lambda\sigma}} & q-1 & (q-1)(q-2) & 0 \\ T_{M_{\lambda\sigma,\lambda\rho}} & 0 & 0 & q(q-1) \end{array}\right).$$

- Lastly, let $g \in \mathcal{M}_{\lambda,\mu}$. If $\xi \in \mathcal{S}_\sigma$ then $g\xi \in \mathcal{M}_{\lambda\sigma,\mu\sigma}$, and if $\xi \in \mathcal{J}_\sigma$ then $g\xi \in \mathcal{M}_{\lambda\sigma,\mu\sigma}$ as well. If $\xi \in \mathcal{M}_{\sigma,\rho}$, then $g\xi \in \mathcal{M}_{\lambda\sigma,\mu\rho}$ if $\lambda\sigma \neq \mu\rho$ and otherwise $g\xi \in \mathcal{S}_{\lambda\sigma}$ precisely for $g = \lambda\sigma\xi^{-1}$ and else $g\xi \in \mathcal{J}_{\lambda\sigma}$. Hence we see that

$$Z_\pi(L_{\mathcal{M}_{\lambda,\mu}}) = \left(\begin{array}{c|cccc} & T_{S_\sigma} & T_{J_\sigma} & T_{M_{\sigma,\rho}} & T_{M_{\sigma',\rho'}} \\ \hline T_{S_{\lambda\sigma}} & 0 & 0 & 0 & [\mathcal{M}_{\sigma',\rho'} \times \mathbb{U}_2] \\ T_{J_{\lambda\sigma}} & 0 & 0 & 0 & [\mathcal{M}_{\sigma',\rho'} \times \mathbb{U}_2]([\mathcal{M}_{\lambda,\mu}] - 1) \\ T_{M_{\lambda\sigma,\lambda\rho}} & [\mathcal{S}_\sigma \times \mathbb{U}_2][\mathcal{M}_{\lambda,\mu}] & [\mathcal{J}_\sigma \times \mathbb{U}_2][\mathcal{M}_{\lambda,\mu}] & [\mathcal{M}_{\sigma,\rho} \times \mathbb{U}_2][\mathcal{M}_{\lambda,\mu}] & 0 \end{array}\right)$$

$$= q(q-1)^2 \left(\begin{array}{c|cccc} & T_{S_\sigma} & T_{J_\sigma} & T_{M_{\sigma,\rho}} & T_{M_{\sigma',\rho'}} \\ \hline T_{S_{\lambda\sigma}} & 0 & 0 & 0 & q \\ T_{J_{\lambda\sigma}} & 0 & 0 & 0 & q(q-1) \\ T_{M_{\lambda\sigma,\lambda\rho}} & q & q(q-1) & q^2 & 0 \end{array}\right),$$

with $\lambda\sigma \neq \mu\rho$ and $\lambda\sigma' = \mu\rho'$.

Now for the reduced TQFT, $\tilde{Z}(L) = Z_\pi(L) \circ \eta^{-1}$ looks like

$$\tilde{Z}(L) = q^3(q-1)^4 A \begin{pmatrix} 1 & 0 & 0 \\ 0 & (q-1)^2 & 0 \\ 0 & 0 & (q-1)^2 \end{pmatrix} A^{-1}, \quad \text{with } A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & q-1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which yields

$$\tilde{Z}(L^g) = q^{3g-1}(q-1)^{4g} \left(\begin{array}{c|ccc} & T_{S_\lambda} & T_{J_\lambda} & T_{M_{\lambda,\mu}} \\ \hline T_{S_\lambda} & (q-1)((q-1)^{2g-1}+1) & (q-1)^{2g}-1 & 0 \\ T_{J_\lambda} & (q-1)((q-1)^{2g}-1) & (q-1)^{2g+1}+1 & 0 \\ T_{M_{\lambda,\mu}} & 0 & 0 & q(q-1)^{2g} \end{array}\right). \tag{5.1}$$

In particular,

$$[\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)] = \frac{1}{[\mathbb{U}_2]^g} \tilde{Z}(L^g)(T_{S_1})|_{T_{S_1}} = q^{2g-1}(q-1)^{2g+1}((q-1)^{2g-1}+1). \tag{5.2}$$

Remark 5.1. For small values of g, we find

$$\begin{aligned} [\mathfrak{X}_{\mathbb{U}_2}(\Sigma_1)] &= q^2(q-1)^3, \\ [\mathfrak{X}_{\mathbb{U}_2}(\Sigma_2)] &= q^4(q-1)^5(q^2-3q+3), \\ [\mathfrak{X}_{\mathbb{U}_2}(\Sigma_3)] &= q^6(q-1)^7(q^4-5q^3+10q^2-10q+5), \\ [\mathfrak{X}_{\mathbb{U}_2}(\Sigma_4)] &= q^8\,(q-1)^9\left(q^6-7q^5+21q^4-35q^3+35q^2-21q+7\right). \end{aligned}$$

Note the factor $(q-1)^{2g+1}$, which can be explained as follows. There is a free action of $(\mathbb{C}^*)^{2g}$ on $\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)$ given by scaling the A_i, B_i (notation as in (4.1)), yielding a factor $(q-1)^{2g}$. For the remaining factor, let $D \subset \mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)$ be the subvariety where all A_i, B_i are diagonal. Then $[D] = (q-1)^{4g}$ and there is a free action of $\mathbb{C}^*$ on $\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g) \backslash D$ given by conjugation with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)$, $x \in \mathbb{C}^*$.

Furthermore, we have

$$\tilde{Z}(L_{\mathcal{S}_\lambda}) = q(q-1)^2 \left(\begin{array}{c|ccc} & T_{S_\sigma} & T_{J_\sigma} & T_{M_{\sigma,\rho}} \\ \hline T_{S_{\lambda\sigma}} & 1 & 0 & 0 \\ T_{J_{\lambda\sigma}} & 0 & 1 & 0 \\ T_{M_{\lambda\sigma,\mu\sigma}} & 0 & 0 & 1 \end{array}\right), \tag{5.3}$$

$$\tilde{Z}(L_{\mathcal{J}_\lambda}) = q(q-1)^2 \left(\begin{array}{c|ccc} & T_{S_\sigma} & T_{J_\sigma} & M_{\sigma,\rho} \\ \hline T_{S_{\lambda\sigma}} & 0 & 1 & 0 \\ T_{J_{\lambda\sigma}} & q-1 & q-2 & 0 \\ T_{M_{\lambda\sigma,\mu\sigma}} & 0 & 0 & q-1 \end{array}\right), \tag{5.4}$$

$$\tilde{Z}(L_{\mathcal{M}_{\lambda,\mu}}) = q(q-1)^2 \left(\begin{array}{c|cccc} & T_{S_\sigma} & T_{J_\sigma} & T_{M_{\sigma,\rho}} & T_{M_{\sigma',\rho'}} \\ \hline T_{S_{\lambda\sigma}} & 0 & 0 & 0 & 1 \\ T_{J_{\lambda\sigma}} & 0 & 0 & 0 & q-1 \\ T_{M_{\lambda\sigma,\mu\rho}} & q & q & q & 0 \end{array}\right), \tag{5.5}$$

with $\lambda\sigma \neq \mu\rho$ but $\lambda\sigma' = \mu\rho'$.

Theorem 5.2. *Let Σ_g be surface of genus g, with parabolic data $Q = \{(\star, \mathcal{J}_{\lambda_1}), \ldots, (\star, \mathcal{J}_{\lambda_k}), (\star, \mathcal{M}_{\mu_1,\sigma_1}), \ldots, (\star, \mathcal{M}_{\mu_\ell,\sigma_\ell})\}$.*

(i) If $\prod_{i=1}^k \lambda_i \prod_{j=1}^\ell \mu_j \neq 1$ or $\prod_{i=1}^k \lambda_i \prod_{j=1}^\ell \sigma_j \neq 1$, then

$$[\mathfrak{X}_{\mathbb{U}_2}(X, Q)] = 0.$$

(ii) Otherwise, and if $\ell = 0$, then

$$[\mathfrak{X}_{\mathbb{U}_2}(X, Q)] = q^{2g-1}(q-1)^{2g}((-1)^k(q-1) + (q-1)^{2g+k}),$$

(iii) and if $\ell > 0$, then

$$[\mathfrak{X}_{\mathbb{U}_2}(X, Q)] = q^{2g+\ell-1}(q-1)^{4g+k}.$$

Proof. First note that (Σ_g, Q) can be seen as the composition

$$D^\dagger \circ L^g \circ L_{\mathcal{J}_{\lambda_1}} \circ \cdots \circ L_{\mathcal{J}_{\lambda_k}} \circ L_{\mathcal{M}_{\mu_1,\sigma_1}} \circ \cdots L_{\mathcal{M}_{\mu_\ell,\sigma_\ell}} \circ D.$$

(i) From expressions (5.1), (5.4) and (5.5) we can see that $Z(L^g \circ L_{\mathcal{J}_{\lambda_1}} \circ \cdots \circ L_{\mathcal{J}_{\lambda_k}} \circ L_{\mathcal{M}_{\mu_1,\sigma_1}} \circ \cdots \circ L_{\mathcal{M}_{\mu_\ell,\sigma_\ell}})(T_{S_1})|_{T_{S_1}} = 0$, and hence $[\mathfrak{X}_{\mathbb{U}_2}(X, Q)] = 0$.

(ii) Using (5.4) and the diagonalization

$$\begin{pmatrix} 0 & 1 & 0 \\ q-1 & q-2 & 0 \\ 0 & 0 & q-1 \end{pmatrix} = A \begin{pmatrix} -1 & 0 & 0 \\ 0 & q-1 & 0 \\ 0 & 0 & q-1 \end{pmatrix} A^{-1} \text{ with } A = \begin{pmatrix} -1 & \frac{1}{q-1} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we find that

$$\begin{aligned}&\tilde{Z}(L_{\mathcal{J}_{\lambda_1}} \circ \cdots \circ L_{\mathcal{J}_{\lambda_k}})(T_{S_1}) \\ &\quad = q^{k-1}(q-1)^{2k}\left((-1)^k(q-1) + (q-1)^k\right) T_{S_\lambda} + \left((-1)^{k+1}(q-1) + (q-1)^{k+1}\right) T_{J_\lambda},\end{aligned}$$

where $\lambda = \prod_{i=0}^k \lambda_i$. Then, using (5.1) and that $\lambda = 1$ we have

$$\tilde{Z}(L^g \circ L_{\mathcal{J}_{\lambda_1}} \circ \cdots \circ L_{\mathcal{J}_{\lambda_k}})(T_{S_1})|_{T_{S_1}} = q^{3g+k-1}(q-1)^{4g+2k}\left((-1)^k(q-1) + (q-1)^{2g+k}\right).$$

So finally

$$\begin{aligned}
[\mathfrak{X}_{\mathbb{U}_2}(X, Q)] &= \frac{1}{[\mathbb{U}_2]^{g+k}} \tilde{Z}(L^g \circ L_{\mathcal{J}_{\lambda_1}} \circ \cdots \circ L_{\mathcal{J}_{\lambda_k}})(T_{S_1})|_{T_{S_1}} \\
&= q^{2g-1}(q-1)^{2g}\left((-1)^k(q-1) + (q-1)^{2g+k}\right).
\end{aligned}$$

Note that this is in accordance with (5.2) for $k = 0$.

(iii) Note that $\prod_{i=0}^{\ell} \mu_i = \prod_{i=0}^{\ell} \sigma_i$. In combination with (5.5) it follows that

$$\tilde{Z}(L_{\mathcal{M}_{\mu_1,\sigma_1}} \circ \cdots \circ L_{\mathcal{M}_{\mu_\ell,\sigma_\ell}})(T_{S_1}) = q^{2\ell-1}(q-1)^{2\ell}(T_{S_\mu} + (q-1)T_{J_\mu}),$$

where $\mu = \prod_{i=0}^{\ell} \mu_i$. Similar as before, we use (5.4) to obtain

$$\begin{aligned}
&[\mathfrak{X}_{\mathbb{U}_2}(X, Q)] \\
&\quad = \frac{1}{[\mathbb{U}_2]^{g+k+\ell}} \tilde{Z}(L^g \circ L_{\mathcal{J}_{\lambda_1}} \circ \cdots \circ L_{\mathcal{J}_{\lambda_k}} \circ L_{\mathcal{M}_{\mu_1,\sigma_1}} \circ \cdots \circ L_{\mathcal{M}_{\mu_\ell,\sigma_\ell}})(T_{S_1})|_{T_{S_1}} \\
&\quad = q^{2g+\ell-1}(q-1)^{4g+k}.
\end{aligned}$$

□

5.2 Character varieties and moduli spaces

Let X be a path-connected topological space with finitely generated fundamental group and G an algebraic group over a field k. There is an action of G on the representation variety $\mathfrak{X}_G(X)$ given by conjugation, and one can look at the *geometric invariant theory* (GIT) quotient

$$\mathcal{M}_G(X) = \mathfrak{X}_G(X) /\!\!/ G,$$

which is known as the *moduli space of G-representations.*

Definition 5.3. Let X be an affine variety over k with an action of a group G. There is an induced action of G on the algebra of regular functions $\mathcal{O}_X(X)$, and we write $\mathcal{O}_X(X)^G$ for the subring of G-invariant elements of $\mathcal{O}_X(X)$. The *GIT quotient* of X by G is defined as the map

$$X \to X /\!\!/ G = \operatorname{Spec} \mathcal{O}_X(X)^G$$

given by the inclusion of rings.

Although the quotient $X /\!\!/ G$ is defined as an affine scheme, we see that it is reduced since X is reduced. Moreover, a theorem by Nagata [24] shows that if G is reductive, then the ring $\mathcal{O}_X(X)^G$ is a finitely generated k-algebra, so $X /\!\!/ G$ will be a variety. For non-reductive groups, such as $\mathbb{U}_2$, this may not always be the case. The key point of the GIT quotient is that it is a categorical quotient [26]:

Definition 5.4. Let X be a variety with a group action of G. A *categorical quotient* of X by G is a G-invariant morphism $\pi : X \to Y$ with the universal property that for any G-invariant morphism

$f : X \to Z$ there exists a unique morphism $g : Y \to Z$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Z \\ Y \arrow[ur, dashed, "g"'] & \end{tikzcd} \tag{5.6}$$

For $k = \mathbb{C}$, a closely related concept is the G-character variety of X. Write $\Gamma = \pi_1(X)$, which was assumed to be finitely generated. The *character* of a representation $\rho \in \mathfrak{X}_G(X)$ is defined as the map

$$\chi_\rho : \Gamma \to \mathbb{C} : \gamma \mapsto \mathrm{tr}(\rho(\gamma)),$$

and the *character map* as

$$\chi : \mathfrak{X}_G(X) \to \mathbb{C}^\Gamma : \rho \mapsto \chi_\rho.$$

The image of χ is called the *G-character variety*, denoted $\chi_G(X)$. By results from [7], there exists a finite set of elements $\gamma_1, \ldots, \gamma_a \in \pi_1(X)$ such that χ_ρ is determined by $(\chi_\rho(\gamma_1), \ldots \chi_\rho(\gamma_a))$ for any ρ. This way $\chi_G(X)$ can be identified with the image of the map $\mathfrak{X}_G(X) \to \mathbb{C}^a : \rho \mapsto (\chi_\rho(\gamma_1), \ldots, \chi_\rho(\gamma_a))$, which gives it the structure of a variety. This structure is independent of the chosen γ_i.

Note that the character map χ is a G-invariant morphism: indeed the trace map is invariant under conjugation. By the universal property of the categorical quotient, there is an induced map

$$\overline{\chi} : \mathcal{M}_{\mathbb{U}_2}(X) \to \chi_G(X).$$

When G is a linear reductive group, this map is actually an isomorphism [7]. However, this may fail when G is not reductive. For example, in Remark 5.6 we will see this is not the case for $G = \mathbb{U}_2$.

In the remaining part of this section we will take $k = \mathbb{C}$ and describe how the group $\mathbb{U}_2$ acts by conjugation on $\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)$, and then describe the categorical quotient of $\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)$ by $\mathbb{U}_2$. First, conjugation in $\mathbb{U}_2$ is given by

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}^{-1} = \begin{pmatrix} a & \frac{bx + y(c-a)}{z} \\ 0 & c \end{pmatrix}.$$

We see that $\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$ stabilizes $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ precisely if $y(a - c) = b(x - z)$. Hence, every element in $\mathbb{U}_2$ is stabilized by scalar matrices (indeed they form the center of $\mathbb{U}_2$). Furthermore, an element $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ is stabilized by some $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$ with $y \neq 0$ precisely if $a = c$, and it is stabilized by some $\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$ with $x \neq z$ precisely if $b = \frac{y}{x-z}(a - c)$. This leads us to look at the following stratification of $\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)$. Using the notation of (4.4), let

$$\mathfrak{X}^{\mathrm{s}}_{\mathbb{U}_2}(\Sigma_g) = \Big\{ A \in \mathfrak{X}_{\mathbb{U}_2}(\Sigma_g) : \text{all } A_i,\, B_i \text{ are scalar matrices} \Big\},$$

whose points have stabilizer equal to $\mathbb{U}_2$, i.e. the group action is trivial on this stratum. Let

$$\mathfrak{X}^{\mathrm{j}}_{\mathbb{U}_2}(\Sigma_g) = \Big\{ A \in \mathfrak{X}_{\mathbb{U}_2}(\Sigma_g) : \text{all } A_i,\, B_i \text{ of the form } \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right), \text{ and } A \notin \mathfrak{X}^{\mathrm{s}}_{\mathbb{U}_2}(\Sigma_g) \Big\},$$

whose points have stabilizer equal to $\{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)\}$. Hence on this stratum the action of $\mathbb{U}_2$ is actually equivalent to an action of $\{\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)\} \subset \mathbb{U}_2$, in the sense that it leads to the same quotient. Let

$$\begin{aligned}\mathfrak{X}^{\mathrm{m}}_{\mathbb{U}_2}(\Sigma_g) &= \{A \in \mathfrak{X}_{\mathbb{U}_2}(\Sigma_g) : \text{for some } \alpha \in \mathbb{C}, \text{ all } A_i,\, B_i \text{ of the form } \left(\begin{smallmatrix} a & \alpha(a-c) \\ 0 & c \end{smallmatrix}\right), \text{ and } A \notin \mathfrak{X}^{\mathrm{s}}_{\mathbb{U}_2}(\Sigma_g) \ \} \\ &= \{(A, \alpha) \in \mathfrak{X}_{\mathbb{U}_2}(\Sigma_g) \times \mathbb{C} : \text{all } A_i,\, B_i \text{ of the form } \left(\begin{smallmatrix} a & \alpha(a-c) \\ 0 & c \end{smallmatrix}\right) \text{ and } A \notin \mathfrak{X}^{\mathrm{s}}_{\mathbb{U}_2}(\Sigma_g)\},\end{aligned}$$

whose points have stabilizer equal to $\left\{ \left(\begin{smallmatrix} x & \alpha(x-z) \\ 0 & z \end{smallmatrix}\right) \right\}$, so on this stratum the action of $\mathbb{U}_2$ is equivalent to an action of $\left\{ \left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) \right\} \subset \mathbb{U}_2$. Finally, let

$$\mathfrak{X}^{\mathrm{f}}_{\mathbb{U}_2}(\Sigma_g) = \mathfrak{X}_{\mathbb{U}_2}(\Sigma_g) - \mathfrak{X}^{\mathrm{s}}_{\mathbb{U}_2}(\Sigma_g) - \mathfrak{X}^{\mathrm{j}}_{\mathbb{U}_2}(\Sigma_g) - \mathfrak{X}^{\mathrm{m}}_{\mathbb{U}_2}(\Sigma_g).$$

By construction, the stabilizer of any $A \in \mathfrak{X}^{\mathrm{f}}_{\mathbb{U}_2}(\Sigma_g)$ are the scalar matrices, i.e. the center of $\mathbb{U}_2$. Hence on this stratum, the action of $\mathbb{U}_2$ is equivalent to an action of $\left\{ \left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right) \right\} \subset \mathbb{U}_2$.

In particular, if we take a point

$$A = \left(\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}, \dots, \begin{pmatrix} a_{2g} & b_{2g} \\ 0 & a_{2g} \end{pmatrix} \right) \in \mathfrak{X}^{\mathrm{j}}_{\mathbb{U}_2}(\Sigma_g),$$

then all the conjugate points

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \left(\begin{pmatrix} a_1 & xb_1 \\ 0 & a_1 \end{pmatrix}, \dots, \begin{pmatrix} a_{2g} & xb_{2g} \\ 0 & a_{2g} \end{pmatrix} \right)$$

should be identified under the quotient. But note that taking the limit $x \to 0$ gives us the point

$$\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix}, \dots, \begin{pmatrix} a_{2g} & 0 \\ 0 & a_{2g} \end{pmatrix} \right) \in \mathfrak{X}^{\mathrm{s}}_{\mathbb{U}_2}(\Sigma_g)$$

which is in the closure of the orbit of A. As is typical for the GIT quotient, this point will be identified with A as well. A similar thing holds for the other strata, and this observation lead us to the following construction.

Lemma 5.5. *Let $\pi : X \to Y$ be a G-invariant morphism of varieties over $\mathbb{C}$, and $\sigma : Y \to X$ a morphism such that $\pi \circ \sigma = id_Y$. If for any $x \in X$ the Zariski-closure of the orbit of x contains $\sigma(\pi(x))$, then π is a categorical quotient.*

Proof. Let $f : X \to Z$ be a G-invariant morphism. We need to show there exists a unique G-invariant morphism $g : Y \to Z$ such that $f = g \circ \pi$. If such a morphism exists, it must be given by $g = f \circ \sigma$ since $g(y) = g(\pi(\sigma(y))) = f(\sigma(y))$ for all $y \in Y$, which already shows uniqueness. Take $x \in X$ and note that as f is G-invariant, $f(\tilde{x}) = f(x)$ for any $\tilde{x}$ in the orbit of x. By continuity we find that $f(\sigma(\pi(x))) = f(x)$ as well, so indeed $f = g \circ \pi$ for this choice of g. □

We can apply this lemma as follows. Let

$$\mathfrak{X}^{\mathrm{d}}_{\mathbb{U}_2}(\Sigma_g) = \Big\{ (A_1, B_1, \dots, A_g, B_g) \in \mathfrak{X}_{\mathbb{U}_2} : \text{all } A_i, B_i \text{ are diagonal} \Big\},$$

and consider the morphism $\pi : \mathfrak{X}_{\mathbb{U}_2}(\Sigma_g) \to \mathfrak{X}^{\mathrm{d}}_{\mathbb{U}_2}(\Sigma_g)$ that sends every component $\left(\begin{smallmatrix} a_i & b_i \\ 0 & c_i \end{smallmatrix}\right)$ to the component $\left(\begin{smallmatrix} a_i & 0 \\ 0 & c_i \end{smallmatrix}\right)$. As $\sigma : \mathfrak{X}^{\mathrm{d}}_{\mathbb{U}_2}(\Sigma_g) \to \mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)$ we take the inclusion. Now indeed π is $\mathbb{U}_2$-invariant, and for any

$$A = \left(\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a_{2g} & b_{2g} \\ 0 & c_{2g} \end{pmatrix} \right) \in \mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)$$

we find that

$$\lim_{x\to 0}\begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix} A \begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix}^{-1} = \left(\begin{pmatrix} a_1 & 0\\ 0 & c_1\end{pmatrix}, \dots, \begin{pmatrix} a_{2g} & 0\\ 0 & c_{2g}\end{pmatrix}\right) = \sigma(\pi(A)),$$

hence $\sigma(\pi(A))$ lies in the analytic-closure (so also Zariski-closure) of the orbit of A. By the lemma we conclude that $\pi : \mathfrak{X}_{\mathbb{U}_2}(\Sigma_g) \to \mathfrak{X}^{\mathrm{d}}_{\mathbb{U}_2}(\Sigma_g)$ is the categorical quotient of $\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)$ by the action of $\mathbb{U}_2$, and hence we will write $\mathcal{M}_{\mathbb{U}_2}(\Sigma_g)$ for the moduli space $\mathfrak{X}^{\mathrm{d}}_{\mathbb{U}_2}(\Sigma_g)$. Note that $\mathcal{M}_{\mathbb{U}_2}(\Sigma_g) \simeq (\mathbb{C}^*)^{4g}$ since every $A = \left(\begin{pmatrix} a_1 & 0\\ 0 & c_1\end{pmatrix}, \dots, \begin{pmatrix} a_{2g} & 0\\ 0 & c_{2g}\end{pmatrix}\right)$ lies in $\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)$.

Remark 5.6. As mentioned before, the character map $\chi : \mathfrak{X}_{\mathbb{U}_2}(\Sigma_g) \to \chi_{\mathbb{U}_2}(\Sigma_g)$ is G-invariant, so there is an induced map

$$\overline{\chi} : \mathcal{M}_{\mathbb{U}_2}(\Sigma_g) \to \chi_{\mathbb{U}_2}(\Sigma_g).$$

Arguing as in Lemma 5.5, the map $\overline{\chi}$ must be given by $\overline{\chi}(A) = \chi_A$. But this cannot be an isomorphism: for general $A \in \mathcal{M}_{\mathbb{U}_2}(\Sigma_g)$ one can consider $B = \left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right) A \left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)^{-1}$ (where the diagonal entries of A are interchanged), and we have $\chi_A = \chi_B$, even though in general $A \neq B$. Therefore, the moduli space $\mathcal{M}_{\mathbb{U}_2}(\Sigma_g)$ is not isomorphic to the character variety $\chi_{\mathbb{U}_2}(\Sigma_g)$ through the natural map.

Finally, we remark that this argument can easily be generalized to the case $G = \mathbb{U}_n$ for any n. Namely, similar to before take

$$\mathfrak{X}^{\mathrm{d}}_{\mathbb{U}_n}(\Sigma_g) = \left\{ (A_1, B_1, \dots, A_{2g}, B_{2g}) \in \mathbb{U}_n^{2g} : \text{all } A_i, B_i \text{ are diagonal}\right\}$$

and $\pi : \mathfrak{X}_{\mathbb{U}_n}(\Sigma_g) \to \mathfrak{X}^{\mathrm{d}}_{\mathbb{U}_n}(\Sigma_g)$ the map that sets all off-diagonal entries to zero, and $\sigma : \mathfrak{X}^{\mathrm{d}}_{\mathbb{U}_n}(\Sigma_g) \to \mathfrak{X}_{\mathbb{U}_n}(\Sigma_g)$ the inclusion. Indeed $\sigma \circ \pi = \mathrm{id}$ and one easily checks that $\sigma(\pi(A))$ lies in the closure of the orbit of A for any $A \in \mathfrak{X}_{\mathbb{U}_n}(\Sigma_g)$ (e.g. conjugate A with $\begin{pmatrix} x^{n-1} & & & \\ & x^{n-2} & & \\ & & \ddots & \\ & & & 1\end{pmatrix}$ and take the limit $x \to 0$). This way we find that $\mathcal{M}_{\mathbb{U}_n}(\Sigma_g) \simeq (\mathbb{C}^*)^{2ng}$. Again note that the natural map $\mathcal{M}_{\mathbb{U}_n}(\Sigma_g) \to \chi_{\mathbb{U}_n}(\Sigma_g)$ cannot be an isomorphism since there are symmetries (permuting diagonal entries) that are invariant under the character map.

5.3 Upper triangular 3×3 matrices

Now consider the case where $G = \mathbb{U}_3$, the group of upper triangular 3×3 matrices, i.e.

$$\mathbb{U}_3 = \left\{ \begin{pmatrix} a & b & c\\ 0 & d & e\\ 0 & 0 & f\end{pmatrix} : a, d, f \neq 0 \right\}.$$

For simplicity we will just consider the representation varieties without parabolic data. As any commutator $[g_1, g_2]$ in $\mathbb{U}_3$ has ones on the diagonal, we only need to consider the conjugacy classes of such elements. There are fives such conjugacy classes, and representatives are given by

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{5.7}$$

Explicitly, the five conjugacy classes are given by

$$\mathcal{C}_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \qquad \mathcal{C}_2 = \left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} : \alpha, \gamma \neq 0 \right\}, \qquad \mathcal{C}_3 = \left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \alpha \neq 0 \right\},$$

$$\mathcal{C}_4 = \left\{ \begin{pmatrix} 1 & 0 & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} : \alpha \neq 0 \right\}, \qquad \mathcal{C}_5 = \left\{ \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \alpha \neq 0 \right\}.$$

We have maps $\pi_i : \mathcal{C}_i \to C_i$, where all C_i are points. Technically one should stratify $G \backslash \cup_i \mathcal{C}_i$ as well, but as everything happens over $\cup_i \mathcal{C}_i$ we omit this. We write $T_i = [C_i]_{C_i} \in \mathrm{K}(\mathbf{Var}/C_i)$, and consider $V = \langle T_1, \ldots, T_5 \rangle$. In what follows, all matrices and vectors will be written with respect to the basis $\{T_1, \ldots, T_5\}$. Similar to the case of $\mathbb{U}_2$, we can satisfy (iv) of Lemma 4.8 by picking $\sigma : \mathcal{C}_i \to G$ such that $g = \sigma(g) \xi_i \sigma(g)^{-1}$ for all $g \in \mathcal{C}_i$.

The map $\eta = \pi_! \pi^*$ is simply given by

$$\eta = \begin{pmatrix} [\mathcal{C}_1] & & & & \\ & [\mathcal{C}_2] & & & \\ & & [\mathcal{C}_3] & & \\ & & & [\mathcal{C}_4] & \\ & & & & [\mathcal{C}_5] \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & q(q-1)^2 & & & \\ & & q(q-1) & & \\ & & & q(q-1) & \\ & & & & q-1 \end{pmatrix}.$$

Now we will compute $Z_\pi(L) = \pi_! \circ Z(L) \circ \pi^*$, starting with $Z_\pi(L)(T_1)$. Since the commutator $[g_1, g_2]$ has ones on the diagonal for all $g_1, g_2 \in \mathbb{U}_3$, indeed $Z_\pi(L)(T_1) \in \langle T_1, \ldots, T_5 \rangle$. We write $g_i = \left(\begin{smallmatrix} a_i & b_i & c_i \\ 0 & d_i & e_i \\ 0 & 0 & f_i \end{smallmatrix} \right)$.

- We have that $Z_\pi(L)(T_1)|_{T_1}$ is the class of $\{(g_1, g_2, h) \in \mathbb{U}_3^3 : g_1 g_2 = g_2 g_1\}$, which is given by the equations
$$\left\{ \begin{array}{c} a_1 b_2 - a_2 b_1 + b_1 d_2 - b_2 d_1 = 0 \\ a_1 c_2 - a_2 c_1 + b_1 e_2 - b_2 e_1 + c_1 f_2 - c_2 f_1 = 0 \\ d_1 e_2 - d_2 e_1 + e_1 f_2 - e_2 f_1 = 0 \end{array} \right\}.$$
We will not solve these equations by hand, but using Algorithm 2.16, this evaluates to $q^3(q-1)^4(q^2+q-1)[\mathbb{U}_3]$.

- We have that $Z_\pi(L)(T_1)|_{T_2}$ is the class of $\{(g_1, g_2, h) \in \mathbb{U}_3^3 : [g_1, g_2] \in \mathcal{C}_2\}$, which is given by the equations
$$\left\{ \begin{array}{c} a_1 b_2 - a_2 b_1 + b_1 d_2 - b_2 d_1 \neq 0 \\ d_1 e_2 - d_2 e_1 + e_1 f_2 - e_2 f_1 \neq 0 \end{array} \right\}.$$
This evaluates to $q^6(q-2)^2(q-1)^4[\mathbb{U}_3]$.

- We have that $Z_\pi(L)(T_1)|_{T_3}$ is the class of $\{(g_1, g_2, h) \in \mathbb{U}_3^3 : [g_1, g_2] \in \mathcal{C}_3\}$, which is given by the equations
$$\left\{ \begin{array}{c} a_1 b_2 - a_2 b_1 + b_1 d_2 - b_2 d_1 \neq 0 \\ d_1 e_2 - d_2 e_1 + e_1 f_2 - e_2 f_1 = 0 \end{array} \right\}.$$
This evaluates to $q^6(q-2)(q-1)^4[\mathbb{U}_3]$.

- We have that $Z_\pi(L)(T_1)|_{T_4}$ is the class of $\{(g_1, g_2, h) \in \mathbb{U}_3^3 : [g_1, g_2] \in \mathcal{C}_4\}$, which is given by the equations
$$\left\{ \begin{array}{c} a_1b_2 - a_2b_1 + b_1d_2 - b_2d_1 = 0 \\ d_1e_2 - d_2e_1 + e_1f_2 - e_2f_1 \neq 0 \end{array} \right\}.$$
This is symmetric to the previous case, so it also evaluates to $q^6(q-2)(q-1)^4[\mathbb{U}_3]$.

- We have that $Z_\pi(L)(T_1)|_{T_5}$ is the class of $\{(g_1, g_2, h) \in \mathbb{U}_3^3 : [g_1, g_2] \in \mathcal{C}_5\}$, which is given by the equations
$$\left\{ \begin{array}{c} a_1b_2 - a_2b_1 + b_1d_2 - b_2d_1 = 0 \\ d_1e_2 - d_2e_1 + e_1f_2 - e_2f_1 = 0 \\ -a_1b_2d_2e_1 - a_1b_2e_2f_1 + a_1c_2d_1d_2 + a_2b_1d_2e_1 + a_2b_1e_2f_1 - a_2c_1d_1d_2 \\ +b_1d_1d_2e_2 - b_1d_2^2e_1 - b_1d_2e_2f_1 + b_2d_1e_2f_1 + c_1d_1d_2f_2 - c_2d_1d_2f_1 \neq 0 \end{array} \right\}.$$
This evaluates to $q^3(q-1)^6(q+1)[\mathbb{U}_3]$.

So far we have computed the first column of the matrix of $Z_\pi(L)$. As a check, indeed we have that the sum of the entries of this column equals $[\mathbb{U}_3]$:
$$\begin{aligned} q^3(q-1)^4(q^2+q-1)[\mathbb{U}_3] + q^6(q-2)^2(q-1)^4[\mathbb{U}_3] + q^6(q-2)(q-1)^4[\mathbb{U}_3] \\ + q^6(q-2)(q-1)^4[\mathbb{U}_3] + q^3(q-1)^6(q+1)[\mathbb{U}_3] = [\mathbb{U}_3]^3. \end{aligned}$$

We will use a similar strategy as in the previous section to determine $Z_\pi(L)(T_i)$ for $i = 2, 3, 4, 5$ from the case $i = 1$. Write ξ_i for the representative of $\mathcal{C}_i$ as in (5.7). One can check that, as mentioned before, for each i there is a (non-unique) morphism $\sigma_i : \mathcal{C}_i \to \mathbb{U}_3$ such that $\sigma_i(g)\xi_i\sigma_i(g)^{-1} = g$. We have
$$Z_\pi(L)(T_j)|_{T_i} = [X_{ij}] \cdot [\mathbb{U}_3] \qquad \text{with} \qquad X_{ij} = \{(g, g_1, g_2) \in \mathcal{C}_j \times \mathbb{U}_3^2 : g[g_1, g_2] \in \mathcal{C}_i\}.$$
We can stratify X_{ij} by
$$X_{ijk} = \{(g, g_1, g_2) \in \mathcal{C}_j \times \mathbb{U}_3^2 : g[g_1, g_2] \in \mathcal{C}_i \text{ and } [g_1, g_2] \in \mathcal{C}_k\} \qquad \text{for } k = 1, \dots, 5.$$
Note that for each i, j, k we have an isomorphism
$$\begin{array}{ccc} X_{ijk} & \xrightarrow{\sim} & \{g \in \mathcal{C}_j : g\xi_k \in \mathcal{C}_i\} \times \{(g_1, g_2) \in G^2 : [g_1, g_2] \in \mathcal{C}_k\} \\ (g, g_1, g_2) & \mapsto & (\sigma_k([g_1, g_2])^{-1} g \sigma_k([g_1, g_2]), g_1, g_2) \end{array}$$
so we find that
$$Z_\pi(L)(T_j)|_{T_i} = \sum_{k=1}^{5} F_{ijk} \cdot Z_\pi(L)(T_1)|_{T_k} \qquad \text{with} \qquad F_{ijk} = [\{g \in \mathcal{C}_j : g\xi_k \in \mathcal{C}_i\}]. \tag{5.8}$$
Although there are about $5^3 = 125$ computations to be done to determine the coefficients F_{ijk}, all of them are quite simple. E.g. it is clear that $F_{i,1,k} = \delta_{ik}$, the Kronecker delta. For $j = 2$, take

any $g = \left(\begin{smallmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{smallmatrix}\right) \in \mathcal{C}_2$. Then $g\xi_1, g\xi_5 \in \mathcal{C}_2$. We have $g\xi_3 \in \mathcal{C}_4$ if $\alpha = -1$ and $g\xi_3 \in \mathcal{C}_2$ otherwise. Similarly, $g\xi_4 \in \mathcal{C}_3$ if $\gamma = -1$ and $g\xi_4 \in \mathcal{C}_2$ otherwise. Finally,

$$g\xi_2 \in \begin{cases} \mathcal{C}_1 & \text{if } \alpha, \gamma = -1 \text{ and } \beta = 0, \\ \mathcal{C}_2 & \text{if } \alpha, \gamma \neq -1, \\ \mathcal{C}_3 & \text{if } \alpha \neq -1 \text{ and } \gamma = -1, \\ \mathcal{C}_4 & \text{if } \alpha = -1 \text{ and } \gamma \neq -1, \\ \mathcal{C}_5 & \text{if } \alpha, \gamma = -1 \text{ and } \beta \neq 0. \end{cases}$$

This gives

$$F_{i,2,k} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ q(q-1)^2 & q(q-2)^2 & q(q-2)(q-1) & q(q-2)(q-1) & q(q-1)^2 \\ 0 & q(q-2) & 0 & q(q-1) & 0 \\ 0 & q(q-2) & q(q-1) & 0 & 0 \\ 0 & q-1 & 0 & 0 & 0 \end{pmatrix},$$

with i the row index and k the column index. By completely similar arguments, one can show that

$$F_{i,3,k} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & q(q-2) & 0 & q(q-1) & 0 \\ q(q-1) & 0 & q(q-2) & 0 & q(q-1) \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & q-1 & 0 & 0 \end{pmatrix}, \qquad F_{i,4,k} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & q(q-2) & q(q-1) & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ q(q-1) & 0 & 0 & q(q-2) & q(q-1) \\ 0 & 0 & 0 & q-1 & 0 \end{pmatrix}$$

$$\text{and } F_{i,5,k} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & q-1 & 0 & 0 & 0 \\ 0 & 0 & q-1 & 0 & 0 \\ 0 & 0 & 0 & q-1 & 0 \\ q-1 & 0 & 0 & 0 & q-2 \end{pmatrix}.$$

Applying (5.8) now yields

$$Z_\pi(L) = q^6(q-1)^7 \begin{pmatrix} q^2+q-1 & q^3(q-2)^2 & q^3(q-2) & q^3(q-2) & (q-1)^2(q+1) \\ q^3(q-2)^2 & q^4\left(q^2-3q+3\right)^2 & q^4(q-2)\left(q^2-3q+3\right) & q^4(q-2)\left(q^2-3q+3\right) & q^3(q-2)^2(q-1) \\ q^3(q-2) & q^4(q-2)\left(q^2-3q+3\right) & q^4\left(q^2-3q+3\right) & q^4(q-2)^2 & q^3(q-2)(q-1) \\ q^3(q-2) & q^4(q-2)\left(q^2-3q+3\right) & q^4(q-2)^2 & q^4\left(q^2-3q+3\right) & q^3(q-2)(q-1) \\ (q-1)^2(q+1) & q^3(q-2)^2(q-1) & q^3(q-2)(q-1) & q^3(q-2)(q-1) & (q-1)\left(q^3-q^2+1\right) \end{pmatrix}$$

and precomposing with η^{-1} gives

$$\tilde{Z}(L) = q^6(q-1)^5$$
$$\begin{pmatrix} (q-1)^2\left(q^2+q-1\right) & q^2(q-2)^2 & q^2(q-2)(q-1) & q^2(q-2)(q-1) & (q-1)^3(q+1) \\ q^3(q-2)^2(q-1)^2 & q^3\left(q^2-3q+3\right)^2 & q^3(q-2)(q-1)\left(q^2-3q+3\right) & q^3(q-2)(q-1)\left(q^2-3q+3\right) & q^3(q-2)^2(q-1)^2 \\ q^3(q-2)(q-1)^2 & q^3(q-2)\left(q^2-3q+3\right) & q^3(q-1)\left(q^2-3q+3\right) & q^3(q-2)^2(q-1) & q^3(q-2)(q-1)^2 \\ q^3(q-2)(q-1)^2 & q^3(q-2)\left(q^2-3q+3\right) & q^3(q-2)^2(q-1) & q^3(q-1)\left(q^2-3q+3\right) & q^3(q-2)(q-1)^2 \\ (q-1)^4(q+1) & q^2(q-2)^2(q-1) & q^2(q-2)(q-1)^2 & q^2(q-2)(q-1)^2 & (q-1)^2\left(q^3-q^2+1\right) \end{pmatrix}.$$

This matrix can be diagonalized as follows:

$$\tilde{Z}(L) = q^6(q-1)^5 A \begin{pmatrix} q^3 & & & & \\ & q(q-1)^2 & & & \\ & & q^3(q-1)^2 & & \\ & & & q^3(q-1)^2 & \\ & & & & q^3(q-1)^4 \end{pmatrix} A^{-1}$$

$$\text{with } A = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ q & 0 & 0 & -q(q-1) & q(q-1)^2 \\ -q & 0 & 1 & 0 & q(q-1) \\ -q & 0 & -1 & q(q-2) & q(q-1) \\ q-1 & -1 & 0 & q-1 & q-1 \end{pmatrix}.$$

Remark 5.7. We see that $Z_\pi(L)$ is symmetric. This can be explained by the fact that for each i and j we have an isomorphism:

$$\begin{array}{ccc} \{(g, g_1, g_2) \in \mathcal{C}_i \times G^2 : g[g_1, g_2] \in \mathcal{C}_j\} & \xleftrightarrow{\sim} & \{(g, g_1, g_2) \in \mathcal{C}_j \times G^2 : g[g_1, g_2] \in \mathcal{C}_i\} \\ (g, g_1, g_2) & \mapsto & (g[g_1, g_2], g_2, g_1) \\ (g[g_1, g_2], g_2, g_1) & \leftarrow\!\shortmid & (g, g_1, g_2) \end{array}$$

Hence the classes of both sides are equal in $\mathrm{K}(\mathbf{Var}_\mathbb{C})$, and so $Z_\pi(L)(T_i)|_{T_j} = Z_\pi(L)(T_j)|_{T_i}$. This does not hold for any TQFT, but it relies on the fact that the C_i are points.

Theorem 5.8. *The virtual class of the $\mathbb{U}_3$-representation variety $\mathfrak{X}_{\mathbb{U}_3}(\Sigma_g)$ is*

$$[\mathfrak{X}_{\mathbb{U}_3}(\Sigma_g)] = q^{3g-3}(q-1)^{2g}\left(q^2(q-1)^{2g+1} + q^{3g}(q-1)^2 + q^{3g}(q-1)^{4g} + 2q^{3g}(q-1)^{2g+1}\right).$$

Proof. One can check that

$$A^{-1} = \frac{1}{q^3}\begin{pmatrix} (q-1)^2 & 1 & 1-q & 1-q & (q-1)^2 \\ q^2(q-1) & 0 & 0 & 0 & -q^2 \\ q(q-2)(q-1) & -q(q-2) & q^3-2q^2+2q & -2q(q-1) & q^3-3q^2+2q \\ 2q-2 & -2 & q-2 & q-2 & 2q-2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

By matrix multiplication, we find that

$$\begin{aligned} \mathfrak{X}_{\mathbb{U}_3}(\Sigma_g) &= \frac{1}{[\mathbb{U}_3]^g}\tilde{Z}(L)^g(T_1)|_{T_1} \\ &= q^{3g-3}(q-1)^{2g}\left(q^2(q-1)^{2g+1} + q^{3g}(q-1)^2 + q^{3g}(q-1)^{4g} + 2q^{3g}(q-1)^{2g+1}\right). \end{aligned}$$

□

Remark 5.9. In particular, for small values of g, we find

$$\begin{aligned} [\mathfrak{X}_{\mathbb{U}_3}(\Sigma_1)] &= q^3(q-1)^4(q^2+q-1), \\ [\mathfrak{X}_{\mathbb{U}_3}(\Sigma_2)] &= q^7\,(q-1)^6\left(q^8-6q^7+15q^6-18q^5+9q^4+q^3-3q^2+3q-1\right), \\ [\mathfrak{X}_{\mathbb{U}_3}(\Sigma_3)] &= q^{11}\,(q-1)^8\left(q^{14}-10q^{13}+45q^{12}-120q^{11}+210q^{10}-250q^9+200q^8\right. \\ &\qquad \left.-100q^7+25q^6+q^5-5q^4+10q^3-10q^2+5q-1\right). \end{aligned}$$

As in Remark 5.1, the factor $(q-1)^{2g+2}$ can be explained from the actions of $(\mathbb{C}^*)^{2g}$ (given by scaling the A_i, B_i) and $(\mathbb{C}^*)^2$ (given by conjugating with $\begin{pmatrix} 1 & & \\ & x & \\ & & y \end{pmatrix}$, $x, y \in \mathbb{C}^*$).

5.4 Upper triangular 4×4 matrices

The last case we will treat is the group $\mathbb{U}_4$ of upper triangular 4×4 matrices. We can use the same strategies as in the previous case of $\mathbb{U}_3$, but all computations are done using Algorithm 2.16. Source code for these computations is given in [29]. The group $\mathbb{U}_4$ contains sixteen unipotent conjugacy classes [3]. We consider the following representatives of these classes

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & 1 & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix},$$

which we denote in order by $\xi_1, \ldots, \xi_{16}$. Explicitly, the conjugacy classes are given by

$$\begin{aligned}
\mathcal{C}_1 &= \{a_{0,1} = a_{0,2} = a_{0,3} = a_{1,2} = a_{1,3} = a_{2,3} = 0\}, \\
\mathcal{C}_2 &= \{a_{1,2} = a_{1,3} = a_{2,3} = 0, a_{0,1} \neq 0\}, \\
\mathcal{C}_3 &= \{a_{0,1} = a_{1,2} = a_{1,3} = a_{2,3} = 0, a_{0,2} \neq 0\}, \\
\mathcal{C}_4 &= \{a_{0,1} = a_{0,2} = a_{1,2} = a_{1,3} = a_{2,3} = 0, a_{0,3} \neq 0\}, \\
\mathcal{C}_5 &= \{a_{0,1} = a_{2,3} = a_{0,3}a_{1,2} - a_{0,2}a_{1,3} = 0, a_{1,2} \neq 0\}, \\
\mathcal{C}_6 &= \{a_{0,1} = a_{0,2} = a_{1,2} = a_{2,3} = 0, a_{1,3} \neq 0\}, \\
\mathcal{C}_7 &= \{a_{0,1} = a_{0,2} = a_{1,2} = 0, a_{2,3} \neq 0\}, \\
\mathcal{C}_8 &= \{a_{2,3} = 0, a_{0,1} \neq 0, a_{1,2} \neq 0\}, \\
\mathcal{C}_9 &= \{a_{1,2} = a_{2,3} = 0, a_{0,1} \neq 0, a_{1,3} \neq 0\}, \\
\mathcal{C}_{10} &= \{a_{1,2} = a_{0,2}a_{2,3} + a_{0,1}a_{1,3} = 0, a_{0,1} \neq 0, a_{2,3} \neq 0\}, \\
\mathcal{C}_{11} &= \{a_{0,1} = a_{1,2} = a_{2,3} = 0, a_{0,2} \neq 0, a_{1,3} \neq 0\}, \\
\mathcal{C}_{12} &= \{a_{0,1} = a_{1,2} = 0, a_{0,2} \neq 0, a_{2,3} \neq 0\}, \\
\mathcal{C}_{13} &= \{a_{0,1} = 0, a_{1,2} \neq 0, a_{2,3} \neq 0\}, \\
\mathcal{C}_{14} &= \{a_{0,1} = a_{2,3} = 0, a_{1,2} \neq 0, a_{0,3}a_{1,2} - a_{0,2}a_{1,3} \neq 0\}, \\
\mathcal{C}_{15} &= \{a_{0,1} \neq 0, a_{1,2} \neq 0, a_{2,3} \neq 0\}, \\
\mathcal{C}_{16} &= \{a_{1,2} = 0, a_{0,1} \neq 0, a_{2,3} \neq 0, a_{0,2}a_{2,3} + a_{0,1}a_{1,3} \neq 0\}.
\end{aligned}$$

As before, we write $T_i = [C_i]_{C_i} \in \mathrm{K}(\mathbf{Var}/C_i)$ and consider $V = \langle T_1, \ldots, T_{16} \rangle$. We wish to compute $Z_\pi(L)(T_1)$ first and deduce the other columns from this column as we did before. We have

$$Z_\pi(L)(T_1)|_{T_i} = [\{(g_1, g_2) \in \mathbb{U}_4 : [g_1, g_2] \in \mathcal{C}_i\}]$$

In terms of coordinates, this will yield systems of equations in 20 variables. One can make some clever stratifications to simplify these computations, but we will not elaborate on this. We obtain

$$Z_\pi(L)(T_1) = q^{12}(q-1)^9 \begin{pmatrix} q^3+4q^2-6q+4 \\ q^3(q-2)\left(q^2+q-1\right) \\ q\left(q^2-q+1\right)\left(q^2+q-4\right) \\ (q-1)\left(q^3+3q^2-6q+4\right) \\ q^2(q-2)\left(q^3+q-1\right) \\ q\left(q^4+q^3-8q^2+9q-4\right) \\ q^2(q-2)\left(q^3+2q^2-4q+2\right) \\ q^6(q-2)^2 \\ q^3(q-2)(q-1)^2(q+1) \\ q^3(q-2)(q-1)\left(q^2-q-1\right) \\ q\left(q^5-2q^4-2q^3+9q^2-9q+4\right) \\ q^2(q-2)(q-1)\left(q^3-2q+2\right) \\ q^6(q-2)^2 \\ q^2(q-2)(q-1)^2\left(q^2+q+1\right) \\ q^6(q-2)^3 \\ q^3(q-2)\left(q^4-3q^3+2q^2-1\right) \end{pmatrix}.$$

Completely analogous to the previous section, the other columns can be computed from this result via

$$Z_\pi(L)(T_j)|_{T_i} = \sum_{k=1}^{16} F_{ijk} \cdot Z_\pi(L)(T_1|_{T_k}).$$

where

$$F_{ijk} = [\{g \in \mathcal{C}_j : g\xi_k \in \mathcal{C}_i\}].$$

Again, see [29] for the actual computations. As usual, the map $\eta = \pi_! \pi^*$ is diagonal, with $\eta(T_i) = [\mathcal{C}_i] T_i$. After diagonalizing the reduced TQFT $\tilde{Z}(L) = Z_\pi(L) \circ \eta^{-1}$, we obtain the following result.

Theorem 5.10. *The virtual class of the $\mathbb{U}_4$-representation variety $\mathfrak{X}_{\mathbb{U}_4}(\Sigma_g)$ is*

$$\begin{aligned} [\mathfrak{X}_{\mathbb{U}_4}(\Sigma_g)] &= q^{8g-2}(q-1)^{4g+2} + q^{8g-2}(q-1)^{6g+1} + q^{10g-4}(q-1)^{2g+3} \\ &\quad + q^{10g-4}(q-1)^{4g+1}\left(2q^2-6q+5\right)^g + 3q^{10g-4}(q-1)^{4g+2} + q^{10g-4}(q-1)^{6g+1} \\ &\quad + q^{12g-6}(q-1)^{8g} + q^{12g-6}(q-1)^{2g+3} + 3q^{12g-6}(q-1)^{4g+2} + 3q^{12g-6}(q-1)^{6g+1}. \end{aligned}$$

Remark 5.11. For small values of g, we have

$$\begin{aligned} [\mathfrak{X}_{\mathbb{U}_4}(\Sigma_1)] &= q^6(q-1)^5\left(q^3+4q^2-6q+4\right), \\ [\mathfrak{X}_{\mathbb{U}_4}(\Sigma_2)] &= q^{15}(q-1)^7\left(q^{12}-9q^{11}+36q^{10}-81q^9+108q^8-76q^7-11q^6+124q^5-219q^4\right. \\ &\quad \left.+222q^3-126q^2+36q-3\right), \\ [\mathfrak{X}_{\mathbb{U}_4}(\Sigma_3)] &= q^{23}(q-1)^9\left(q^{22}-15q^{21}+105q^{20}-455q^{19}+1365q^{18}-3000q^{17}+4975q^{16}\right. \\ &\quad -6300q^{15}+6075q^{14}-4366q^{13}+2136q^{12}-93q^{11}-2139q^{10}+5157q^9 \\ &\quad \left.-8101q^8+8885q^7-6746q^6+3465q^5-1196q^4+329q^3-110q^2+35q-5\right). \end{aligned}$$

6 Conclusion

In this thesis we looked at the construction of a TQFT that computes virtual classes of character varieties. Along the way we studied the Grothendieck ring, techniques for computing classes of varieties and general properties of TQFTs. We computed the class of $\mathfrak{X}_{\mathbb{U}_n}(\Sigma_g)$ for $n = 2, 3, 4$ and any genus g. Also we found expressions for the family of moduli spaces $\mathcal{M}_{\mathbb{U}_n}(\Sigma_g) \simeq (\mathbb{C}^*)^{2ng}$ for all n and g, as categorical quotients, and showed that these are not isomorphic to the G-character varieties for $n \geq 2$ (through the canonically induced morphism). To our knowledge, such results have not been shown before for groups of such high dimension, and this reflects in some sense the power of the TQFT method.

In future work, we will explore whether this method is applicable to different groups G. In principle, the strategy used in sections 5.3 and 5.4 can be applied to compute the class of the $\mathbb{U}_5$-representation variety as well. However, as the group $\mathbb{U}_5$ contains 120 unipotent conjugacy classes [3] this would require quite some computational power. We will also look into the unitary groups $\mathrm{U}(n)$ or $\mathrm{SU}(n)$, which are interesting in light of the Narasimhan–Seshadri theorem [25] that relates stable holomorphic vector bundles over a Riemann surface to irreducible unitary representations of its fundamental group.

Furthermore, we note that the representation varieties are not proper over $\mathbb{C}$, and one can search for natural compactifications of these varieties.

Finally, in Section 5.2 we saw that a lot of information on how $\mathbb{U}_2$ acts on different parts of the $\mathbb{U}_2$-character variety is lost when passing to the moduli space $\mathcal{M}_{\mathbb{U}_2}(\Sigma_g)$. For this reason it might be interesting to look at the moduli space of representations as a quotient stack instead.

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