Motivic invariants of character stacks

Proefschrift

ter verkrijging van
de graad van doctor aan de Universiteit Leiden,
op gezag van rector magnificus prof. dr. ir. H. Bijl,
volgens besluit van het college voor promoties
te verdedigen op donderdag 13 juni 2024
klokke 12.30 uur

door

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geboren te Loppersum, Nederland
in 1996

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Introduction

The theory of representations of groups is a rich and fascinating subject in mathematics. For certain classes of groups, the representation theory is fairly well understood. For example, for finite groups, the representation theory is largely described by their character table, and for connected compact Lie groups, the representation theory is given by the theorem of the highest weight. However, the representation theory of finitely generated groups, lying somewhere in between, is not so easily described. For a finitely generated group Γ, the set of n-dimensional representations $\rho\colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$, denoted

$$\mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{C})),$$

defines a complex variety, called the *representation variety* of Γ. Recall that two representations $\rho, \rho'\colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$ are isomorphic if $\rho'(\gamma) = g\rho(\gamma)g^{-1}$ for some $g \in \mathrm{GL}_n(\mathbb{C})$ and all $\gamma \in \Gamma$. In other words, the group $\mathrm{GL}_n(\mathbb{C})$ acts by conjugation on the representation variety $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{C}))$, and the quotient $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})$, known as the *character variety* of Γ, can be thought of as a geometric analogue of the character table. A subtle point is that it is not completely clear how to take this quotient. Using geometric invariant theory [Mum65], one arrives at the classical definition of the character variety. Another possibility is to enter the realm of algebraic stacks, to arrive at the quotient stack

$$[\mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})],$$

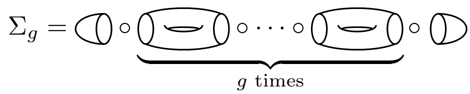
known as the *character stack* of Γ, for which the character variety is a coarse moduli space. More generally, one may replace $\mathbb{C}$ by any field k, and GL_n by any algebraic group G over k. As an example, when $\Gamma = \mathbb{Z}$, a representation from Γ into G is simply the choice of an element of G, so the representation variety is isomorphic to G, and the character variety, or character stack, is the appropriate quotient of G by the action of G by conjugation on itself. In general, the geometry of these spaces can be quite complicated and is a wide field of study. The goal of this thesis is to provide a better understanding of the geometry of these spaces.

Many finitely generated groups arise as the fundamental group $\Gamma = \pi_1(M, *)$ of a connected compact manifold M with a basepoint $*$. In this case, representations of Γ into G correspond to G-local systems on M, and isomorphic representations correspond to isomorphic local systems [Sza09, Corollary 2.6.2]. In this sense, the character variety (or stack) of Γ can be seen as the moduli space of G-local systems on M, and is in the literature also known as the *Betti moduli space* of M. In the particular case that M is the underlying manifold of a complex smooth projective curve C, this space appears in the geometric Langlands program [BD96, BN18] and plays a major role in non-abelian Hodge theory [Cor88, Don87, Sim91, Sim94], where it is strongly related to a moduli space of Higgs bundles on C and a moduli space of flat connections on C. The study of these moduli spaces motivated the $P = W$ conjecture [CHM12], which was recently proved [MS22, Hau+22]. The main focus of this thesis will be the case where M has dimension 2. Such manifolds M are either orientable and classified by their genus, or non-orientable and classified by their demigenus.

The geometry of the representation variety (and of its quotients) can be studied in many ways, for instance by computing their invariants. When k is a finite field, one could count the number of k-rational points, and when $k = \mathbb{C}$, one could compute the singular cohomology of the analytification. In this thesis, we focus on invariants χ that are *additive* and *multiplicative* in the sense that $\chi(X) = \chi(Z) + \chi(X \setminus Z)$ and $\chi(X \times Y) = \chi(X)\chi(Y)$ whenever X and Y are varieties over k and $Z \subseteq X$ a closed subvariety. We call these *motivic invariants*, and they include the point count when k is finite, and the Euler characteristic of the analytification when $k = \mathbb{C}$. Another such invariant for $k = \mathbb{C}$, which is central in this thesis, is the *E-polynomial*, a refinement of the Euler characteristic. The E-polynomial of a complex variety is a polynomial in two variables whose coefficients reflect the mixed Hodge structure on its cohomology. In this thesis we discuss various such invariants, and develop tools for computing them. In particular, we focus on the universal such invariant, called the *virtual class*, which takes values in the *Grothendieck ring of varieties*.

The computation of motivic invariants for representation varieties of orientable surfaces started with Hausel and Rodriguez-Villegas [HR08], who studied the representation variety by counting the number of points over finite fields $\mathbb{F}_q$. They could express these counts in terms of the representation theory of the finite groups $G(\mathbb{F}_q)$, and moreover, infer from these counts the E-polynomial of the representation variety. This approach, which we will call the *arithmetic method*, has led many to study the E-polynomials of character varieties for various Γ and G [HLR11, Mer15, Let15, MR15, Cam17, BH17, LR22, BK22].

A few years later, Logares, Muñoz and Newstead [LMN13] initiated the *geometric* *method*: a geometric approach to compute the E-polynomial of the representation variety, making use of its additive and multiplicative property and clever stratifications. González-Prieto, Logares and Muñoz [GLM20] showed that the geometric method can be phrased in terms of a *Topological Quantum Field Theory (TQFT)*, a concept originating from physics. In particular, an orientable surface of genus g can be considered as a composite of manifolds with boundaries, known as *bordisms*, as follows:



In short, a TQFT (over some commutative ring R) assigns to every boundary (possibly empty) an R-module, and to every bordism between boundaries a linear map between the corresponding modules, such that composition of bordisms corresponds to composition of the linear maps. In other words, a TQFT is a certain functor from the category of bordisms to the category of R-modules. Now, the idea of the geometric method is that the E-polynomial of the representation variety corresponds to the image of Σ_g under the TQFT, and so the computation of this E-polynomial can be broken down into a simpler computation for each bordism. It was shown later [Gon20] that the same construction can be used to compute the virtual class of the representation variety in the Grothendieck ring of varieties.

Both the arithmetic and geometric method are discussed in detail in Chapter 4. One of the main results of this chapter, which is based on [GHV23], is that the two methods can be unified into a single framework. In particular, we show how the arithmetic method can be translated into the language of TQFTs, and moreover, we show that the TQFTs, for the geometric and arithmetic method, are related through natural transformations. As a consequence, we describe how parts of the character tables of the finite groups $G(\mathbb{F}_q)$, specifically the dimensions of the irreducible representations of $G(\mathbb{F}_q)$, are related to the eigenvalues of the image of the bordism  under the TQFT corresponding to the geometric method.

Another aim of this thesis, besides giving theoretical descriptions, is to apply the above methods to explicitly compute invariants of the representation varieties and character stacks of surfaces, for certain algebraic groups G. In Chapter 5 we focus on the group $G = \mathrm{SL}_2$, generalizing the results of [LMN13, MM16, LR22] where the E-polynomials of the representation varieties were computed. Lifting these computations from E-polynomials to the Grothendieck ring of varieties introduces many subtle problems that have to be dealt with. In Chapter 6, based on [HV22, Vog24], we concentrate on the groups of $n \times n$ upper triangular matrices and unipotent upper triangular matrices. By means of computer-assisted calculations, we compute the virtual classes of the character stacks of Σ_g for $n \leq 5$ through the geometric method, and their E-polynomials for $n \leq 10$ through the arithmetic method.

Finally, in Chapter 7, we turn our attention to the representation varieties and character stacks of the free groups F_n and free abelian groups $\mathbb{Z}^n$. These spaces, parametrizing tuples (resp. commuting tuples) of elements of G, have also been widely studied [Bai07, AC07, FL11, PS13, FL14, RS19, FS21]. When considering the homology of these spaces, an interesting phenomenon emerges: as shown in [RS21], the homology groups of these spaces (and many variations thereof) stabilize as n tends to infinity, in a well-defined sense due to [CF13] known as *representation stability*. In Chapter 7, we will combine the notion of representation stability with that of *motivic stability* [VW15] to define an analogous notion of *motivic representation stability* for stability in the Grothendieck ring of varieties. As an application, we will show that the character stacks of F_n and $\mathbb{Z}^n$ stabilize in this sense for the linear groups $G = \mathrm{GL}_r$.

These explicit applications and computations have led to a number of new computational techniques. For instance, the study of equivariant motivic invariants, in Section 3.6, describes how motivic invariants, in particular the virtual class, behave with respect to finite group actions. The results in this section are crucial to the computations for the SL_2-character stacks, and to the definition of motivic representation stability. Other new computational techniques include the introduction of *algebraic representatives*, in Section 6.1, and the development of an algorithm for computing virtual classes, in Section 3.4. Without these techniques, the computations for the character stacks for upper triangular matrices of high rank would not have been possible.

Chapter 1

Stacks

Algebraic stacks were first introduced by Deligne and Mumford to study the moduli space of curves [DM69], and later their definition was generalized by Artin [Art74]. Roughly speaking, an algebraic stack can be thought of as a generalization of a scheme. If we view a scheme as a functor of points, its points form a set, whereas for an algebraic stack they form a groupoid. In other words, the points of a stack are allowed to have automorphisms. The notion of a stack is not specific to algebraic geometry, that is, stacks can also be defined in the context of manifolds, analytic spaces, topological spaces, or, in general, for any site, that is, category with a Grothendieck topology.

The goal of this chapter is to give a concise overview of (algebraic) stacks, with a focus on quotient stacks, which should be sufficient to understand the later chapters. For the curious reader who wishes to read more in-depth expositions of (algebraic) stacks, we refer to [Fan01, Beh14, Ols16, LM00, Stacks], in order from introductory and intuitive to detailed and rigorous.

1.1 Groupoids

Crucial to the subject of stacks is the concept of a groupoid, that is, a category in which every morphism is an isomorphism.

Definition 1.1.1. A groupoid is *finite* if it has finitely many morphisms. A groupoid is *finitely generated* if there exists a finite collection of morphisms, called *generators*, such that every morphism of the groupoid can be written as a composite of generators and inverses of generators. In particular, any finite or finitely generated groupoid has finitely many objects, because every object has at least an identity morphism. A groupoid is *essentially finite* if it is equivalent to a finite groupoid, and similarly, a groupoid is *essentially finitely generated* if it is equivalent to a finitely generated groupoid.

Denote by **Grpd** the 2-category of groupoids, whose objects are groupoids, 1-morphisms are functors, and 2-morphisms are natural transformations. Similarly, denote by **FinGrpd** and **FGGrpd** the full sub-2-categories of essentially finite groupoids and essentially finitely generated groupoids, respectively.

Definition 1.1.2. Let G be a group acting on a set X. The *action groupoid*, denoted $[X/G]$, is the groupoid whose objects are elements of X, morphisms $x \to y$ are given by elements $g \in G$ such that $y = g \cdot x$. Composition of $g\colon x \to y$ and $h\colon y \to z$ is given by $hg\colon x \to z$.

Definition 1.1.3. Let Γ be an essentially finite groupoid. The *groupoid cardinality* of Γ is defined as

$$|\Gamma| = \sum_{[x] \in \Gamma/\sim} \frac{1}{|\operatorname{Aut}(x)|} \in \mathbb{Q}$$

where $\Gamma/\sim$ denotes the set of isomorphism classes of Γ.

Example 1.1.4. Let G be a finite group acting on a finite set X. Then the groupoid cardinality of the action groupoid $\Gamma = [X/G]$ is $|X|/|G|$. Indeed, from the orbit-stabilizer theorem it follows that

$$|\Gamma| = \sum_{[x] \in [X/G]/\sim} \frac{1}{|\operatorname{Aut}(x)|} = \sum_{x \in X} \frac{1}{|Gx|} \frac{1}{|\operatorname{Aut}(x)|} = \sum_{x \in X} \frac{1}{|G|} = \frac{|X|}{|G|}.$$

Definition 1.1.5. Let $f\colon B \to A$ and $g\colon C \to A$ be morphisms of groupoids. The *fiber product* of B and C over A, denoted $B \times_A C$, is the groupoid whose objects are triples (x, y, α) with x an object of B, y an object of C and $\alpha\colon f(x) \to g(y)$ a morphism in A. A morphism from (x', y', α') to (x, y, α) is given by a pair of morphisms $(\beta\colon x' \to x, \gamma\colon y' \to y)$ such that $g(\gamma) \circ \alpha' = \alpha \circ f(\beta)$.

Note that the diagram

$$\begin{array}{ccc} B \times_A C & \xrightarrow{\pi_C} & C \\ {\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{f} & A \end{array}$$

with π_B and π_C the obvious projections, does not strictly commute whenever there are non-trivial morphisms $\alpha\colon f(x) \to g(y)$. However, there is a natural isomorphism $f \circ \pi_B \Rightarrow g \circ \pi_C$, whose component at (x, y, α) is given by α. This is the correct notion of commutativity for 2-categories, and we say this diagram *2-commutes.*

This also shows what the correct universal property of the fiber product is. For every groupoid D with morphisms $i\colon D \to B$ and $j\colon D \to C$ such that $f \circ i$ is naturally isomorphic to $g \circ j$, there exists, up to a unique natural isomorphism, a unique morphism $h\colon D \to B \times_A C$ and natural isomorphisms $\pi_B \circ h \cong i$ and $\pi_C \circ h \cong j$. One can easily verify that the above definition of the fiber product for groupoids satisfies this universal property.

Definition 1.1.6. Let $f\colon A \to B$ be a functor between groupoids, and let b be an object of B. The *fiber* of f over b is the groupoid

$$f^{-1}(b) = A \times_B \{b\}$$

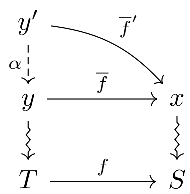
where $\{b\}$ is the groupoid with a single object b and one (identity) morphism, and $\{b\} \to B$ the natural map.

1.2 Categories fibered in groupoids

Throughout the following sections, let $\mathfrak{S}$ be a site, that is, a category equipped with a Grothendieck topology.

Definition 1.2.1. A *category over* $\mathfrak{S}$ is a category $\mathfrak{X}$ with a functor $p\colon \mathfrak{X} \to \mathfrak{S}$. An object x of $\mathfrak{X}$ is said to *lie over* an object S of $\mathfrak{S}$, or x is said to be a *lift* of S, if $p(x) = S$, and similarly for morphisms. If S is an object of $\mathfrak{S}$, the *fiber* of $\mathfrak{X}$ over S, denoted $\mathfrak{X}_S$, is the subcategory of $\mathfrak{X}$ of objects over S and morphisms over id_S. A morphism of categories over $\mathfrak{S}$ is a functor that respects the functor to $\mathfrak{S}$. If $p\colon \mathfrak{X} \to \mathfrak{S}$ and $q\colon \mathfrak{Y} \to \mathfrak{S}$ are categories over $\mathfrak{S}$, and f and g morphisms from $\mathfrak{X}$ to $\mathfrak{Y}$, then a *2-morphism* $f \to g$ is a natural transformation $\mu\colon f \Rightarrow g$ such that all components $\mu_x\colon f(x) \to g(x)$ lie over $\mathrm{id}_{p(x)}$. The categories over $\mathfrak{S}$ form a 2-category. An *isomorphism* of categories over $\mathfrak{S}$ is a morphism which is an equivalence of categories.

Definition 1.2.2. A category $\mathfrak{X}$ over $\mathfrak{S}$ is called a *category fibered in groupoids* over $\mathfrak{S}$ if for any morphism $f\colon T \to S$ in $\mathfrak{S}$ and object x lying over S, there exists a lift $\overline{f}\colon y \to x$ of f which is unique up to unique isomorphism. That is, for any other lift $\overline{f}'\colon y' \to x$ of f, there exists a unique isomorphism $\alpha\colon y' \to y$ such that $\overline{f}' = \overline{f} \circ \alpha$.



As a motivation for the terminology, consider the following lemma.

Lemma 1.2.3. *Let $\mathfrak{X}$ be a category fibered in groupoids over $\mathfrak{S}$. Then every morphism $\varphi\colon y \to x$ of $\mathfrak{X}$ that lies over an isomorphism $f\colon T \to S$ of $\mathfrak{S}$, is an isomorphism as well. In particular, for every object S of $\mathfrak{S}$ the fiber $\mathfrak{X}_S$ is groupoid.*

Proof. Write g for the inverse of f, and choose a lift $\overline{g}\colon z \to y$ of g. As $\varphi \circ \overline{g}\colon z \to x$ lies over $f \circ g = \mathrm{id}_S$, it is a lift of id_S with target x. Since id_x is so as well, there exists a (unique) isomorphism $\alpha\colon z \to x$ such that $\varphi \circ \overline{g} = \alpha$. Now $\psi = \overline{g} \circ \alpha^{-1}$ is a right inverse of φ which lies over g. Repeating the argument, replacing φ by ψ, one shows ψ also has a right inverse, which must be φ. □

In particular, every 2-morphism between morphisms of categories over $\mathfrak{S}$ is automatically an isomorphism.

Example 1.2.4. Any object X of $\mathfrak{S}$ can be regarded as a category fibered in groupoids $p\colon \mathfrak{X} \to \mathfrak{S}$ where $\mathfrak{X}$ is the slice category $\mathfrak{S}/X$ and p simply forgets the morphism to X. Indeed, for any $f\colon T \to S$ in $\mathfrak{S}$ and $x\colon S \to X$ in $\mathfrak{X}$, there is a unique lift of f, given by $T \xrightarrow{x \circ f} X$. Hence, we can think of a category fibered in groupoids (and as we shall see later, a stack) $\mathfrak{X}$ over $\mathfrak{S}$ as a generalization of an object of $\mathfrak{S}$, and the fibers $\mathfrak{X}_S$ can be interpreted as the groupoid of S-points of $\mathfrak{X}$.

For convenience, we usually assume that for every morphism $f\colon T \to S$ in $\mathfrak{S}$ and object x over S, we have chosen a lift $f^*x \to x$ of f with target x. Depending on the context, this can be done either by direct construction, or by using a suitable version of the axiom of choice. Note that it is not required that $g^*(f^*x)$ equals $(f \circ g)^*x$, but the two are naturally isomorphic. While such a choice of lifts is not necessary, it makes it easier to write down the definition of a stack. We refer to the object f^*x as the *pullback* of x along f. When the morphism $f\colon T \to S$ is clear from context, we will also write $x|_T$ for f^*x.

Remark 1.2.5. Let $\alpha\colon x' \to x$ be a morphism in the fiber over some object S of $\mathfrak{S}$ (in particular, α is an isomorphism). Given a morphism $f\colon T \to S$, there exists a unique isomorphism $f^*\alpha\colon f^*x' \to f^*x$ such that the diagram

$$\begin{array}{ccc} f^*x' & \longrightarrow & x' \\ {\scriptstyle f^*\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ f^*x & \longrightarrow & x \end{array}$$

commutes. Namely, $f^*x' \to x' \to x$ is also a lift of f with target x. When the morphism f is clear from context, we will also write $\alpha|_T$ for $f^*\alpha$.

Notation 1.2.6. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two categories fibered in groupoids over $\mathfrak{S}$. Note that the morphisms from $\mathfrak{X}$ to $\mathfrak{Y}$ form a category, which we denote by $\mathfrak{Y}(\mathfrak{X})$, where morphisms between morphisms are given by 2-morphisms. Moreover, since every 2-morphism is an isomorphism, this category is a groupoid. When $\mathfrak{X} = \mathfrak{S}/X$ for an object X of $\mathfrak{S}$, as in Example 1.2.4, this groupoid is in fact equivalent to the fiber $\mathfrak{Y}_X$.

Definition 1.2.7. Let $f\colon \mathfrak{X} \to \mathfrak{Z}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ be morphisms of categories fibered in groupoids over $\mathfrak{S}$. The *fiber product* of $\mathfrak{X}$ and $\mathfrak{Y}$ over $\mathfrak{Z}$ is the following category fibered in groupoids. Its objects over S are triples (x, y, α) with x an object of $\mathfrak{X}_S$, y an object of $\mathfrak{Y}_S$ and $\alpha\colon f(x) \to g(y)$ an isomorphism in the fiber $\mathfrak{Z}_S$. Given a morphism $f\colon S' \to S$, a morphism from (x', y', α') to (x, y, α) over f is given by a pair of morphisms $(\beta\colon x' \to x, \gamma\colon y' \to y)$ over f such that $g(\gamma) \circ \alpha' = \alpha \circ f(\beta)$. The induced diagram

$$\begin{array}{ccc} \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} & \xrightarrow{\pi_{\mathfrak{Y}}} & \mathfrak{Y} \\ {\scriptstyle \pi_{\mathfrak{X}}}\downarrow & & \downarrow{\scriptstyle g} \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Z} \end{array}$$

with $\pi_{\mathfrak{X}}$ and $\pi_{\mathfrak{Y}}$ the projections, need not strictly commute, but it *2-commutes*. That is, the two composites $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \to \mathfrak{Z}$ are related by a natural 2-morphism. Observe the similarity with Definition 1.1.5, the fiber product for groupoids.

1.3 Descent data and stacks

Informally speaking, a stack is a category fibered in groupoids where objects can be glued uniquely from local data. Let $\{S_i \to S\}$ be a covering of an object S of $\mathfrak{S}$, and let x be an object over S. Denote by x_i the pullback of x to S_i, and by S_{ij} the intersection $S_i \times_S S_j$, and similarly for S_{ijk}. The object x cannot be reconstructed solely from the x_i, also the induced isomorphisms $\alpha_{ij}\colon x_i|_{S_{ij}} \to x_j|_{S_{ij}}$, which satisfy the cocycle condition on S_{ijk}, are needed. In a stack, we want to be able to glue the x_i on the intersections via the α_{ij}. This motivates the following definition.

Definition 1.3.1. Let $\mathfrak{X}$ be a category fibered in groupoids over $\mathfrak{S}$. A *descent datum* for $\mathfrak{X}$ over an object S of $\mathfrak{S}$ is given by

(i) a covering $\{S_i \to S\}$,

(ii) for every i a lift x_i of S_i in $\mathfrak{X}$,

(iii) for every i and j an isomorphism $\alpha_{ij}\colon x_i|_{S_{ij}} \to x_j|_{S_{ij}}$ in $\mathfrak{X}_{S_{ij}}$, satisfying the cocycle condition $\alpha_{ik}|_{S_{ijk}} = \alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}}$ in $\mathfrak{X}_{S_{ijk}}$.

Such a descent datum is called *effective* if there exists a lift x of S in $\mathfrak{X}$ together with isomorphisms $\alpha_i\colon x|_{S_i} \to x_i$ in $\mathfrak{X}_{S_i}$ such that $\alpha_{ij} = \alpha_j|_{S_{ij}} \circ \alpha_i|_{S_{ij}}^{-1}$ in $\mathfrak{X}_{S_{ij}}$. In this case, one says that the x_i over S_i *descend* to x over S.

Furthermore, in a stack, we want such a gluing to be unique (up to unique isomorphism). That is, for any other gluing (x', α_i') there should be a unique isomorphism $\beta\colon x' \to x$ such that $\alpha_i' = \alpha_i \circ \beta|_{S_i}$ over S_i. To have this property, we will require that isomorphisms in fibers can be reconstructed uniquely from local data. This idea is expressed in the following definition.

Definition 1.3.2. Let $\mathfrak{X}$ be a category fibered in groupoids over $\mathfrak{S}$. We say that *isomorphisms are a sheaf* for $\mathfrak{X}$ if, for any object S of $\mathfrak{S}$, any objects x and y in $\mathfrak{X}_S$, every covering $\{S_i \to S\}$ of S, and every collection of isomorphisms $\alpha_i\colon x|_{S_i} \to y|_{S_i}$ in $\mathfrak{X}_{S_i}$ such that $\alpha_i|_{S_{ij}} = \alpha_j|_{S_{ij}}$, there exists a unique isomorphism $\alpha\colon x \to y$ such that $\alpha_i = \alpha|_{S_i}$.

Remark 1.3.3. Alternatively, the above definition can be expressed as follows. For any two objects x and y in $\mathfrak{X}$ lying over an object S in $\mathfrak{S}$, one can define a presheaf

$$\mathrm{Isom}(x, y)\colon (\mathfrak{S}/S)^{\mathrm{op}} \to \mathbf{Set}$$

on the slice category $\mathfrak{S}/S$, by assigning to $f\colon T \to S$ the set $\mathrm{Hom}_{\mathfrak{X}_T}(f^*x, f^*y)$ of isomorphisms from f^*x to f^*y in $\mathfrak{X}_T$, and to a morphism $g\colon T' \to T$ from $f'\colon T' \to S$ to $f\colon T \to S$ the map that is given by pullback along g, that is,

$$\mathrm{Hom}_{\mathfrak{X}_T}(f^*x, f^*y) \to \mathrm{Hom}_{\mathfrak{X}_{T'}}(g^*f^*x, g^*f^*y) \cong \mathrm{Hom}_{\mathfrak{X}_{T'}}((f')^*x, (f')^*y),$$

where the latter isomorphism is induced by the natural isomorphisms $g^*f^*x \cong (f')^*x$ and $g^*f^*y \cong (f')^*y$. Now, saying that isomorphisms are a sheaf for $\mathfrak{X}$ is equivalent to saying that $\mathrm{Isom}(x, y)$ is a sheaf for all x, y and S. Note that, while it looks as if $\mathrm{Isom}(x, y)$ depends on the choice of f^*x and f^*y, any other choice would yield a presheaf that is naturally isomorphic.

Definition 1.3.4. A *stack* over $\mathfrak{S}$ is a category fibered in groupoids $\mathfrak{X}$ over $\mathfrak{S}$ such that every descent datum for $\mathfrak{X}$ is effective and isomorphisms are a sheaf for $\mathfrak{X}$. A morphism of stacks over $\mathfrak{S}$ is simply a morphism of categories over $\mathfrak{S}$, and similarly for 2-morphisms and isomorphisms. Fiber products of stacks can be computed as fiber products of categories over groupoids.

Remark 1.3.5. As in Example 1.2.4, any object X of $\mathfrak{S}$ can be considered a category fibered in groupoids over $\mathfrak{S}$ as the slice category $\mathfrak{S}/X$, where $\mathfrak{S}/X \to \mathfrak{S}$ forgets the morphism to X. Unfortunately, this does not always give a stack, it depends on the topology on $\mathfrak{S}$. However, for most of the examples of interest it will give a stack and will be easy to prove, e.g. for schemes, manifolds, analytic spaces, topological spaces, etc. with the usual topologies [Fan01].

Definition 1.3.6. A stack $\mathfrak{X}$ over $\mathfrak{S}$ is *representable* if it is isomorphic to the stack $\mathfrak{S}/X$ for some object X of $\mathfrak{S}$. A morphism of stacks $\mathfrak{X} \to \mathfrak{Y}$ is *representable* if, for every morphism $S \to \mathfrak{X}$ with S in $\mathfrak{S}$, the fiber product $S \times_{\mathfrak{Y}} \mathfrak{X}$ is representable.

Intuitively, this says that a morphism of stacks is representable if all of its fibers are representable.

From now on, we will simply write X for the category $\mathfrak{S}/X$ as well.

1.4 Algebraic stacks

An algebraic stack, over a fixed base scheme S, is a special type of stack over the site $\mathfrak{S} = \mathbf{Sch}_S$, where $\mathfrak{S}$ is usually equipped with the étale or fppf topology. To give a precise definition, one needs the notion of an *algebraic space*. Informally speaking, whereas a scheme is locally an affine scheme in the Zariski topology, an algebraic space is locally an affine scheme in the étale topology. For an in-depth treatment on algebraic spaces, see [LM00, Ols16, Stacks]. For our purposes, it suffices to think of an algebraic space as a geometric object slightly more general than a scheme, and to know that, just as for schemes, any algebraic space over S can naturally be considered as a category fibered in groupoids over $\mathfrak{S}$ (recall from Remark 1.3.5 that any scheme X over S can be identified with the slice category $\mathfrak{S}/X$). A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is said to be *representable by algebraic spaces* if for every scheme T and morphism $T \to \mathfrak{Y}$, the fiber product $T \times_{\mathfrak{Y}} \mathfrak{X}$ is representable by an algebraic space.

Before giving the definition of an algebraic stack, we first need to introduce some properties of representable morphisms.

Definition 1.4.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of categories fibered in groupoids over $\mathfrak{S}$ which is representable by algebraic spaces. Let P be a property of morphisms of algebraic spaces which is stable under base change and fppf-local on the base, such as being *smooth*, *étale*, *unramified*, *flat*, *surjective*, *(quasi-)separated*, *affine*, *proper*, *(locally) of finite type*, *(locally) of finite presentation*, or *an (open or closed) immersion*. Then f is said to have the property P if for every scheme T over S and $T \to \mathfrak{Y}$ the base change $T \times_{\mathfrak{Y}} \mathfrak{X} \to T$ has the property P.

Definition 1.4.2. A stack $\mathfrak{X}$ over $\mathfrak{S}$ is an *Artin stack* (resp. *Deligne–Mumford stack*) if the diagonal $\Delta_{\mathfrak{X}/S} : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is representable by algebraic spaces and there exists a smooth (resp. étale) and surjective morphism $X \to \mathfrak{X}$ for some scheme X. Such a morphism $X \to \mathfrak{X}$ is called a *presentation* of $\mathfrak{X}$.

An *algebraic stack* over $\mathfrak{S}$ will simply be an Artin stack over $\mathfrak{S}$.

Remark 1.4.3. Note that, in Definition 1.4.2, $\Delta_{\mathfrak{X}/S}$ being representable automatically implies the morphism $X \to \mathfrak{X}$ is representable, so that it makes sense to talk about this morphism being surjective, smooth or étale. Indeed, for every scheme T and morphism $T \to \mathfrak{X}$, we have that $T \times_{\mathfrak{X}} X \cong \mathfrak{X} \times_{\mathfrak{X} \times_S \mathfrak{X}} (X \times_S T)$ is representable by an algebraic space.

What follows now is a list of definitions of properties for algebraic stacks and for morphisms thereof. The general philosophy is that the common properties for schemes or algebraic spaces (and morphisms thereof) translate directly to the setting of algebraic stacks by making use of some kind of representability and the way these properties behave (e.g. often they are local on the source or target in some topology). We adopt the definitions as used by [Stacks], as indicated in the definitions. This list is by far not complete, but should cover all properties that are needed in the later chapters. For a more elaborate discussion on these properties, we refer to [Beh14, LM00, Ols16, Stacks].

Definition 1.4.4 ([Stacks, Tag 04YF])**.** Let P be a property of schemes which is local in the smooth topology, such as being *reduced*, *locally noetherian*, *normal* or *regular*. An algebraic stack $\mathfrak{X}$ is said to have P if there exists a smooth surjective morphism $X \to \mathfrak{X}$ with X a scheme having property P.

Definition 1.4.5 ([Stacks, Tag 04YC, Tag 050U])**.** An algebraic stack $\mathfrak{X}$ is *quasi-compact* if there exists a smooth and surjective morphism $X \to \mathfrak{X}$ with X a quasi-compact scheme.

A morphism of algebraic stacks $f\colon \mathfrak{X} \to \mathfrak{Y}$ is *quasi-compact* if for every quasi-compact algebraic stack $\mathfrak{Z}$ and morphism $\mathfrak{Z} \to \mathfrak{Y}$, the fiber product $\mathfrak{Z} \times_{\mathfrak{Y}} \mathfrak{X}$ is quasi-compact.

Definition 1.4.6 ([Stacks, Tag 0CHQ, Tag 0CHU, Tag 04YL])**.** Let P be any of the properties of being *affine*, *finite*, or *an (open or closed) immersion*. Then a morphism of algebraic stacks $f\colon \mathfrak{X} \to \mathfrak{Y}$ is said to have the property P if it is representable and has property P in the sense of Definition 1.4.1.

Definition 1.4.7 ([Stacks, Tag 06FM, Tag 0CIF])**.** Let P be a property of morphisms of algebraic spaces which is local on the source and target in the smooth (resp. étale) topology, such as being *locally of finite type*, *locally of finite presentation*, *flat*, or *smooth* (resp. or *unramified* or *étale*). Then a morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of algebraic stacks is said to have the property P if there exists a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f'} & V \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

with U and V algebraic spaces such that the vertical morphisms are smooth, $U \to \mathfrak{X} \times_{\mathfrak{Y}} V$ is smooth (resp. étale) and f' has the property P.

Definition 1.4.8 ([Stacks, Tag 04YW, Tag 06FS, Tag 06Q2])**.** A morphism of algebraic stacks $f : \mathfrak{X} \to \mathfrak{Y}$ is

- *separated* if the diagonal is proper in the sense of Definition 1.4.1,
- *quasi-separated* if the diagonal is quasi-compact and quasi-separated in the sense of Definition 1.4.1,
- of *finite type* if it is locally of finite type and quasi-compact,
- of *finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

1.5 Quotient stacks

A rich source of examples of algebraic stacks is given by quotients of schemes by group actions. For example, many moduli spaces are constructed in this way: one first describes a scheme X overparametrizing the objects of interest, and then describes an equivalence relation on the objects via the action of a group G on X. The moduli space should then be the quotient of X by G.

To get an intuition for what this quotient should look like, imagine a group G acting on some kind of geometric object X (e.g. a manifold or topological space). If the group action is sufficiently nice (i.e., free), the quotient $X \to X/G$ is expected to be a G-torsor, also known as a principal G-bundle. In particular, the pullback of X along any map $T \to X$ will be a G-torsor over T, and the projection $X \times_{X/G} T \to X$ will be G-equivariant. Moreover, any G-torsor $P \to T$ with an equivariant map to X conversely induces a map from T to the quotient X/G. This motivates the following definition.

Definition 1.5.1. Let G be a smooth group scheme acting on a scheme X over S. The *quotient stack* of X by G, denoted $[X/G]$, is the category over $\mathfrak{S}$ whose objects over T are diagrams

$$\begin{array}{ccc} P & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle p} & & \\ T & & \end{array}$$

where $P \xrightarrow{p} T$ is a G-torsor and $P \xrightarrow{\phi} X$ is a G-equivariant morphism. The morphisms from $T' \xleftarrow{p'} P' \xrightarrow{\phi'} X$ to $T \xleftarrow{p} P \xrightarrow{\phi} X$ over $f\colon T' \to T$ are G-equivariant morphisms $\alpha\colon P' \to P$ such that $p \circ \alpha = f \circ p'$ and $\phi \circ \alpha = \phi'$. Since G-torsors can be glued from local data, it is easy to verify that $[X/G]$ is indeed a stack over $\mathfrak{S}$. There is a natural quotient map $\pi\colon X \to [X/G]$ corresponding to the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ {\scriptstyle \pi_X}\downarrow & & \\ X & & \end{array}$$

More generally, one can replace the scheme X by an algebraic stack $\mathfrak{X}$ to define the quotient stack $[\mathfrak{X}/G]$.

Example 1.5.2. The quotient stack $[S/G]$ corresponding to the trivial action of G on S is also known as the *classifying stack* of G and is denoted $\mathrm{B}G$.

Remark 1.5.3. For any morphism $T \xrightarrow{f} [X/G]$, the corresponding G-torsor over T with G-equivariant map to X can be recovered via pullback along π, as depicted in the following diagram.

$$\begin{array}{ccc} P & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow{f} & [X/G] \end{array}$$

Indeed, by definition of the fiber product, the objects of $T \times_{[X/G]} X$ over T' are triples of morphisms $(f\colon T' \to T, h\colon T' \to X, \alpha\colon G \times T' \to P \times_T T')$ with α being G-equivariant such that

$$\begin{array}{ccccc} & & G \times T' & & \\ & \swarrow & & \searrow{\scriptstyle \sigma\circ(\mathrm{id}_G \times h)} & \\ T' & & \downarrow{\scriptstyle \alpha} & & X \\ & \nwarrow & & \nearrow{\scriptstyle \phi\circ\pi_P} & \\ & & P \times_T T' & & \end{array}$$

commutes. Since α is G-equivariant, it must be of the form $\alpha(g, t') = (g \cdot \beta(t'), t')$ for $\beta\colon T' \to P$ given by $\beta(t') = \pi_P(\alpha(1, t'))$. But then $f = p \circ \beta$, $h = \phi \circ \beta$ and α can all be expressed in terms of β. Hence, $(T \times_{[X/G]} X)(T') = P(T')$ and this provides a canonical isomorphism $T \times_{[X/G]} X \cong P$.

Remark 1.5.4. The above remark shows that the quotient stack $[X/G]$ is an Artin stack with presentation $\pi\colon X \to [X/G]$. Indeed, the morphism π is smooth and surjective since $P \xrightarrow{p} T$ is smooth and surjective, as G was assumed to be smooth. Similarly, if G is a finite group, one shows that the quotient stack $[X/G]$

is a Deligne–Mumford stack. For representability of the diagonal, see [Ols16, Example 8.1.12].

Remark 1.5.5. The quotient stack $[X/G]$ indeed satisfies the quotient property, that is, for any G-invariant morphism $f\colon X \to Y$ there is an induced morphism $\overline{f}\colon [X/G] \to Y$. Indeed, for any diagram $T \leftarrow P \xrightarrow{\phi} X$, the composite $f \circ \phi$ is G-invariant, so there is an induced morphism $T \to Y$, which defines a T-point of Y.

Remark 1.5.6. The quotient stack construction is functorial in the following sense. Let G and H be smooth group schemes acting on schemes X and Y, respectively, over S. Suppose $f\colon X \to Y$ is a morphism of schemes over S, and $\varphi\colon G \to H$ a morphism of group schemes over S, such that $f(g \cdot x) = \varphi(g) \cdot f(x)$. Then there is an induced morphism of quotient stacks $\overline{f}\colon [X/G] \to [Y/H]$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ [X/G] & \xrightarrow{\overline{f}} & [Y/H] \end{array}$$

2-commutes. The morphism $\overline{f}$ is given by sending a diagram $T \xleftarrow{p} P \xrightarrow{\phi} X$ to the diagram $T \xleftarrow{p \circ \pi_P} H \times_G P \xrightarrow{\psi} Y$ where $\psi(h,p) = h \cdot f(\phi(p))$. The commutativity of the diagram follows from the natural isomorphism $H \times_G (G \times T) \cong H \times T$ of H-torsors over T.

Lemma 1.5.7. *Let G and H be smooth group schemes acting on schemes X and Y, respectively, over S. Then there is a natural isomorphism $[X/G] \times [Y/H] \cong [X \times Y/G \times H]$ given by sending a pair of diagrams $T \leftarrow P \xrightarrow{\phi} X$ and $T \leftarrow Q \xrightarrow{\psi} Y$ to the diagram $T \leftarrow P \times Q \xrightarrow{\phi \times \psi} X \times Y$.*

Proof. The described map is clearly functorial. Conversely, for any $(G \times H)$-torsor R over T, the natural isomorphism $R \cong R/H \times_T R/G$ of $(G \times H)$-torsors over T yields an inverse. □

Lemma 1.5.8. *If G acts freely on X, then $[X/G]$ is representable by an algebraic space.*

Proof. To prove this, we will use the characterization of an algebraic space as an algebraic stack whose objects all have trivial automorphism groups [Stacks, Tag 03YR]. Any point of $[X/G]$ over any T corresponds to a G-torsor over T with an equivariant morphism $\phi\colon P \to X$. An automorphism of this point is an automorphism $\alpha\colon P \to P$ over T such that $\phi \circ \alpha = \phi$. Étale-locally, $P \cong G \times T$, and ϕ is determined by its restriction $\phi'\colon S \times T \to X$ along the unit $e : S \to G$. Furthermore, $\alpha\colon G \times T \to G \times T$ is given by multiplication by some element $g \in G$. Now $g \cdot \phi'(t) = \phi'(t)$ for all $t \in T$, and since G acts freely on X we have $g = 1$, that is, $\alpha = \mathrm{id}_P$. Since this holds étale-locally, we also have $\alpha = \mathrm{id}_P$ globally, and thus this automorphism is trivial. □

Remark 1.5.9. As is reflected in the notation, the quotient stack can be thought of as a geometric analogue of the action groupoid. However, in general we have

$$[X/G](T) \not\simeq [X(T)/G(T)].$$

For example, the classifying stack BG of $G = \mathbb{Z}/2\mathbb{Z}$ has up to isomorphism precisely two $\mathbb{F}_q$-points: the trivial G-torsor $\mathbb{F}_q \to (\mathbb{F}_q)^2$ and the non-trivial G-torsor $\mathbb{F}_q \to \mathbb{F}_{q^2}$ whose G-action is given by the Frobenius automorphism, both having an automorphism group of $\mathbb{Z}/2\mathbb{Z}$. On the other side, the action groupoid has only one object with automorphism group $\mathbb{Z}/2\mathbb{Z}$.

In special cases, this discrepancy can be resolved.

Proposition 1.5.10. *Let G be an algebraic group acting on a scheme X over a field k. If (i) k is separably closed, or (ii) k is finite and G is connected, then there is an equivalence of groupoids*

$$[X/G](k) \simeq [X(k)/G(k)].$$

Proof. In both cases, any G-torsor over $\operatorname{Spec} k$ is trivial. For *(i)* because $\operatorname{Spec} k$ does not have a non-trivial étale cover, and for *(ii)* by Lang's theorem [Lan56]. Hence, the objects of the groupoid $[X/G](k)$ are G-equivariant morphisms $G \xrightarrow{\phi} X$, which are completely determined by the value $\phi(1) \in X(k)$, and morphisms $\phi \to \phi'$ are given by an element $g \in G(k)$ such that $\phi(1) = \phi'(g) = g \cdot \phi'(1)$. But this is precisely (equivalent to) $[X(k)/G(k)]$. □

1.6 Stabilizers

Definition 1.6.1. Let $\mathfrak{X}$ be an algebraic stack over $\mathfrak{S}$, and $x\colon \operatorname{Spec} K \to \mathfrak{X}$ a K-point of $\mathfrak{X}$ for some field K. The *stabilizer* of x is the fiber product

$$\mathrm{Stab}_{\mathfrak{X}}(x) = \operatorname{Spec} K \times_{\mathfrak{X}} \operatorname{Spec} K$$

as a group scheme (or more precisely, group algebraic space) over K. Indeed, for any $T \to \operatorname{Spec} K$, the T-points of $\mathrm{Stab}_{\mathfrak{X}}(x)$ can be identified with the automorphism group in $\mathfrak{X}$ of the T-point $T \to \operatorname{Spec} K \xrightarrow{x} \mathfrak{X}$. We say $\mathfrak{X}$ *has affine stabilizers* if $\mathrm{Stab}_{\mathfrak{X}}(x)$ is an affine group scheme for every x. We say $\mathfrak{X}$ *has finite stabilizers* if $\mathrm{Stab}_{\mathfrak{X}}(x)$ is a finite group scheme for every x.

Lemma 1.6.2. *Let G be a smooth group scheme acting on a scheme X. The quotient stack $\mathfrak{X} = [X/G]$ has affine stabilizers (resp. finite stabilizers) if G is affine (resp. finite).*

Proof. A point $x\colon \operatorname{Spec} K \to \mathfrak{X}$ corresponds to a G-torsor $P \xrightarrow{p} \operatorname{Spec} K$ with a G-equivariant map $P \xrightarrow{\phi} X$. As P is étale-locally trivial, we have $P \times_{\operatorname{Spec} K} \operatorname{Spec} L \cong G \times \operatorname{Spec} L$ for some finite separable field extension L/K. The G-equivariant morphism $G \times \operatorname{Spec} L \xrightarrow{\psi} X$ induced by ϕ corresponds to a point $x' = \psi(1) \in X(L)$. Now consider the base change $\operatorname{Stab}_{\mathfrak{X}}(x) \times_{\operatorname{Spec} K} \operatorname{Spec} L$. Its T-points are given by G-equivariant isomorphisms $\alpha\colon G \times T \to G \times T$ over T such that $\psi \circ \alpha = \psi$. Hence, we obtain a fiber product:

$$\begin{array}{ccc} \operatorname{Stab}_{\mathfrak{X}}(x) \times_{\operatorname{Spec} K} \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} L \\ \downarrow & & \downarrow{\scriptstyle x'} \\ G \times \operatorname{Spec} L & \xrightarrow{\quad\psi\quad} & X \end{array}$$

This shows that $\operatorname{Stab}_{\mathfrak{X}}(x) \times_{\operatorname{Spec} K} \operatorname{Spec} L$ is a subgroup of $G \times \operatorname{Spec} L$, which is, just like $G \times \operatorname{Spec} L$, affine (resp. finite). Since being affine (resp. finite) is local in the étale topology [Stacks, Tag 02L5, Tag 02LA], it follows that $\operatorname{Stab}_{\mathfrak{X}}(x)$ is also affine (resp. finite). □

Lemma 1.6.3. *Let $f\colon \mathfrak{X} \to \mathfrak{Z}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ be morphisms between algebraic stacks with affine (resp. finite) stabilizers. Then the fiber product $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ also has affine (resp. finite) stabilizers.*

Proof. Pick any point $(x, y, \alpha) \in (\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y})(K)$. An automorphism of (x, y, α) consists of morphisms $\beta\colon x \to x$ and $\gamma\colon y \to y$ such that $\alpha \circ f(\beta) = g(\gamma) \circ \alpha$. That is, the automorphism group of (x, y, α) is precisely the stabilizer of α for the action of $\operatorname{Aut}_{\mathfrak{X}}(x) \times \operatorname{Aut}_{\mathfrak{Y}}(y)$ on $\operatorname{Hom}_{\mathfrak{Z}}(f(x), g(y))$, given by $(\beta, \gamma) \cdot \alpha = g(\gamma) \circ \alpha \circ f(\beta)$. Also note that $\operatorname{Hom}_{\mathfrak{Z}}(f(x), g(y)) \cong \operatorname{Aut}_{\mathfrak{Z}}(z)$ for any object z of $\mathfrak{Z}$ isomorphic to $f(x) \cong g(y)$. This reasoning shows that the stabilizer of (x, y, α) can be identified as the fiber product in the following cartesian square

$$\begin{array}{ccc} \operatorname{Stab}_{\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}}(x, y, \alpha) & \longrightarrow & \operatorname{Spec} K \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ \operatorname{Stab}_{\mathfrak{X}}(x) \times \operatorname{Stab}_{\mathfrak{Y}}(y) & \longrightarrow & \operatorname{Stab}_{\mathfrak{Z}}(z) \end{array}$$

where z is again any object of $\mathfrak{Z}$ isomorphic to $f(x) \cong g(y)$. By assumption, all of $\operatorname{Stab}_{\mathfrak{X}}(x)$, $\operatorname{Stab}_{\mathfrak{Y}}(y)$ and $\operatorname{Stab}_{\mathfrak{Z}}(z)$ are affine (resp. finite), and therefore, the fiber product $\operatorname{Stab}_{\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}}(x, y, \alpha)$ is also affine (resp. finite). □

Definition 1.6.4. Let $\mathfrak{S}$ be an algebraic stack with affine stabilizers. Let $\mathbf{Stck}_{\mathfrak{S}}$ be the full subcategory of algebraic stacks of finite type over $\mathfrak{S}$ with affine stabilizers. By Lemma 1.6.3, this category is closed under pullbacks.

The algebraic stacks that appear in this thesis all have affine stabilizers. The following proposition shows that we can think of such algebraic stacks, at least locally, as quotient stacks of quasi-projective schemes by linear groups.

Proposition 1.6.5 ([Kre99, Proposition 3.5.9])**.** *Let $\mathfrak{X}$ be a reduced Artin stack of finite type over a field with affine stabilizers. Then $\mathfrak{X}$ admits a stratification by quotient stacks $[X_i/\mathrm{GL}_{n_i}]$ where X_i is a quasi-projective scheme.*

Chapter 2

Character stacks

In this chapter we will define, and describe various properties of, character stacks, which are the main objects of study in this thesis. Roughly speaking, they are the moduli space of representations of a finitely generated group Γ into a linear algebraic group G. While Γ can be any finitely generated group, it most commonly arises as the fundamental group $\pi_1(M, *)$ of a compact manifold M. In fact, every finitely presented group arises in this way. In this case, it is well known that representations $\pi_1(M, *) \to G$ correspond to G-local systems on M [Sza09, Corollary 2.6.2]. Moreover, isomorphic local systems correspond to conjugate representations. Therefore, one is interested in the quotient of the space parametrizing *all* representations $\Gamma \to G$ (this space will be called the 'representation variety'), by the action of conjugation by G. This quotient will be the G-character stack of Γ.

2.1 Representation varieties

Fix a base scheme S. Typically, S will be $\operatorname{Spec} k$ where k is a field or a finitely generated $\mathbb{Z}$-algebra. Let G be a linear algebraic group over S, by which we understand a closed subgroup of the group scheme GL_r over S for some $r \geq 0$.

Definition 2.1.1. Let Γ be a finitely generated group. The *G-representation variety* of Γ is the scheme $R_G(\Gamma)$ over S whose functor of points is given by

$$R_G(\Gamma)(T) = \mathrm{Hom}(\Gamma, G(T)).$$

Let us explain why $R_G(\Gamma)$ is indeed representable. After choosing a presentation

$$\Gamma = \langle \gamma_1, \ldots, \gamma_n \mid r_i(\gamma_1, \ldots, \gamma_n) = 1 \text{ for } i \in I \rangle,$$

any representation $\rho \colon \Gamma \to G(T)$ can be identified with the image of its generators, that is, the tuple $(\rho(\gamma_1), \ldots, \rho(\gamma_n)) \in G(T)^n$. However, not all tuples in $G(T)^n$ define such a representation because of the relations r_i between the generators. Every such relation r_i, which is a word in the symbols γ_i, defines a morphism $r_i \colon G^n \to G$ given on points by $(g_1, \ldots, g_n) \mapsto r_i(g_1, \ldots, g_n)$, and hence a closed subscheme $X_i \subseteq G^n$ as in the pullback diagram

$$\begin{array}{ccc} X_i & \longrightarrow & G^n \\ \downarrow & & \downarrow{\scriptstyle r_i} \\ S & \xrightarrow{e} & G \end{array}$$

where e is the unit of G, a closed immersion [Stacks, Tag 047G]. Now, the intersection of all X_i over G^n realizes $R_G(\Gamma)$ as a closed subscheme of G^n. This closed subscheme corresponds to the sheaf of ideals in $\mathcal{O}_{G^n}$ that is generated by the sheaves of ideals $\mathcal{I}_i \subseteq \mathcal{O}_{G^n}$ corresponding to the X_i. Indeed, we have

$$R_G(\Gamma)(T) = \bigcap_{i \in I} \big\{ t \in G(T)^n \mid r_i(t) = 1 \big\} = \bigcap_{i \in I} X_i(T) = \Big(\bigcap_{i \in I} X_i \Big)(T).$$

Remark 2.1.2. The G-representation variety $R_G(\Gamma)$ will always be separated and of finite type over S, as it is a closed subscheme of G^n, which itself is separated and of finite type over S. Moreover, $R_G(\Gamma)$ is affine over S, as G^n is affine over S. However, the G-representation variety may be non-reduced. For example, it was shown in [LM85, (2.10.4)] that for the von Dyck group $\Gamma = \langle a, b, c \mid a^3 = b^3 = c^3 = abc = 1 \rangle \cong \mathbb{Z}^2 \rtimes S_3$ and $G = \mathrm{GL}_2$ over $S = \operatorname{Spec} \mathbb{C}$, the G-representation variety $R_G(\Gamma)$ is non-reduced.

For us, the main example of a finitely generated group Γ is the fundamental group of a compact manifold.

Proposition 2.1.3. *Let M be a connected compact manifold with a basepoint x. Then $\pi_1(M, x)$ is finitely presented.*

Proof. Every compact manifold M is homotopy equivalent to a finite CW-complex [Whi40]. Since M is connected, this finite CW-complex can be chosen to consist of a single 0-cell corresponding to x. It follows that the fundamental group of M has a presentation with a generator for every 1-cell and a relation for every 2-cell, and is therefore finitely presented. □

When M is a connected compact manifold, we will simply write $R_G(M)$ instead of $R_G(\pi_1(M, x))$ and call it the *G-representation variety* of M. Note that this scheme is, up to isomorphism, independent of the chosen basepoint x since the fundamental group $\pi_1(M, x)$ is, up to isomorphism, independent of x.

Example 2.1.4.
- The circle S^1 has fundamental group $\pi_1(S^1, *) \cong \mathbb{Z}$, from which follows that $R_G(S^1) \cong G$.

- The fundamental group of a closed orientable surface Σ_g of genus g can be presented as $\pi_1(\Sigma_g, *) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$, where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ denotes the commutator. Therefore, $R_G(\Sigma_g)$ is the closed subscheme of G^{2g} given by $\prod_{i=1}^{g} [A_i, B_i] = 1$.
- Let N_r be the connected sum of r projective planes, that is, the non-orientable closed surface of demigenus r. Its fundamental group can be presented as $\pi_1(N_r, *) = \langle a_1, \ldots, a_r \mid a_1^2 \cdots a_r^2 = 1 \rangle$. Hence, $R_G(N_r)$ is the closed subvariety of G^r given by $\prod_{i=1}^{r} A_i^2 = 1$.

While the G-representation variety $R_G(\Gamma)$ is an interesting object on its own, it cannot quite be regarded as the moduli space of representations of Γ into G. Namely, two different points of $R_G(\Gamma)$ might represent isomorphic representations, that is, representations that are related through conjugation. Formulated differently, the linear algebraic group G acts on the G-representation variety by conjugation

$$(g \cdot \rho)(\gamma) = g \rho(\gamma) g^{-1}$$

for all $g \in G(T)$ and $\gamma \in R_G(\Gamma)(T)$. In this sense, the correct moduli space should be the quotient of $R_G(\Gamma)$ by the action of G. Unfortunately, quotients are famously hard in algebraic geometry, and it is not always clear which quotient one wants to take.

One possibility is to take the *Geometric Invariant Theory (GIT)* quotient as developed by Mumford [Mum65]. Given an affine variety $X = \operatorname{Spec} R$ over a field k with an action of a linear algebraic group G over k, encoded by a ring morphism $\hat{\sigma}\colon R \to R \otimes_k \mathcal{O}_G(G)$, the (affine) GIT quotient of X by G is

$$X /\!\!/ G = \operatorname{Spec} R^G$$

where $R^G = \{ r \in R \mid \hat{\sigma}(r) = r \otimes 1 \}$ denotes the subring of invariants of R. The projection $X \to X /\!\!/ G$ corresponds to the inclusion $R^G \subseteq R$. Even though the GIT quotient can be constructed as a scheme, it was shown by Nagata that in general the resulting scheme need not be of finite type over k [Nag59]. However, he also showed that if G is reductive, the ring of invariants will be finitely generated over k [Nag64].

Definition 2.1.5. Let G be a reductive linear algebraic group over a field k, and Γ a finitely generated group. The *G-character variety* of Γ is the GIT quotient

$$X_G(\Gamma) = R_G(\Gamma) /\!\!/ G.$$

Remark 2.1.6. In the literature, the term 'G-character variety' is also used for a notion which is different, but related, to the above definition. Given a linear algebraic group $G \subseteq \mathrm{GL}_n$ over $k = \mathbb{C}$, one defines

$$\chi_G(\Gamma) = \operatorname{Spec} \mathbb{C}[\tau_\gamma \mid \gamma \in \Gamma]$$

to be the spectrum of the complex algebra generated by the functions $\tau_\gamma \colon \rho \mapsto \mathrm{tr}(\rho(\gamma))$ on $R_G(\Gamma)$. Since the functions τ_γ are invariant under the action of G, there is a canonical morphism

$$X_G(\Gamma) \to \chi_G(\Gamma).$$

While this morphism is known to be an isomorphism for various G, such as SL_n, GL_n, Sp_{2n} and O_n, see [FL11, Theorem A.1] and [Pro76], it fails to be so for other groups, such as SO_{2n} [Sik13].

Besides the GIT quotient, there are other ways to construct quotients. In the following sections we will apply the theory of quotient stacks, as encountered in Section 1.5, to take the quotient in the category of stacks, defining the *G-character stack*. One advantage to this approach is that the quotient remembers the automorphisms of the representations. Another advantage is that we do not need to assume that G is reductive.

2.2 Character groupoids

Before we properly introduce the G-character stack, we will first forget all geometry, and let G be an ordinary group. Furthermore, we will allow for a more general setup, with Γ being a groupoid, rather than a group.

Definition 2.2.1. Let G be a group. For any groupoid Γ, the *G-character groupoid* of Γ, denoted $\mathfrak{X}_G(\Gamma)$, is the groupoid whose objects are functors $\rho \colon \Gamma \to G$ (where G is seen as a groupoid with a single object), and whose morphisms $\rho_1 \to \rho_2$ are given by natural transformations $\mu \colon \rho_1 \Rightarrow \rho_2$.

The map $\mathfrak{X}_G$ can naturally be extended to a 2-functor $\mathfrak{X}_G \colon \mathbf{Grpd} \to \mathbf{Grpd}^{\mathrm{op}}$. Explicitly:

- For any functor $f \colon \Gamma' \to \Gamma$ between groupoids, let $\mathfrak{X}_G(f) \colon \mathfrak{X}_G(\Gamma) \to \mathfrak{X}_G(\Gamma')$ be the functor given by precomposition $\mathfrak{X}_G(f)(\rho) = \rho \circ f$ for any $\rho \in \mathfrak{X}_G(\Gamma)$, and $\mathfrak{X}_G(f)(\mu) = \mu f$ for any morphism $\mu \colon \rho_1 \to \rho_2$.
- For any natural transformation $\eta \colon f_1 \Rightarrow f_2$ between functors $f_1, f_2 \colon \Gamma' \to \Gamma$, let $\mathfrak{X}_G(\eta) \colon \mathfrak{X}_G(f_1) \Rightarrow \mathfrak{X}_G(f_2)$ be the natural transformation given by $(\mathfrak{X}_G(\eta)_\rho)_{x'} = \rho(\eta_{x'})$ for all $\rho \in \mathfrak{X}_G(\Gamma)$ and $x' \in \Gamma'$. Indeed, this defines a natural transfor-

mation as the square

$$\begin{array}{ccc} \rho(f_1(x')) & \xrightarrow{\rho(\eta_{x'})} & \rho(f_2(x')) \\ {\scriptstyle \rho(f_1(\gamma'))}\downarrow & & \downarrow{\scriptstyle \rho(f_2(\gamma'))} \\ \rho(f_1(y')) & \xrightarrow[\rho(\eta_{y'})]{} & \rho(f_2(y')) \end{array}$$

commutes for every $\gamma' \colon x' \to y'$ in Γ' by naturality of η, and this is natural in ρ.

Note that $\mathfrak{X}_G$ strictly preserves composition of 1-morphisms and 2-morphisms, and therefore defines a strict 2-functor.

Corollary 2.2.2. *An equivalence between groupoids* Γ *and* Γ' *naturally induces an equivalence between the* G*-character groupoids* $\mathfrak{X}_G(\Gamma)$ *and* $\mathfrak{X}_G(\Gamma')$. □

Let us apply the above corollary as follows in the case that G is a finite group. If Γ is a finitely generated groupoid, then it can easily be seen that the groupoid $\mathfrak{X}_G(\Gamma)$ is finite. But now it follows from Corollary 2.2.2 that $\mathfrak{X}_G(\Gamma)$ is essentially finite if Γ is essentially finitely generated. Therefore, for G finite, we can restrict $\mathfrak{X}_G$ to a 2-functor

$$\mathfrak{X}_G \colon \mathbf{FGGrpd} \to \mathbf{FinGrpd}^{\mathrm{op}}.$$

As before, the main example of a finitely generated groupoid Γ for us comes from a compact manifold.

Definition 2.2.3. Let M be a compact manifold. The *fundamental groupoid* of M is the groupoid $\Pi(M)$ whose objects are the points of M, and morphisms $x \to y$ are given by homotopy classes of paths from x to y.

For any smooth map of manifolds $f \colon M \to N$, there is an induced a functor $\Pi(f) \colon \Pi(M) \to \Pi(N)$. In particular, one can think of Π as a functor $\Pi \colon \mathbf{Mnfd} \to \mathbf{Grpd}$ from the category of manifolds to the category of groupoids. Moreover, Π can be promoted to a 2-functor if one considers **Mnfd** as a 2-category where 2-morphisms are given by smooth homotopies.

Note that the fundamental groupoid $\Pi(M)$ is essentially finitely generated when M is a compact manifold. Namely, choosing a basepoint $x_1, \ldots, x_n$ on each of the finitely many connected component of M, we find that $\Pi(M)$ is equivalent to $\pi_1(M, x_1) \sqcup \cdots \sqcup \pi_1(M, x_n)$, which is finitely generated by Proposition 2.1.3.

Definition 2.2.4. Let G be a group and let M be a compact manifold. The *G-character groupoid* of M, denoted $\mathfrak{X}_G(M)$, is defined as $\mathfrak{X}_G(\Pi(M))$, where $\Pi(M)$ is the fundamental groupoid of M. In particular, if G is finite, $\mathfrak{X}_G(M)$ is essentially finite.

Let us elaborate a bit more on the groupoid $\mathfrak{X}_G(M)$. Its objects $\rho\colon \Pi(M) \to G$ assign to every homotopy class of paths γ an element $\rho(\gamma)$ of G. A morphism from ρ_1 to ρ_2 is a natural transformation $\mu\colon \rho_1 \Rightarrow \rho_2$. Such a natural transformation can be thought of as a function $\mu\colon M \to G$ such that $\rho_2(\gamma) = \mu(y)\rho_1(\gamma)\mu(x)^{-1}$ for any path $\gamma\colon x \to y$ in $\Pi(M)$. Such transformations are known in physics as *local gauge transformations.*

With this characterization, the G-character groupoid can be defined in an alternative way. Let $\mathcal{G}_\Gamma = \prod_{x \in \Gamma} G$ be the *group of local gauge transformations*, which acts on the set $X = \operatorname{Hom}(\Gamma, G)$ via

$$((g_x)_{x\in\Gamma} \cdot \rho)(\gamma) = g_y \rho(\gamma) g_x^{-1}$$

for any $\rho \in X$ and $\gamma\colon x \to y$ in Γ. Now, the G-character groupoid $\mathfrak{X}_G(\Gamma)$ is equivalent to the action groupoid $[X/\mathcal{G}_\Gamma]$. This alternative description will be of crucial importance in defining the G-character stacks.

2.3 Character stacks

The G-character stack will be defined as the geometric analogue of the G-character groupoid, replacing the action groupoid by the quotient stack. Fix a base scheme S and let G be a linear algebraic group over S.

Definition 2.3.1. Let Γ be a finitely generated groupoid. The *G-representation variety* of Γ is the scheme over S whose functor of points is given by

$$R_G(\Gamma)(T) = \operatorname{Hom}(\Gamma, G(T)),$$

where $G(T)$ is seen as a groupoid with a single object. Completely analogous to the discussion below Definition 2.1.1, the G-representation variety is representable by a closed subscheme of G^n for some n.

Importantly, note that $R_G(\Gamma)$ is not well-defined up to equivalence of Γ. That is, $R_G(\Gamma)$ need not be isomorphic to $R_G(\Gamma')$ even when Γ is equivalent to Γ'. This problem will be fixed once we pass to the G-character stack.

Analogous to the previous section, for a finitely generated groupoid Γ, we define the *group of local gauge transformations* to be the group scheme

$$\mathcal{G}_\Gamma = \prod_{x\in\Gamma} G$$

which, as a finite product of linear algebraic groups, is again a linear algebraic group over S. It acts naturally on $R_G(\Gamma)$, and the action is pointwise given by

$$((g_x)_{x\in\Gamma} \cdot \rho)(\gamma) = g_y \rho(\gamma) g_x^{-1}$$

for all $(g_x)_{x \in \Gamma} \in \mathcal{G}_\Gamma(T)$ and $\rho \in R_G(\Gamma)(T)$ and $\gamma\colon x \to y$ in Γ.

Definition 2.3.2. Let Γ be a finitely generated groupoid. The *G-character stack* of Γ is the quotient stack

$$\mathfrak{X}_G(\Gamma) = [R_G(\Gamma)/\mathcal{G}_\Gamma] \, .$$

As for the G-character groupoids, we want to extend $\mathfrak{X}_G(-)$ to essentially finitely generated groupoids, and promote it to a 2-functor $\mathbf{FGGrpd} \to \mathbf{Stck}_S^{\mathrm{op}}$, where $\mathbf{Stck}_S$ is the category of algebraic stacks of finite type over S with affine stabilizers, as defined in Definition 1.6.4.

Let $f\colon \Gamma' \to \Gamma$ be a functor between finitely generated groupoids. Such a functor induces a morphism between the representation varieties, given by pullback

$$f^*\colon R_G(\Gamma) \to R_G(\Gamma'), \quad \rho \mapsto \rho \circ f \quad \text{for all } \rho \in R_G(\Gamma)(T),$$

and also a morphism of algebraic groups

$$\mathcal{G}_f\colon \mathcal{G}_\Gamma \to \mathcal{G}_{\Gamma'}, \quad (g_x)_{x\in\Gamma} \mapsto (g_{f(x')})_{x'\in\Gamma'}.$$

In particular, as described in Remark 1.5.6, there is an induced map on character stacks $\mathfrak{X}_G(f)\colon \mathfrak{X}_G(\Gamma) \to \mathfrak{X}_G(\Gamma')$ that sends a $\mathcal{G}_\Gamma$-torsor P to the $\mathcal{G}_{\Gamma'}$-torsor $\mathcal{G}_{\Gamma'} \times_{\mathcal{G}_\Gamma} P$. Note that this construction is functorial in f.

Next, let $\eta\colon f_1 \Rightarrow f_2$ be a natural transformation between functors $f_1, f_2\colon \Gamma' \to \Gamma$. We want to assign a 2-morphism $\mathfrak{X}_G(\eta)\colon \mathfrak{X}_G(f_1) \Rightarrow \mathfrak{X}_G(f_2)$ to this natural transformation, which amounts to, for every $\mathcal{G}_\Gamma$-torsor P over T with $\mathcal{G}_\Gamma$-equivariant map $\rho\colon P \to R_G(\Gamma)$, a morphism of $\mathcal{G}_{\Gamma'}$-torsors (as indicated by the dashed arrow) such that the diagram

$$\begin{array}{ccc} \mathcal{G}_{\Gamma'} \times_{\mathcal{G}_\Gamma} P & \xrightarrow{(g',p)\mapsto g'\cdot f_1^*(\rho(p))} & \\ \Big\downarrow & & R_G(\Gamma') \\ \mathcal{G}_{\Gamma'} \times_{\mathcal{G}_\Gamma} P & \xrightarrow{(g',p)\mapsto g'\cdot f_2^*(\rho(p))} & \end{array}$$

commutes. Analogous to the case for G-character groupoids, this morphism is given by $(g', p) \mapsto (g'\rho(p)(\eta_{x'}), p)$. One easily sees that this map is well-defined, that is, respects the $\mathcal{G}_\Gamma$-action on both sides.

Corollary 2.3.3. *Any equivalence between finitely generated groupoids Γ and Γ' naturally induces an isomorphism between the G-character stacks $\mathfrak{X}_G(\Gamma)$ and $\mathfrak{X}_G(\Gamma')$.* □

This corollary allows us to extend the definition of the G-character stack to groupoids Γ which are only essentially finitely generated, but only up to a natural isomorphism. In particular, we obtain a 2-functor

$$\mathfrak{X}_G(-)\colon \mathbf{FGGrpd} \to \mathbf{Stck}_S^{\mathrm{op}}.$$

We are now able to define the G-character stack of a compact manifold.

Definition 2.3.4. Let M be a compact manifold (possibly with boundary). It was shown that the fundamental groupoid $\Pi(M)$ of M is essentially finitely generated, that is, is equivalent to a finitely generated groupoid Γ. The *G-character stack* of M is defined as

$$\mathfrak{X}_G(M) = \mathfrak{X}_G(\Gamma).$$

This definition is, up to isomorphism, independent of the choice of Γ by the above corollary.

Remark 2.3.5. It might be tempting to define the G-character stack of Γ, similar to the G-representation variety, as the category fibered in groupoids over $\mathfrak{S} = \mathbf{Sch}_S$ whose fiber over an object T is the G-character groupoid $\mathfrak{X}_{G(T)}(\Gamma)$. However, these groupoids are different as explained in Remark 1.5.9.

Lemma 2.3.6. *$\mathfrak{X}_G(-)$ sends finite colimits in* $\mathbf{FGGrpd}$ *to limits in* $\mathbf{Stck}_S$*.*

Proof. Let $\Gamma = \operatorname{colim}_{i \in I} \Gamma_i$ be a colimit in $\mathbf{FGGrpd}$. Up to equivalence, we can assume all Γ_i and Γ are finitely generated groupoids. A T-point of $\lim_{i \in I} \mathfrak{X}_G(\Gamma_i)$ is a collection of $\mathcal{G}_{\Gamma_i}$-torsors P_i over T with $\mathcal{G}_{\Gamma_i}$-equivariant morphisms $\rho_i\colon P_i \to R_G(\Gamma_i)$, which are compatible in the sense that there are natural isomorphisms $\mathcal{G}_{\Gamma_i} \times_{\mathcal{G}_{\Gamma_j}} P_j \cong P_i$ in $\mathfrak{X}_G(\Gamma_i)$ for every $i \to j$ in I. On the other hand, a T-point of $\mathfrak{X}_G(\Gamma)$ is a $\mathcal{G}_\Gamma$-torsor P over T with a $\mathcal{G}_\Gamma$-equivariant morphism $\rho\colon P \to R_G(\Gamma)$. Note that ρ, on T'-points, is given by

$$\rho\colon P(T') \to R_G(\Gamma)(T') = \operatorname{Hom}(\operatorname*{colim}_{i \in I} \Gamma_i, G(T')) = \lim_{i \in I} \operatorname{Hom}(\Gamma_i, G(T'))$$

so ρ is equivalently described by compatible morphisms $\rho_i\colon P \to R_G(\Gamma_i)$ which are $\mathcal{G}_{\Gamma_i}$-equivariant, where $\mathcal{G}_{\Gamma_i}$ acts on P via $\mathcal{G}_\Gamma$.

These two descriptions are related as follows. From the $\mathcal{G}_\Gamma$-torsor P, one constructs the $\mathcal{G}_{\Gamma_i}$-torsors $P_i = \mathcal{G}_{\Gamma_i} \times_{\mathcal{G}_\Gamma} P$, which are naturally compatible. Conversely, from the P_i one constructs $\lim_{i \in I} P_i$, where the limit is taken as schemes over T, which naturally comes with the structure of a $(\lim_{i \in I} \mathcal{G}_{\Gamma_i})$-torsor, and one puts $P = \mathcal{G}_\Gamma \times_{(\lim_{i \in I} \mathcal{G}_{\Gamma_i})} \lim_{i \in I} P_i$. This induces the desired isomorphism between $\lim_{i \in I} \mathfrak{X}_G(\Gamma_i)$ and $\mathfrak{X}_G(\Gamma)$. $\square$

Chapter 3

Motivic invariants

When studying a geometric object, say a compact manifold X, one can try to understand X by means of its invariants. One of the simplest invariants is the *Euler characteristic* of X, a topological invariant, which is an integer $\chi(X) \in \mathbb{Z}$ given by the alternating sum of its Betti numbers

$$\chi(X) = \sum_{k \geq 0} (-1)^k \dim_{\mathbb{C}} H^k(X; \mathbb{C}).$$

There are many ways in which the Euler characteristic can be refined. For instance, when X is (the analytification of) a smooth projective complex variety, the cohomology groups $H^k(X; \mathbb{C})$ admit a Hodge structure by the Hodge decomposition theorem [PS08, Theorem 1.8]. The *Hodge polynomial* of X,

$$P_{\text{Hodge}}(X) = \sum_{p,q \geq 0} \dim_{\mathbb{C}} H^{p,q}(X)\, u^p v^q \in \mathbb{Z}[u, v] \tag{3.1}$$

specializes to the Euler characteristic for $u = v = -1$. One may replace $H^k(X; \mathbb{C})$ by the compactly supported cohomology groups $H^k_c(X; \mathbb{C})$ in order to extend the Euler characteristic to non-compact X. Analogously, as explained in Section 3.1, by work of Deligne [Del71b, Del74] the Hodge polynomial can be extended to an invariant for all complex varieties, possibly non-smooth and non-projective, called the *E-polynomial* $e(X) \in \mathbb{Z}[u, v]$, also known as the *Hodge–Deligne polynomial* or *Serre polynomial*. This invariant is additive and multiplicative in the sense that $e(X) = e(Z) + e(X \setminus Z)$ and $e(X \times Y) = e(X)\, e(Y)$ for all complex varieties X and Y and closed subvarieties $Z \subseteq X$.

The goal of this chapter is to discuss various such invariants, and to give tools for computing them. Our main focus will be on the invariant that takes values in the *Grothendieck ring of varieties*, defined in Section 3.2, which is universal among all additive and multiplicative invariants.

3.1 Mixed Hodge structures

Let X be a complex variety. It was shown by Deligne [Del71a, Del71b] that the singular cohomology groups with compact support $H^k_c(X;\mathbb{Q})$ naturally admit the structure of a mixed Hodge structure. Let us recall the definition of a (mixed) Hodge structure.

Definition 3.1.1. A *Hodge structure of weight* $k \in \mathbb{Z}$ is a pair $(H, F^\bullet H)$ consisting of a finite-dimensional rational vector space H and a decreasing filtration $F^\bullet H$ on $H_\mathbb{C} = H \otimes_\mathbb{Q} \mathbb{C}$,

$$H_\mathbb{C} \supseteq \cdots \supseteq F^p H \supseteq F^{p+1} H \supseteq \cdots \supseteq 0,$$

such that $H_\mathbb{C} = F^p \oplus \overline{F^q}$ for $p+q = k+1$. A morphism of Hodge structures of the same weight $(H, F^\bullet H) \to (H', F^\bullet H')$ is a linear map $f \colon H \to H'$ which preserves the filtration, that is, $f_\mathbb{C}(F^p H) \subseteq F^p H'$ for all p. A *mixed Hodge structure* is a triple $(H, W_\bullet H, F^\bullet H)$ consisting of a finite-dimensional rational vector space H, an increasing filtration $W_\bullet H$ on H, called the *weight filtration*,

$$0 \subseteq \cdots \subseteq W_k H \subseteq W_{k+1} H \subseteq \cdots \subseteq H,$$

and a decreasing filtration $F^\bullet H$ on $H_\mathbb{C}$,

$$H_\mathbb{C} \supseteq \cdots \supseteq F^p H \supseteq F^{p+1} H \supseteq \cdots \supseteq 0,$$

such that the induced filtration of $F^\bullet H$ on the graded pieces $(\operatorname{Gr}^W_k H) \otimes_\mathbb{Q} \mathbb{C} = (W_k H / W_{k+1} H) \otimes_\mathbb{Q} \mathbb{C}$ are Hodge structures of weight k. A morphism of mixed Hodge structures is a linear map which preserves both the increasing and decreasing filtration. The categories of Hodge structures and of mixed Hodge structures are denoted by **HS** and **MHS**, respectively.

Now, more precisely, Deligne showed that the cohomology groups $H^k_c(X;\mathbb{Q})$ and their complexification $H^k_c(X;\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C} = H^k_c(X;\mathbb{C})$ can naturally be equipped with weight filtrations $W_\bullet$ and decreasing filtrations $F^\bullet$, respectively, such that the triples $H^k_c(X) = (H^k_c(X;\mathbb{Q}), W_\bullet, F^\bullet)$ are mixed Hodge structures. Moreover, the construction is functorial in X, agrees with the usual Hodge decomposition when X is smooth and projective, and is compatible with various classical exact sequences in cohomology. For the explicit construction and more details, we refer to [Del71b, Del74, PS08].

There is an exact functor from the category of mixed Hodge structures to the category of finite-dimensional bigraded complex vector spaces [Del71b, Theorem 1.2.10]:

$$\operatorname{Gr}^*_F \operatorname{Gr}^W_* : \mathbf{MHS} \to (\mathbf{Vect}^{\mathbb{Z}\times\mathbb{Z}}_\mathbb{C})_{\text{fin}}, \quad H \mapsto \bigoplus_{p,q\in\mathbb{Z}} \operatorname{Gr}^p_F \operatorname{Gr}^W_{p+q} H_\mathbb{C} \tag{3.2}$$

In the case of a mixed Hodge structure on $H_c^k(X;\mathbb{Q})$, we denote its bigraded pieces by

$$H_c^{k;p,q}(X) = \mathrm{Gr}_F^p \, \mathrm{Gr}_{p+q}^W \, H_c^k(X;\mathbb{C}).$$

In fact, $H_c^{k;p,q}(X)$ is non-zero only if $p, q \geq 0$. The dimensions of these vector spaces can be collected as the coefficients of a polynomial. This way we obtain the following definition, as first introduced in [DK86].

Definition 3.1.2. Let X be a complex variety. The *E-polynomial* of X (also known as the *Hodge–Deligne polynomial* or the *Serre polynomial*) is the polynomial $e(X) \in \mathbb{Z}[u,v]$ given by

$$e(X) = \sum_{k,p,q \in \mathbb{Z}} (-1)^k \dim_{\mathbb{C}} H_c^{k;p,q}(X) \, u^p v^q.$$

In particular, when X is smooth and projective, the E-polynomial $e(X)$ coincides with the Hodge polynomial (3.1), up to the change of signs induced by $u \mapsto -u$ and $v \mapsto -v$.

Amazingly, the E-polynomial is additive and multiplicative, in the sense that

$$e(X) = e(Z) + e(X \setminus Z) \quad \text{and} \quad e(X \times Y) = e(X)\, e(Y) \tag{3.3}$$

for complex varieties X and Y, and $Z \subseteq X$ a closed subvariety. These properties follow from the long exact sequence

$$\cdots \to H_c^k(X \setminus Z;\mathbb{C}) \to H_c^k(X;\mathbb{C}) \to H_c^k(Z;\mathbb{C}) \to H_c^{k+1}(X \setminus Z;\mathbb{C}) \to \cdots \tag{3.4}$$

of mixed Hodge structures [PS08, p.138], and the Künneth formula [Del74, Proposition 8.2.10], respectively, together with the fact that (3.2) is exact.

3.2 Grothendieck ring of varieties

As seen in the previous section, the E-polynomial is an additive and multiplicative invariant (3.3). In this section, we will define the *Grothendieck ring of varieties*: the ring in which the universal invariant, among all additive and multiplicative invariants, takes values. This means, in particular, that when computing the E-polynomial of some complex variety using only these properties, one might as well compute the invariant in the Grothendieck ring of varieties, to obtain a more refined invariant. One of the advantages to the Grothendieck ring of varieties is that, as opposed to other invariants, it can be defined for varieties over any field k, and also more generally in the relative setting for varieties over a base variety S.

The Grothendieck ring of varieties $\mathrm{K}_0(\mathbf{Var}_k)$ was originally introduced in a letter from Grothendieck to Serre [CS01, 16 Aug. 1964], and came with a hypothetical morphism

$$\mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathbf{M}(k)) \tag{3.5}$$

to the 'Grothendieck group of the abelian category of motives'. For this reason, we refer to these invariants as *motivic invariants*. To gain some understanding about this morphism, we first introduce the Grothendieck group of an abelian or triangulated category.

Definition 3.2.1. The *Grothendieck group* of an abelian category $\mathcal{A}$, denoted $\mathrm{K}_0(\mathcal{A})$, is the free abelian group on isomorphism classes $[A]$ of objects A of $\mathcal{A}$, modulo the relations

$$[B] = [A] + [C]$$

for all short exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{A}$. Similarly, the *Grothendieck group* of a triangulated category $\mathcal{A}$, also denoted $\mathrm{K}_0(\mathcal{A})$, is the free abelian group on isomorphism classes $[A]$ of objects A of $\mathcal{A}$, modulo the relations

$$[B] = [A] + [C]$$

for all distinguished triangles $A \to B \to C \to A[1]$ in $\mathcal{A}$. When $\mathcal{A}$ is a tensor triangulated category, the tensor product $\otimes$ induces the structure of a commutative ring on $\mathrm{K}_0(\mathcal{A})$ given on generators by

$$[A][B] = [A \otimes B].$$

Remark 3.2.2. The Grothendieck group of an abelian category $\mathcal{A}$ is naturally isomorphic to that of its derived category $D^b(\mathcal{A})$ as triangulated category. In particular, the functor $\mathcal{A} \to D^b(\mathcal{A})$, which assigns to any object A the complex with A concentrated in degree 0, induces a morphism $\mathrm{K}_0(\mathcal{A}) \to \mathrm{K}_0(D^b(\mathcal{A}))$. It is an easy exercise in homological algebra to show that an inverse is given by $[A^\bullet] \mapsto \sum_{i \in \mathbb{Z}} (-1)^i [H^i(A^\bullet)]$.

Even though the category of varieties is neither abelian nor triangulated (not even additive), the Grothendieck ring of varieties is defined similarly, where exact sequences are replaced by closed immersions with open complements. For many invariants, these notions can be related through a long exact sequence such as (3.4).

Definition 3.2.3. Let S be a variety over a field k. The *Grothendieck ring of varieties over S*, denoted $\mathrm{K}_0(\mathbf{Var}_S)$, is the free abelian group on isomorphism classes $[X]$ of varieties X over S, modulo the relations

$$[X] = [Z] + [X \setminus Z]$$

for all closed immersions $Z \to X$ of varieties over S. It admits the structure of a commutative ring, where multiplication is given on generators by

$$[X][Y] = [(X \times_S Y)^{\mathrm{red}}].$$

In particular, the classes $[\varnothing]$ and $[S]$ are the zero and unit of this ring, respectively. For any variety X over S, the element $[X]$ in $\mathrm{K}_0(\mathbf{Var}_S)$ is also known as the *virtual class* of X.

Remark 3.2.4. Although the Grothendieck ring is generated by isomorphism classes of varieties X, one could allow for X to be non-reduced without affecting the ring. Indeed, $X^{\mathrm{red}} \subseteq X$ is a closed subscheme with complement $\varnothing$, so that $[X^{\mathrm{red}}] = [X]$. Also, in this case, one can define multiplication simply by $[X][Y] = [X \times_S Y]$. Similarly, we can omit the condition that X be separated since any scheme X of finite type over S can be partitioned into finitely many separated subschemes $X_1, \dots, X_n$, so that $[X] = [X_1] + \cdots + [X_n]$. However, we cannot permit any X which is not quasi-compact over S. For example, if $X = \bigsqcup_{\mathbb{Z}} S$ and $Z = S$, then $X \setminus Z \cong X$ which would imply $1 = [Z] = 0$, collapsing the ring to the trivial ring. Indeed, Grothendieck originally defined his ring allowing for isomorphism classes of all schemes X of finite type over S [CS01, 16 Aug. 1964].

Notation 3.2.5. To distinguish between virtual classes over different bases, we sometimes write $[X]_S$ to emphasize the virtual class lives in $\mathrm{K}_0(\mathbf{Var}_S)$. When the base S is clear from context, or when $S = \operatorname{Spec} k$, we simply write $[X]$.

Definition 3.2.6. The virtual class of the affine line $\mathbb{A}^1_S$ over S in $\mathrm{K}_0(\mathbf{Var}_S)$ is called *Lefschetz class* and is denoted by $\mathbb{L}$.

Example 3.2.7.
- The virtual class of affine n-space is $[\mathbb{A}^n_S]_S = \mathbb{L}^n$ for any $n \geq 0$.
- Since $\mathbb{P}^n_S \setminus \mathbb{P}^{n-1}_S \cong \mathbb{A}^n_S$, it follows by induction on n that $[\mathbb{P}^n_S]_S = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + 1$ for all $n \geq 0$.

Example 3.2.8. The following invariants are additive and multiplicative, and hence factor through the Grothendieck ring of varieties.

- For $S = \operatorname{Spec} \mathbb{C}$, the E-polynomial (see Definition 3.1.2) factors through $\mathrm{K}_0(\mathbf{Var}_{\mathbb{C}})$, which gives a ring morphism
$$e \colon \mathrm{K}_0(\mathbf{Var}_{\mathbb{C}}) \to \mathbb{Z}[u, v], \quad [X] \mapsto e(X). \tag{3.6}$$
- For any point $\operatorname{Spec} \mathbb{F}_q \to S$, one can count $\mathbb{F}_q$-rational points
$$\#_{\mathbb{F}_q} \colon \operatorname{Ob}(\mathbf{Var}_S) \to \mathbb{Z}, \quad X \mapsto |X(\mathbb{F}_q)|.$$
This map, being additive and multiplicative, factors through $\mathrm{K}_0(\mathbf{Var}_S)$.

- Let $S = \operatorname{Spec} k$ for a field k with $\operatorname{char}(k) = 0$. Then there is a ring morphism

  $$\mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q}))$$

  to the Grothendieck group of the $\mathbb{Q}$-linearization of Voevodsky's triangulated category of effective geometric motives [BD07, Appendix A]. This morphism sends the virtual class $[X]$ of a variety X over k to the class $[M^c_{\mathrm{gm}}(X)]$ of its motive with compact support. Viewing $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{Q})$ as a substitute for the derived category of the 'abelian category of motives', this map would be the morphism (3.5) that Grothendieck had in mind in his letter.

- Again, let $S = \operatorname{Spec} k$ for a field k with $\operatorname{char}(k) = 0$. Then there is a ring morphism

  $$\mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathbf{CHMot}_k)$$

  to the Grothendieck group of the category of Chow motives over k with rational coefficients [GN02, (5.5)].

Let us describe some formal and functorial properties of the Grothendieck ring of varieties. Given a morphism $f \colon X \to Y$ of varieties over S, there is an induced ring morphism

$$f^* \colon \mathrm{K}_0(\mathbf{Var}_Y) \to \mathrm{K}_0(\mathbf{Var}_X), \quad [W]_Y \mapsto [W \times_Y X]_X.$$

Indeed, this map is well-defined since, for any variety W over Y and closed subvariety $Z \subseteq W$, we have $[W \times_Y X]_X = [Z \times_Y X]_X + [(W \setminus Z) \times_Y X]_X$. Similarly, f^* respects multiplication as $(W \times_Y W') \times_Y X \cong (W \times_Y X) \times_X (W' \times_Y X)$ for any varieties W and W' over Y. The morphism f^* turns $\mathrm{K}_0(\mathbf{Var}_X)$ into a $\mathrm{K}_0(\mathbf{Var}_Y)$-algebra, and in particular $\mathrm{K}_0(\mathbf{Var}_X)$ is a $\mathrm{K}_0(\mathbf{Var}_S)$-algebra for every variety X over S. Moreover, f^* is a morphism of $\mathrm{K}_0(\mathbf{Var}_S)$-algebras.

Similarly, the morphism $f \colon X \to Y$ induces a map

$$f_! \colon \mathrm{K}_0(\mathbf{Var}_X) \to \mathrm{K}_0(\mathbf{Var}_Y), \quad [W]_X \mapsto [W]_Y$$

which is a morphism of $\mathrm{K}_0(\mathbf{Var}_S)$-modules. However, note that $f_!$ is generally not a morphism of rings.

Remark 3.2.9. The maps f^* and $f_!$ can more generally be seen as functors

$$\mathbf{Var}_X \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathbf{Var}_Y$$

given by pulling back along f and post-composing with f, respectively, forming an adjoint pair $f_! \dashv f^*$. Indeed, for any varieties U over X and V over Y there is a natural bijection

$$\mathrm{Hom}_Y(U, V) \cong \mathrm{Hom}_X(U, V \times_Y X).$$

Example 3.2.10. Let X and Y be varieties over S. There is a natural morphism of $\mathrm{K}_0(\mathbf{Var}_S)$-algebras

$$\mathrm{K}_0(\mathbf{Var}_X) \otimes_{\mathrm{K}_0(\mathbf{Var}_S)} \mathrm{K}_0(\mathbf{Var}_Y) \to \mathrm{K}_0(\mathbf{Var}_{X\times_S Y})$$

given, on generators, by $[U]_X \otimes [V]_Y \mapsto [U \times_S V]_{X\times_S Y}$ for all varieties U over X and V over Y. This map is generally not surjective. For example, let $X = Y = \mathbb{A}^1_k$ over $S = \operatorname{Spec} k$ for a finite field $k = \mathbb{F}_q$. Consider the class $[\Delta_{\mathbb{A}^1_k}]$ of the diagonal in $X \times Y = \mathbb{A}^2_k$, and suppose $[\Delta_{\mathbb{A}^1_k}]$ is equal to the image of $\sum_{i=1}^n u_i \otimes v_i$ under this map for some $u_i, v_i \in \mathrm{K}_0(\mathbf{Var}_{\mathbb{A}^1_k})$. Note that every $\mathbb{F}_{q^m}$-rational point $x \in \mathbb{A}^1_k(\mathbb{F}_{q^m})$ induces a ring morphism

$$\#_x \colon \mathrm{K}_0(\mathbf{Var}_{\mathbb{A}^1_k}) \to \mathbb{Z}, \quad [U \xrightarrow{f} \mathbb{A}^1_k] \mapsto |f^{-1}(x)|$$

counting the number of $\mathbb{F}_{q^m}$-rational points in the fiber over x. The same construction works for $\mathbb{A}^2_k$, and together they induce the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{K}_0(\mathbf{Var}_{\mathbb{A}^1_k}) \otimes_{\mathrm{K}_0(\mathbf{Var}_k)} \mathrm{K}_0(\mathbf{Var}_{\mathbb{A}^1_k}) & \longrightarrow & \mathrm{K}_0(\mathbf{Var}_{\mathbb{A}^2_k}) \\
\downarrow & & \downarrow \\
\left(\prod_{x\in\mathbb{A}^1_k(\mathbb{F}_{q^m})} \mathbb{Z}\right) \otimes \left(\prod_{x\in\mathbb{A}^1_k(\mathbb{F}_{q^m})} \mathbb{Z}\right) & & \prod_{x\in\mathbb{A}^2_k(\mathbb{F}_{q^m})} \mathbb{Z} \\
\| & & \| \\
\mathbb{Z}^{q^m} \otimes \mathbb{Z}^{q^m} & \xrightarrow{\sim} & \mathbb{Z}^{q^m\times q^m}
\end{array}$$

Now, the image of $[\Delta_{\mathbb{A}^1_k}]$ in $\mathbb{Z}^{q^m\times q^m}$ corresponds to the $q^m \times q^m$ identity matrix, which has rank q^m, while the image of $\sum_{i=1}^n u_i \otimes v_i$ has rank at most n. This yields a contradiction for sufficiently large m.

3.3 Stratifications and fibrations

Let S be a variety over a field k.

Definition 3.3.1. Let X be a variety over S. A *stratification* of X is a collection of disjoint locally closed subvarieties $\{X_i\}_{i\in I}$ of X such that $X = \bigcup_{i\in I} X_i$.

Lemma 3.3.2 ([Bri12, Lemma 2.2])**.** *Let X be a variety over S and $\{X_i\}_{i\in I}$ a stratification of X. Then only finitely many of the X_i are non-empty and $[X] = \sum_{i\in I}[X_i]$ in $\mathrm{K}_0(\mathbf{Var}_S)$.*

Proof. Proof by induction on the dimension of X. If $\dim X = 0$, then X is a finite set of points, and the result is clear. Now assume that $\dim X > 0$ and that the result holds for all varieties of dimension less than $\dim X$.

We prove the result for X by induction on the number of irreducible components of X. If this number is 1, that is, X is irreducible, then some $U = X_i$ contains the generic point of X and is therefore open. The complement $Z = X \setminus U$ is of smaller dimension than X and is stratified by the other X_i. Since $[X] = [Z]+[U]$, the result follows from the induction hypothesis (on $\dim X$).

Now suppose that X is reducible. Take an irreducible component and remove the intersections with the other irreducible components, which gives an irreducible open subset $U \subseteq X$. The complement $Z = X \setminus U$ is a closed subvariety with fewer irreducible components than X. Note that $\{Z \cap X_i\}_{i\in I}$ and $\{U \cap X_i\}_{i\in I}$ are stratifications of Z and U, respectively, so only finitely many strata are non-empty, and we have

$$[U] = \sum_{i\in I}[U \cap X_i] \quad \text{and} \quad [Z] = \sum_{i\in I}[Z \cap X_i].$$

This follows, if $\dim Z$ (resp. $\dim U$) is less than $\dim X$, from the induction hypothesis on the dimension, or, if $\dim Z$ (resp. $\dim U$) is equal to $\dim X$, from the induction hypothesis on the number of irreducible components. Finally, $[X_i] = [U \cap X_i] + [Z \cap X_i]$ implies that $[X] = [U] + [Z] = \sum_{i\in I}[X_i]$. □

Lemma 3.3.3. *Let $f\colon Y \to X$ be a fiber bundle of varieties over S with fiber F which is locally trivial in the Zariski topology. That is, there exists an open cover $Y = \cup_{i\in I}U_i$ such that $f^{-1}(U_i)$ is isomorphic to $F \times U_i$ over U_i for each $i \in I$. Then $[Y]_S = [F] \cdot [X]_S$ in $\mathrm{K}_0(\mathbf{Var}_S)$.*

Proof. From the given open cover, we construct a stratification of Y as follows. Let $Z_0 = Y$ and inductively construct Z_{j+1} for $j \geq 0$: if $Z_j \neq \varnothing$, there exists some $i \in I$ such that $Z_j \cap U_i \neq \varnothing$, and set $Z_{j+1} = Z_j \setminus (Z_j \cap U_i)$. As Y is noetherian, this results in a finite descending chain of closed sets

$$Y = Z_0 \supsetneq Z_1 \supsetneq \ldots \supsetneq Z_n \supsetneq Z_{n+1} = \varnothing$$

and the locally closed sets $Y_j = Z_j \setminus Z_{j+1}$ for $j = 0, 1, \ldots, n$ form a stratification of Y. Moreover, since $Y_j \subseteq U_i$ for some i by construction, f is trivial over each Y_j, that is, $f^{-1}(Y_j) \cong F \times Y_j$. Using Lemma 3.3.2 we conclude

$$[X]_S = \sum_{j=0}^{n} \left[f^{-1}(Y_j)\right]_S = \sum_{j=0}^{n}[F] \cdot [Y_j]_S = [F] \cdot [Y]_S. \qquad \square$$

Example 3.3.4. For any $n \geq 0$, the natural morphism $\mathbb{A}^{n+1}_S \setminus \{0\} \to \mathbb{P}^n_S$ is a fiber bundle with fiber $\mathbb{G}_m$ which is locally trivial in the Zariski topology. Indeed, we have

$$[\mathbb{G}_m] \cdot [\mathbb{P}^n_S]_S = (\mathbb{L} - 1)(\mathbb{L}^n + \cdots + \mathbb{L} + 1) = \mathbb{L}^{n+1} - 1 = [\mathbb{A}^{n+1}_S \setminus \{0\}]_S.$$

Example 3.3.5. Consider the general linear group GL_n of rank n over a field k. The morphism $\mathrm{GL}_n \to \operatorname{Spec} k$ factors as

$$\mathrm{GL}_n = Y_n \to Y_{n-1} \to \cdots \to Y_0 = \operatorname{Spec} k$$

where $Y_m \subseteq \mathrm{Mat}_{n\times m}$ denotes the locally closed subvariety of m-linearly independent vectors over k, and the morphisms are given by forgetting the last vector. Now, for any $m = 1, \dots, n$, the variety Y_m can be regarded as the open complement in $Y_{m-1} \times \mathbb{A}^n_k$ of the closed subvariety $Y_{m-1} \times \mathbb{A}^{m-1}_k$. In particular, $[Y_m] = (\mathbb{L}^n - \mathbb{L}^{m-1})[Y_{m-1}]$. Therefore, by induction on m, we obtain

$$[\mathrm{GL}_n] = \prod_{m=1}^{n} (\mathbb{L}^n - \mathbb{L}^{m-1}).$$

Proposition 3.3.6. *Let S be a variety with stratification $\{S_i\}_{i\in I}$ and write $f_i \colon S_i \to S$ for the immersion of S_i into S. Then the map*

$$\mathrm{K}_0(\mathbf{Var}_S) \to \bigoplus_{i\in I} \mathrm{K}_0(\mathbf{Var}_{S_i}), \quad X \mapsto (f_i^* X)_{i\in I}$$

is an isomorphism of $\mathrm{K}_0(\mathbf{Var}_k)$-algebras.

Proof. Every f_i^* is a morphism of $\mathrm{K}_0(\mathbf{Var}_k)$-algebras, so this map is as well. Its inverse is given by $(X_i)_{i\in I} \mapsto \sum_{i\in I} (f_i)_! X_i$, which is well-defined because only finitely many S_i are non-empty by Lemma 3.3.2. Indeed, it is a right inverse to the given map because

$$f_i^*(f_j)_! = \begin{cases} \mathrm{id}_{\mathrm{K}_0(\mathbf{Var}_{S_i})} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It is a left inverse because any variety T over S is stratified by $\{T \times_S S_i\}_{i\in I}$, so that

$$[T]_S = \sum_{i\in I} [T \times_S S_i]_S = \sum_{i\in I} (f_i)_! f_i^* [T]_S. \qquad \square$$

Notation 3.3.7. For any $X \in \mathrm{K}_0(\mathbf{Var}_S)$, we will write $X|_{S_i} \in \mathrm{K}_0(\mathbf{Var}_{S_i})$ for the components of the image of X under this isomorphism.

Inclusion-exclusion matrix

Let X be a variety over S with stratification $\{X_i\}_{i\in I}$ and let Y be a variety over X. The goal of this subsection is to show that, in order to compute the virtual classes $[Y \times_X X_i]$ in $\mathrm{K}_0(\mathbf{Var}_S)$ for all i, it is sufficient to compute the virtual classes $[Y \times_X \overline{X}_i]$ for all i instead, where $\overline{X}_i$ denotes the Zariski closure of X_i in X, making use of an inclusion-exclusion principle.

Example 3.3.8. Suppose X is stratified by a closed subvariety $X_0 \subseteq X$ and its open complement $X_1 = X \backslash X_0$. If we were to compute $[X_0]$ and $[X_1]$ in $\mathrm{K}_0(\mathbf{Var}_S)$, computing the latter would likely result in the computation $[X]-[X_0]$, so that the result of the computation of $[X_0]$ can be reused. Therefore, instead of computing $[X_0]$ and $[X_1]$, one can compute $[\overline{X}_0] = [X_0]$ and $[\overline{X}_1] = [X]$, from which formally follows that $[X_1] = [\overline{X}_1] - [\overline{X}_0]$.

Lemma 3.3.9. *Let X be a variety over S with stratification $\{X_i \neq \varnothing\}_{i \in I}$. Then $\overline{X}_i = \overline{X}_j$ if and only if $i = j$.*

Proof. For each $i \in I$, write $X_i = Z_i \cap U_i$ for some closed $Z_i \subseteq X$ and open $U_i \subseteq X$. Without loss of generality, we may assume $Z_i = \overline{X}_i$. Now, if $\overline{X}_i = \overline{X}_j$ for some $i, j \in I$, then both X_i and X_j are open and dense in $\overline{X}_i = \overline{X}_j \neq \varnothing$, so they must intersect. But this contradicts the assumption that X_i and X_j are disjoint, since they are part of the stratification. □

Definition 3.3.10. Let X be a variety over S with a finite stratification $\{X_i \neq \varnothing\}_{i \in I}$. Put a partial order on I where $i \leq j$ if and only if $\overline{X}_i \subseteq \overline{X}_j$. Reflexivity and transitivity are clear, and anti-symmetry follows from the above lemma. The virtual classes $[X_i]$ and $[\overline{X}_i]$ in $\mathrm{K}_0(\mathbf{Var}_S)$ are now linearly related through

$$[\overline{X}_i] = \sum_{j \in I} A_{ij} [X_j]$$

where $A_{ij} = 1$ for $j \leq i$ and $A_{ij} = 0$ for $i < j$. Hence, the A_{ij} define a linear map $A \colon \mathbb{Z}^I \to \mathbb{Z}^I$ with determinant 1. The inverse $C = A^{-1}$ is called the *inclusion-exclusion matrix* of the stratification, and satisfies

$$[X_i] = \sum_{j \in I} C_{ij} [\overline{X}_j].$$

Corollary 3.3.11. *Let X be a variety over S with finite stratification $\{X_i\}_{i \in I}$ and corresponding inclusion-exclusion matrix C. Then for any variety Y over X, we have*

$$[Y \times_X X_i]_S = \sum_{j \in I} C_{ij} [Y \times_X \overline{X}_j]_S. \qquad \square$$

Special algebraic groups

Special algebraic groups were first introduced by Serre [Ser58]. In this section, we describe some basic properties of these groups, and show why they are extremely useful in the context of computing virtual classes.

Definition 3.3.12. An algebraic group G over a field k is *special* if any G-torsor in the étale topology is locally trivial in the Zariski topology.

Lemma 3.3.13. *Let G be a special algebraic group. Then for every G-torsor of varieties $P \to X$ in the étale topology over S, we have $[P]_S = [G] \cdot [X]_S$ in* $\mathrm{K}_0(\mathbf{Var}_S)$.

Proof. The G-torsor $P \to X$ is Zariski-locally trivial, so the result follows from Lemma 3.3.3. □

Example 3.3.14. In general, the equality $[P]_S = [G] \cdot [X]_S$ fails to hold when G is not special. Consider for instance the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ and the G-torsor $P = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \to \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} = X$ given by $x \mapsto x^n$. Then $[P] = \mathbb{L} - 1 \neq n(\mathbb{L} - 1) = [G][X]$ for $n \geq 2$, showing $\mathbb{Z}/n\mathbb{Z}$ is not special for $n \geq 2$.

Corollary 3.3.15 (Motivic orbit-stabilizer theorem)**.** *Let G be an algebraic group over k acting on a variety X. For any point $\xi \in X(k)$, if the stabilizer* $\mathrm{Stab}(\xi)$ *is special, then*

$$[G] = [\mathrm{Stab}(\xi)][\mathrm{Orbit}(\xi)]$$

in $\mathrm{K}_0(\mathbf{Var}_k)$.

Proof. Since the map $G \to \mathrm{Orbit}(\xi)$ given by $g \mapsto g \cdot \xi$ is a $\mathrm{Stab}(\xi)$-torsor, the result follows from Lemma 3.3.13. □

Proposition 3.3.16. *Let $1 \to N \hookrightarrow G \xrightarrow{\pi} H \to 1$ be an exact sequence of algebraic groups.*

(i) If N and H are special, then so is G.

(ii) If the sequence splits and G is special, then so is H.

(iii) If the sequence splits and G is special, then so is N.

Proof. (i) Any G-torsor $X \to S$ can be written as the composite of the N-torsor $X \to X/N$ and the H-torsor $X/N \to X/G \cong S$. As H is special, there exist opens $S_i \subseteq S$ such that $(X/N) \times_S S_i \cong H \times S_i$. Pulling back the N-torsor $X \times_S S_i \to H \times S_i$ along $S_i \xrightarrow{(1,\mathrm{id})} H \times S_i$ gives an N-torsor $Y_i \to S_i$, which is also Zariski-locally trivial as N is special. Hence, there exist opens $S_{ij} \subseteq S_i$ such that $Y_i \times_{S_i} S_{ij} \cong N \times S_{ij}$. There is now a natural morphism $G \times S_{ij} \to X \times_S S_{ij}$ of G-torsors over S_{ij}, which must be an isomorphism. Therefore, $X \to S$ is Zariski-locally trivial.

(ii) As sequence splits, there exists a section $\sigma \colon H \to G$ to π, i.e., $\pi \circ \sigma = \mathrm{id}_H$. Let $X \to S$ be an H-torsor, and consider the G-torsor $G \times_H X := (G \times X)/H \to S$, where H acts on $G \times X$ via $h \cdot (g, x) = (g\sigma(h)^{-1}, h \cdot x)$. This G-torsor factors

through the N-torsor $G \times_H X \to X$ given by $(g, x) \mapsto \pi(g) \cdot x$. Hence, any trivialization of $G \times_H X$ induces a trivialization of X, and such a trivialization exists as G is special.

(iii) Let $X \to S$ be an N-torsor, and consider the G-torsor $G \times_N X \coloneqq (G \times X)/N \to S$, where N acts on $G \times X$ via $n \cdot (g, x) = (gn^{-1}, n \cdot x)$. As G is special, there exist opens $S_i \subseteq S$ and G-equivariant isomorphism $\varphi_i \colon (G \times_N X) \times_S S_i \to G \times S_i$. These induce N-equivariant isomorphisms $X \times_S S_i \to N \times S_i$ given by $(x, s) \mapsto (g\,\sigma(\pi(g^{-1})), s)$, where $(g, s) = \varphi_i((1, x), s)$, showing the S_i also trivialize X. □

Example 3.3.17.
- By Hilbert's Theorem 90, the general linear groups GL_n are special over any field k [Mil80, Proposition III.4.9, Lemma III.4.10].
- The exact sequence $1 \to \mathrm{SL}_n \to \mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_m \to 1$ splits, so it follows from Proposition 3.3.16 *(iii)* that SL_n is also special over any field k.
- The additive group $\mathbb{G}_a$ is special over any field k [Mil80, Proposition III.3.7].
- The projective linear group PGL_n is not special for $n \geq 2$. In fact, the PGL_n-torsors over a variety X which are not Zariski-locally trivial are classified by the Brauer group of X, which is in general non-trivial [Mil80, IV §2].

3.4 Algorithmic computations

Let k be a field. In this section, we describe, from a practical and computational point of view, various strategies for computing the virtual class of varieties in $\mathrm{K}_0(\mathbf{Var}_k)$, in terms of the classes of some simple varieties, such as $\mathbb{L} = [\mathbb{A}^1_k]$. These strategies are combined in a recursive algorithm, Algorithm 3.4.3. We remark already that the algorithm will not be a general recipe for computing virtual classes in $\mathrm{K}_0(\mathbf{Var}_k)$: it is allowed to fail. In fact, whenever the algorithm does not fail, it will return the virtual class of the given variety as a polynomial in $\mathbb{L}$. Of course, there exist varieties whose virtual class is not of this form, but it turns out that this algorithm is sufficiently general for the purposes of the later chapters.

In order to algorithmically manipulate varieties, we will encode them as follows. While not all varieties can be encoded in such a way, this should not be too much of a restriction since any variety can be stratified into varieties of this form.

Notation 3.4.1. Let $A = \{x_1, \ldots, x_n\}$ be a finite set, and let F and G be finite subsets of $k[A]$. Then we write

$$X(A, F, G)$$

for the reduced locally closed subvariety of $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ given by $f = 0$ for all $f \in F$ and $g \neq 0$ for all $g \in G$.

Furthermore, we will introduce a notation for the evaluation of polynomials.

Notation 3.4.2. Given an element $x \in A$ and polynomials $f \in k[A]$ and $u \in k[A \setminus \{x\}]$, denote by $\operatorname{eval}^x_u(f)$ the evaluation of f in $x = u$. For polynomials $u, v \in k[A \setminus \{x\}]$, write $\operatorname{eval}^x_{u/v}(f)$ for the evaluation of f in $x = u/v$ multiplied by $v^{\deg_x(f)}$, so that $\operatorname{eval}^x_{u/v}(f) \in k[A \setminus \{x\}]$. Similarly, for subsets $F \subseteq k[A]$, write $\operatorname{eval}^x_u(F) = \{\operatorname{eval}^x_u(f) : f \in F\}$ and $\operatorname{eval}^x_{u/v}(F) = \{\operatorname{eval}^x_{u/v}(f) : f \in F\}$.

An implementation of this algorithm can be found at [Vog22].

Algorithm 3.4.3. Input: Finite sets A, F and G as in Notation 3.4.1.

Output: The virtual class $[X] \in \mathrm{K}_0(\mathbf{Var}_k)$ of $X = X(A, F, G)$ as a polynomial in $\mathbb{L} = [\mathbb{A}^1_k]$.

1. If F contains a non-zero constant or if $0 \in G$, then $X = \varnothing$, so return $[X] = 0$.

2. If $F = G = \varnothing$ or $A = \varnothing$, then $X = \mathbb{A}^{|A|}_k$, so return $[X] = \mathbb{L}^{|A|}$.

3. If $F, G \subseteq k[A \setminus \{x\}]$ for some $x \in A$, then $X \cong \mathbb{A}^1_k \times X'$ with $X' = X(A \setminus \{x\}, F, G)$, so return $[X] = \mathbb{L}\,[X']$.

4. If $f = u^n$ (with $n > 1$) for some $f \in F$ and $u \in k[A]$, then we replace f with u without changing X, that is, $X = X(A, (F \setminus \{f\}) \cup \{u\}, G)$. Similarly, if $g = u^n$ (with $n > 1$) for some $g \in G$ and $u \in k[A]$, then $X = X(S, F, (G \setminus \{g\}) \cup \{u\})$. Continue with this new presentation.

5. If $f \in k[x]$ for some $f \in F$ and $x \in A$, and if f factors as $f = c(x - a_1) \cdots (x - a_m)$ for some $c \in k^\times$ and $a_i \in k$, then return $[X] = \sum_{i=1}^m [X_i]$ with
$$X_i = X\left(A \setminus \{x\}, \operatorname{eval}^x_{a_i}(F \setminus \{f\}), \operatorname{eval}^x_{a_i}(G)\right).$$

6. Suppose $f = uv$ for some $f \in F$ and non-constant $u, v \in k[A]$. Then X is stratified by its closed subvariety given by $u = 0$ and its open complement given by $u \neq 0$ and $v = 0$. Hence, return $[X] = [X_1] + [X_2]$ with
$$X_1 = X(A, (F \setminus \{f\}) \cup \{u\}, G),$$
$$X_2 = X(A, (F \setminus \{f\}) \cup \{v\}, G \cup \{u\}).$$

7. Suppose $f = ux + v$ for some element $x \in A$ and polynomials $f \in F$ and $u, v \in k[A \setminus \{x\}]$, with u non-zero. Then X is stratified by its closed

subvariety given by $u = v = 0$ and its open complement given by $u \neq 0$ and $a = -v/u$. Hence, return $[X] = [X_1] + [X_2]$ with

$$X_1 = X(A, (F \setminus \{f\}) \cup \{u, v\}, G),$$
$$X_2 = X(A, \mathrm{eval}_a(F \setminus \{f\}, -v/u), \mathrm{eval}_a(G, -v/u) \cup \{u\}).$$

8. Suppose $\mathrm{char}(k) \neq 2$ and $f = ux^2 + vx + w$ for some element $x \in A$ and polynomials $f \in F$ and $u, v, w \in k[A \setminus \{a\}]$ with u non-zero. Moreover, suppose that the discriminant $D = v^2 - 4uw$ is a square, that is, $D = d^2$ for some $d \in k[A \setminus \{a\}]$. Then return $[X] = [X_1] + [X_2] + [X_3] + [X_4]$, where X is stratified by the following varieties:

$$X_1 = X(A, (F \setminus \{f\}) \cup \{u, vx + w\}, G),$$
$$X_2 = X(A, \mathrm{eval}^x_{-v/2u}(F \setminus \{f\}) \cup \{d\}, \mathrm{eval}^x_{-v/2u}(G) \cup \{u\}),$$
$$X_3 = X(A, \mathrm{eval}^x_{(-v-d)/2u}(F \setminus \{f\}), \mathrm{eval}^x_{(-v-d)/2u}(G) \cup \{u, d\}),$$
$$X_4 = X(A, \mathrm{eval}^x_{(-v+d)/2u}(F \setminus \{f\}), \mathrm{eval}^x_{(-v+d)/2u}(G) \cup \{u, d\}).$$

9. If $G \neq \varnothing$, pick any $g \in G$, and return $[X] = [X_1] - [X_2]$ with

$$X_1 = X(A, F, G \setminus \{g\}),$$
$$X_2 = X(A, F \cup \{g\}, G).$$

10. If none of the above rules apply, fail.

Remark 3.4.4. Of course, it is possible to replace Step 10 with:

10'. If none of the above rules apply, create a new symbol for the variety $X(A, F, G)$ and return that.

However, this raises the question of what it means to 'compute the virtual class' of a variety. For the purpose of computing motivic invariants, an expression for the virtual class of a variety in terms of the classes of other varieties is only useful if the motivic invariants of those other varieties are known. As far as the applications in this thesis go, the varieties to which this algorithm will be applied all have a virtual class that is a polynomial in $\mathbb{L}$.

3.5 Grothendieck ring of stacks

In order to study motivic invariants of stacks, we would like to have an analogue of the Grothendieck ring of varieties for stacks. A number of constructions have been proposed by various authors, such as in [Joy07, Toë05, BD07]. We will follow the construction by Ekedahl as in [Eke09a, Eke09b], since its definition is closest to Definition 3.2.3. As in Definition 1.6.4, we will restrict to algebraic (Artin) stacks which are of finite type over a field with affine stabilizers.

Definition 3.5.1. Let $\mathfrak{S}$ be an algebraic stack of finite type over a field k with affine stabilizers. The *Grothendieck ring of stacks over* $\mathfrak{S}$, denoted $\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{S}})$, is the free abelian group on isomorphism classes of algebraic stacks of finite type over $\mathfrak{S}$ with affine stabilizers, modulo the relations

(1) $[\mathfrak{X}] = [\mathfrak{Z}] + [\mathfrak{X} \setminus \mathfrak{Z}]$ for all closed immersions $\mathfrak{Z} \to \mathfrak{X}$ over $\mathfrak{S}$,

(2) $[\mathfrak{E}] = \mathbb{L}^n[\mathfrak{X}]$ for any vector bundle $\mathfrak{E}$ over $\mathfrak{X}$ of rank n, where $\mathbb{L} = [\mathbb{A}^1_{\mathfrak{S}}]$.

Multiplication is given on generators by $[\mathfrak{X}][\mathfrak{Y}] = [\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}]$.

Remark 3.5.2. If S is a variety over k, then the inclusion $\mathbf{Var}_S \to \mathbf{Stck}_S$ induces a ring morphism

$$\mathrm{K}_0(\mathbf{Var}_S) \to \mathrm{K}_0(\mathbf{Stck}_S).$$

In particular, any relation that holds in $\mathrm{K}_0(\mathbf{Var}_S)$ also holds in $\mathrm{K}_0(\mathbf{Stck}_S)$.

Example 3.5.3. Consider the classifying stack $\mathrm{B}\mathbb{G}_m = [\mathrm{Spec}(k)/\mathbb{G}_m]$ over a field k. The natural morphism $[\mathbb{A}^1_k/\mathbb{G}_m] \to \mathrm{B}\mathbb{G}_m$ is a vector bundle of rank one, so $[\mathbb{A}^1_k/\mathbb{G}_m] = \mathbb{L}[\mathrm{B}\mathbb{G}_m]$ in $\mathrm{K}_0(\mathbf{Stck}_k)$. On the other hand, the closed subscheme $\mathrm{B}\mathbb{G}_m \subseteq [\mathbb{A}^1_k/\mathbb{G}_m]$, given by the origin, yields the relation $[\mathbb{A}^1_k/\mathbb{G}_m] = [\mathrm{B}\mathbb{G}_m] + [\mathbb{G}_m/\mathbb{G}_m] = [\mathrm{B}\mathbb{G}_m]+1$. Therefore, $(\mathbb{L}-1)[\mathrm{B}\mathbb{G}_m] = 1$ and hence $[\mathrm{B}\mathbb{G}_m]$ is invertible with inverse $(\mathbb{L} - 1) = [\mathbb{G}_m]$.

The above example can be generalized to other algebraic groups, and more general quotient stacks. The following proposition treats the case $G = \mathrm{GL}_n$.

Proposition 3.5.4. *For any $n \geq 0$, the element $[\mathrm{GL}_n]$ in $\mathrm{K}_0(\mathbf{Stck}_k)$ is invertible, and $[\mathfrak{X}/\mathrm{GL}_n]_{\mathfrak{S}} = [\mathrm{GL}_n]^{-1} \cdot [\mathfrak{X}]_{\mathfrak{S}}$ in $\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{S}})$ for any algebraic stack $\mathfrak{X}$ over $\mathfrak{S}$ in $\mathbf{Stck}_{\mathfrak{S}}$ with an action of GL_n such that the map $\mathfrak{X} \to \mathfrak{S}$ is G-invariant.*

Proof. As in Example 3.3.5, for any $0 \leq m \leq n$, let $Y_m \subseteq \mathrm{Mat}_{n\times m}$ be the subscheme of m-linearly independent vectors. The group GL_n acts naturally on each Y_m, and we construct $\mathfrak{Y}_m = [(Y_m \times \mathfrak{X})/\mathrm{GL}_n]$. Now, the quotient $\mathfrak{X} \to [\mathfrak{X}/\mathrm{GL}_n]$ factors as

$$\mathfrak{X} = \mathfrak{Y}_n \to \mathfrak{Y}_{n-1} \to \cdots \to \mathfrak{Y}_0 = [\mathfrak{X}/\mathrm{GL}_n].$$

For any $1 \leq m \leq n$, the scheme Y_m can be identified with the open complement of $Y_{m-1} \times \mathbb{A}^{m-1}_k$ inside $Y_{m-1} \times \mathbb{A}^n_k$. Hence, $[\mathfrak{Y}_m]_{\mathfrak{S}} = (\mathbb{L}^n - \mathbb{L}^{m-1})[\mathfrak{Y}_{m-1}]_{\mathfrak{S}}$ for all

$1 \leq m \leq n$, and thus $[\mathfrak{X}]_{\mathfrak{S}} = \left(\prod_{m=0}^{n-1}(\mathbb{L}^n - \mathbb{L}^m)\right)[\mathfrak{X}/\mathrm{GL}_n]_{\mathfrak{S}} = [\mathrm{GL}_n]\cdot[\mathfrak{X}/\mathrm{GL}_n]_{\mathfrak{S}}$. Specializing to the case $\mathfrak{X} = \mathfrak{S} = \operatorname{Spec} k$, we find that $[\mathrm{GL}_n]$ is invertible with inverse $[\mathrm{BGL}_n]$. Therefore, $[\mathfrak{X}/\mathrm{GL}_n]_{\mathfrak{S}} = [\mathrm{GL}_n]^{-1}\cdot[\mathfrak{X}]_{\mathfrak{S}}$. □

Proposition 3.5.5. *Let G be a special algebraic group over a field k. Then $[G]$ is invertible in $\mathrm{K}_0(\mathbf{Stck}_k)$, and for any G-torsor of $\mathfrak{P} \to \mathfrak{X}$ in $\mathbf{Stck}_{\mathfrak{S}}$, one has $[\mathfrak{X}]_{\mathfrak{S}} = [G]^{-1}\cdot[\mathfrak{P}]_{\mathfrak{S}}$ in $\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{S}})$.*

Proof. Using Proposition 1.6.5, we can reduce to the case where $\mathfrak{X} = [X/\mathrm{GL}_n]$ for a quasi-projective scheme X. Now form the following cartesian diagram.

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathfrak{P} & \longrightarrow & [X/\mathrm{GL}_n] \end{array}$$

Then P is a GL_n-torsor over $\mathfrak{P}$, as described in Section 1.5, and a G-torsor over X. By Proposition 3.5.4 and Lemma 3.3.13, we have $[\mathrm{GL}_n]\cdot[\mathfrak{P}]_{\mathfrak{S}} = [P]_{\mathfrak{S}} = [G]\cdot[X]_{\mathfrak{S}} = [G][\mathrm{GL}_n]\cdot[\mathfrak{X}]_{\mathfrak{S}}$, and hence $[\mathfrak{P}]_{\mathfrak{S}} = [G]\cdot[\mathfrak{X}]_{\mathfrak{S}}$. In the special case that $\mathfrak{P} = \mathfrak{S} = \operatorname{Spec} k$ and $\mathfrak{X} = \mathrm{B}G$, we find that $[G]$ is invertible with inverse $[\mathrm{B}G]$ and thus $[\mathfrak{X}]_{\mathfrak{S}} = [G]^{-1}\cdot[\mathfrak{P}]_{\mathfrak{S}}$. □

Example 3.5.6. In general, it need not be the case that $[\mathrm{B}G] = [G]^{-1}$. For example, consider the group $G = \mu_n$ of n-th roots of unity. The morphism $\mathbb{G}_m \to \mathrm{B}\mu_n$, corresponding to the μ_n-torsor $\mathbb{G}_m \to \mathbb{G}_m$ given by $x \mapsto x^n$, is a $\mathbb{G}_m$-torsor itself, so it follows that $[\mathrm{B}\mu_n] = [\mathbb{G}_m]/[\mathbb{G}_m] = 1$. It was shown by Ekedahl that also $[\mathrm{B}S_n] = 1$ for the symmetric groups S_n for all $n \geq 0$ [Eke09b]. He also showed that there are finite groups G for which $[\mathrm{B}G] \neq 1$.

From Proposition 3.5.4 and the expression of $[\mathrm{GL}_n]$ in terms of $\mathbb{L}$, see Example 3.3.5, it follows that the elements $\mathbb{L}$ and $\mathbb{L}^n - 1$ for all $n \geq 1$ are invertible in $\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{S}})$. Hence, if $\mathfrak{S} = S$ is a variety over k, there is a natural map from the localization $\mathrm{K}_0(\mathbf{Var}_S)[\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1} : n \geq 1]$ (where we adjoined inverses of $\mathbb{L}$ and $\mathbb{L}^n - 1$ for all $n \geq 1$) to $\mathrm{K}_0(\mathbf{Stck}_S)$. In fact, this map is an isomorphism.

Theorem 3.5.7 ([Eke09a, Theorem 1.2])**.** *The map $\mathrm{K}_0(\mathbf{Var}_S)[\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1} : n \geq 1] \to \mathrm{K}_0(\mathbf{Stck}_S)$ is an isomorphism of rings.*

Remark 3.5.8. The isomorphism of Theorem 3.5.7 allows us to extend any invariant $\chi\colon \mathrm{K}_0(\mathbf{Var}_S) \to R$ to $\mathrm{K}_0(\mathbf{Stck}_S)$, possibly after inverting $\chi(\mathbb{L})$ and $\chi(\mathbb{L}^n - 1)$ in R, for all $n \geq 1$, provided they are not zero-divisors in R. In particular, this extends the E-polynomial to all algebraic stacks $\mathfrak{X}$ of finite type over $\mathbb{C}$ with affine stabilizers,

$$e\colon \mathrm{K}_0(\mathbf{Stck}_{\mathbb{C}}) \to \mathbb{Z}[\![u, v]\!][u^{-1}, v^{-1}]. \tag{3.7}$$

This approach is taken for example in [Joy07, Theorem 4.10]. Alternatively, given a presentation $X \to \mathfrak{X}$, one can construct a simplicial scheme $X^\bullet$ resolving $\mathfrak{X}$, given by $X^n = X \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X$ ($n+1$ times). Now, Deligne's construction applies in fact to simplicial schemes [Del74], so that the cohomology groups $H_c^k(\mathfrak{X}, \mathbb{C})$ of the geometric realization of the analytification of $X^\bullet$ admit a mixed Hodge structure, which can be shown to be independent of the presentation X. The corresponding E-polynomial $e(\mathfrak{X})$ agrees with (3.7). For details, see [BD07] or [Toë05]. In the particular case of a quotient stack $\mathfrak{X} = [X/G]$ with G a connected group, one has $e(\mathfrak{X}) = e(X)/e(G)$.

3.6 Equivariant motivic invariants

Let G be a finite group acting on a complex variety X. The action of G turns the cohomology groups $H_c^k(X;\mathbb{C})$ into representations of G, by functoriality of cohomology. Moreover, the action of G, being algebraic, respects the mixed Hodge structure [PS08, FS21], so the graded pieces $H_c^{k;p,q}(X) = \mathrm{Gr}_F^p \, \mathrm{Gr}_{p+q}^W \, H_c^k(X;\mathbb{C})$, see (3.2), turn into representations of G as well. From this, one constructs the *G-equivariant E-polynomial*

$$e^G(X) = \sum_{k,p,q} (-1)^k u^p v^q \otimes [H_c^{k;p,q}(X)] \in \mathbb{Z}[u,v] \otimes R_{\mathbb{C}}(G),$$

where $R_{\mathbb{C}}(G)$ denotes the representation ring of G. The G-equivariant E-polynomial is still additive and multiplicative, that is,

$$e^G(X) = e^G(Z) + e^G(X \setminus Z) \quad \text{and} \quad e^G(X \times Y) = e^G(X)\, e^G(Y)$$

for complex varieties X and Y with a G-action, and $Z \subseteq X$ a G-invariant closed subvariety [FS21]. The original E-polynomial $e(X)$ can be obtained from $e^G(X)$ via the map $\dim\colon R_{\mathbb{C}}(G) \to \mathbb{Z}$.

In this section, we investigate to which extent other invariants can be made G-equivariant, with a special focus on the virtual class in the Grothendieck ring of varieties.

Definition 3.6.1. Let G be an algebraic group over a field k, and S a variety over k. A *G-variety* over S is a variety X over S with an action of G such that $X \to S$ is G-invariant and X admits a cover by G-invariant affine opens. Denote by $\mathbf{Var}_S^G$ the category of G-varieties over S and G-equivariant morphisms over S between them. The *Grothendieck ring of G-varieties over S*, denoted $\mathrm{K}_0(\mathbf{Var}_S^G)$, is defined, analogous to Definition 3.2.3, as the free abelian group on isomorphism classes $[X]$ of G-varieties X over S modulo the relations $[X] = [Z] + [X \setminus Z]$ for all G-invariant closed subvarieties $Z \subseteq X$. Multiplication is given on generators by $[X][Y] = [X \times_S Y]$, where G acts diagonally on $X \times_S Y$.

Now, more precisely, we investigate whether an invariant $\chi\colon \mathrm{Ob}(\mathbf{Var}_k) \to R$, for some commutative ring R, can be promoted to some $\chi^G\colon \mathrm{Ob}(\mathbf{Var}_k^G) \to R \otimes R_{\mathbb{C}}(G)$ such that χ is obtained from χ^G via the map $\dim\colon R_{\mathbb{C}}(G) \to \mathbb{Z}$, while remaining additive or multiplicative. We show this is possible in many cases, such as for $R = \mathrm{K}_0(\mathbf{MHS})$ or $R = \mathrm{K}_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k, \mathbb{C}))$. However, we also show this is not possible for $R = \mathrm{K}_0(\mathbf{Var}_k)$. Nevertheless, under certain assumptions on G, we will provide a construction which, although far from ideal, provides a new tool for computations in $\mathrm{K}_0(\mathbf{Var}_k)$.

Let us start with a positive result.

Proposition 3.6.2. *Let G be a finite group with splitting field K. Let $\mathcal{A}$ be an idempotent complete K-linear tensor category, whose unit object is denoted by K. Suppose G acts on an object X of $\mathcal{A}$. Then X decomposes in $\mathcal{A}$ as*

$$X \cong \bigoplus_{\rho} X_\rho \otimes [\rho],$$

for some objects X_ρ of $\mathcal{A}$, where ρ ranges over the irreducible representations of G, and $[\rho] \coloneqq K^{\oplus \dim \rho}$. Moreover, the isomorphism is G-equivariant when G acts trivially on X_ρ and via ρ on $[\rho]$, and the objects X_ρ are uniquely determined up to isomorphism.

Proof. Denote by $\rho_1, \dots, \rho_n$ the irreducible representations of G over K. The Artin–Wedderburn theorem gives an isomorphism $K[G] \cong \prod_{i=1}^n \mathrm{Mat}_{d_i \times d_i}(K)$, where $d_i = \dim \rho_i$, given by $g \mapsto (\rho_i(g))_{i=1}^n$ [Ser77, Proposition 10]. This decomposition corresponds to a sequence $e_1, \dots, e_n \in K[G]$ of pairwise orthogonal central idempotents such that $\sum_{i=1}^n e_i = 1$. For every $i = 1, \dots, n$, the idempotents e_i induce idempotent morphisms $X \xrightarrow{e_i} X$ which, by assumption, split as $X \xrightarrow{r_i} Y_i \xrightarrow{s_i} X$ with $s_i \circ r_i = e_i$ and $r_i \circ s_i = \mathrm{id}_{Y_i}$. Orthogonality of the idempotents implies, for $i \neq j$, that $s_i \circ r_i \circ s_j \circ r_j = 0$ and hence $r_i \circ s_j = 0$. Therefore, we have an isomorphism

$$\bigoplus_{i=1}^n Y_j \underset{\coprod_i r_i}{\overset{\coprod_i s_i}{\rightleftarrows}} X.$$

Since the e_i are central in $K[G]$, the action of G restricts to Y_i for every i.

Every idempotent e_i corresponds to a factor $\mathrm{Mat}_{d_i \times d_i}(K)$. Write E_{jk} for the $d_i \times d_i$ matrix which is zero everywhere, except at position (j, k), where the entry is one. Then $e_i = \sum_{j=1}^{d_i} e_{ij}$ for the pairwise orthogonal idempotents $e_{ij} = E_{jj}$. As above, this yields a decomposition

$$Y_i \cong \bigoplus_{j=1}^{d_i} Z_{ij}.$$

Moreover, the Z_{ij} are isomorphic for all j since E_{jk} defines an isomorphism from Z_{ik} to Z_{ij}, with inverse E_{kj}, so $Y_i \cong Z_i \otimes K^{\oplus d_i}$. Under this isomorphism, G acts trivially on Z_i and on $K^{\oplus d_i}$ via ρ_i, because of the isomorphism $K[G] \cong \prod_{i=1}^n \mathrm{Mat}_{d_i \times d_i}(K)$. Therefore, $Y_i \cong Z_i \otimes [\rho_i]$. Finally, the $X_{\rho_i} \coloneqq Z_i$ are uniquely determined up to isomorphism, as they correspond to the idempotents e_{ij}. □

Remark 3.6.3. Suppose G acts on objects X and Y in $\mathcal{A}$ as in Proposition 3.6.2. Then it follows from the uniqueness statement that

$$(X \otimes Y)_{\rho_k} \cong \bigoplus_{i,j=1}^n (X_{\rho_i} \otimes Y_{\rho_j})^{\oplus a_{ij}^k},$$

where $a_{ij}^k \in \mathbb{Z}_{\geq 0}$ are the *Clebsch–Gordan series*, given by $\rho_i \otimes \rho_j \cong \bigoplus_{k=1}^n \rho_k^{\oplus a_{ij}^k}$.

Now, let $\mathcal{A}$ be as in Proposition 3.6.2, and suppose we are given a functor $\mathcal{X} \colon \mathbf{Var}_k \to \mathcal{A}$. If $\mathcal{A}$ is an abelian or triangulated category, we obtain an invariant $\chi \colon \mathrm{Ob}(\mathbf{Var}_k) \to \mathrm{K}_0(\mathcal{A})$ which, using Proposition 3.6.2, promotes to an invariant

$$\chi^G \colon \mathrm{Ob}(\mathbf{Var}_k^G) \to \mathrm{K}_0(\mathcal{A}) \otimes R_K(G)$$

where $\mathcal{X}(X) \cong \bigoplus_\rho X_\rho \otimes [\rho]$ is sent to $\sum_\rho [X_\rho] \otimes [\rho]$. One can obtain χ from χ^G via $\dim \colon R_K(G) \to \mathbb{Z}$ as the image of $[\rho] = K^{\oplus \dim \rho}$ in $\mathrm{K}_0(\mathcal{A})$ equals $\dim \rho$.

Furthermore, if G-invariant closed subvarieties $Z \subseteq X$ induce exact sequences $0 \to \mathcal{X}(X \setminus Z) \to \mathcal{X}(X) \to \mathcal{X}(Z) \to 0$ (when $\mathcal{A}$ is abelian) or distinguished triangles $\mathcal{X}(X \setminus Z) \to \mathcal{X}(X) \to \mathcal{X}(Z) \to \mathcal{X}(X \setminus Z)[1]$ (when $\mathcal{A}$ is triangulated) with G-equivariant maps in $\mathcal{A}$, then χ^G will also be additive, that is, $\chi^G(X) = \chi^G(Z) + \chi^G(X \setminus Z)$. In this case, χ^G descends to a group morphism

$$\chi^G \colon \mathrm{K}_0(\mathbf{Var}_k^G) \to \mathrm{K}_0(\mathcal{A}) \otimes R_K(G). \tag{3.8}$$

Moreover, when $\mathcal{A}$ is tensor triangulated and there are natural isomorphisms $\mathcal{X}(X \times Y) \cong \mathcal{X}(X) \otimes \mathcal{X}(Y)$ for all G-varieties X and Y, where G acts diagonally on $X \times Y$, it follows from Remark 3.6.3 that χ^G is multiplicative, that is, $\chi^G(X \times Y) = \chi^G(X)\, \chi^G(Y)$. In this case, (3.8) is a ring morphism.

Example 3.6.4.
- Let $\mathcal{A} = D^b(\mathbf{MHS})$ be the derived category of mixed Hodge structures. The assignment of the mixed Hodge structures $H_c^k(X)$ to a complex variety X can be promoted to a functor $\mathcal{X} = R\Gamma(-, \mathbb{Q}) \colon \mathbf{Var}_\mathbb{C} \to \mathcal{A}$ such that $H_c^k(X)$ is the k-th cohomology group of $\mathcal{X}(X)$ [Beĭ86]. The resulting G-equivariant invariant χ^G is additive by the G-equivariant long exact sequence (3.4). It is also multiplicative by the Künneth formula, and hence induces a ring morphism

$$\chi^G \colon \mathrm{K}_0(\mathbf{Var}_\mathbb{C}^G) \to \mathrm{K}_0(\mathbf{MHS}) \otimes R_\mathbb{C}(G).$$

- Extending the previous example, note that the exact functor $\mathrm{Gr}_F^* \mathrm{Gr}_*^W$ in (3.2) induces an exact functor $D^b(\mathbf{MHS}) \to D^b((\mathbf{Vect}_{\mathbb{C}}^{\mathbb{Z}\times\mathbb{Z}})_{\mathrm{fin}})$, and let $\mathcal{X}$ be the composite

$$\mathbf{Var}_{\mathbb{C}} \xrightarrow{R\Gamma(-,\mathbb{Q})} D^b(\mathbf{MHS}) \xrightarrow{\mathrm{Gr}_F^* \mathrm{Gr}_*^W} D^b((\mathbf{Vect}_{\mathbb{C}}^{\mathbb{Z}\times\mathbb{Z}})_{\mathrm{fin}}).$$

  The induced invariant χ^G is still additive and multiplicative, and hence induces ring morphism

$$\chi^G \colon \mathrm{K}_0(\mathbf{Var}_{\mathbb{C}}^G) \to \mathrm{K}_0((\mathbf{Vect}_{\mathbb{C}}^{\mathbb{Z}\times\mathbb{Z}})_{\mathrm{fin}}) \otimes R_{\mathbb{C}}(G) \cong \mathbb{Z}[u^{\pm 1}, v^{\pm 1}] \otimes R_{\mathbb{C}}(G)$$

  which is precisely the G-equivariant E-polynomial.

- Let $\mathcal{A} = \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k, K)$ be the K-linearization of Voevodsky's triangulated category of effective geometric motives, with k a field of characteristic zero, and let $\mathcal{X} \colon \mathbf{Var}_k \to \mathcal{A}$ be the motive M_{gm} or the motive with compact support M_{gm}^c. The induced invariant χ^G is multiplicative, and if $\mathcal{X} = M_{\mathrm{gm}}^c$ also additive.

Grothendieck ring of varieties

Unfortunately, the Grothendieck ring of varieties $\mathrm{K}_0(\mathbf{Var}_k)$ is not given by the Grothendieck group of an abelian (or triangulated) category $\mathcal{A}$. To get an idea of how an analogous construction could work for $\mathrm{K}_0(\mathbf{Var}_k)$, we first consider some properties of the G-equivariant E-polynomial.

Let G be a finite group and $H \subseteq G$ a subgroup. Denote by

$$\mathrm{Res}_H^G \colon R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(H) \quad \text{and} \quad \mathrm{Ind}_H^G \colon R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(G)$$

the restriction and induction maps [Ser77, p.28]. Using the same symbols, we define restriction and induction for G-varieties.

Definition 3.6.5. Let G be an algebraic group over k with a subgroup $H \subseteq G$, and S a variety over k. Define the functors

$$\mathrm{Res}_H^G \colon \mathbf{Var}_S^G \to \mathbf{Var}_S^H \quad \text{and} \quad \mathrm{Ind}_H^G \colon \mathbf{Var}_S^H \to \mathbf{Var}_S^G$$

where Res_H^G restricts the action from G to H (in fact, Res_H^G is defined for any morphism $H \to G$ of algebraic groups), and $\mathrm{Ind}_H^G(Y) = (G \times Y) /\!\!/ H$, where H acts on $G \times Y$ via $h \cdot (g, y) = (gh^{-1}, h \cdot y)$ and G acts on the resulting quotient by left multiplication on the factor of G. Note that, by [PV89, Theorem 4.19], the quotient $(G \times Y) /\!\!/ H$ is a variety, even when H is non-reductive. It is easy to see that these functors descend to the Grothendieck ring of varieties

$$\mathrm{Res}_H^G \colon \mathrm{K}_0(\mathbf{Var}_S^G) \to \mathrm{K}_0(\mathbf{Var}_S^H) \quad \text{and} \quad \mathrm{Ind}_H^G \colon \mathrm{K}_0(\mathbf{Var}_S^H) \to \mathrm{K}_0(\mathbf{Var}_S^G).$$

When G and H are finite, the underlying variety of $\mathrm{Ind}_H^G(Y)$ is simply $\bigsqcup_{G/H} Y$.

Lemma 3.6.6. *Let G be a finite group and $H \subseteq G$ a subgroup.*

(i) $e^H(\mathrm{Res}^G_H(X)) = \mathrm{Res}^G_H(e^G(X))$ *for all objects X of $\mathbf{Var}^G_k$,*

(ii) $e^G(\mathrm{Ind}^G_H(Y)) = \mathrm{Ind}^G_H(e^H(Y))$ *for all objects Y of $\mathbf{Var}^H_k$,*

(iii) $e(X /\!\!/ G) = \langle T, e^G(X)\rangle$, *where $T \in R_{\mathbb{C}}(G)$ corresponds to the trivial representation, and $\langle -, -\rangle$ denotes the inner product of characters.*

Proof. *(i)* and *(ii)* directly follow from the definitions of Res^G_H and Ind^G_H for representations and varieties. *(iii)* follows from [FS21, Proposition 4.3]. □

The following example shows how these properties can be used to compute the G-equivariant E-polynomials in some simple cases.

Example 3.6.7. Consider $G = \mathbb{Z}/2\mathbb{Z}$ and denote by $T, N \in R_{\mathbb{C}}(G)$ the trivial and non-trivial character of G. For any G-variety X, we have

$$e^G(X) = \alpha \otimes T + \beta \otimes N$$

for some $\alpha, \beta \in \mathbb{Z}[u, v]$. The properties of Lemma 3.6.6 imply that $e(X /\!\!/ G) = \alpha$ and $e(X) = \mathrm{Res}^G_1(e^G(X)) = \alpha + \beta$. Therefore,

$$e^G(X) = e(X /\!\!/ G) \otimes T + (e(X) - e(X /\!\!/ G)) \otimes N.$$

Example 3.6.8. Consider $G = S_3$ and denote by $T, S, D \in R_{\mathbb{C}}(G)$ the trivial, sign and standard representation. For any G-variety X, we have

$$e^G(X) = \alpha \otimes T + \beta \otimes S + \gamma \otimes D$$

for some $\alpha, \beta, \gamma \in \mathbb{Z}[u, v]$. For $\tau = (1\ 2)$ and $\rho = (1\ 2\ 3)$ in S_3, we find

$$e(X) = \alpha + \beta + 2\gamma,\ e(X /\!\!/ \langle\tau\rangle) = \alpha + \gamma,\ e(X /\!\!/ \langle\rho\rangle) = \alpha + \beta,\ e(X /\!\!/ G) = \alpha.$$

In particular, it follows that

$$\alpha = e(X /\!\!/ G),\ \beta = e(X) - 2 \cdot e(X /\!\!/ \langle\tau\rangle) + e(X /\!\!/ G),\ \gamma = e(X /\!\!/ \langle\tau\rangle) - e(X /\!\!/ G).$$

Note that, since there are more subgroups than irreducible representations, the relation

$$e(X) - 2 \cdot e(X /\!\!/ \langle\tau\rangle) - e(X /\!\!/ \langle\rho\rangle) + 2 \cdot e(X /\!\!/ G) = 0 \tag{3.9}$$

will always hold.

Let us return to the Grothendieck ring of varieties. Given a G-variety X, we want to define $[X]^G \in \mathrm{K}_0(\mathbf{Var}_k) \otimes R_{\mathbb{C}}(G)$ such that

$$\langle T_H, \operatorname{Res}^G_H [X]^G \rangle = [X /\!\!/ H] \tag{3.10}$$

for every subgroup $H \subseteq G$. Unfortunately, here we run into trouble trying to define $[X]^G$. The following example shows that the analogue of (3.9) need not hold in $\mathrm{K}_0(\mathbf{Var}_k)$ in general.

Example 3.6.9 ([Saw22])**.** Let $G = S_3$ and let X be a complex smooth projective curve of genus $6g + 1$ for some $g \geq 1$ with a free action of S_3. By the Riemann–Hurwitz formula, the quotients $X /\!\!/ \langle \tau \rangle$, $X /\!\!/ \langle \rho \rangle$ and $X /\!\!/ S_3$ have genera $3g+1, 2g+1$ and $g+1$, respectively. Hence, none of these quotients are stably birational to X or to each other. Now, the isomorphism $\mathrm{K}_0(\mathbf{Var}_{\mathbb{C}})/(\mathbb{L}) \cong \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$ by Larsen and Lunts [LL03] shows there is no $\mathbb{Z}$-linear relation between their classes in $\mathrm{K}_0(\mathbf{Var}_{\mathbb{C}})$.

It seems that having too many subgroups results in $[X]^G$ being ill-defined. A possible remedy could be to fix a set of subgroups of G. On the other hand, having too few subgroups could also be a problem, e.g. for $G = \mathbb{Z}/3\mathbb{Z}$, which has 3 irreducible representations but only 2 subgroups. For this reason, we will focus only on rational representations of G. This makes sense in analogy with the G-equivariant E-polynomial, since $H^k_c(X; \mathbb{C}) = H^k_c(X; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. Finally, note that the quotient $X /\!\!/ H$ only depends on the conjugacy class of H.

Definition 3.6.10. Let G be a finite group, and let $\mathcal{H}$ be a set of conjugacy classes of subgroups of G. Define the map

$$\Psi^{\mathcal{H}}_G \colon R_{\mathbb{Q}}(G) \to \bigoplus_{[H] \in \mathcal{H}} \mathbb{Z}, \quad V \mapsto \left(\langle T_H, \operatorname{Res}^G_H V \rangle \right)_{[H] \in \mathcal{H}}.$$

Lemma 3.6.11. *If $\mathcal{H}$ contains the conjugacy classes of all subgroups of G, then $\Psi^{\mathcal{H}}_G$ is injective.*

Proof. Take any $V \in R_{\mathbb{Q}}(G)$ such that $\langle T_H, \operatorname{Res}^G_H V \rangle = 0$ for all H. By Frobenius reciprocity, this is the same as $\langle \operatorname{Ind}^G_H T_H, V \rangle = 0$ for all H. Now, by [Ser77, Theorem 30], the elements $\operatorname{Ind}^G_H T_H$ generate $R_{\mathbb{Q}}(G)$, so $V = 0$. □

Shrinking $\mathcal{H}$ appropriately, the map $\Psi^{\mathcal{H}}_G$ will still be injective, and its image will have rank equal to $|\mathcal{H}|$. In particular, $\Psi^{\mathcal{H}}_G \otimes \mathbb{Q}$ will be an isomorphism, so further tensoring with $\mathrm{K}_0(\mathbf{Var}_k)$ shows the existence and uniqueness of an element $[X]^G \in \mathrm{K}_0(\mathbf{Var}_k) \otimes R_{\mathbb{Q}}(G) \otimes \mathbb{Q}$ satisfying (3.10) for all $[H] \in \mathcal{H}$. We end up with the following definition.

Definition 3.6.12. Let G be a finite group and $\mathcal{H}$ a set of conjugacy classes of subgroups of G such that

$$\Psi_G^{\mathcal{H}} \otimes \mathbb{Q} \colon R_{\mathbb{Q}}(G) \otimes \mathbb{Q} \to \bigoplus_{H \in \mathcal{H}} \mathbb{Q} \tag{3.11}$$

is an isomorphism. In this case we say that $\mathcal{H}$ is a *good* set of conjugacy classes of subgroups of G. Then for any G-variety X over S, the *G-virtual class* of X is the unique element $[X]^G \in \mathrm{K}_0(\mathbf{Var}_S) \otimes R_{\mathbb{Q}}(G) \otimes \mathbb{Q}$ such that

$$\langle T_H, \mathrm{Res}_H^G [X]^G \rangle = [X /\!\!/ H]$$

in $\mathrm{K}_0(\mathbf{Var}_S)$ for all $[H] \in \mathcal{H}$.

Remark 3.6.13. The G-virtual class is clearly additive, that is, $[X]^G = [Z]^G + [X \setminus Z]^G$ for all G-invariant closed subvarieties $Z \subseteq X$. Hence, it induces a group morphism

$$[-]^G \colon \mathrm{K}_0(\mathbf{Var}_S^G) \to \mathrm{K}_0(\mathbf{Var}_S) \otimes R_{\mathbb{Q}}(G) \otimes \mathbb{Q}.$$

Example 3.6.14. Let $G = \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 1$ and let $\mathcal{H}$ be the set of conjugacy classes of all subgroups of G. Then $\Psi_G^{\mathcal{H}} \otimes \mathbb{Q}$ is injective by Lemma 3.6.11, and an isomorphism because $|\mathcal{H}|$ equals the number of divisors of n, which is equal to the rank of $R_{\mathbb{Q}}(G)$.

Example 3.6.15. Consider the symmetric group $G = S_n$ with the set $\mathcal{H} = \{S_{\lambda_1} \times \cdots \times S_{\lambda_k} : \lambda \text{ a partition of } n\}$ of *Young subgroups*. From the representation theory of S_n [FH91, Lecture 4] it can be shown that (3.11) is an isomorphism. In particular, the irreducible representations of S_n are parametrized by the partitions λ of n. Denote by V_λ the irreducible representation of S_n corresponding to such a partition λ. Now, for any $V = \sum_\lambda a_\lambda [V_\lambda] \in R_{\mathbb{Q}}(S_n)$, we find that

$$\begin{aligned}
\Psi_G^{\mathcal{H}}(V) &= \left(\left\langle T_{S_\lambda}, \mathrm{Res}_{S_\lambda}^{S_n} V \right\rangle \right)_{S_\lambda \in \mathcal{H}} \\
&= \left(\left\langle \mathrm{Ind}_{S_\lambda}^{S_n} T_{S_\lambda}, V \right\rangle \right)_{S_\lambda \in \mathcal{H}} \\
&= \left(\textstyle\sum_\mu a_\mu K_{\mu\lambda} \right)_{S_\lambda \in \mathcal{H}}
\end{aligned}$$

where $K_{\mu\lambda}$ are the *Kostka numbers*, by Young's rule [FH91, Corollary 4.39]. Since $K_{\lambda\lambda} = 1$ and $K_{\mu\lambda} = 0$ for $\mu < \lambda$ (for the lexicographical order on partitions), it follows that $\Psi_G^{\mathcal{H}}$ is invertible.

Example 3.6.16. Suppose G_1 and G_2 are finite groups with good sets of conjugacy classes of subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then

$$\mathcal{H} = \{[H_1 \times H_2] : [H_1] \in \mathcal{H}_1 \text{ and } [H_2] \in \mathcal{H}_2\}$$

is a good set of conjugacy classes of subgroups of $G_1 \times G_2$ if $R_{\mathbb{Q}}(G_1) = R_{\mathbb{C}}(G_1)$ or $R_{\mathbb{Q}}(G_2) = R_{\mathbb{C}}(G_2)$. In particular, this provides good sets of conjugacy classes of subgroups for all Young subgroups $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$.

Even though the G-virtual class is additive, the following example shows that in general, already for $G = \mathbb{Z}/2\mathbb{Z}$, the G-virtual class is not multiplicative.

Example 3.6.17. Let $G = \mathbb{Z}/2\mathbb{Z}$ and take A and B elliptic curves over $k = \mathbb{C}$ with $[A] \neq [B]$ and $A \times A \cong B \times B$ as abelian varieties, as in [Poo02, Lemma 3]. Equip the elliptic curves A and B with the G-action of negation, $P \mapsto -P$. Now suppose that the G-virtual class is multiplicative. Then, using the notation $[X]^G = [X]_+ \otimes T + [X]_- \otimes N$ with $[X]_+ = [X /\!\!/ G]$ and $[X]_- = [X] - [X /\!\!/ G]$, we find that

$$\begin{aligned}[A \times A]_+ &= [A]_+^2 + [A]_-^2 = [\mathbb{P}^1_k]^2 + ([A] - [\mathbb{P}^1_k])^2 = [A]^2 + 2[\mathbb{P}^1_k]^2 - 2[A][\mathbb{P}^1_k] \\ [B \times B]_+ &= [B]_+^2 + [B]_-^2 = [\mathbb{P}^1_k]^2 + ([B] - [\mathbb{P}^1_k])^2 = [B]^2 + 2[\mathbb{P}^1_k]^2 - 2[B][\mathbb{P}^1_k]\end{aligned}$$

where $A /\!\!/ G \cong B /\!\!/ G \cong \mathbb{P}^1_k$. Since the isomorphism $A \times A \cong B \times B$ is G-equivariant, we have $[A \times A]_+ = [B \times B]_+$, and hence $([A] - [B])[\mathbb{P}^1_k] = 0$ in $\mathrm{K}_0(\mathbf{Var}_k)$. However, the Albanese map $\mathrm{K}_0(\mathbf{Var}_k) \to \mathbb{Z}[\mathrm{AV}_k]$, described in [Poo02, Section 4], from the Grothendieck ring of varieties to the monoid ring of abelian varieties over k sends $([A] - [B])[\mathbb{P}^1_k]$ to $[A] - [B]$, which is non-zero in $\mathbb{Z}[\mathrm{AV}_k]$. Therefore, the G-virtual class cannot be multiplicative.

Nevertheless, we present the following construction, to measure to which extent the G-virtual class is multiplicative.

Lemma 3.6.18. *Let G be a finite group and $\mathcal{H}$ a good set of conjugacy classes of subgroups of G. Let $\mathcal{V}_G^{\mathcal{H}} \subseteq \mathrm{K}_0(\mathbf{Var}_S^G)$ be the subset of elements X such that $[XY]^G = [X]^G[Y]^G$ for all $Y \in \mathrm{K}_0(\mathbf{Var}_S^G)$. Then $\mathcal{V}_G^{\mathcal{H}}$ is a $\mathrm{K}_0(\mathbf{Var}_S)$-subalgebra of $\mathrm{K}_0(\mathbf{Var}_S^G)$.*

Proof. Note that $\mathcal{V}_G^{\mathcal{H}}$ is the left radical of the $\mathrm{K}_0(\mathbf{Var}_S)$-bilinear form

$$\begin{aligned}\mathrm{K}_0(\mathbf{Var}_S^G) \times \mathrm{K}_0(\mathbf{Var}_S^G) &\to \mathrm{K}_0(\mathbf{Var}_S) \otimes R_{\mathbb{Q}}(G) \otimes \mathbb{Q} \\ (X, Y) &\mapsto [XY]^G - [X]^G[Y]^G\end{aligned}$$

and is therefore a subgroup of $\mathrm{K}_0(\mathbf{Var}_S)$. Furthermore, $\mathcal{V}_G^{\mathcal{H}}$ is closed under multiplication, because for all $X, Y \in \mathcal{V}_G^{\mathcal{H}}$ and $Z \in \mathrm{K}_0(\mathbf{Var}_S^G)$ we have

$$[(XY)Z]^G = [X(YZ)]^G = [X]^G[YZ]^G = [X]^G[Y]^G[Z]^G = [XY]^G[Z]^G. \qquad \square$$

Theorem 3.6.19. *Let G be a finite group, $\mathcal{H}$ a good set of conjugacy classes of subgroups of G, and suppose that k is a splitting field for G.*

(i) If G acts linearly on $\mathbb{A}^1_k$, then $\mathcal{V}^{\mathcal{H}}_G$ contains $[\mathbb{A}^1_k]$.

(ii) If G acts diagonally on $\mathbb{A}^n_k$, then $\mathcal{V}^{\mathcal{H}}_G$ contains $[\mathbb{A}^n_k]$.

(iii) If G acts diagonally on $\mathbb{P}^n_k$, then $\mathcal{V}^{\mathcal{H}}_G$ contains $[\mathbb{P}^n_k]$.

(iv) Let $[H] \in \mathcal{H}$ be such that $[H' \cap gHg^{-1}] \in \mathcal{H}$ for all $[H'] \in \mathcal{H}$ and $g \in G$. Then the set G/H of cosets with the natural action of G lies in $\mathcal{V}^{\mathcal{H}}_G$.

Proof. (i) As $[\mathbb{A}^1_k \mathbin{/\!\!/} H] = \mathbb{L}$ for any finite group H acting linearly on $\mathbb{A}^1_k$, we have $[\mathbb{A}^1_k]^G = \mathbb{L} \otimes T$ where $T \in R_{\mathbb{Q}}(G)$ corresponds to the trivial representation. Hence, it suffices to show that $[(\mathbb{A}^1_k \times Y) \mathbin{/\!\!/} H]_S = \mathbb{L}[Y \mathbin{/\!\!/} H]_S$ for all $Y \in \mathbf{Var}^G_S$ and $[H] \in \mathcal{H}$. Take such Y and H, write $\tau \colon H \to \mathrm{GL}_1(k)$ for the representation via which H acts on $\mathbb{A}^1_k$, and let $N = \ker \tau$. Since

$$(\mathbb{A}^1_k \times Y) \mathbin{/\!\!/} H = (\mathbb{A}^1_k \times (Y \mathbin{/\!\!/} N)) \mathbin{/\!\!/} (H/N)$$

we may, replacing H by H/N and Y by $Y \mathbin{/\!\!/} N$, assume that H is a finite cyclic group, that is, $H = \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 1$.

Also, we may assume $Y = \operatorname{Spec} R$ is affine. Write $R^H \subseteq R$ for the subring of H-invariants. Then R is finitely generated as R^H-module [Mon80, Corollary 5.9], so it can be written as

$$R = R^H\langle \sigma_{1,1}, \dots, \sigma_{1,m_1} \rangle \oplus \cdots \oplus R^H\langle \sigma_{n-1,1}, \dots, \sigma_{n-1,m_{n-1}} \rangle$$

for some $\sigma_{i,j} \in R$ such that H acts via $(a \bmod n) \cdot \sigma_{i,j} \mapsto \zeta_n^{ai}\sigma_{i,j}$, where $\zeta_n \in k$ is a primitive n-th root of unity. Note that, for any $1 \leq i \leq n-1$, we have $\sigma_{i,1}^n = r$ for some $r \in R^H$, and for any $2 \leq j \leq m_i$, we have $\sigma_{i,1}^{n-1}\sigma_{i,j} = s$ for some $s \in R^H$. But then, over the closed subvariety of Y given by $r = 0$, we have $\sigma_{i,1}^n = r = 0$, so we can omit $\sigma_{i,1}$ from the generators. Similarly, over the open complement where r is invertible (so $\sigma_{i,1}$ is invertible as well), we can remove $\sigma_{i,j} = \frac{s}{r}\sigma_{i,1}$ from the generators. Hence, after sufficiently many stratifications, we may reduce to the case that

$$R = R^H \oplus R^H\langle \sigma_d \rangle \oplus R^H\langle \sigma_{2d} \rangle \oplus \cdots \oplus R^H\langle \sigma_{n-d} \rangle$$

for some $d \geq 1$ dividing n, and some $\sigma_i \in R^\times$, such that H acts via $(a \bmod n) \cdot \sigma_i = \zeta_n^{ai} \cdot \sigma_i$. In particular, for any $1 \leq m \leq n/d$, we have $\sigma_d^m = r_m\sigma_{md}$ for some $r_m \in (R^H)^\times$, that is, $\sigma_{md} = \sigma_d^m/r_m$, and hence

$$R = R^H[\sigma]/(\sigma^{n/d} - r)$$

with $\sigma = \sigma_d$ and $r = r_{n/d}$.

Note that $\tau\colon H \to \mathrm{GL}_1(k)$ must be of the form $\tau(a \bmod n) = \zeta_n^{ac}$ for some $0 \le c \le n-1$. Stratifying $\mathbb{A}^1_k$ as $\{0\} \sqcup (\mathbb{A}^1_k \setminus \{0\})$, we find that

$$(\{0\} \times Y) /\!\!/ H = Y /\!\!/ H$$

and

$$\begin{aligned}((\mathbb{A}^1_k \setminus \{0\}) \times Y) /\!\!/ H &\cong \operatorname{Spec} R[x^{\pm 1}]^H \\ &\cong \operatorname{Spec} \Big(R^H[\sigma, x^{\pm 1}]/(\sigma^{n/d} - r) \Big)^H \\ &\cong \operatorname{Spec} R^H \left\langle x^i \sigma^j : (i,j) \in L \right\rangle,\end{aligned}$$

where $L = \big\{(i,j) \in \mathbb{Z}^2 \mid ci + dj \equiv 0 \bmod n\big\}$ is a lattice. Take some $(i_0, j_0) \in L$ such that $i_0 > 0$ is minimal, and write $w = x^{i_0}\sigma^{j_0}$. Then, for any other $(i,j) \in L$, we must have $i = mi_0$ for some $m \in \mathbb{Z}$, and hence $(x^i\sigma^j)/w^m$ is an element of R^H. Therefore,

$$((\mathbb{A}^1_k \setminus \{0\}) \times Y) /\!\!/ H \cong \operatorname{Spec} R^H[w^{\pm 1}] \cong \big(\mathbb{A}^1_k \setminus \{0\}\big) \times (Y /\!\!/ H).$$

Finally, we find that

$$\begin{aligned}[(\mathbb{A}^1_k \times Y) /\!\!/ H]_S &= [(\{0\} \times Y) /\!\!/ H]_S + [((\mathbb{A}^1_k \setminus \{0\}) \times Y) /\!\!/ H]_S \\ &= [Y /\!\!/ H]_S + (\mathbb{L} - 1)[Y /\!\!/ H]_S \\ &= \mathbb{L}[Y /\!\!/ H]_S\end{aligned}$$

as desired.

Since $\mathcal{V}^{\mathcal{H}}_G$ is closed under multiplication, *(ii)* follows from *(i)*. For *(iii)*, stratify $\mathbb{P}^n_S$ as $\mathbb{A}^n_S \sqcup \mathbb{P}^{n-1}_S$, so that the result follows from *(i)* and by induction on n.

For *(iv)*, note that $[G/H]^G = 1 \otimes \operatorname{Ind}^G_H(T_H)$, where $T_H \in R_{\mathbb{Q}}(H)$ corresponds to the trivial representation. Now, for any $[H'] \in \mathcal{H}$, we can choose representatives gH for the points of the quotient $(G/H) /\!\!/ H'$. Note that the stabilizer of gH for the action of H' is $H' \cap gHg^{-1}$, and therefore

$$((G/H) \times Y) /\!\!/ H' = \bigsqcup_{[gH] \in (G/H) /\!\!/ H'} Y /\!\!/ (H' \cap gHg^{-1}).$$

Since $[H' \cap gHg^{-1}] \in \mathcal{H}$ by assumption, it follows that the coefficients of $[(G/H) \times Y]^G_S$ can be written naturally in terms of the coefficients of $[Y]^G_S$, and therefore $[G/H]$ must be contained in $\mathcal{V}^{\mathcal{H}}_G$. □

Remark 3.6.20. The condition in *(iv)* of the above theorem is trivially satisfied when $\mathcal{H}$ contains the conjugacy classes of all subgroups of G. Also, it is satisfied in the case of Example 3.6.15 with $G = S_n$. That is, the intersection of conjugates of Young subgroups is again the conjugate of a Young subgroup.

We conclude this section with two examples, both for the group $G = \mathbb{Z}/2\mathbb{Z}$. Let $T, N \in R_{\mathbb{Q}}(G)$ correspond to the trivial and non-trivial irreducible representation of G, respectively. For any G-variety X, the G-virtual class is, similar to Example 3.6.7, given by

$$[X]^{\mathbb{Z}/2\mathbb{Z}} = [X /\!\!/ (\mathbb{Z}/2\mathbb{Z})] \otimes T + ([X] - [X /\!\!/ (\mathbb{Z}/2\mathbb{Z})]) \otimes N. \tag{3.12}$$

Example 3.6.21. Let k be an algebraically closed field with $\operatorname{char}(k) \neq 2$. Consider the subvariety $M = \{A \in \mathrm{SL}_2 \mid \operatorname{tr} A \neq \pm 2\}$ of SL_2, over k, of diagonalizable non-scalar matrices. Note that we have a cartesian diagram

$$\begin{array}{ccc} \mathrm{GL}/D \times (\mathbb{A}^1_k \setminus \{0, \pm 1\}) & \longrightarrow & M \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{tr}} \\ \mathbb{A}^1_k \setminus \{0, \pm 1\} & \longrightarrow & \mathbb{A}^1_k \setminus \{\pm 2\} \end{array}$$

where $D \subseteq \mathrm{GL}_2$ is the subgroup of diagonal matrices, and GL_2/D the left coset space. The bottom morphism is given by $\lambda \mapsto \lambda + \lambda^{-1}$, and the top morphism by $(P, \lambda) \mapsto P \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix} \right) P^{-1}$. The group $G = \mathbb{Z}/2\mathbb{Z}$ acts both on GL_2/D and $\mathbb{A}^1_k \setminus \{0, \pm 1\}$, via $P \mapsto P \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ and $\lambda \mapsto \lambda^{-1}$, respectively, and we can identify M with $(\mathrm{GL}_2/D \times (\mathbb{A}^1_k \setminus \{0, \pm 1\})) /\!\!/ G$. Since $\mathbb{A}^1_k \setminus \{0, \pm 1\}$ is a projective line minus some points, its class lies in $\mathcal{V}_G^{\mathcal{H}}$ using Theorem 3.6.19, so we can compute $[M] \in \mathrm{K}_0(\mathbf{Var}_k)$ from the G-virtual classes $[\mathrm{GL}_2/D]^G$ and $[\mathbb{A}^1_k \setminus \{0, \pm 1\}]^G$. Using $(\mathbb{A}^1_k \setminus \{0, \pm 1\}) /\!\!/ G \cong \mathbb{A}^1_k \setminus \{\pm 2\}$ and (3.12), we find that

$$[\mathbb{A}^1_k \setminus \{0, \pm 1\}]^G = (\mathbb{L} - 2) \otimes T - 1 \otimes N.$$

Similarly, from $[(\mathrm{GL}_2/D) /\!\!/ G] = \mathbb{L}^2$ follows that

$$[\mathrm{GL}_2/D]^G = \mathbb{L}^2 \otimes T + \mathbb{L} \otimes N$$

and hence

$$[\mathrm{GL}_2/D \times (\mathbb{A}^1_k \setminus \{0, \pm 1\})]^G = (\mathbb{L}^3 - 2\mathbb{L}^2 + \mathbb{L}) \otimes T + (2\mathbb{L}^2 - 2\mathbb{L}) \otimes N.$$

Therefore, $[M] = \mathbb{L}^3 - 2\mathbb{L}^2 + \mathbb{L}$.

Example 3.6.22. Consider $G = \mathbb{Z}/2\mathbb{Z}$ acting on $X = \mathbb{G}_m$ via $x \mapsto x^{-1}$, over any field k. As $[X] = \mathbb{L} - 1$ and $[X /\!\!/ G] = \mathbb{L}$, we obtain $[X]^G = \mathbb{L} \otimes T - 1 \otimes N$. Since X can be seen as a projective line minus two points, its class lies in $\mathcal{V}_G^{\mathcal{H}}$, so we find that

$$[X^n /\!\!/ G] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{L} & -1 \\ -1 & \mathbb{L} \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{(\mathbb{L} - 1)^n + (\mathbb{L} + 1)^n}{2}.$$

Chapter 4

Topological Quantum Field Theories

The aim of this chapter is to study motivic invariants of character stacks associated to closed manifolds. One of the first approaches in this direction was by Hausel and Rodriguez-Villegas [HR08], whose idea was to study the G-representation variety by counting the number of points over finite fields $\mathbb{F}_q$. They could express these counts in terms of the representation theory of the finite groups $G(\mathbb{F}_q)$, and moreover, determine from these counts the E-polynomial of the G-representation variety. We call this approach the *arithmetic method*.

A few years later, Logares, Muñoz and Newstead [LMN13] initiated the *geometric method*, a geometric approach to compute the same invariant, making use of clever stratifications of the G-representation variety. González-Prieto, Logares and Muñoz [GLM20] showed that the geometric method can be phrased in terms of a *Topological Quantum Field Theory (TQFT)*.

TQFTs, originating from physics, describe the topological aspects of a quantum field theory. Atiyah [Ati88] was the first to mathematically axiomatize the notion of a TQFT, defining a TQFT as a monoidal functor from the category of bordisms to the category of vector spaces. The idea that TQFTs can be used to compute invariants of geometric objects is not a new idea. For instance, Witten, in his seminal paper [Wit89], constructed a TQFT that computes the Jones polynomial of knots.

In this chapter, we will describe both the arithmetic and geometric method, and show how they can be unified using the framework of TQFTs. Specifically, we will show that both methods can be formulated as TQFTs, and that these TQFTs can be related through natural transformations.

4.1 Monoidal categories

Central to the theory of TQFTs is the notion of a *monoidal category*. Monoidal categories were defined by Mac Lane [Mac63] under the name 'bicategory', and by Bénabou [Bén63] under the name 'categories with multiplication'.

Definition 4.1.1. A *monoidal category* is a category $\mathcal{C}$ with a functor $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, called the *tensor product*, an object 1 in $\mathcal{C}$, called the *unit object*, and natural isomorphisms

$$\begin{array}{ccc} \alpha\colon - \otimes (- \otimes -) \Rightarrow (- \otimes -) \otimes - & \lambda\colon 1 \otimes - \Rightarrow \mathrm{id}_{\mathcal{C}} & \rho\colon - \otimes 1 \Rightarrow \mathrm{id}_{\mathcal{C}} \\ \text{(the } \textit{associator}\text{)} & \text{(the } \textit{left unitor}\text{)} & \text{(the } \textit{right unitor}\text{)} \end{array}$$

such that the triangle

$$\begin{array}{ccc} (X \otimes 1) \otimes Y & \xrightarrow{\alpha_{X,1,Y}} & X \otimes (1 \otimes Y) \\ & {\scriptstyle \rho_X \otimes \mathrm{id}_Y} \searrow \quad X \otimes Y \quad \swarrow {\scriptstyle \mathrm{id}_X \otimes \lambda_Y} & \end{array}$$

and the pentagon

$$\begin{array}{ccccc} & & (X \otimes Y) \otimes (Z \otimes W) & & \\ & {\scriptstyle \alpha_{X\otimes Y,Z,W}} \nearrow & & \searrow {\scriptstyle \alpha_{X,Y,Z\otimes W}} & \\ ((X \otimes Y) \otimes Z) \otimes W & & & & X \otimes (Y \otimes (Z \otimes W)) \\ \downarrow {\scriptstyle \alpha_{X,Y,Z} \otimes \mathrm{id}_W} & & & & \uparrow {\scriptstyle \mathrm{id}_X \otimes \alpha_{Y,Z,W}} \\ (X \otimes (Y \otimes Z)) \otimes W & & \xrightarrow{\alpha_{X,Y\otimes Z,W}} & & X \otimes ((Y \otimes Z) \otimes W) \end{array}$$

commute for all objects X, Y, Z and W in $\mathcal{C}$. A *symmetric monoidal category* is a monoidal category $\mathcal{C}$ together with natural isomorphisms

$$\tau_{X,Y}\colon X \otimes Y \to Y \otimes X$$

such that

$$\tau_{Y,X} \circ \tau_{X,Y} = \mathrm{id}_{X \otimes Y}$$

and the diagrams

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{\alpha_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{\tau_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X \\ {\scriptstyle \tau_{X,Y} \otimes \mathrm{id}_Z} \downarrow & & & & \downarrow {\scriptstyle \alpha_{Y,Z,X}} \\ (Y \otimes X) \otimes Z & \xrightarrow{\alpha_{Y,X,Z}} & Y \otimes (X \otimes Z) Y & \xrightarrow{\mathrm{id}_Y \otimes \tau_{X,Z}} & Y \otimes (Z \otimes X) \end{array}$$

and

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & \xrightarrow{\alpha^{-1}_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{\tau_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) \\ {\scriptstyle \mathrm{id}_X \otimes \tau_{Y,Z}} \downarrow & & & & \downarrow {\scriptstyle \alpha^{-1}_{Z,X,Y}} \\ X \otimes (Y \otimes Z) & \xrightarrow{\alpha^{-1}_{X,Z,Y}} & (X \otimes Z) \otimes Y & \xrightarrow{\tau_{X,Z} \otimes \mathrm{id}_Y} & (Z \otimes X) \otimes Y \end{array}$$

commute for all X, Y and Z in $\mathcal{C}$.

Example 4.1.2. A basic example of a monoidal category is the category **Set** with tensor product $\times$ and unit object $\{1\}$. More generally, any category $\mathcal{C}$ with finite products can naturally be promoted to a monoidal category with tensor product $\times$ and unit object a terminal object. Such a monoidal category is called a *cartesian monoidal category*. Dually, a category with finite coproducts can be promoted to a monoidal category with tensor product $\sqcup$ and unit object an initial object, which is called a *cocartesian monoidal category*. For example, **Set** with $\sqcup$ and $\varnothing$, or the category of R-algebras $\mathbf{Alg}_R$ with $\otimes_R$ and R, for a commutative ring R. Another typical example of a monoidal category is the category of R-modules $\mathbf{Mod}_R$ with tensor product $\otimes_R$ and unit object R. All of these examples are naturally also symmetric monoidal categories.

Definition 4.1.3. A *monoidal functor* is a functor $F\colon \mathcal{C} \to \mathcal{D}$ between monoidal categories together with a natural isomorphism

$$\mu\colon F(-) \otimes_{\mathcal{D}} F(-) \Rightarrow F(- \otimes_{\mathcal{C}} -)$$

and an isomorphism $\varepsilon\colon 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$, such that the diagrams

$$\begin{array}{ccc}
(F(X) \otimes_{\mathcal{D}} F(Y)) \otimes_{\mathcal{D}} F(Z) & \xrightarrow{\alpha^{\mathcal{D}}_{F(X),F(Y),F(Z)}} & F(X) \otimes_{\mathcal{D}} (F(Y) \otimes_{\mathcal{D}} F(Z)) \\
{\scriptstyle \mu_{X,Y} \otimes \mathrm{id}_{F(Z)}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{F(X)} \otimes \mu_{Y,Z}} \\
F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} F(Z) & & F(X) \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \\
{\scriptstyle \mu_{X \otimes_{\mathcal{C}} Y, Z}} \downarrow & & \downarrow {\scriptstyle \mu_{X, Y \otimes_{\mathcal{C}} Z}} \\
F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) & \xrightarrow{F(\alpha^{\mathcal{C}}_{X,Y,Z})} & F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z))
\end{array}$$

$$\begin{array}{ccc}
1_{\mathcal{D}} \otimes_{\mathcal{D}} F(X) & \xrightarrow{\varepsilon \otimes \mathrm{id}_{F(X)}} & F(1_{\mathcal{C}}) \otimes_{\mathcal{D}} F(X) \\
{\scriptstyle \lambda^{\mathcal{D}}_{F(X)}} \downarrow & & \downarrow {\scriptstyle \mu_{1_{\mathcal{C}},X}} \\
F(X) & \xleftarrow[F(\lambda^{\mathcal{C}}_X)]{} & F(1_{\mathcal{C}} \otimes_{\mathcal{C}} X)
\end{array}
\qquad
\begin{array}{ccc}
F(X) \otimes_{\mathcal{D}} 1_{\mathcal{D}} & \xrightarrow{\mathrm{id}_{F(X)} \otimes \varepsilon} & F(X) \otimes_{\mathcal{D}} F(1_{\mathcal{C}}) \\
{\scriptstyle \rho^{\mathcal{D}}_{F(X)}} \downarrow & & \downarrow {\scriptstyle \mu_{X,1_{\mathcal{C}}}} \\
F(X) & \xleftarrow[F(\rho^{\mathcal{C}}_X)]{} & F(X \otimes_{\mathcal{C}} 1_{\mathcal{C}})
\end{array}$$

commute for all objects X, Y and Z in $\mathcal{C}$. If μ is only a natural transformation, and ε only a morphism, then such a functor is called a *lax monoidal functor*. A (lax) monoidal functor between symmetric monoidal categories is said to be *symmetric* if it respects the symmetric structure, that is, the diagram

$$\begin{array}{ccc}
F(X) \otimes_{\mathcal{D}} F(Y) & \xrightarrow{\tau^{\mathcal{D}}_{F(X),F(Y)}} & F(Y) \otimes_{\mathcal{D}} F(X) \\
{\scriptstyle \mu_{X,Y}} \downarrow & & \downarrow {\scriptstyle \mu_{Y,X}} \\
F(X \otimes_{\mathcal{C}} Y) & \xrightarrow{F(\tau^{\mathcal{C}}_{X,Y})} & F(Y \otimes_{\mathcal{C}} X)
\end{array}$$

commutes for all X and Y in $\mathcal{C}$.

4.2 Bordisms

The monoidal category that is central in the theory of TQFTs is the *category of bordisms*. By convention, we consider all manifolds to be smooth.

Definition 4.2.1. Let $n \geq 1$. Given two closed $(n-1)$-dimensional manifolds M_1 and M_2, a *bordism* from M_1 to M_2 is a compact n-dimensional manifold W with boundary ∂W together with inclusions

$$M_2 \xrightarrow{i_2} W \xleftarrow{i_1} M_1$$

such that $\partial W = i_1(M_1) \sqcup i_2(M_2)$.

Definition 4.2.2. The *category of n-bordisms*, denoted $\mathbf{Bord}_n$, is the category defined as follows.

- Its objects are closed $(n-1)$-dimensional manifolds.
- A morphism $M_1 \to M_2$ is an equivalence class of bordisms from M_1 to M_2, where two such bordisms W and W' are called equivalent if there is a diffeomorphism $f\colon W \to W'$ such that the diagram

$$\begin{array}{ccccc} & \overset{i_2}{\nearrow} & W & \overset{i_1}{\nwarrow} & \\ M_2 & & \downarrow f & & M_1 \\ & \underset{i_2'}{\searrow} & W' & \underset{i_1'}{\swarrow} & \end{array} \tag{4.1}$$

  commutes. We will also refer to such equivalences classes as *bordisms*, with the understanding that it is only up to diffeomorphism.
- The composite of morphisms $W\colon M_1 \to M_2$ and $W'\colon M_2 \to M_3$ is given by $W \sqcup_{M_2} W'\colon M_1 \to M_3$. While this operation is not well-defined on bordisms (there can be multiple manifold structures on $W \sqcup_{M_2} W'$ such that the inclusions of W and W' are smooth), such a structure is unique up to diffeomorphism, making it a well-defined operation on equivalence classes of bordisms [Mil65].
- For any closed $(n-1)$-dimensional manifold M, the identity on M is given by (the equivalence class of) the cylinder $M \times [0,1]$, with the inclusions $M \times \{0\} \to M \times [0,1] \leftarrow M \times \{1\}$.

The category $\mathbf{Bord}_n$ naturally carries the structure of a symmetric monoidal category, whose tensor product is the disjoint union operator and whose unital object is the empty manifold $\varnothing$.

Definition 4.2.3. Let R be a commutative ring. An *n-dimensional Topological Quantum Field Theory (TQFT)* over R is a monoidal functor

$$Z\colon \mathbf{Bord}_n \to \mathbf{Mod}_R$$

where $\mathbf{Mod}_R$ is monoidal with tensor product $\otimes_R$ and unit object R. If such a functor is only lax monoidal it is called a *lax n-dimensional TQFT*, and similarly if it is symmetric.

Interestingly, observe that $Z(\varnothing)$ is by definition naturally isomorphic to R for any TQFT $Z\colon \mathbf{Bord}_n \to \mathbf{Mod}_R$. Hence, any closed n-dimensional manifold W, viewed as a bordism $W\colon \varnothing \to \varnothing$, induces a morphism $Z(W)\colon R \to R$ that is multiplication by $Z(W)(1) \in R$. The element $Z(W)(1)$ is an invariant associated to W, that is, it is the same for all W' diffeomorphic to W.

Definition 4.2.4. Let χ be an R-valued invariant of closed n-dimensional manifolds. An n-dimensional TQFT Z is said to *quantize* χ if $Z(W)(1) = \chi(W)$ for all closed n-dimensional manifolds W.

There are many variations on the category of bordisms, by equipping the manifolds with extra data. One common is to equip them with an orientation.

Definition 4.2.5. Let $i\colon M \to \partial W$ be an embedding of a closed oriented $(n-1)$-dimensional manifold M into the boundary of a compact oriented n-dimensional manifold W. Then i is said to be an *in-boundary* (resp. *out-boundary*) if for all $x \in M$, positively oriented bases $v_1, \ldots, v_{n-1}$ for T_xM, and $w \in T_{i(x)}W$ pointing inwards (resp. outwards) compared to W, the basis $di_x(v_1), \ldots, di_x(v_{n-1}), w$ for $T_{i(x)}W$ is positively oriented.

Given two closed oriented $(n-1)$-dimensional manifolds M_1 and M_2, an *oriented bordism* from M_1 to M_2 is a bordism

$$M_2 \xrightarrow{i_2} W \xleftarrow{i_1} M_1$$

with an orientation on W such that i_1 an *in-boundary* and i_2 is an *out-boundary.*

The *category of oriented n-bordisms*, denoted $\mathbf{Bord}_n^{\mathrm{or}}$, is the category whose objects are closed oriented $(n-1)$-dimensional manifolds, and morphisms are equivalence classes of oriented bordisms, with composition given as for $\mathbf{Bord}_n$.

Finally, an *n-dimensional oriented TQFT* over R is a monoidal functor

$$Z\colon \mathbf{Bord}_n^{\mathrm{or}} \to \mathbf{Mod}_R.$$

While any TQFT induces an oriented TQFT, simply by forgetting the orientation, not every oriented TQFT can be extended to a TQFT, as will be shown in Section 4.4.

Remark 4.2.6. Although in Definition 4.2.2 the category of bordisms $\mathbf{Bord}_n$ was defined as a 1-category, it can naturally be promoted to a 2-category: the objects still being closed $(n-1)$-dimensional manifolds, a 1-morphism being a bordism (rather than an equivalence class of bordisms), and a 2-morphism between bordisms being an equivalence as in (4.1). Now, if we view $\mathbf{Bord}_n$ as a 2-category, it is only natural to promote $\mathbf{Mod}_R$ to a 2-category as well and require a TQFT to be a 2-functor. We will promote $\mathbf{Mod}_R$ in the trivial way, where the only 2-morphisms are identity morphisms. Note that, in essence, this does not change the definition of a TQFT, since two bordisms which are equivalent must be sent to the same R-linear map. Therefore, in the context of TQFTs as in Definition 4.2.3, it does not matter whether we view $\mathbf{Bord}_n$ as a 2-category or simply as a 1-category.

For the correct notions of monoidal categories and monoidal functors in the context of 2-categories, see [KV94, BN96].

4.3 Physical interpretation

As mentioned, the notion of a TQFT originates from physics, and was first mathematically axiomatized by Atiyah [Ati88]. While not strictly necessary, we believe it is helpful to discuss the physical interpretation of these objects for a better intuition of the remainder of this chapter.

A TQFT describes a quantum mechanical system, specifically a quantum field theory. Space, at some point in time, is represented by a closed manifold, that is, an object of $\mathbf{Bord}_n$. A morphism in this category, a bordism connecting such manifolds, represents a part of spacetime, where the extra dimension corresponds to the dimension of time.

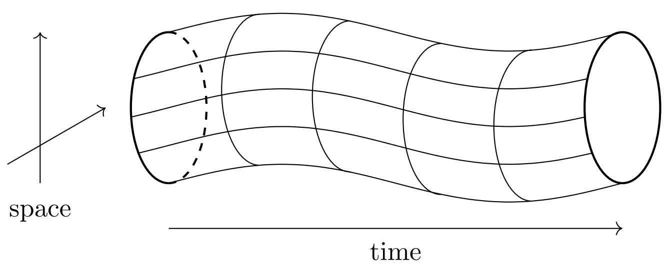


Figure 4.1: A bordism connecting two boundaries, representing a part of spacetime connecting space at two points in time.

For simplicity, let us take $R = \mathbb{C}$. Then a TQFT assigns to a space M a complex vector space $\mathcal{H} = Z(M)$ and to a spacetime $W\colon M_1 \to M_2$ a linear map $Z(W)\colon \mathcal{H}_1 \to \mathcal{H}_2$. We can think of the vector space $\mathcal{H}$ as the Hilbert space associated to M, that is, the vector space of all quantum states on this space. The linear map $Z(W)$ describes the time-evolution of the system.

What makes a TQFT '*topological*', is that the system it describes has no actual dynamics. That is, the Hamiltonian of the system is zero, and only topological effects come into play. For instance, the cylinder $M \times [0, 1]$, being topologically trivial, induces the identity on $\mathcal{H}$, and hence does not change the state of the system.

It is not uncommon for Hilbert spaces to be infinite-dimensional, and in this case the tensor product of Hilbert spaces is not simply the tensor product of the underlying vector spaces: it should be completed. For this reason, the Hilbert space $\mathcal{H}$ associated to a disjoint union $M_1 \sqcup M_2$ is not necessarily expected to be equal to the tensor product (as vector spaces) of the Hilbert spaces associated to M_1 and M_2, but at least there should be a natural morphism $\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2 \to \mathcal{H}$. Although this might be an indication that the category of vector spaces is not quite the correct target for a TQFT, we will take it as motivation for the definition of a lax TQFT.

A common way to construct a TQFT is as the composite of two functors, a *field theory* and a *quantization functor*. This corresponds to describing a classical field theory, followed by a quantization procedure. The field theory $\mathcal{F}$ assigns to a manifold M a *phase space* $\mathcal{F}(M)$: a geometric object parametrizing all possible classical states, or field configuration, of the system on M. For simplicity, we can think of such a state or field as a vector bundle or a local system on M. Note that, given a bordism $W\colon M_1 \to M_2$, a field over W can be restricted to a field over any of the boundaries. In particular, we obtain the following diagram.

$$\mathcal{F}(M_1) \xleftarrow{\;i_1^*\;} \mathcal{F}(W) \xrightarrow{\;i_2^*\;} \mathcal{F}(M_2)$$

Such a diagram is known as a *correspondence* from $\mathcal{F}(M_1)$ to $\mathcal{F}(M_2)$. The field theory $\mathcal{F}$ should therefore be a functor from $\mathbf{Bord}_n$ to the category of correspondences, whose objects are some kind of geometric objects and whose morphisms are correspondences between them. Now, let us consider what happens to a composite of bordisms. Two bordisms $W\colon M_1 \to M_2$ and $W' : M_2 \to M_3$ induce the following diagram.

$$\begin{array}{ccccccc} & & & \mathcal{F}(W \sqcup_{M_2} W') & & & \\ & & \swarrow & & \searrow & & \\ & \mathcal{F}(W) & & & & \mathcal{F}(W') & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ \mathcal{F}(M_1) & & & \mathcal{F}(M_2) & & & \mathcal{F}(M_3) \end{array}$$

A field over $W \sqcup_{M_2} W'$ is essentially the same as a field over W and a field over W' that agree over M_2, so the middle square will be cartesian. This is precisely how the composition of correspondences is defined, so $\mathcal{F}$ will indeed be a functor.

Next, let us consider the quantization functor $\mathcal{Q}$. This functor assigns to the phase space $\mathcal{F}(M)$ a complex vector space, its Hilbert space, whose vectors represent the *quantum states* of the system on M. For simplicity, we can think of a quantum state as a complex-valued function (a *wave function*) on $\mathcal{F}(M)$, which describes a distribution or superposition of classical states. Furthermore, on correspondences, $\mathcal{Q}$ is commonly given by a 'pull-push' construction. Given a quantum state $\psi_1 \in \mathcal{Q}(\mathcal{F}(M_1))$, that is, a complex-valued function on $\mathcal{F}(M_1)$, one can pull back ψ_1 along i_1^* to obtain $\Psi = \psi_1 \circ i_1^* \in \mathcal{Q}(\mathcal{F}(W))$, a complex-valued function on $\mathcal{F}(W)$. Next, one can push forward Ψ along i_2^*, by integrating along fibers, to obtain $\psi_2 \in \mathcal{Q}(\mathcal{F}(M_2))$ given by

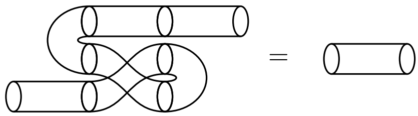
$$\psi_2(y) = \int_{(i_2^*)^{-1}(y)} \Psi(x)\, dx,$$

provided such an integral exists. The resulting map $\mathcal{Q}(\mathcal{F}(M_1)) \to \mathcal{Q}(\mathcal{F}(M_2))$ corresponds, roughly speaking, to a *(Feynman) path integral*. That is, the amplitude corresponding to state y is determined by considering all possible paths to state y (points on $\mathcal{F}(W)$ over y), and their amplitudes are added.

More generally, one could replace the above integral by a weighted integral with weight $e^{iS(x)}$, where S is a function on $\mathcal{F}(W)$ called the *action*. For TQFTs there is no such weighting, since only the topology of the bordisms is considered, and no other extra data. However, equipping the bordisms with extra data can result in QFTs with non-trivial actions. For example, one obtains a conformal field theory by equipping the bordisms with a conformal structure.

4.4 Low-dimensional TQFTs

Let us discuss some properties of an oriented TQFT $Z\colon \mathbf{Bord}_n^{\mathrm{or}} \to \mathbf{Mod}_R$. Given a closed oriented $(n-1)$-dimensional manifold M, denote by $\overline{M}$ the same manifold but with opposite orientation, and by $U_M : \varnothing \to M \sqcup \overline{M}$ and $U_M^\dagger : M \sqcup \overline{M} \to \varnothing$ be the bordisms with underlying manifold $M \times [0,1]$. Note that the map $Z(U_M)\colon R \to Z(M) \otimes_R Z(\overline{M})$ is completely determined by the element $Z(U_M)(1) = \sum_{i=1}^m v_i \otimes \overline{v}_i$ with $v_i \in Z(M)$ and $\overline{v}_i \in Z(\overline{M})$. From the equality $(U_M^\dagger \sqcup \mathrm{id}_M) \circ (\mathrm{id}_M \sqcup \tau_{M,\overline{M}}) \circ (\mathrm{id}_M \sqcup U_M) = \mathrm{id}_M$, depicted pictorially as



it follows that

$$v = \sum_{i=1}^m Z(U_M^\dagger)(v \otimes \overline{v}_i)\, v_i$$

for all $v \in Z(M)$. In particular, $Z(M)$ is generated by $v_1, \ldots, v_m$. Similarly, from the equality $(U_M^\dagger \sqcup \mathrm{id}_{\overline{M}}) \circ (\tau_{\overline{M},M} \sqcup \mathrm{id}_{\overline{M}}) \circ (\mathrm{id}_{\overline{M}} \sqcup U_M) = \mathrm{id}_{\overline{M}}$ it follows that

$$\overline{v} = \sum_{i=1}^m Z(U_M^\dagger)(v_i \otimes \overline{v})\, \overline{v}_i$$

for all $\overline{v} \in Z(\overline{M})$, so $Z(\overline{M})$ is generated by $\overline{v}_1, \ldots, \overline{v}_m$. This shows that $Z(M)$ is a *dualizable object* with dual $Z(\overline{M})$, unit $Z(U_M)$ and counit $Z(U_M^\dagger)$.

Remark 4.4.1. If Z is only a lax TQFT, the image of $Z(U_M)$ need not necessarily lie in the tensor product $Z(M) \otimes Z(\overline{M})$, and consequently, the module $Z(M)$ need not be finitely generated. This will be the case for the TQFT constructed in Section 4.7.

In the category of R-modules, the dualizable objects are precisely the finitely generated projective modules [PS14]. In dimension $n = 1$, this completely characterizes the TQFT.

Proposition 4.4.2. *Let R be a commutative ring. There is an equivalence of categories*

$$\mathbf{1\text{-}TQFT}_R^{\mathrm{or}} \simeq \mathbf{FGProjMod}_R$$

between the category of 1-dimensional oriented TQFTs over R and the category of finitely generated projective R-modules, which assigns to a TQFT Z the R-module $Z(p)$, where p is the point with orientation $+1$.

Proof. As shown above, an oriented 1-TQFT Z over R determines a dualizable (that is, finitely generated projective) R-module $M = Z(p)$. From a morphism between such TQFTs (a natural transformation) we obtain a morphism between the corresponding modules. This gives a functor $\mathbf{1\text{-}TQFT}_R^{\mathrm{or}} \to \mathbf{FGProjMod}_R$.

Conversely, let M be a dualizable R-module. The objects of $\mathbf{Bord}_1^{\mathrm{or}}$ are finite disjoint unions of p and $\overline{p}$. As shown above, $Z(\overline{p})$ is dual to $Z(p)$, so by monoidality, specifying $Z(p) = M$ determines Z on objects, with $Z(\overline{p}) = \mathrm{Hom}_R(M, R)$. The only connected bordisms in $\mathbf{Bord}_1^{\mathrm{or}}$ are

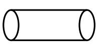
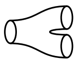
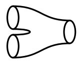
$$\mathrm{id}_p \colon p \to p, \quad \mathrm{id}_{\overline{p}} \colon \overline{p} \to \overline{p}$$

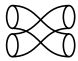
$$U_p \colon \varnothing \to p \sqcup \overline{p}, \quad U_p^\dagger \colon p \sqcup \overline{p} \to \varnothing$$

and $S^1 = U_p^\dagger \circ U_p \colon \varnothing \to \varnothing$, all of whose image under Z is canonically determined by the unit and counit of the dualizable module M. This construction, being natural in M, defines a functor $\mathbf{FGProjMod}_R \to \mathbf{1\text{-}TQFT}_R^{\mathrm{or}}$.

These functors are easily seen to be pseudo-inverses of each other, establishing the equivalence of categories. □

A similar characterization of oriented TQFTs can be given in dimension $n = 2$. The objects of $\mathbf{Bord}_2^{\mathrm{or}}$ are disjoint unions of S^1, the circle, where we fix an orientation of S^1. Using the classification of oriented surfaces, one can show the morphisms in $\mathbf{Bord}_2^{\mathrm{or}}$ are 'generated' by the following bordisms:

 $\colon S^1 \to S^1$,  $\colon S^1 \sqcup S^1 \to S^1$,  $\colon S^1 \to S^1 \sqcup S^1$,

 $\colon S^1 \to \varnothing$,  $\colon \varnothing \to S^1$ and  $\colon S^1 \sqcup S^1 \to S^1 \sqcup S^1$. (4.2)

That is, any bordism in $\mathbf{Bord}_2^{\mathrm{or}}$ is isomorphic to a composite of disjoint unions of these bordisms [Koc04, Proposition 1.4.13]. Hence, we expect 2-dimensional oriented TQFTs to correspond to dualizable modules with some extra algebraic structure. For $R = k$ a field, the correct algebraic structure turns out to be that of a *Frobenius algebra*.

Definition 4.4.3. A *Frobenius algebra* over a field k is an algebra A over k, whose multiplication and unit we denote by $\mu \colon A \otimes_k A \to A$ and $\eta \colon k \to A$, equipped with a bilinear form $\beta \colon A \otimes_k A \to k$, which is

- associative, that is, $\beta(\mu(a \otimes b) \otimes c) = \beta(a \otimes \mu(b \otimes c))$ for all $a, b, c \in A$,
- non-degenerate, that is, there exists a k-linear map $\gamma \colon k \to A \otimes_k A$ such that $(\beta \otimes \mathrm{id}_A)(a \otimes \gamma(1)) = a = (\mathrm{id}_A \otimes \beta)(\gamma(1) \otimes a)$ for all $a \in A$.

Remark 4.4.4. For any Frobenius algebra A, writing $\gamma(1) = \sum_i a_i \otimes b_i$ for some $a_i, b_i \in A$, we find that $a = (\mathrm{id}_A \otimes \beta)(\sum_i a_i \otimes b_i \otimes a) = \sum_i a_i \, \beta(b_i \otimes a)$ for all $a \in A$. In particular, A is finite-dimensional and generated by the a_i. This equality also shows that β is non-degenerate in the usual sense: if $\beta(b \otimes a) = 0$ for all $b \in A$, then $a = 0$. Similarly, one shows non-degeneracy in the other argument. Furthermore, this implies γ must be unique. Namely, if γ and γ' both satisfy the condition, then write $\gamma(1) - \gamma'(1) = \sum_i a_i \otimes b_i$ with $a_i, b_i \in A$ and the a_i are linearly independent. Since $0 = (\mathrm{id}_A \otimes \beta)(\sum_i a_i \otimes b_i \otimes a) = \sum_i a_i \, \beta(b_i \otimes a)$ for all $a \in A$, it follows that $b_i = 0$ for all i, so $\gamma(1) = \gamma'(1)$.

A Frobenius algebra naturally carries the structure of a k-coalgebra, see [Koc04, Section 2.3], where the comultiplication δ and counit ε are given by

$$\delta = (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \gamma) \quad \text{and} \quad \varepsilon = \beta \circ (\mathrm{id}_A \otimes \eta). \tag{4.3}$$

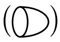
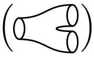
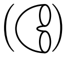
A *morphism of Frobenius algebras* is a morphism of k-algebras which is also a morphism of k-coalgebras.

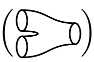
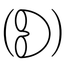
The following theorem makes a precise correspondence between 2-dimensional oriented TQFTs and Frobenius algebras. It was initially proved by Dijkgraaf [Dij89], and later reproved in more detail by others, such as [Abr96, Koc04].

Theorem 4.4.5. *Let k be a field. There is an equivalence of categories*

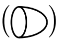
$$\mathbf{2\text{-}TQFT}_k^{\mathrm{or}} \simeq \mathbf{CFrobAlg}_k$$

between the category of 2-dimensional oriented TQFTs over k and the category of commutative Frobenius algebras over k, which assigns to a TQFT Z the Frobenius algebra $A = Z(S^1)$ and

Z () $= \eta$, Z () $= \mu$, Z () $= \beta$,

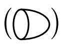
Z () $= \varepsilon$, Z () $= \delta$, Z () $= \gamma$.

Example 4.4.6. Let S be a finite set and let $A = k^S$ be the k-algebra of k-valued functions on S, where multiplication is given pointwise. Then A admits the structure of a Frobenius algebra with $\beta(f \otimes g)(s) = \sum_{s \in S} f(s)g(s)$ and $\gamma(1) = \sum_{s \in S} \mathbb{1}_s \otimes \mathbb{1}_s$, where $\mathbb{1}_s$ denotes the indicator function. From (4.3), we find that the coalgebra structure is given by $\varepsilon(f) = \sum_{s \in S} f(s)$ and $\delta(f) = \sum_{s \in S} f(s) \mathbb{1}_s \otimes \mathbb{1}_s$. For any closed surface Σ_g of genus g, we have

$Z(\Sigma_g)(1) = Z$ () $\circ Z$ ()$^g \circ Z$ () $= (\varepsilon \circ (\mu \circ \delta)^g \circ \eta)(1) = |S|$.

Therefore, $Z(M)(1) = |S|^{\pi_0(M)}$ for any general closed oriented surface M, that is, Z quantizes the number of connected components of M.

Example 4.4.7. The complex numbers $A = \mathbb{C}$ are a Frobenius algebra over $k = \mathbb{R}$ with $\beta(z_1 \otimes z_2) = \mathrm{Re}(z_1 z_2)$ and $\gamma(1) = 1 \otimes 1 - i \otimes i$. One quickly finds that $\delta(z) = z \otimes 1 - iz \otimes i$ and $\varepsilon(z) = \mathrm{Re}(z)$. The corresponding TQFT yields

$Z(\Sigma_g)(1) = Z$ () $\circ Z$ ()$^g \circ Z$ () $= (\varepsilon \circ (\mu \circ \delta)^g \circ \eta)(1) = 2^g$.

In this sense, this TQFT quantizes the genus of the surface.

Remark 4.4.8. Let us make a note about the difference between oriented and unoriented TQFTs. Clearly, via the forgetful functor $\mathbf{Bord}_n^{\mathrm{or}} \to \mathbf{Bord}_n$, which forgets the orientation, any TQFT induces an oriented TQFT. In this sense, an unoriented TQFT can be seen as an oriented TQFT with extra structure. However, not every oriented TQFT arises in such a way. This follows from Proposition 4.4.2 and the fact that not every finitely generated projective module (that is, dualizable module) is isomorphic to its dual.

4.5 Representation ring as TQFT

Let G be a finite group, and denote by $A = R_{\mathbb{C}}(G)$ the representation ring of G, that is, the complex algebra generated by $\mathbb{C}$-valued class functions on G. Of great importance in the representation theory of G is the inner product that is defined on A, which we will denote by $\beta\colon A \otimes_{\mathbb{C}} A \to \mathbb{C}$, and which is given by

$$\beta(a \otimes b) = \frac{1}{|G|} \sum_{g \in G} a(g) b(g^{-1}) \quad \text{for } a, b \in A.$$

A lesser known but equally important operation on A is the convolution operation $\mu\colon A \otimes_{\mathbb{C}} A \to A$ on A, which is given by

$$\mu(a \otimes b)(g) = \sum_{h \in G} a(h) b(h^{-1} g) \quad \text{for } a, b \in A,$$

and is related to the inner product via $\beta(a \otimes b) = \mu(a \otimes b)(1)$ for $a, b \in A$. The unit $\eta\colon \mathbb{C} \to A$ with respect to μ is given by $\eta(1)(1) = 1$ and $\eta(1)(g) = 0$ for $g \neq 1$. Alternatively, η can be expressed as

$$\eta(1) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \chi(1) \chi, \tag{4.4}$$

where $\hat{G}$ denotes the set of irreducible complex characters of G. These operations give $R_{\mathbb{C}}(G)$ the structure of a commutative Frobenius algebra over $\mathbb{C}$.

Proposition 4.5.1. *The representation ring $R_{\mathbb{C}}(G)$ is a commutative Frobenius algebra over $\mathbb{C}$ with multiplication μ and bilinear form β.*

Proof. First note that μ is associative as

$$\begin{aligned} \mu(a \otimes \mu(b \otimes c))(g) &= \sum_{h_1, h_2 \in G} a(h_1) b(h_2) c(h_2^{-1} h_1^{-1} g) \\ &= \sum_{h_1', h_2' \in G} a(h_2') b({h_2'}^{-1} h_1') c({h_1'}^{-1} g) = \mu(\mu(a \otimes b) \otimes c)(g) \end{aligned}$$

for all $a, b, c \in A$ and $g \in G$, where $h_1' = h_1 h_2$ and $h_2' = h_1$, and μ is commutative as

$$\mu(a \otimes b)(g) = \sum_{h \in G} a(h) b(h^{-1} g) = \sum_{h' \in G} b(h') a(h'^{-1} g) = \mu(b \otimes a)(g)$$

for all $a, b \in A$ and $g \in G$, where $h' = h^{-1} g$. Furthermore, β is associative as

$$\begin{aligned} \beta(\mu(a \otimes b) \otimes c) &= \frac{1}{|G|} \sum_{g,h \in G} a(h) b(h^{-1} g) c(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g,h \in G} a(g) b(g^{-1} h) c(h^{-1}) = \beta(a \otimes \mu(b \otimes c)) \end{aligned}$$

for all $a, b, c \in A$. Finally, β is non-degenerate as $\gamma \colon \mathbb{C} \to A \otimes_{\mathbb{C}} A$ given by $\gamma(1) = \sum_{\chi \in \hat{G}} \chi \otimes \chi$ satisfies

$$(\beta \otimes \mathrm{id}_A)(a \otimes \gamma(1)) = \sum_{\chi \in \hat{G}} \beta(a \otimes \chi) \chi = a$$

by the first orthogonality theorem [Ser77, Theorem 3], and by the same argument $(\mathrm{id}_A \otimes \beta)(\gamma(1) \otimes a) = a$, for all $a \in A$. □

Remark 4.5.2. The copairing $\gamma \colon \mathbb{C} \to A \otimes_{\mathbb{C}} A$, or rather $\gamma(1)$, can be seen as an inner product on the conjugacy classes of G. As a function $G \times G \to \mathbb{C}$, it is given by

$$\gamma(1)(g_1, g_2) = |\{h \in G \mid h g_1 h^{-1} = g_2\}|.$$

Under the equivalence of Theorem 4.4.5, the representation ring $A = R_{\mathbb{C}}(G)$ corresponds to a 2-dimensional oriented TQFT

$$Z_G \colon \mathbf{Bord}_2^{\mathrm{or}} \to \mathbf{Vect}_{\mathbb{C}}.$$

From (4.3), we find that the comultiplication $\delta \colon A \to A \otimes_{\mathbb{C}} A$ and counit $\varepsilon \colon A \to \mathbb{C}$ on A are given by

$$\delta(a) = \sum_{\chi \in \hat{G}} \mu(a \otimes \chi) \otimes \chi \quad \text{and} \quad \varepsilon(a) = \frac{1}{|G|} a(1). \tag{4.5}$$

The convolution of irreducible characters $\chi, \chi' \in \hat{G}$ is well known [Isa76, Theorem 2.13] to be given by

$$\mu(\chi \otimes \chi') = \begin{cases} \frac{|G|}{\chi(1)} \chi & \text{if } \chi = \chi', \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

This implies that, for any irreducible character $\chi \in \hat{G}$,

$Z_G\big($  $\big)(\chi) = (\mu \circ \delta)(\chi) = \dfrac{|G|^2}{\chi(1)^2}\chi.$ (4.7)

In other words, the irreducible characters of G form a basis of eigenvectors for the map $Z_G\big($  $\big)$. The following theorem describes the invariant that this TQFT quantizes.

Theorem 4.5.3. *The TQFT Z_G quantizes the groupoid cardinality $|\mathfrak{X}_G(\Sigma_g)| = |R_G(\Sigma_g)|/|G|$. In particular,*

$$Z_G(\Sigma_g)(1) = \sum_{\chi \in \hat{G}} \left(\frac{|G|}{\chi(1)}\right)^{2g-2} = |\mathfrak{X}_G(\Sigma_g)|.$$

Proof. The first equality follows from (4.7) and the expressions for η and ε. For the second equality, let $f\colon G \to \mathbb{C}$ be the class function given by $f(g) = |\{(A,B) \in G^2 \mid [A,B] = g\}|$. From the explicit presentation of $R_G(\Sigma_g)$,

$$R_G(\Sigma_g) = \left\{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] = 1 \right\},$$

and the definition of the convolution operator on $R_{\mathbb{C}}(G)$, it is clear that

$$|R_G(\Sigma_g)| = (\underbrace{f * \cdots * f}_{g \text{ times}})(1),$$

where $f * f = \mu(f \otimes f)$. Therefore, it suffices to show that f is equal to

$Z_G\big($  $\circ$  $\big)(1) = (\mu \circ \delta \circ \eta)(1) = \displaystyle\sum_{\chi \in \hat{G}} \frac{|G|}{\chi(1)}\chi,$

or equivalently, that $\beta(f \otimes \chi) = |G|/\chi(1)$ for every irreducible complex character χ of G. Note that $\beta(f \otimes \chi)$ is equal to

$$\frac{1}{|G|}\sum_{g \in G} f(g)\chi(g^{-1}) = \frac{1}{|G|}\sum_{A,B \in G} \chi([A,B]^{-1}) = \frac{1}{|G|}\sum_{A,B \in G} \chi(BAB^{-1}A^{-1}).$$

Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation with character χ. Schur's lemma implies that, for any $A \in G$, the operator $T_A = \sum_{B \in G} \rho(BAB^{-1})$ is a scalar multiple of the identity, that is, $T_A = \mathrm{tr}(T_A)/\chi(1) = |G|\chi(A)/\chi(1)$. Hence, it follows that

$$\beta(f \otimes \chi) = \frac{1}{|G|}\sum_{A,B \in G} \mathrm{tr}(T_A A^{-1}) = \frac{1}{|G|}\sum_{A \in G} \frac{|G|}{\chi(1)}\chi(A)\chi(A^{-1}) = \frac{|G|}{\chi(1)}. \qquad \square$$

Example 4.5.4. When G is abelian, all irreducible representations of G are of one-dimensional, so $Z_G\big($  $\big)$ is simply multiplication by $|G|^2$. Therefore, $|R_G(\Sigma_g)| = |G|^{2g}$ as expected.

4.6 Arithmetic method

Let us elaborate on the arithmetic method from [HR08]. Given a complex algebraic group G, typically a linear algebraic group such as GL_n or SL_n, the goal of this method is to compute the E-polynomial of the G-representation variety $R_G(\Sigma_g)$. It tries to accomplish this using the following theorem, which is a consequence of [HR08, Theorem 6.1.2].

Theorem 4.6.1 (Katz' theorem)**.** *Let X be a complex variety with a spreading-out $\tilde{X}$ over a finitely generated $\mathbb{Z}$-algebra $R \subseteq \mathbb{C}$. If there exists a polynomial $P \in \mathbb{Z}[q]$ such that $|(\tilde{X} \times_R \mathbb{F}_q)(\mathbb{F}_q)| = P(q)$ for all ring morphisms $R \to \mathbb{F}_q$, then the E-polynomial of X is given by $P(uv) \in \mathbb{Z}[u, v]$.*

Most common linear algebraic groups G, such as GL_n and SL_n, can be defined over $\mathbb{Z}$, which determine a natural spreading-out of $R_G(\Sigma_g)$ over $R = \mathbb{Z}$. In this case, we find that

$$\begin{aligned} |R_G(\Sigma_g)(\mathbb{F}_q)| &= |\operatorname{Hom}(\pi_1(\Sigma_g, *), G(\mathbb{F}_q))| \\ &= |G(\mathbb{F}_q)|\, |\mathfrak{X}_{G(\mathbb{F}_q)}(\Sigma_g)| \\ &= |G(\mathbb{F}_q)| Z_{G(\mathbb{F}_q)}(\Sigma_g)(1). \end{aligned}$$

Hence, to compute this point count, we can apply Theorem 4.5.3. This reduces the problem to studying the representation theory of the finite groups $G(\mathbb{F}_q)$, or more specifically, to studying the dimensions of their irreducible representations. This was originally done for the groups $G = \mathrm{GL}_n$ in [HR08], and later also for the groups $G = \mathrm{SL}_n$ in [Mer15].

However, note that it is not clear at all why these point counts should be polynomial in q. It will be, by Theorem 4.5.3, when $|G(\mathbb{F}_q)|$ is polynomial in q and the irreducible representations χ of $G(\mathbb{F}_q)$ come in families in which both $\chi(1)$ and the size of the family is polynomial in q. This is the case in [HR08, Mer15], but fails already when G is not a linear algebraic group, such as an elliptic curve.

An amazing result by [BK22] shows that the above quantities are polynomial in q when G is a connected split reductive group, using Lusztig's Jordan decomposition to describe the irreducible representations of $G(\mathbb{F}_q)$. More precisely, these quantities are polynomial in q after fixing a congruence condition $q \equiv i \bmod d$, where d is an integer depending on the root datum of G, and i can be any integer. By appropriate choice of finitely generated $\mathbb{Z}$-algebra R, one can enforce the congruence condition $q \equiv 1 \bmod d$, implying that the E-polynomial of the character stack $\mathfrak{X}_G(\Sigma_g)$ is polynomial in uv [BK22, Corollary 4].

4.7 Character stack TQFT

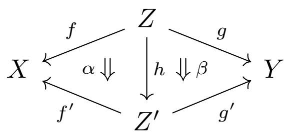
In this section, we will construct a lax TQFT quantizing the virtual class of the G-character stack in the Grothendieck ring of stacks. The construction of this TQFT is an adaptation of the work of González-Prieto, Logares and Muñoz [GLM20], the main differences being that we will not fix a set of basepoints on our manifolds, and that we focus on the character stack rather than the representation variety.

Fix a base scheme S and let G be a linear algebraic group over S. Like described in Section 4.3, the TQFT will be constructed as the composite of two functors, a field theory $\mathcal{F}_G$ and a quantization functor $\mathcal{Q}$,

$$\mathbf{Bord}_n \xrightarrow{\mathcal{F}_G} \mathrm{Corr}(\mathbf{Stck}_S) \xrightarrow{\mathcal{Q}} \mathrm{K}_0(\mathbf{Stck}_S)\text{-}\mathbf{Mod}$$

where the category $\mathrm{Corr}(\mathbf{Stck}_S)$ is defined as follows. Recall from Definition 1.6.4 that $\mathbf{Stck}_S$ is the 2-category of algebraic stacks of finite type over S with affine stabilizers, which has pullbacks by Lemma 1.6.3.

Definition 4.7.1. Let $\mathcal{C}$ be a 2-category with pullbacks. The *category of correspondences* over $\mathcal{C}$ is the 2-category, denoted by $\mathrm{Corr}(\mathcal{C})$, defined as follows. Its objects are the objects of $\mathcal{C}$. A 1-morphism from X to Y is a *correspondence*, that is, a diagram $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathcal{C}$. A 2-morphism between correspondences $X \xleftarrow{f} Z \xrightarrow{g} Y$ and $X \xleftarrow{f'} Z' \xrightarrow{g'} Y$ is an isomorphism $h\colon Z \to Z'$ in $\mathcal{C}$ together with 2-isomorphisms $\alpha\colon f \Rightarrow f' \circ h$ and $\beta\colon g \Rightarrow g' \circ h$.



The composition of correspondences $X \xleftarrow{f} Y \xrightarrow{g} X'$ and $X' \xleftarrow{f'} Y' \xrightarrow{g'} X''$ is given by $X \xleftarrow{f \circ \pi_Y} Y \times_{X'} Y' \xrightarrow{g \circ \pi_{Y'}} X''$. If $\mathcal{C}$ is monoidal, then so is $\mathrm{Corr}(\mathcal{C})$.

Remark 4.7.2. Correspondences over $\mathcal{C}$ can be viewed as an extension of morphisms in $\mathcal{C}$, since any morphism $f\colon X \to Y$ in $\mathcal{C}$ can be seen as the correspondence $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y$.

Let us start with the field theory. For this we want to consider $\mathbf{Bord}_n$ as a 2-category, as in Remark 4.2.6.

Definition 4.7.3. Let $\mathcal{F}_G\colon \mathbf{Bord}_n \to \mathrm{Corr}(\mathbf{Stck}_S)$ be the 2-functor that assigns to a closed manifold M the character stack $\mathfrak{X}_G(M)$, to a bordism $W\colon M_1 \to M_2$ the correspondence

$$\mathfrak{X}_G(M) \leftarrow \mathfrak{X}_G(W) \to \mathfrak{X}_G(M')$$

induced by the inclusions $M_i \to W$, and finally to an equivalence of bordisms $f\colon W \to W'$ the diagram

$$\begin{array}{ccccc} & & \mathfrak{X}_G(W) & & \\ & \swarrow & \downarrow\wr & \searrow & \\ \mathfrak{X}_G(M_1) & & & & \mathfrak{X}_G(M_2) \\ & \nwarrow & \downarrow & \nearrow & \\ & & \mathfrak{X}_G(W') & & \end{array}$$

where the vertical isomorphism is induced by f.

Proposition 4.7.4. *$\mathcal{F}_G$ defines a symmetric monoidal functor.*

Proof. As $M\times[0,1]$ is homotopy equivalent to M, it follows that $\mathfrak{X}_G(M\times[0,1]) \cong \mathfrak{X}_G(M)$ for any closed manifold M, so $\mathcal{F}_G$ preserves identity morphisms. For any two bordisms $W\colon M_1 \to M_2$ and $W'\colon M_2 \to M_3$, the diagram

$$\begin{array}{ccccccccc} & & & & \Pi(W' \circ W) & & & & \\ & & & \nearrow & & \nwarrow & & & \\ & & \Pi(W) & & & & \Pi(W') & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \Pi(M_1) & & & & \Pi(M_2) & & & & \Pi(M_3) \end{array}$$

naturally commutes, and the square is a pushout square by the Seifert–van Kampen theorem for fundamental groupoids [Bro67]. Lemma 2.3.6 implies that the resulting square on G-character stacks is a cartesian square, which shows that $\mathcal{F}_G$ is functorial. The same lemma also shows $\mathfrak{X}_G(M_1 \sqcup M_2)$ is naturally isomorphic to $\mathfrak{X}_G(M_1) \times \mathfrak{X}_G(M_2)$, and this isomorphism clearly respects the symmetric monoidal structure, that is, $\mathcal{F}_G$ is symmetric monoidal. ☐

Next, we define the quantization functor. As in Remark 4.2.6, we view the category $\mathrm{K}_0(\mathbf{Stck}_S)$-**Mod** as a 2-category in the trivial way, and define $\mathcal{Q}$ as a 2-functor. Equivalently, one can think of $\mathcal{Q}$ as a 1-functor after identifying isomorphic 1-morphisms in $\mathrm{Corr}(\mathbf{Stck}_S)$.

Definition 4.7.5. Let $\mathcal{Q}\colon \mathrm{Corr}(\mathbf{Stck}_S) \to \mathrm{K}_0(\mathbf{Stck}_S)$-**Mod** be the 2-functor that assigns to an object $\mathfrak{X}$ the $\mathrm{K}_0(\mathbf{Stck}_S)$-module

$$\mathcal{Q}(\mathfrak{X}) = \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X}})$$

and to a correspondence $\mathfrak{X} \xleftarrow{f} \mathfrak{Z} \xrightarrow{g} \mathfrak{Y}$ the morphism

$$\mathcal{Q}(\mathfrak{X} \xleftarrow{f} \mathfrak{Z} \xrightarrow{g} \mathfrak{Y}) = g_! \circ f^*\colon \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X}}) \to \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{Y}})$$

with f^* and $g_!$ as in Section 3.2. Note that two correspondences connected by a 2-morphism are indeed assigned to the same $\mathrm{K}_0(\mathbf{Stck}_S)$-module morphism.

Proposition 4.7.6. *$\mathcal{Q}$ is a symmetric lax monoidal functor.*

Proof. For any object $\mathfrak{X}$ of $\mathbf{Stck}_S$, it is immediate from the definition that $\mathcal{Q}(\mathrm{id}_{\mathfrak{X}}) = \mathrm{id}_{\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X}})}$. Consider a composite of correspondences

$$\begin{array}{ccccccc} & & & \mathfrak{Z} & & & \\ & & \overset{h}{\swarrow} & & \overset{j}{\searrow} & & \\ & \mathfrak{Y}_1 & & & & \mathfrak{Y}_2 & \\ \overset{f_1}{\swarrow} & & \overset{g_1}{\searrow} & & \overset{f_2}{\swarrow} & & \overset{g_2}{\searrow} \\ \mathfrak{X}_1 & & & \mathfrak{X}_2 & & & \mathfrak{X}_3 \end{array}$$

for which the square is a 2-cartesian square. To show $\mathcal{Q}$ respects composition, it suffices to show $f_2^* \circ (g_1)_! = j_! \circ h^*$ as morphisms $\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{Y}_1}) \to \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{Y}_2})$. This is true for formal reasons: for any $\mathfrak{U} \to \mathfrak{Y}_1$, the diagram

$$\begin{array}{ccccc} \mathfrak{U} \times_{\mathfrak{Y}_1} \mathfrak{Z} & \longrightarrow & \mathfrak{Z} & \xrightarrow{j} & \mathfrak{Y}_2 \\ \downarrow & & \downarrow h & & \downarrow f_2 \\ \mathfrak{U} & \longrightarrow & \mathfrak{Y}_1 & \xrightarrow{g_1} & \mathfrak{X}_2 \end{array}$$

is a 2-cartesian rectangle as both squares are 2-cartesian squares. Therefore, $(f_2^* \circ (g_1)_!)([\mathfrak{U} \to \mathfrak{Y}_1]) = [\mathfrak{U} \times_{\mathfrak{Y}_1} \mathfrak{Z}] = (j_! \circ h^*)([\mathfrak{U} \to \mathfrak{Y}_1])$. The fact that $\mathcal{Q}$ is lax monoidal and symmetric follows from the natural morphism

$$\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X}}) \otimes_{\mathrm{K}_0(\mathbf{Stck}_S)} \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{Y}}) \to \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X} \times_S \mathfrak{Y}}). \qquad \square$$

Remark 4.7.7. Note that $\mathcal{Q}$ is only lax monoidal, and not monoidal, since the above morphism is not necessarily an isomorphism, see Example 3.2.10.

Finally, let us show that the TQFT obtained through the composition of $\mathcal{F}_G$ and $\mathcal{Q}$ indeed quantizes the virtual class of the G-character stack.

Theorem 4.7.8. *There exists a lax TQFT*

$$Z_G \colon \mathbf{Bord}_n \to \mathrm{K}_0(\mathbf{Stck}_S)\text{-}\mathbf{Mod}$$

given by the composite $Z_G = \mathcal{Q} \circ \mathcal{F}_G$, quantizing the virtual class of the G-character stack. That is, $[\mathfrak{X}_G(W)] = Z_G(W)(1)$ in $\mathrm{K}_0(\mathbf{Stck}_S)$ for any closed manifold W.

Proof. For any closed manifold W, viewed as a bordism from and to $\varnothing$, the corresponding field theory $\mathcal{F}_G(W)$ is given by the correspondence

$$S \xleftarrow{t} \mathfrak{X}_G(W) \xrightarrow{t} S$$

where t is the terminal morphism. Applying the quantization functor $\mathcal{Q}$, it follows that

$$Z_G(W)(1) = t_! t^*(1) = [\mathfrak{X}_G(W)] \in \mathrm{K}_0(\mathbf{Stck}_S). \qquad \square$$

4.8 Field theory of surfaces

The goal of this section is to make explicit the field theories $\mathcal{F}_G(W)$ corresponding to various bordisms W in dimension $n = 2$. We focus in particular on the generators (4.2), as any 2-dimensional oriented bordism can be built from these through composition and taking disjoint unions.

Example 4.8.1. The inclusions $S^1 \to S^1 \times [0,1]$ of the in- and out-boundary into the cylinder induce an equivalence of groupoids $\Pi(S^1) \simeq \Pi(S^1 \times [0,1])$. Therefore, the field theory $\mathcal{F}_G(S^1 \times [0,1])$ is given by the identity on $\mathcal{F}_G(S^1)$, as expected.

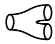
Proposition 4.8.2. *The field theory of the bordism*  *from* $\varnothing$ *to* S^1 *is given by*

$$S \longleftarrow \mathrm{B}G \xrightarrow{\;e\;} [G/G] \;,$$

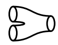
where e *is induced by the unit of* G*, and* G *acts on itself by conjugation. Similarly, the field theory of the bordism*  *from* S^1 *to* $\varnothing$ *is given by*

$$[G/G] \xleftarrow{\;e\;} \mathrm{B}G \longrightarrow S \;.$$

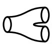
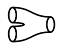
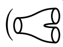
Proof. Since the fundamental group of the disk is trivial, its G-character stack is given by $\mathfrak{X}_G(D^1) = \mathrm{B}G$. The inclusion of S^1 into the disk induces the trivial homomorphism $\pi_1(S^1,*) = \mathbb{Z} \to 1 = \pi_1(D^1,*)$ between fundamental groups, and consequently the corresponding map $\mathrm{B}G \to [G/G]$ is given by the inclusion of the identity. ☐

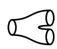
Proposition 4.8.3. *The field theory of the bordism*  *from* $S^1 \sqcup S^1$ *to* S^1 *is given by*

$$[G/G]^2 \xleftarrow{\;\pi_1\times\pi_2\;} [G^2/G] \xrightarrow{\;m\;} [G/G]$$

where $\pi_1, \pi_2 \colon [G^2/G] \to [G/G]$ *are induced by the projections, and* m *by multiplication on* G*. Similarly, the field theory of the bordism*  *from* S^1 *to* $S^1 \sqcup S^1$ *is given by*

$$[G/G] \xleftarrow{\;m\;} [G^2/G] \xrightarrow{\;\pi_1\times\pi_2\;} [G/G]^2 \;.$$

Proof. We will compute the field theory for , and the field theory for  can be computed completely analogous. Choose a basepoint x on the out-boundary, basepoints y and z on the in-boundary, a path γ_1 from x to y, a path γ_2 from x to z, and let α and β be generators of the fundamental group π_1 (, x) $\cong F_2$ as depicted in the figure below.

Under the inclusion of the in-boundary $S^1 \sqcup S^1$ into , the generators of $\pi_1(S^1, y) \cong \mathbb{Z}$ and $\pi_1(S^1, z) \cong \mathbb{Z}$ are sent to $\gamma_1\alpha\gamma_1^{-1}$ and $\gamma_2\beta\gamma_2^{-1}$, respectively.

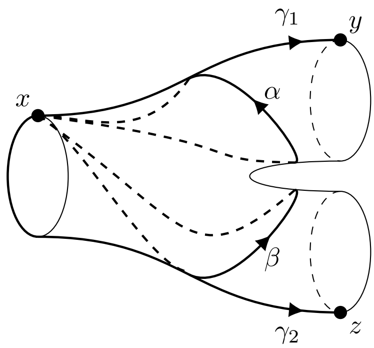
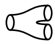


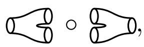
Figure 4.2: The pair of pants as a bordism from $S^1 \sqcup S^1$ to S^1. A basepoint x on the out-boundary is chosen, and basepoints y and z on the in-boundary. Also are chosen paths γ_1 from x to y and γ_2 from x to z, and two generators α and β of the fundamental group at x.

This determines the map $[G^2/G] \to [G/G]^2$ as claimed. Under the inclusion of the out-boundary S^1 into , the generator of $\pi_1(S^1, x) \cong \mathbb{Z}$ is sent to the loop $\alpha\beta$, which determines the map $[G^2/G] \to [G/G]$ as claimed. $\square$

Proposition 4.8.4. *The field theory of the bordism*  *from S^1 to S^1 is given by*

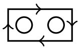
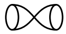
$$[G/G] \xleftarrow{\ \pi_1\ } [G^3/G] \xrightarrow{\ \theta\ } [G/G]$$

where π_1 is induced by the first projection $G^3 \to G$, and θ is induced by $G^3 \to G$ given by $(C, A, B) \mapsto C[A, B]$.

Proof. Since  is equal to the composite , it suffices to compute the composite of the correspondences as given by Proposition 4.8.3. Hence, let us describe the fiber product $\mathfrak{X} = [G^2/G] \times_{[G/G]^2} [G^2/G]$. By definition of fiber products of stacks, the objects of $\mathfrak{X}$ over T are tuples $(P, Q, g_1, h_1, g_2, h_2, \alpha, \beta)$, where P and Q are G-torsors over T with G-equivariant morphisms $P \xrightarrow{(g_1,h_1)} G^2$ and $Q \xrightarrow{(g_2,h_2)} G^2$, and $\alpha, \beta \colon P \to Q$ are morphisms of G-torsors such that $g_1 = g_2 \circ \alpha$ and $h_1 = h_2 \circ \beta$. A morphism from $(P', Q', g_1', h_1', g_2', h_2', \alpha', \beta')$ to $(P, Q, g_1, h_1, g_2, h_2, \alpha, \beta)$ is a pair of morphisms of G-torsors $(\gamma_1 : P' \to P, \gamma_2 : Q' \to Q)$ such that $g_i' = g_i \circ \gamma_i$ and $h_i' = h_i \circ \gamma_i$ for $i = 1, 2$ and $\alpha = \gamma_2 \circ \alpha' \circ \gamma_1^{-1}$ and $\beta = \gamma_2 \circ \beta' \circ \gamma_1^{-1}$. Note that every object is isomorphic one with $P = Q$ and $\beta = \mathrm{id}_P$. Therefore, we can equivalently describe this category as the category whose objects over T are tuples (P, C, A, B), where P is a G-torsor over T and $C = m \circ (g_1, h_1)$, $A = h_1^{-1}$ and $B = \alpha$, and a morphism $(P', C', A', B') \to (P, C, A, B)$ is a morphism $\gamma \colon P' \to P$ such that $C' = C \circ \gamma$, $A' = A \circ \gamma$ and

$B' = \gamma^{-1} \circ B \circ \gamma$. With this description it is clear that $\mathfrak{X} \cong [G^3/G]$. Unfolding the definitions, the morphism $\mathfrak{X} \to [G/G]$ corresponding to the in-boundary is indeed given by $(C, A, B) \mapsto C$. The morphism $\mathfrak{X} \to [G/G]$ corresponding to the out-boundary is given by $(C, A, B) \mapsto B^{-1}CABA^{-1}$, which is naturally isomorphic to $C[A, B]$. □

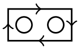
Besides orientable bordisms, there are also non-orientable bordisms, which our field theory $\mathcal{F}_G$ allows. Of interest to us are the bordisms

 $: S^1 \to S^1$ and  $: S^1 \to S^1$ (4.8)

corresponding to the projective plane with two punctures and the cylinder which reverses the orientation of S^1, respectively. The field theory $\mathcal{F}_G$ () is easily seen to be the correspondence

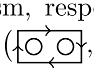
$$[G/G] \xleftarrow{\;i\;} [G/G] \xrightarrow{\;\mathrm{id}\;} [G/G]$$

where i is induced by the inversion $g \mapsto g^{-1}$. The field theory of the punctured projective plane is described by the following proposition.

Proposition 4.8.5. *The field theory of the bordism*  *from S^1 to S^1 is given by*

$$[G/G] \xleftarrow{\;\pi_1\;} [G^2/G] \xrightarrow{\;v\;} [G/G]$$

where v is given by $(B, A) \mapsto BA^2$.

Proof. Choose basepoints x and y on the in- and out-boundary of the bordism, respectively, and let α and β be generators of the fundamental group π_1 ( $, x) \cong F_2$ as depicted in the figure below.

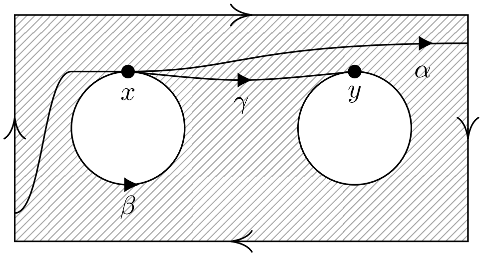
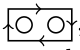


Figure 4.3: The projective plane with two punctures as a bordism from S^1 to S^1. Basepoints x and y are chosen on in- and out-boundary, respectively. Also, generators α and β of the fundamental group at x are chosen, and path γ connecting x and y.

Under the inclusion of the in-boundary S_1 into , the generator of $\pi_1(S^1, x) \cong \mathbb{Z}$ is sent to β, and under the inclusion of the out-boundary S^1, the generator of $\pi_1(S^1, y) \cong \mathbb{Z}$ is sent to $\gamma\beta\alpha^2\gamma^{-1}$. These determine the maps $[G^2/G] \to [G/G]$ as claimed. □

4.9 Arithmetic TQFT

In this section we will construct a higher-dimensional analogue of the TQFT of Section 4.5. To be precise, for any finite group G, we will construct an *arithmetic TQFT*

$$Z_G^{\#} \colon \mathbf{Bord}_n \to \mathbf{Vect}_{\mathbb{C}}$$

which will agree (up to natural isomorphism) with the TQFT of Section 4.5. Whereas the construction of the TQFT of Section 4.5 is very ad-hoc, in terms of specific operations on the representation ring $R_{\mathbb{C}}(G)$, the construction of $Z_G^{\#}$ will be very much like that of the character stack TQFT: as the composite of a field theory and a quantization functor.

Field theory and quantization

Fix a finite group G.

Definition 4.9.1. The *arithmetic field theory* is the 2-functor

$$\mathcal{F}_G^{\#} \colon \mathbf{Bord}_n \to \mathrm{Corr}(\mathbf{FinGrpd})$$

which assigns to a closed $(n-1)$-dimensional manifold M the G-character groupoid

$$\mathcal{F}_G^{\#}(M) = \mathfrak{X}_G(M)$$

and to a bordism $W \colon M_1 \to M_2$ the correspondence

$$\mathcal{F}_G^{\#}(W) = \left(\mathfrak{X}_G(M_1) \xleftarrow{i_1} \mathfrak{X}_G(W) \xrightarrow{i_2} \mathfrak{X}_G(M_2) \right).$$

Proposition 4.9.2. *$\mathcal{F}_G^{\#}$ is a symmetric monoidal functor.*

Proof. The proof is completely analogous to that of Proposition 4.7.4, where $\mathfrak{X}_G(-)$ is also sends finite colimits in **FGGrpd** to limits in **FinGrpd**. □

Definition 4.9.3. Given a groupoid A, denote by $\mathbb{C}^A$ the complex vector space of complex-valued functions on the objects of A which are invariant under isomorphism.

This construction admits some functoriality. Given a functor $f\colon A \to B$ between groupoids, we can pull back functions via

$$f^*\colon \mathbb{C}^B \to \mathbb{C}^A, \quad \varphi \mapsto \varphi \circ f.$$

Pullback is functorial in the sense that $(g \circ f)^* = f^* \circ g^*$ for functors $f\colon A \to B$ and $g\colon B \to C$. Moreover, if $\mu\colon f \Rightarrow g$ is a natural transformation between functors $f, g\colon A \to B$, then $f^* = g^*$. In particular, if A and B are equivalent groupoids, then $\mathbb{C}^A$ and $\mathbb{C}^B$ are naturally isomorphic.

Furthermore, if $f\colon A \to B$ is a functor between essentially finite groupoids, we define pushforward along f as

$$f_!\colon \mathbb{C}^A \to \mathbb{C}^B, \quad \varphi \mapsto \left(b \mapsto \sum_{[(a,\beta)] \in f^{-1}(b)/\sim} \frac{\varphi(a)}{|\operatorname{Aut}(a,\beta)|} \right)$$

where $f^{-1}(b)$ denotes the fiber product $A \times_B \{b\}$ as in Definition 1.1.6. It is an easy exercise to show that pushforward is also functorial in the sense that $(g \circ f)_! = g_! \circ f_!$ for functors $f\colon A \to B$ and $g\colon B \to C$.

Example 4.9.4. For any groupoid A, let $f\colon A \to \{*\}$ be the final morphism and let $\varphi \in \mathbb{C}^A$ be the constant function $\varphi(a) = 1$. Then $(f_! \varphi)(*) = |A|$ is the groupoid cardinality of A.

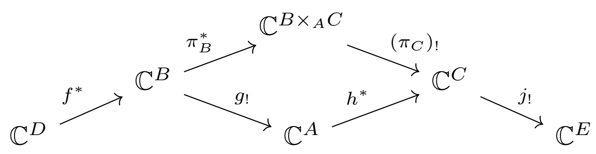
Definition 4.9.5. The *arithmetic quantization functor* is the functor

$$\mathcal{Q}^\#\colon \operatorname{Corr}(\mathbf{FinGrpd}) \to \mathbf{Vect}_\mathbb{C}$$

which assigns to a groupoid A the vector space $\mathbb{C}^A$, and which assigns to a correspondence of groupoids $A \xleftarrow{f} B \xrightarrow{g} C$ the morphism $g_! \circ f^*\colon \mathbb{C}^A \to \mathbb{C}^C$. Note that two correspondences connected by a 2-morphism are indeed assigned to the same linear map.

Lemma 4.9.6. *$\mathcal{Q}^\#$ is a symmetric monoidal functor.*

Proof. Let $D \xleftarrow{f} B \xrightarrow{g} A$ and $A \xleftarrow{h} C \xrightarrow{i} E$ be correspondences of essentially finite groupoids. The relevant diagram in $\mathbf{Vect}_\mathbb{C}$ is given by



where $\pi_B\colon B \times_A C \to B$ and $\pi_C\colon B \times_A C \to C$ are the projections. To show $\mathcal{Q}^\#$ respects composition, it suffices to show that $h^* \circ g_! = (\pi_C)_! \circ \pi_B^*$.

First note that, for any $x \in C$, the groupoids $\pi_C^{-1}(x) = (B \times_A C) \times_C \{x\}$ and $g^{-1}(h(x)) = B \times_A \{h(x)\}$ are equivalent. Explicitly, an object of $\pi_C^{-1}(x)$ is a tuple (b, c, α, γ) with $(b, c, \alpha) \in B \times_A C$ and $\gamma \colon c \to x$ a morphism in C. A morphism $(b', c', \alpha', \gamma') \to (b, c, \alpha, \gamma)$ is given by a tuple of morphisms $(\beta \colon b' \to b, \zeta \colon c' \to c)$ such that $\alpha \circ g(\beta) = h(\zeta) \circ \alpha'$ and $\gamma \circ \zeta = \gamma'$. By appropriate choice of ζ, this is equivalent to the groupoid whose objects are (b, α) with $b \in B$ and $\alpha \colon g(b) \to h(x)$ and morphisms $(b', \alpha') \to (b, \alpha)$ are morphisms $\beta \colon b' \to b$ such that $\alpha' \circ g(\beta) = \alpha$. But this is precisely $g^{-1}(h(x))$.

Now, for any $\varphi \in \mathbb{C}^B$ and any $c \in C$, it follows that

$$\begin{aligned} ((\pi_C)_! \pi_B^* \varphi)(c) &= \sum_{[(b,c,\alpha,\gamma)] \in \pi_C^{-1}(c)/\sim} \frac{\varphi(b)}{|\operatorname{Aut}(b, c, \alpha, \gamma)|} \\ &= \sum_{[(b,\alpha)] \in g^{-1}(h(c))/\sim} \frac{\varphi(b)}{|\operatorname{Aut}(b, \alpha)|} = (h^* g_! \varphi)(c). \end{aligned}$$

□

Definition 4.9.7. The *arithmetic TQFT* $Z_G^{\#} \colon \mathbf{Bord}_n \to \mathbf{Vect}_{\mathbb{C}}$ is the composite $\mathcal{Q}^{\#} \circ \mathcal{F}_G^{\#}$.

Proposition 4.9.8. *The arithmetic TQFT quantizes the groupoid cardinality of the G-character groupoid, that is,*

$$Z_G^{\#}(W)(1) = |\mathfrak{X}_G(W)|$$

for any closed n-dimensional manifold W, seen as a bordism $\varnothing \to \varnothing$.

Proof. The field theory $\mathcal{F}_G^{\#}(W)$ is given by

$$\{*\} \xleftarrow{\;t\;} \mathfrak{X}_G(W) \xrightarrow{\;t\;} \{*\}$$

where t is the final morphism. Applying the quantization functor $\mathcal{Q}^{\#}$, we find

$$Z_G^{\#}(W)(1) = t_! t^*(1) = \sum_{[x] \in t^{-1}(*)/\sim} \frac{1}{|\operatorname{Aut}(x)|} = |\mathfrak{X}_G(W)|,$$

using that $t^{-1}(*) = \mathfrak{X}_G(W)$. □

Remark 4.9.9. The arithmetic TQFT can be seen as a special case of the Dijkgraaf–Witten TQFT [DW90] with $\alpha = 0 \in H^n(\mathrm{B}G, \mathbb{R}/\mathbb{Z})$. This TQFT is also known as *finite gauge theory*, since the gauge group G is finite.

Comparison with the representation ring

Let us return to case $n = 2$. For a finite group G, we have $\mathfrak{X}_G(S^1) = [G/G]$ where G acts on itself by conjugation. In particular, $Z_G^{\#}(S^1)$ is the complex vector space of complex-valued functions on G which are invariant under conjugation. But this is precisely the underlying vector space of the representation ring $R_{\mathbb{C}}(G)$, that is, there is a canonical isomorphism

$$Z_G^{\#}(S^1) = \mathbb{C}^{[G/G]} \cong R_{\mathbb{C}}(G) = Z_G(S^1). \tag{4.9}$$

Proposition 4.9.10. *Let G be a finite group. For $n = 2$, there is a natural isomorphism*

$$Z_G^{\#} \cong Z_G$$

as functors $\mathbf{Bord}_2^{\mathrm{or}} \to \mathbf{Vect}_{\mathbb{C}}$ from the arithmetic TQFT to the TQFT of Section 4.5.

Proof. Since both $Z_G^{\#}$ and Z_G are monoidal functors, the isomorphism (4.9) naturally extends to isomorphisms $Z_G^{\#}\left(\sqcup_{i=1}^m S^1\right) \cong Z_G\left(\sqcup_{i=1}^m S^1\right)$ for all $m \geq 0$. As the category $\mathbf{Bord}_2^{\mathrm{or}}$ of 2-dimensional oriented bordisms is generated by the bordisms (4.2), it suffices to verify the naturality of the isomorphisms for these generators only.

- Case $W =$ . The field theory $\mathcal{F}_G^{\#}(W)$ is given by

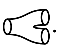
$$\{*\} \xleftarrow{\;t\;} [\{*\}/G] \xrightarrow{\;e\;} [G/G]$$

  where t is the terminal morphism and e is the inclusion of the unit of G. Hence, the morphism $Z_G^{\#}(W) : \mathbb{C} \to \mathbb{C}^{[G/G]}$ sends 1 to $e_! t^* 1$, which is precisely the indicator function on the unit of G and corresponds, under the isomorphism (4.9), to the unit $\eta(1)$ of $R_{\mathbb{C}}(G)$.

- Case $W =$ . Similarly, the field theory $\mathcal{F}_G^{\#}(W)$ is given by

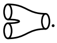
$$[G/G] \xleftarrow{\;e\;} [\{*\}/G] \xrightarrow{\;t\;} \{*\}$$

  so the morphism $Z_G^{\#}(W) : \mathbb{C}^{[G/G]} \to \mathbb{C}$ is given by $f \mapsto t_! e^* f = \frac{1}{|G|} f(1)$, which corresponds, under the isomorphism (4.9), to the counit ε of $R_{\mathbb{C}}(G)$.

- Case $W =$ . The field theory $\mathcal{F}_G^{\#}(W)$ is given by

$$[G/G]^2 \xleftarrow{\;\pi_1 \times \pi_2\;} [G^2/G] \xrightarrow{\;m\;} [G/G]$$

  where m is multiplication on G. Hence, the morphism $Z_G^{\#}(W)\colon R_{\mathbb{C}}(G) \otimes_{\mathbb{C}} R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G)$ maps $f_1 \otimes f_2 \mapsto m_!(\pi_1 \times \pi_2)^*(f_1 \otimes f_2)$ which is precisely $\mu(f_1 \otimes f_2)$.

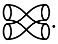
- Case $W =$ . Similarly, the field theory $\mathcal{F}_G^{\#}(W)$ is given by

$$[G/G] \xleftarrow{\quad m \quad} [G^2/G] \xrightarrow{\pi_1 \times \pi_2} [G/G]^2$$

so the morphism $Z_G^{\#}(W)\colon R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G) \otimes_{\mathbb{C}} R_{\mathbb{C}}(G)$ is given by $f \mapsto (\pi_1 \times \pi_2)_! m^* f$. Note that, for any $g_1, g_2 \in G$, the groupoid $(\pi_1 \times \pi_2)^{-1}(g_1, g_2)$ is equivalent to G as a set. Hence, for any irreducible character χ, we find

$$Z_G^{\#}(W)(\chi)(g_1, g_2) = \sum_{h \in G} \chi(g_1 h g_2 h^{-1}) = |G| \chi(g)\chi(h)/\chi(1),$$

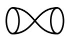
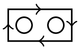
where the last equality is shown as in the proof of Theorem 4.5.3. Therefore, using (4.5) and (4.6), we find that $Z_G^{\#}(W)(\chi)$ is precisely $\delta(\chi)$.

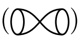
- Case $W =$ . The field theory $\mathcal{F}_G^{\#}(W)$ is given by

$$[G/G]^2 \xleftarrow{\quad \mathrm{id} \quad} [G/G]^2 \xrightarrow{\quad t \quad} [G/G]^2$$

where t switches the two copies of G. Clearly, $Z_G^{\#}(W)$ is given by $f_1 \otimes f_2 \mapsto f_2 \otimes f_1$, which is precisely τ. □

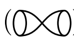
Non-orientable surfaces

Recall that the TQFT of Section 4.5 was given by the Frobenius algebra structure on the representation ring $R_{\mathbb{C}}(G)$. However, $Z_G^{\#}$ is also defined for non-orientable bordisms, so we obtain additional operations on the representation ring. In particular, let us consider the orientation-reversing cylinder : $S^1 \to S^1$ and the projective plane : $S^1 \to S^1$ as in (4.8).

The field theory $\mathcal{F}_G^{\#}$ () is easily seen to be

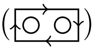
$$[G/G] \xleftarrow{\quad i \quad} [G/G] \xrightarrow{\quad \mathrm{id} \quad} [G/G]$$

where i is induced by the inversion $g \mapsto g^{-1}$. Hence, it follows that

$Z_G^{\#}$ () $: R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G), \quad f \mapsto i^* f = (g \mapsto f(g^{-1})) = \overline{f}$

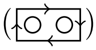
is complex conjugation of class functions.

Regarding the projective plane, we have the following lemma.

Lemma 4.9.11. *The map* $\nu \coloneqq Z_G^{\#}$ () $: R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G)$ *is given by*

$$\nu(\chi) = \varepsilon_\chi \frac{|G|}{\chi(1)} \chi$$

for any irreducible character $\chi \in \hat{G}$, *where* $\varepsilon_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$ *is known as the* Frobenius–Schur indicator *of* χ [FS06].

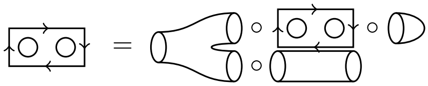
Proof. Analogous to Proposition 4.8.5, the field theory $\mathcal{F}_G^\#$ () is

$$[G/G] \xleftarrow{\pi_1} [G^2/G] \xrightarrow{v} [G/G]$$

where v is induced by $(B, A) \mapsto BA^2$. Applying $\mathcal{Q}^\#$ we find that

$$\nu(f) = v_! \pi_1^* f = \left(g \mapsto \sum_{BA^2 = g} f(B) = \sum_{h \in G} f(gh^2) \right).$$

Note that we have an equality of bordisms



which shows that $\nu(f) = \mu((\nu \circ \eta)(1) \otimes f)$ for all $f \in R_{\mathbb{C}}(G)$. We compute

$$(\nu \circ \eta)(1)(g) = \sum_{h \in G} \eta(1)(gh^2) = |\{h \in G \mid h^2 = g^{-1}\}|$$

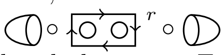
and thus, for any $\chi \in \hat{G}$, we find

$$\beta((\nu \circ \eta)(1) \otimes \chi) = \frac{1}{|G|} \sum_{g \in G} (\nu \circ \eta)(1)(g) \chi(g^{-1}) = \frac{1}{|G|} \sum_{h \in G} \chi(h^2) = \varepsilon_\chi$$

from which we obtain that

$$(\nu \circ \eta)(1) = \sum_{\chi \in \hat{G}} \varepsilon_\chi \chi.$$

Finally, using (4.6) we conclude that $\nu(\chi) = \mu((\nu \circ \eta)(1) \otimes \chi) = \varepsilon_\chi \frac{|G|}{\chi(1)} \chi$. □

This expression can be used to compute the groupoid cardinality $|\mathfrak{X}_G(N_r)|$ of the G-character groupoid of the non-orientable closed surface N_r of demigenus r, that is, the connected sum of r non-projective planes. The decomposition $N_r =$  yields the following proposition. Note that this formula was already known to Frobenius and Schur in [FS06, (9), p.197].

Proposition 4.9.12. *Let N_r be the closed non-orientable surface of demigenus r, that is, the surface obtained as the connected sum of r projective planes. Then*

$$Z_G^\# (N_r)(1) = |\mathfrak{X}_G(N_r)| = \sum_{\chi \in \hat{G}} \varepsilon_\chi^r \left(\frac{|G|}{\chi(1)} \right)^{r-2}.$$ □

4.10 Comparison of TQFTs

Let us summarize the various TQFTs constructed so far. Fix a base scheme S, a linear algebraic group G over S, a finite field $\mathbb{F}_q$, and an $\mathbb{F}_q$-rational point $x\colon \operatorname{Spec}\mathbb{F}_q \to S$ of S. The functors defined in the previous sections, i.e., the field theory and quantization functors, fit nicely together in the following (not necessarily commutative!) diagram. The dashed arrow, completing the diagram, will be defined in this section.

$$\begin{array}{ccccc}
 & & \mathrm{Corr}(\mathbf{Stck}_S) & \xrightarrow{\ \mathcal{Q}\ } & \mathrm{K}_0(\mathbf{Stck}_S)\text{-}\mathbf{Mod} \\
 & \overset{\mathcal{F}_G}{\nearrow} & \Big\downarrow{\scriptstyle (-)(\mathbb{F}_q)} & & \Big\uparrow{\scriptstyle \mu_S^*} \\
\mathbf{Bord}_n & & & & \\
 & \underset{\mathcal{F}^{\#}_{G(\mathbb{F}_q)}}{\searrow} & & & \\
 & & \mathrm{Corr}(\mathbf{FinGrpd}) & \xrightarrow[\ \mathcal{Q}^{\#}\ ]{} & \mathbf{Vect}_{\mathbb{C}}
\end{array}$$

Recall that for any algebraic stack $\mathfrak{X}$ over S, the groupoid $\mathfrak{X}(\mathbb{F}_q)$ is the groupoid of $\mathbb{F}_q$-points $\operatorname{Spec}\mathbb{F}_q \to \mathfrak{X}$ whose composition to S is equal to the fixed point $x\colon \operatorname{Spec}\mathbb{F}_q \to S$. In particular, $S(\mathbb{F}_q) = \{x\}$.

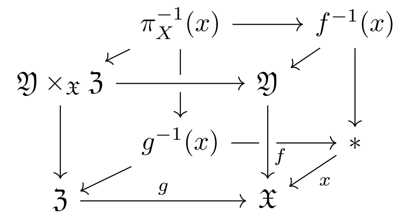
We can see the TQFT of G-character stacks, $Z_G = \mathcal{Q} \circ \mathcal{F}_G$, in the top row, and the arithmetic TQFT, $Z^{\#}_{G(\mathbb{F}_q)} = \mathcal{Q}^{\#} \circ \mathcal{F}^{\#}_{G(\mathbb{F}_q)}$, in the bottom row. Note that one can interpolate between the two: using the field theory $\mathcal{F}_G$, then taking the $\mathbb{F}_q$-rational points, and finally applying the arithmetic quantization functor $\mathcal{Q}^{\#}$, we obtain yet another TQFT given by the composite $\tilde{Z}_G = \mathcal{Q}^{\#} \circ (-)(\mathbb{F}_q) \circ \mathcal{F}_G$.

It turns out all three TQFTs all quantize different invariants. Of course, Z_G quantizes a different type of invariant (an element in the Grothendieck ring of stacks) while $Z^{\#}_G$ and $\tilde{Z}_G$ quantize a complex number. Nevertheless, in this section we will relate these TQFTs through natural transformations. More precisely, there will be a natural transformation in the square on the right in the diagram, and, if G is connected, a natural isomorphism in the triangle on the left. The functor μ_S^* will be defined in order to relate the targets of the geometric and arithmetic TQFT.

Definition 4.10.1. For any object $\mathfrak{X}$ of $\mathbf{Stck}_S$, define

$$\mu_{\mathfrak{X}}\colon \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X}}) \to \mathbb{C}^{\mathfrak{X}(\mathbb{F}_q)}, \quad [\mathfrak{Y} \xrightarrow{f} \mathfrak{X}] \mapsto \big(x \mapsto |f^{-1}(x)|\big)$$

where the groupoid cardinality of $f^{-1}(x) = \mathfrak{Y}(\mathbb{F}_q) \times_{\mathfrak{X}(\mathbb{F}_q)} \{x\}$ was taken. This map is easily seen to be a morphism of rings, where multiplicativity follows from the following diagram, in which all squares are cartesian:



In particular, the morphism $\mu_S\colon \mathrm{K}_0(\mathbf{Stck}_S) \to \mathbb{C}^{S(\mathbb{F}_q)} = \mathbb{C}$ induces the functor

$$\mu_S^*\colon \mathbf{Vect}_{\mathbb{C}} \to \mathrm{K}_0(\mathbf{Stck}_S)\text{-}\mathbf{Mod}$$

given by restriction of scalars.

Proposition 4.10.2. *The maps $\mu_{\mathfrak{X}}$ define a natural transformation*

$$\mu\colon \mathcal{Q} \Rightarrow \mu_S^* \circ \mathcal{Q}^{\#} \circ (-)(\mathbb{F}_q).$$

In particular, this induces a natural transformation of TQFTs

$$Z_G \Rightarrow \mu_S^* \circ \mathcal{Q}^{\#} \circ (-)(\mathbb{F}_q) \circ \mathcal{F}_G.$$

Proof. For any correspondence $\mathfrak{X} \xleftarrow{f} \mathfrak{Z} \xrightarrow{h} \mathfrak{Y}$ in $\mathbf{Stck}_S$, the relevant diagram of $\mathrm{K}_0(\mathbf{Stck}_S)$-modules is:

$$\begin{array}{ccccc}
\mathrm{K}_0(\mathbf{Stck}_{\mathfrak{X}}) & \xrightarrow{f^*} & \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{Z}}) & \xrightarrow{g_!} & \mathrm{K}_0(\mathbf{Stck}_{\mathfrak{Y}}) \\
\downarrow{\scriptstyle \mu_{\mathfrak{X}}} & & \downarrow{\scriptstyle \mu_{\mathfrak{Z}}} & & \downarrow{\scriptstyle \mu_{\mathfrak{Y}}} \\
\mathbb{C}^{\mathfrak{X}(\mathbb{F}_q)} & \xrightarrow{f^*} & \mathbb{C}^{\mathfrak{Z}(\mathbb{F}_q)} & \xrightarrow{g_!} & \mathbb{C}^{\mathfrak{Y}(\mathbb{F}_q)}
\end{array}$$

Let us show that the first square commutes. For any stack $\mathfrak{U} \xrightarrow{h} \mathfrak{X}$ and point $z \in \mathfrak{Z}(\mathbb{F}_q)$, we have

$$\mu_{\mathfrak{Z}}(f^*[\mathfrak{U}])(z) = \sum_{[(u,z,\alpha)] \in (\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{Z})(\mathbb{F}_q)/\sim} \frac{|\operatorname{Aut}(z)|}{|\operatorname{Aut}(u,z,\alpha)|}.$$

As in the proof of Lemma 1.6.3, the group $\operatorname{Aut}(u) \times \operatorname{Aut}(z)$ acts naturally on the set $\operatorname{Hom}_{\mathfrak{X}(\mathbb{F}_q)}(h(u), f(z))$, and the stabilizer of any α in this set is precisely $\operatorname{Aut}(u,z,\alpha)$. Hence, it follows from the orbit-stabilizer theorem that

$$\begin{aligned}
\mu_{\mathfrak{Z}}(f^*[\mathfrak{U}])(z) &= \sum_{\substack{[u] \in \mathfrak{U}(\mathbb{F}_q)/\sim \\ \alpha\colon h(u) \xrightarrow{\sim} f(z)}} \frac{1}{|\operatorname{Aut}(u)|} \\
&= \sum_{[(u,\alpha)] \in h^{-1}(f(z))/\sim} \frac{1}{|\operatorname{Aut}(u)|} \\
&= |h^{-1}(f(z))| = (f^*\mu_{\mathfrak{X}}([\mathfrak{U}]))(z).
\end{aligned}$$

Next, let us show that the second square commutes. For any stack $\mathfrak{U} \xrightarrow{h} \mathfrak{Z}$ and point $y \in \mathfrak{Y}(\mathbb{F}_q)$, we have

$$(g_! \mu_{\mathfrak{Z}}[\mathfrak{U}])(y) = \sum_{[(z,\alpha)] \in g^{-1}(y)/\sim} \frac{|h^{-1}(z)|}{|\operatorname{Aut}(z)|} = |(g \circ h)^{-1}(y)| = (\mu_{\mathfrak{Y}} g_! [\mathfrak{U}])(y). \qquad \square$$

Proposition 4.10.3. *If G is connected, there is a natural isomorphism*

$$(-)(\mathbb{F}_q) \circ \mathcal{F}_G \cong \mathcal{F}^{\#}_{G(\mathbb{F}_q)}.$$

In particular, this induces a natural isomorphism of TQFTs

$$\mathcal{Q}^{\#} \circ (-)(\mathbb{F}_q) \circ \mathcal{F}_G \cong Z^{\#}_{G(\mathbb{F}_q)}.$$

Proof. Proposition 1.5.10 implies that $\mathfrak{X}_G(M)(\mathbb{F}_q)$ is naturally isomorphic to $\mathfrak{X}_{G(\mathbb{F}_q)}(M)$ for any compact manifold M. The statement now follows directly from the definitions of the field theories. $\square$

Remark 4.10.4. For non-connected G, there need not even be a natural transformation $(-)(\mathbb{F}_q) \circ \mathcal{F}_G \Rightarrow \mathcal{F}^{\#}_{G(\mathbb{F}_q)}$. Consider the 2-sphere S^2 as a bordism $\varnothing \to \varnothing$. Since the G-character stack of S^2 is $\mathrm{B}G$, one has $\tilde{Z}_G(S^2)(1) = |\mathrm{B}G(\mathbb{F}_q)|$, whereas $Z^{\#}_{G(\mathbb{F}_q)}(S^2)(1) = |R_G(S^2)(\mathbb{F}_q)|/|G(\mathbb{F}_q)| = |1|/|G(\mathbb{F}_q)|$. Already for $G = \mathbb{Z}/2\mathbb{Z}$, these quantities are different, see Remark 1.5.9.

Corollary 4.10.5. *Suppose G is connected. Then there is a natural transformation between the geometric and arithmetic TQFT*

$$Z_G \Rightarrow \mu_S^* \circ Z^{\#}_{G(\mathbb{F}_q)}. \qquad \square$$

Unfolding the definitions in dimension $n = 2$, we obtain the following theorem, relating the geometric method to the arithmetic method.

Theorem 4.10.6. *Suppose G is connected. Denote by $I \in \mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ the class of $[S/G] \to [G/G]$ induced by the unit of G. If the $\mathrm{K}_0(\mathbf{Stck}_S)$-module $\mathcal{V} = \langle Z_G$ ()^g (I) for $g \in \mathbb{Z}_{\geq 0}\rangle$ is finitely generated, then:*

(i) The sums of equidimensional irreducible complex characters form a basis for the subspace $\mu_{[G/G]}(\mathcal{V}) \subseteq \mathbb{C}^{[G/G](\mathbb{F}_q)}$.

(ii) The dimensions of the irreducible complex characters of $G(\mathbb{F}_q)$ are precisely given by

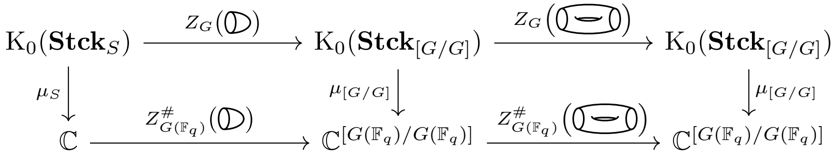
$$d_i = \frac{|G(\mathbb{F}_q)|}{\sqrt{\lambda_i}}$$

for $\lambda_i \in \mathbb{Z}$ the eigenvalues of $\mu_S(A)$, where A is any matrix representing the linear map Z_G () with respect to a generating set of $\mathcal{V}$.

(iii) Write $\mu_{[G/G]}(I) = \sum_i v_i$, where v_i are eigenvectors of $\mu_S(A)$ corresponding to the eigenvalues λ_i. Then each v_i is a scalar multiple of the sum of equidimensional characters, or more precisely,

$$v_i = \frac{d_i}{|G(\mathbb{F}_q)|} \sum_{\substack{\chi \in \hat{G} \text{ s.t.} \\ \chi(1) = d_i}} \chi.$$

Proof. By Corollary 4.10.5, we have the following commutative diagram.



The square on the right shows that

$\mu_{[G/G]}(\mathcal{V}) = \langle Z^{\#}_{G(\mathbb{F}_q)}$  $(\mu_{[G/G]}(I))$ for $g \in \mathbb{Z}_{\geq 0}\rangle$,

and the square on the left shows that $\mu_{[G/G]}(I) \in \mathbb{C}^{[G/G](\mathbb{F}_q)} \cong R_{\mathbb{C}}(G(\mathbb{F}_q))$ corresponds to the unit

$$\eta(1) = \frac{1}{|G(\mathbb{F}_q)|} \sum_{\chi \in \hat{G}} \chi(1)\, \chi = \sum_{d \geq 0} w_d \quad \text{with} \quad w_d = \frac{d}{|G(\mathbb{F}_q)|} \sum_{\substack{\chi \in \hat{G} \text{ s.t.} \\ \chi(1) = d}} \chi. \qquad (*)$$

Clearly, the w_d are linearly independent, and moreover, by (4.7), they are eigenvectors of $Z^{\#}_{G(\mathbb{F}_q)}$  with eigenvalues $|G(\mathbb{F}_q)|^2/d^2$. Since the eigenvalues are distinct, the w_d form a basis for $\mu_{[G/G]}(\mathcal{V})$, proving *(i)*.

For *(ii)*, as the matrix $\mu_S(A)$ represents $Z^{\#}_{G(\mathbb{F}_q)}$ , its eigenvalues are precisely given by $|G(\mathbb{F}_q)|^2/d^2$. Finally, *(iii)* follows from $(*)$. □

4.11 $\mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$-character stacks

Let us illustrate how the arithmetic TQFT and the character stack TQFT are related, and how they differ, by means of an example. Throughout this section, we consider the group $G = \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{G}_m$ via $x \mapsto x^{-1}$, over any field k of characteristic not equal to 2, or more generally, over the finitely generated algebra $R = \mathbb{Z}[\frac{1}{2}]$.

Arithmetic method. Following the arithmetic method, we consider the representation theory of the finite groups $G(\mathbb{F}_q) = \mathbb{F}_q^\times \rtimes \mathbb{Z}/2\mathbb{Z}$ with q is odd. The character table of $G(\mathbb{F}_q)$ can easily be computed, e.g. using [Ser77, Proposition 25]. Fixing any generator $x \in \mathbb{F}_q^\times$, the character table of $G(\mathbb{F}_q)$ is given by

	$\{1\}$	$\{-1\}$	$\{x^\ell, x^{-\ell}\}$	$(\mathbb{F}_q^\times)^2\sigma$	$(\mathbb{F}_q^\times)^2 x\sigma$
$\rho_{\varepsilon,\delta}$	1	$\varepsilon^{\frac{q-1}{2}}$	ε^ℓ	δ	$\varepsilon\delta$
τ_k	2	$2(-1)^k$	$\zeta_{q-1}^{k\ell} + \zeta_{q-1}^{-k\ell}$	0	0

where $1 \leq k, \ell \leq \frac{q-3}{2}$ and $\varepsilon, \delta = \pm 1$, and σ is the non-trivial element in $\mathbb{Z}/2\mathbb{Z}$. Summing characters of the same dimension, the character table reduces to

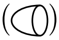
	$\{1\}$	$\{-1\}$	$\{x^\ell, x^{-\ell}\}$	$(\mathbb{F}_q^\times)^2\sigma$	$(\mathbb{F}_q^\times)^2 x\sigma$
$v_1 = \sum_{\varepsilon,\delta} \rho_{\varepsilon,\delta}$	4	$4\alpha_{(q-1)/2}$	$4\alpha_\ell$	0	0
$v_2 = \sum_k \tau_k$	$q-3$	$-2\alpha_{(q-1)/2}$	$-2\alpha_\ell$	0	0

where $\alpha_\ell = 1$ for ℓ even and $\alpha_\ell = 0$ for ℓ odd. Alternatively, this table can be expressed as

	$\{1\}$	$\{t \in G(\mathbb{F}_q) \mid t \neq 1 \text{ a square}\}$	$\{t \in G(\mathbb{F}_q) \mid t \text{ not a square}\}$
v_1	4	4	0
v_2	$q-3$	-2	0

Now, from (4.4), (4.5) and (4.7) follows that the TQFT $Z^{\#}_{G(\mathbb{F}_q)}$ is, with respect to the basis v_1, v_2, given by

$Z^{\#}_{G(\mathbb{F}_q)}$ () $= |G(\mathbb{F}_q)|^2 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix},$

$Z^{\#}_{G(\mathbb{F}_q)}$ () $= \frac{1}{|G(\mathbb{F}_q)|} \begin{pmatrix} 4 & q-3 \end{pmatrix},$ $Z^{\#}_{G(\mathbb{F}_q)}$ () $= \frac{1}{|G(\mathbb{F}_q)|} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$

where, of course, $|G(\mathbb{F}_q)| = 2(q-1)$. Therefore, the number of points of the $G(\mathbb{F}_q)$-representation varieties are given by

$$|R_{G(\mathbb{F}_q)}(\Sigma_g)| = (q-3)(q-1)^{2g-1} + 2^{2g+1}(q-1)^{2g-1}.$$

Applying Theorem 4.6.1 (with $R = \mathbb{Z}[\frac{1}{2}]$), we obtain the E-polynomial

$$e(R_{G \times_R \mathbb{C}}(\Sigma_g)) = (uv-3)(uv-1)^{2g-1} + 2^{2g+1}(uv-1)^{2g-1}.$$

Geometric method. Following the geometric method, the goal is to compute the $\mathrm{K}_0(\mathbf{Stck}_k)$-module morphism Z_G (). Since $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ is not finitely generated as $\mathrm{K}_0(\mathbf{Stck}_k)$-module, it is impossible to compute this map in full, so instead we restrict to a finitely generated submodule of $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ which will be invariant under this map.

Note that, via the natural map $[G/G] \to \mathrm{B}G$, we can view $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ as a $\mathrm{K}_0(\mathbf{Stck}_{\mathrm{B}G})$-module. Moreover, from Proposition 4.8.4 it is not hard to see that Z_G () promotes to a morphism of $\mathrm{K}_0(\mathbf{Stck}_{\mathrm{B}G})$-modules.

Denote by $I \in \mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ the class of the inclusion $[\{1\}/G] \to [G/G]$ of the identity, and denote by $S \in \mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ the class of the morphism $[\mathbb{G}_m/G] \to [G/G]$ induced by the squaring map $x \mapsto x^2$. Furthermore, we will make use of the following classes in $\mathrm{K}_0(\mathbf{Stck}_{\mathrm{B}G})$. Denote by A, B and C the classes $[\mathbb{G}_m/G], [\mathbb{G}_m\sigma/G]$ and $[(\mathbb{Z}/2\mathbb{Z})/G]$, respectively, where $\mathbb{G}_m$, $\mathbb{G}_m\sigma$ (recall that σ denotes the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$) and $\mathbb{Z}/2\mathbb{Z}$ are viewed as subvarieties of G on which G acts by conjugation.

Proposition 4.11.1. *The* $\mathrm{K}_0(\mathbf{Stck}_{\mathrm{B}G})$*-submodule* $\langle I, S\rangle \subseteq \mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ *is invariant under* Z_G ()*, and*

Z_G () $(I) = A^2 \cdot I + 3B \cdot S,$

Z_G () $(S) = (A+B)^2 \cdot S.$

Proof. The image of I is the virtual class of the morphism $[G^2/G] \to [G/G]$ induced by the commutator $[-,-]\colon G^2 \to G$. Stratifying G by $\mathbb{G}_m$ and $\mathbb{G}_m\sigma$, we find

$$[x,y] = 1, \qquad [x, y\sigma] = x^2,$$
$$[x\sigma, y] = y^{-2}, \qquad [x\sigma, y\sigma] = x^2y^{-2}.$$

The first stratum contributes $A^2 \cdot I$. The second and third stratum both contribute $B \cdot S$. After a change of variables $x' = x^2y^{-2}$ and $y' = y$, we find that the fourth stratum contributes $B \cdot S$ as well.

Next, the image of S is the virtual class of the morphism $[(\mathbb{G}_m \times G^2)/G] \to [G/G]$ induced by

$$\mathbb{G}_m \times G^2 \to G, \quad (z, a, b) \mapsto z^2[a, b].$$

Stratifying G as above, this morphism is given by

$$z^2[x,y] = z^2, \qquad z^2[x, y\sigma] = x^2z^2,$$
$$z^2[x\sigma, y] = y^{-2}z^2, \qquad z^2[x\sigma, y\sigma] = x^2y^{-1}z^2.$$

The first stratum contributes $A^2 \cdot S$, the second and third stratum contribute $AB \cdot S$ each, and the fourth stratum contributes $B^2 \cdot S$. $\square$

In order to repeatedly apply Z_G (), we must understand how the scalars A, B and C behave under multiplication.

Lemma 4.11.2. *In* $\mathrm{K}_0(\mathbf{Stck}_{\mathrm{B}G})$, *the following relations hold:*

(i) $A^2 = (\mathbb{L}+2)A - (\mathbb{L}-2)C - (\mathbb{L}+1)$

(ii) $B^2 = AB$

(iii) $C^2 = 2C$

(iv) $AC = (\mathbb{L}-1)C$

Proof. *(ii)* and *(iii)* follow from the G-equivariant isomorphisms

$$\begin{aligned} \mathbb{G}_m\sigma \times \mathbb{G}_m\sigma &\to \mathbb{G}_m\sigma \times \mathbb{G}_m, & (x\sigma, y\sigma) &\mapsto (x\sigma, \tfrac{y}{x}\sigma) \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} &\to \{\pm 1\} \times \mathbb{Z}/2\mathbb{Z}, & (a,b) &\mapsto (ab, b) \end{aligned}$$

where G acts trivially on $\{\pm 1\}$. For *(i)*, the action of G on $\mathbb{G}_m$ by conjugation can be extended to $\mathbb{P}^1_k$, so that $A = [\mathbb{P}^1_k/G] - C$. After a change of variables on $\mathbb{P}^1_k$, the action of G can be described by $\sigma \cdot (x:y) = (-x:y)$. Note that this change of variables uses the assumption that 2 is invertible. Now, $[\mathbb{P}^1_k/G] = [\mathbb{A}^1_k/G] + 1$ where G acts on $\mathbb{A}^1_k$ by $\sigma \cdot x = -x$, and thus $A = [\mathbb{A}^1_k/G] + 1 - C$. One sees, similar to *(ii)* and *(iii)*, that $[\mathbb{A}^1_k/G]^2 = \mathbb{L}[\mathbb{A}^1_k/G]$ and $[\mathbb{A}^1_k/G]C = \mathbb{L}C$. It follows that

$$\begin{aligned} A^2 &= ([\mathbb{A}^1_k/G] + 1 - C)^2 \\ &= (\mathbb{L}+2)[\mathbb{A}^1_k/G] - 2\mathbb{L}C + 1 \\ &= (\mathbb{L}+2)A - (\mathbb{L}-2)C - (\mathbb{L}+1). \end{aligned}$$

Finally, *(iv)* follows as $AC = ([\mathbb{A}^1_k/G] + 1 - C)C = (\mathbb{L}-1)C$. ☐

The above lemma, in combination with Proposition 4.11.1, allows us to obtain the images under Z_G () of the elements

$$I, \quad A \cdot I, \quad C \cdot I, \quad B \cdot S, \quad AB \cdot S, \quad BC \cdot S.$$

Moreover, it follows that the $\mathrm{K}_0(\mathbf{Stck}_k)$-submodule of $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ generated by these elements is invariant under Z_G (). In terms of these generators, the map Z_G () is represented by the following matrix.

$$\begin{bmatrix} -\mathbb{L}-1 & -\mathbb{L}^2-3\mathbb{L}-2 & 0 & 0 & 0 & 0 \\ \mathbb{L}+2 & \mathbb{L}^2+3\mathbb{L}+3 & 0 & 0 & 0 & 0 \\ 2-\mathbb{L} & -2\mathbb{L}^2+3\mathbb{L}+2 & \mathbb{L}^2-2\mathbb{L}+1 & 0 & 0 & 0 \\ 3 & 0 & 0 & -4\mathbb{L}-4 & -4\mathbb{L}^2-12\mathbb{L}-8 & 0 \\ 0 & 3 & 0 & 4\mathbb{L}+8 & 4\mathbb{L}^2+12\mathbb{L}+12 & 0 \\ 0 & 0 & 3 & 8-4\mathbb{L} & -8\mathbb{L}^2+12\mathbb{L}+8 & 4\mathbb{L}^2-8\mathbb{L}+4 \end{bmatrix}$$

One diagonalizes this matrix with eigenvalues

$$1, \quad 4, \quad (\mathbb{L}-1)^2, \quad (\mathbb{L}+1)^2, \quad 4(\mathbb{L}-1)^2, \quad 4(\mathbb{L}+1)^2,$$

and eigenvectors

$$\begin{bmatrix} \mathbb{L}+1 \\ -1 \\ -1 \\ -\mathbb{L}-1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbb{L}+1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ (\mathbb{L}-1)^2 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2(\mathbb{L}+1)^2 \\ -2(\mathbb{L}+1)^2 \\ (\mathbb{L}-2)(\mathbb{L}+1)^2 \\ -2 \\ 2 \\ 2-\mathbb{L} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -2 \\ \mathbb{L}-2 \end{bmatrix},$$

respectively. From the decomposition $\Sigma_g =$  $\circ$  $^g \circ$ , we can now compute the virtual class $[\mathfrak{X}_G(\Sigma_g)]$ in $\mathrm{K}_0(\mathbf{Stck}_{BG})$. Using that

Z_G () $(1) = I,\quad Z_G$ () $(I) = 1$ and Z_G () $(S) = 2$

we obtain the following theorem.

Theorem 4.11.3. *Let $G = \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ over a field of characteristic not equal to 2. The virtual class of the G-character stack $\mathfrak{X}_G(\Sigma_g)$ in $\mathrm{K}_0(\mathbf{Stck}_{BG})$ equals*

$$\begin{aligned} [\mathfrak{X}_G(\Sigma_g)] = {} & \mathbb{L}^{-1}(\mathbb{L}+1-(\mathbb{L}+1)^{2g}) \\ & + \mathbb{L}^{-1}((\mathbb{L}+1)^{2g}-1)A \\ & + 2\mathbb{L}^{-1}(4^g-1)(\mathbb{L}-(\mathbb{L}+1)^{2g-2}+1)B \\ & + \tfrac{1}{2}\mathbb{L}^{-1}(\mathbb{L}(\mathbb{L}-1)^{2g}-(\mathbb{L}-2)(\mathbb{L}+1)^{2g}-2)C \\ & + 2\mathbb{L}^{-1}(4^g-1)((\mathbb{L}+1)^{2g-2}-1)AB \\ & + \mathbb{L}^{-1}(4^g-1)(\mathbb{L}(\mathbb{L}-1)^{2g-2}-(\mathbb{L}-2)(\mathbb{L}+1)^{2g-2}-2)BC. \qquad \square \end{aligned}$$

Finally, in order to obtain the virtual class of the character stack in $\mathrm{K}_0(\mathbf{Stck}_k)$, we simply need to compute the images of $1, A, B, C, AB$ and BC under the morphism

$$c_! \colon \mathrm{K}_0(\mathbf{Stck}_{BG}) \to \mathrm{K}_0(\mathbf{Stck}_k).$$

Lemma 4.11.4. *In $\mathrm{K}_0(\mathbf{Stck}_k)$, the following equalities hold:*

(i) $c_!(1) = \mathbb{L}/(\mathbb{L}^2-1)$

(ii) $c_!(A) = 1$

(iii) $c_!(B) = 1$

(iv) $c_!(C) = (\mathbb{L}-1)^{-1}$

(v) $c_!(AB) = \mathbb{L}$

(vi) $c_!(BC) = 1$

Proof. *(i)* View G as the subgroup of GL_2 generated by $\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. From the GL_2-torsor $[\mathrm{GL}_2/G] \to \mathrm{B}G$ follows that $[\mathrm{B}G] = [\mathrm{GL}_2/G]/[\mathrm{GL}_2]$. Writing $\mathrm{GL}_2 = \operatorname{Spec} k[a, b, c, d, (ad - bc)^{-1}]$, we can identify

$$\begin{aligned} [\mathrm{GL}_2/G] &= \operatorname{Spec} k[ab, cd, ad + bc, (ad - bc)^{-2}] \\ &= \operatorname{Spec} k[x, y, z, (z^2 - 4xy)^{-1}] \end{aligned}$$

whose virtual class is easily seen to be $\mathbb{L}^2(\mathbb{L} - 1)$. Hence, we obtain $c_!(1) = \mathbb{L}^2(\mathbb{L} - 1)/[\mathrm{GL}_2] = \mathbb{L}/(\mathbb{L}^2 - 1)$.

(ii) From the GL_2-torsor $\mathrm{GL}_2 \times_G \mathbb{G}_m \to [\mathbb{G}_m/G]$, it follows that $[\mathbb{G}_m/G] = [\mathbb{G}_m \times_G \mathrm{GL}_2]/[\mathrm{GL}_2]$, where we can identify $[\mathbb{G}_m \times_G \mathrm{GL}_2]$ with

$$\begin{aligned} &\operatorname{Spec} k[ab, cd, ad + bc, x + x^{-1}, (ad - bc)(x - x^{-1}), (ad - bc)^{-2}] \\ &\quad = \operatorname{Spec} k[u, v, w, t, s, (w^2 - 4uv)^{-1}]/(s^2 - (t^2 - 4)(w^2 - 4uv)) \end{aligned}$$

whose virtual class can be computed as $\mathbb{L}(\mathbb{L} - 1)^2(\mathbb{L} + 1)$. Hence, we obtain $c_!(A) = \mathbb{L}(\mathbb{L} - 1)^2(\mathbb{L} + 1)/[\mathrm{GL}_2] = 1$.

(iii) This case is analogous to *(ii)*.

(iv) As $[(\mathbb{Z}/2\mathbb{Z})/G] = \mathrm{B}\mathbb{G}_m$ and $\mathbb{G}_m$ is special, we find $[(\mathbb{Z}/2\mathbb{Z})/G] = (\mathbb{L} - 1)^{-1}$.

(v) Note that $[\mathbb{G}_m\sigma \times \mathbb{G}_m/G] \cong \mathrm{B}(\mathbb{Z}/2\mathbb{Z}) \times [\mathbb{G}_m/\langle\sigma\rangle]$. Since $[\mathrm{B}(\mathbb{Z}/2\mathbb{Z})] = 1$ and $[\mathbb{G}_m/\langle\sigma\rangle] = \mathbb{L}$, we find that $[\mathbb{G}_m\sigma \times \mathbb{G}_m/G] = \mathbb{L}$.

(vi) Finally, $[(\mathbb{G}_m\sigma \times (\mathbb{Z}/2\mathbb{Z}))/G] = [\mathrm{B}(\mathbb{Z}/2\mathbb{Z})] = 1$. □

Corollary 4.11.5. *Let $G = \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$ over a field k of characteristic not equal to 2. The virtual class of the G-character stack $\mathfrak{X}_G(\Sigma_g)$ in $\mathrm{K}_0(\mathbf{Stck}_k)$ is given by*

$$[\mathfrak{X}_G(\Sigma_g)] = \frac{(\mathbb{L} - 1)^{2g-2}\left(2^{2g+1} + \mathbb{L} - 3\right)}{2} + \frac{(\mathbb{L} + 1)^{2g-2}\left(2^{2g+1} + \mathbb{L} - 1\right)}{2}. \qquad \square$$

Indeed, note that $[\mathfrak{X}_G(\Sigma_g)] \neq [R_G(\Sigma_g)]/[G]$ in $\mathrm{K}_0(\mathbf{Stck}_k)$, reflecting the fact that G is not connected.

4.12 Representation variety TQFT

While the construction of the TQFT of Theorem 4.7.8 is quite elegant, using it to explicitly compute the virtual class of character stacks can be rather hard. When M is a connected closed manifold and G a special algebraic group, the virtual class of the G-character stack $\mathfrak{X}_G(M)$ can also be computed as

$[\mathfrak{X}_G(M)] = [R_G(M)]/[G]$ by Proposition 3.5.5. Hence, if there were to exist a (lax) TQFT that quantizes the virtual class of the G-representation variety $R_G(M)$, this would lead to a more practical approach, as more stratifications will be allowed for in computations: stratifications on $R_G(M)$, as opposed to stratifications on $\mathfrak{X}_G(M)$, need not be G-equivariant with respect to the action of G by conjugation. Such a TQFT was proposed by [GLM20], making use of *pointed bordisms* instead of bordisms, that is, bordisms equipped with a choice of basepoints on their boundaries. These basepoints are used to keep track of any non-trivial loops that arise when bordisms are composed. The downside of this TQFT is that it does not quite quantize the virtual class of $R_G(M)$, but rather $[G]^n[R_G(M)]$, where n is the number of basepoints on M. Without these basepoints, no such TQFT exists.

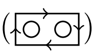
Nevertheless, we can define the following morphisms, which will effectively compute the virtual class of the representation variety. However, we stress that these maps do *not* come from a TQFT.

Definition 4.12.1. Let G be a linear algebraic group over a field k. Define the following $\mathrm{K}_0(\mathbf{Var}_k)$-module morphisms,

Z_G^{rep} () $: \mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathbf{Var}_G), \quad 1 \mapsto \left[\{1\} \to G\right]$

Z_G^{rep} () $: \mathrm{K}_0(\mathbf{Var}_G) \to \mathrm{K}_0(\mathbf{Var}_k), \quad \begin{bmatrix} X \\ \downarrow f \\ G \end{bmatrix} \mapsto \left[f^{-1}(1)\right]$

Z_G^{rep} () $: \mathrm{K}_0(\mathbf{Var}_G) \to \mathrm{K}_0(\mathbf{Var}_G), \quad \begin{bmatrix} X \\ \downarrow f \\ G \end{bmatrix} \mapsto \begin{bmatrix} X \times G^2 & (x, A, B) \\ \downarrow & \mapsto \\ G & f(x)[A, B] \end{bmatrix}$

Z_G^{rep} () $: \mathrm{K}_0(\mathbf{Var}_G) \to \mathrm{K}_0(\mathbf{Var}_G), \quad \begin{bmatrix} X \\ \downarrow f \\ G \end{bmatrix} \mapsto \begin{bmatrix} X \times G & (x, A) \\ \downarrow & \mapsto \\ G & f(x)A^2 \end{bmatrix}$

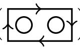
where $\{1\} \to G$ is the inclusion of the unit, and $[A, B] = ABA^{-1}B^{-1}$ denotes the group commutator. Furthermore, we define the $\mathrm{K}_0(\mathbf{Var}_k)$-module morphism

Z_G^{rep} () $: \begin{array}{rcl} \mathrm{K}_0(\mathbf{Var}_G) \otimes \mathrm{K}_0(\mathbf{Var}_G) & \to & \mathrm{K}_0(\mathbf{Var}_G) \\ \left[X \xrightarrow{f} G\right] \otimes \left[Y \xrightarrow{g} G\right] & \mapsto & \left[X \times Y \xrightarrow{m \circ (f \times g)} G\right] \end{array}$

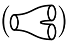
where $m \colon G \times G \to G$ denotes the multiplication map.

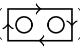
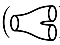
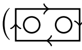
From the explicit presentations of the G-representation varieties of the orientable and non-orientable surfaces, $R_G(\Sigma_g)$ and $R_G(N_r)$, as in Example 2.1.4, it follows that their virtual classes can be computed through the following formulas.

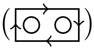
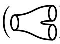
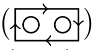
$[R_G(\Sigma_g)] = Z_G^{\mathrm{rep}}\big($  $\big) \circ Z_G^{\mathrm{rep}}\big($  $\big)^g \circ Z_G^{\mathrm{rep}}\big($  $\big)(1) \qquad (4.10)$

$[R_G(N_r)] = Z_G^{\mathrm{rep}}\big($  $\big) \circ Z_G^{\mathrm{rep}}\big($  $\big)^g \circ Z_G^{\mathrm{rep}}\big($  $\big)(1) \qquad (4.11)$

Moreover, it follows immediately from the expressions in Definition 4.12.1 that the following relations hold, for all $X \in \mathrm{K}_0(\mathbf{Var}_G)$:

$Z_G^{\mathrm{rep}}\big($  $\big)(X) = Z_G^{\mathrm{rep}}\big($  $\big)\big(X \otimes \big(Z_G^{\mathrm{rep}}\big($  $\big) \circ Z_G^{\mathrm{rep}}\big($  $\big)\big)(1)\big) \qquad (4.12)$

$Z_G^{\mathrm{rep}}\big($  $\big)(X) = Z_G^{\mathrm{rep}}\big($  $\big)\big(X \otimes \big(Z_G^{\mathrm{rep}}\big($  $\big) \circ Z_G^{\mathrm{rep}}\big($  $\big)\big)(1)\big) \qquad (4.13)$

Let us explain why the above equations are useful. They show that, in order to compute $Z_G^{\mathrm{rep}}\big($  $\big)$ and $Z_G^{\mathrm{rep}}\big($  $\big)$, it suffices to understand only the image of $Z_G^{\mathrm{rep}}\big($  $\big)(1)$ under these maps, and to compute $Z_G^{\mathrm{rep}}\big($  $\big)$. However, this latter is only 'linear' in the two inputs, whereas the original maps $Z_G^{\mathrm{rep}}\big($  $\big)$ and $Z_G^{\mathrm{rep}}\big($  $\big)$ are 'quadratic' in their inputs. This will result in significant simplifications in the concrete computations of the following chapters.

Chapter 5

SL_2-character stacks

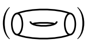
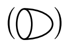
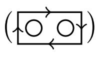
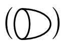
In this chapter, we apply the theory of Chapter 4 to compute the virtual classes of the G-character stacks $\mathfrak{X}_G(M)$, for M equal to both orientable and non-orientable closed surfaces, and G equal to

$$\mathrm{SL}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1 \right\}.$$

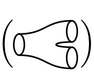
Even though $G = \mathrm{SL}_2$ is one of the simplest non-trivial groups, the resulting computations are quite intricate. Throughout this chapter, we work over an algebraically closed field k with $\mathrm{char}(k) \neq 2$.

Similar computations were performed by [LMN13, MM16] to compute the corresponding E-polynomials. While the scissor relation (3.3) is the main ingredient in these computations, they cannot simply be lifted to the Grothendieck ring of varieties. Instead, many subtle points arise and have to be dealt with, such as the study of $\mathbb{P}^1$-fibrations, equivariant motivic invariants (as in Section 3.6), and non-zero elements in the Grothendieck ring of varieties whose E-polynomial is zero.

As $G = \mathrm{SL}_2$ is a special group, the virtual class of the G-character stack $\mathfrak{X}_G(M)$ is equal to that of the G-representation variety $R_G(M)$ divided by $[\mathrm{SL}_2] = \mathbb{L}(\mathbb{L} - 1)(\mathbb{L} + 1)$. Hence, we can apply the theory of Section 4.12, allowing us to make non-equivariant stratifications. In order to use (4.10), (4.11), (4.12) and (4.13), we will compute

Z_G^{rep} () $\circ\, Z_G^{\mathrm{rep}}$ () and Z_G^{rep} () $\circ\, Z_G^{\mathrm{rep}}$ () (5.1)

in Section 5.2 and Section 5.3, respectively, and in Section 5.4 we compute

Z_G^{rep} () . (5.2)

It is not necessary to compute these maps in full. Rather it suffices to compute their restriction to a certain finitely generated submodule of $\mathrm{K}_0(\mathbf{Var}_G)$. The generators for this submodule are described in Section 5.1. For the computation of some of these maps, we need an extra relation in the Grothendieck ring of varieties regarding $\mathbb{P}^1$-fibrations. This will also be discussed in Section 5.1.

Finally, in Section 5.5 we collect and discuss the results.

5.1 Generators, relations and $\mathbb{P}^1$-fibrations

Let us introduce some notation. The following varieties are all considered naturally as varieties over $G = \mathrm{SL}_2$.

$$\begin{aligned}
I_+ &= \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}, \\
I_- &= \{\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}, \\
J_+ &= \{A \in G \mid A \text{ conjugate to } \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\}, \\
J_- &= \left\{A \in G \mid A \text{ conjugate to } \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)\right\}, \\
M &= \{A \in G \mid \mathrm{tr}(A) \neq \pm 2\}, \\
X_2 &= \{(A, \ell) \in M \times \mathbb{A}^1_k \mid \ell^2 = \mathrm{tr}(A) - 2\}, \\
X_{-2} &= \{(A, \ell) \in M \times \mathbb{A}^1_k \mid \ell^2 = \mathrm{tr}(A) + 2\}, \\
X_{2,-2} &= \{(A, \ell) \in M \times \mathbb{A}^1_k \mid \ell^2 = \mathrm{tr}(A)^2 - 4\}, \\
Y &= \{(A, \omega) \in M \times \mathbb{A}^1_k \setminus \{0\} \mid \mathrm{tr}(A) = \omega^2 + \omega^{-2}\},
\end{aligned} \tag{5.3}$$

By the same symbols, we will also denote their virtual class in $\mathrm{K}_0(\mathbf{Var}_G)$. These elements will be the generators of the $\mathrm{K}_0(\mathbf{Var}_k)$-submodule of $\mathrm{K}_0(\mathbf{Var}_G)$ on which (5.1) and (5.2) will be computed. A useful alternative presentation of the last five generators is as follows:

$$\begin{aligned}
M &\cong (\mathrm{GL}_2/D \times \mathbb{A}^1_k \setminus \{0, \pm 1\}) /\!\!/ S_2 \to G, & (P, \lambda) &\mapsto P \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) P^{-1} \\
X_2 &\cong (\mathrm{GL}_2/D \times \mathbb{A}^1_k \setminus \{0, \pm 1, \pm i\}) /\!\!/ S_2 \to G, & (P, \omega) &\mapsto P \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^{-2} \end{pmatrix} P^{-1} \\
X_{-2} &\cong (\mathrm{GL}_2/D \times \mathbb{A}^1_k \setminus \{0, \pm 1, \pm i\}) /\!\!/ S_2 \to G, & (P, \omega) &\mapsto P \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^{-2} \end{pmatrix} P^{-1} \\
X_{2,-2} &\cong \mathrm{GL}_2/D \times \mathbb{A}^1_k \setminus \{0, \pm 1\} \to G, & (P, \lambda) &\mapsto P \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) P^{-1} \\
Y &\cong \mathrm{GL}_2/D \times \mathbb{A}^1_k \setminus \{0, \pm 1, \pm i\} \to G, & (P, \omega) &\mapsto P \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^{-2} \end{pmatrix} P^{-1}
\end{aligned}$$

where $D \subseteq \mathrm{GL}_2$ is the subgroup of diagonal matrices, and where S_2 acts on the left coset space GL_2/D by $P \mapsto P \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and acts on the coordinates λ and ω by $\lambda \mapsto \lambda^{-1}$ and $\omega \mapsto \omega^{-1}$. The following lemma gives a better understanding of the relation between these generators.

Lemma 5.1.1. *The following relations hold in* $\mathrm{K}_0(\mathbf{Var}_G)$*:*

$$X_2^2 = 2X_2, \quad X_{-2}^2 = 2X_{-2}, \quad X_{2,-2}^2 = 2X_{2,-2}$$
$$\text{and} \quad Y = X_2 X_{-2} = X_2 X_{2,-2} = X_{-2} X_{2,-2}.$$

Proof. The first equality follows from

$$X_2 \times_M X_2 = \{(A, \ell_1, \ell_2) \in M \times \mathbb{A}_k^2 \mid \ell_1^2 = \mathrm{tr}(A) - 2 \text{ and } \ell_2 = \pm \ell_1\} \cong X_2 \sqcup X_2,$$

and similarly for the second and third. The final two equalities follow from the fact that, if $\ell_1^2 = \mathrm{tr}(A) - 2$ and $\ell_2^2 = \mathrm{tr}(A) + 2$, then $(\ell_1 \ell_2)^2 = \mathrm{tr}(A)^2 - 4$. Finally, the fourth equality follows from the isomorphism

$$Y \xrightarrow{\sim} X_2 \times_M X_{-2} = \{(A, \ell_1, \ell_2) \in M \times \mathbb{A}_k^2 \mid \ell_1^2 = \mathrm{tr}(A) - 2 \text{ and } \ell_2^2 = \mathrm{tr}(A) + 2\}$$

which is given by $(A, \omega) \mapsto (A, \omega - \omega^{-1}, \omega + \omega^{-1})$ with inverse $(A, \ell_1, \ell_2) \mapsto (A, \frac{1}{2}(\ell_1 + \ell_2))$. ☐

Remark 5.1.2. The symbols X_2, X_{-2} and $X_{2,-2}$ were adopted from [Gon20] and reflect the monodromy action of these spaces as covering spaces over M. They are double covers of M, and have non-trivial monodromy for loops around $\mathrm{tr}\, A = 2$ or $\mathrm{tr}\, A = -2$ as indicated by the subscript of the symbol. More precisely, write T for the trivial representation of $\pi_1(M, *)$ and N_2 (resp. N_{-2}) for the 1-dimensional representation that sends a loop around $\mathrm{tr}\, A = 2$ (resp. $\mathrm{tr}\, A = -2$) to -1. Then the monodromy representations of X_2, X_{-2} and $X_{2,-2}$ are $T \oplus N_2$, $T \oplus N_{-2}$ and $T \oplus N_2 \otimes N_{-2}$, respectively. Since $Y \cong X_2 \times_M X_{-2}$, it follows that Y is a 4-to-1 cover of M with monodromy representation $T \oplus N_2 \oplus N_{-2} \oplus N_2 \otimes N_{-2}$. In particular, the monodromy representation of $M \sqcup M \sqcup Y$ is equal to that of $X_2 \sqcup X_{-2} \sqcup X_{2,-2}$. This is also the case for their *Hodge monodromy representation* [LMN13, (5)], and for this reason, the generator Y is not needed in the E-polynomial computations of [LMN13, MM16]. However, the analogous equality does not (necessarily) hold in $\mathrm{K}_0(\mathbf{Var}_G)$ as $M \sqcup M \sqcup Y$ is not isomorphic to $X_2 \sqcup X_{-2} \sqcup X_{2,-2}$ over M: the former has two sections over M whereas the latter has none.

Finally, when computing the images of the generators (5.3) under the maps (5.1) and (5.2), we will encounter some non-trivial $\mathbb{P}^1$-fibrations. Recall, a $\mathbb{P}^1$-*fibration* is a morphism $P \to X$ which is étale-locally of the form $X \times \mathbb{P}_k^1 \xrightarrow{\pi_X} X$, where π_X denotes the projection to X. However, as many motivic invariants $\chi \colon \mathrm{K}_0(\mathbf{Var}_k) \to R$ satisfy $\chi(P) = \chi(\mathbb{P}_k^1)\chi(X)$ for all $\mathbb{P}^1$-fibrations $P \to X$, we will impose this relation as well. This includes the point-count over finite fields, and the E-polynomial and Euler characteristic over $\mathbb{C}$ [MOV09, Lemma 2.4].

Definition 5.1.3. Let S be a variety over k. Denote by $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_S)$ the quotient of $\mathrm{K}_0(\mathbf{Var}_S)$ by relations of the form

$$[P]_S = [\mathbb{P}^1_k] \cdot [X]_S \tag{5.4}$$

for all $\mathbb{P}^1$-fibrations $P \to X$ over S. Similarly, denote by $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Stck}_S)$ the quotient of $\mathrm{K}_0(\mathbf{Stck}_S)$ by the same relations. Furthermore, if G is a finite group, denote by $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_S^G)$ the quotient of $\mathrm{K}_0(\mathbf{Var}_S^G)$ by the same relations, for all G-equivariant $\mathbb{P}^1$-fibrations $P \to X$ over S.

We will need the G-equivariant version when dealing with varieties of the form $X /\!\!/ G$, and we want to stratify X in a G-equivariant manner. In that case, it is important that taking the quotient with respect to G respects the relation (5.4).

Proposition 5.1.4. *Let S be variety over k, and let G be a finite group. The morphism $\mathrm{K}_0(\mathbf{Var}_S^G) \to \mathrm{K}_0(\mathbf{Var}_S)$ given by $[X]_S \mapsto [X /\!\!/ G]_S$ descends to a morphism*

$$\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_S^G) \to \mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_S).$$

Proof. It must be shown that for every G-equivariant $\mathbb{P}^1$-fibration $P \to X$ over S, we have $[P /\!\!/ G]_S = [\mathbb{P}^1_k] \cdot [X /\!\!/ G]_S$ in $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_S)$.

If G does not act faithfully on X, then $N = \{g \in G \mid g \cdot x = x \text{ for all } x \in X\}$ is a normal subgroup of G which acts trivially on X. Since $X /\!\!/ G = X /\!\!/ (G/N)$ and $P /\!\!/ G = (P /\!\!/ N) /\!\!/ (G/N)$, we may replace G by G/N and P by $P' = P /\!\!/ N$ (still a $\mathbb{P}^1$-fibration over X) and assume that G does act faithfully on X.

Next, let $\mathcal{H}$ be a set of representatives for the conjugacy classes of subgroups of G. Stratify $X = \bigsqcup_{H \in \mathcal{H}} X_H$, where $X_H = \{x \in X \mid \mathrm{Stab}(x) \text{ is conjugate to } H\}$. Note that the action of G restricts to X_H since $\mathrm{Stab}(g \cdot x) = g \,\mathrm{Stab}(x) g^{-1}$ for all $g \in G$ and $x \in X$. Furthermore, we have $X_H /\!\!/ G = Y_H /\!\!/ N_G(H)$, where $Y_H = \{x \in X \mid \mathrm{Stab}(x) = H\}$ and $N_G(H)$ is the normalizer of H in G, and similarly $(P \times_X X_H) /\!\!/ G = (P \times_X Y_H) /\!\!/ N_G(H)$. Hence, replacing G by $N_G(H)$ and X by Y_H, we may assume $\mathrm{Stab}(x)$ is constant and normal in G. Moreover, since we could assume G to act faithfully on X, we can assume the action of G on X to be free.

After stratifying X into smooth strata, the quotient map $X \to X /\!\!/ G$ is étale [Dré04, Proposition 4.11], so from the cartesian diagram

$$\begin{array}{ccc} P & \longrightarrow & P /\!\!/ G \\ \downarrow & & \downarrow \\ X & \longrightarrow & X /\!\!/ G \end{array}$$

it follows that $P /\!\!/ G \to X /\!\!/ G$ is a $\mathbb{P}^1$-fibration over S as well. Therefore, $[P /\!\!/ G]_S = [\mathbb{P}^1_k] \cdot [X /\!\!/ G]_S$, as desired. $\square$

5.2 Orientable surfaces

The goal of this section is to prove the following proposition, which completely characterizes the first map of (5.1). The stratifications used are similar to those in [LMN13], but adapted to the setting of $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_G)$.

Proposition 5.2.1. *The virtual class of $G^2 \to G$ given by $(A, B) \mapsto [A, B]$ in $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Var}_G)$ is equal to*

$\Big(Z_G^{\mathrm{rep}}\big($  $\big) \circ Z_G^{\mathrm{rep}}\big($  $\big)\Big)(1)$

$$\begin{aligned} &= \mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)(\mathbb{L}+4)I_+ + \mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)I_- \\ &\quad + \mathbb{L}(\mathbb{L}-3)(\mathbb{L}+1)J_+ + \mathbb{L}^2(\mathbb{L}+3)J_- + (\mathbb{L}-1)^2(\mathbb{L}+1)M \\ &\quad + 2\mathbb{L}(\mathbb{L}+1)X_2 - \mathbb{L}(\mathbb{L}+1)X_{-2} - (\mathbb{L}-1)^2 X_{2,-2} + \mathbb{L}(\mathbb{L}-2)Y. \end{aligned}$$

Proof. Write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ and stratify based on the conjugacy class of $[A, B]$.

- If $[A, B] = 1$, we consider the following cases.
  - Case $A = \pm 1$. Since any B commutes with A, this stratum has a virtual class equal to $2[G]I_+ = 2\mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)I_+$.
  - Case $A \in J_\pm$. Conjugate A to $\left(\begin{smallmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{smallmatrix}\right)$ to find that B must be of the form $\left(\begin{smallmatrix} \pm 1 & x \\ 0 & \pm 1 \end{smallmatrix}\right)$. Hence, we obtain $4\mathbb{L}[J_+]I_+ = 4\mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)I_+$.
  - Case $A \in M$. Note that A can be conjugated to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ for some $\lambda \neq 0, \pm 1$, after which B must be diagonal. Hence, this stratum can be identified with

    $$\left(\left\{(P, \lambda, \mu) \in \mathrm{GL}_2/D \times (\mathbb{A}^1_k \setminus \{0, \pm 1\}) \times (\mathbb{A}^1_k \setminus \{0\})\right\}\right) /\!\!/ S_2$$

    where $A = P \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) P^{-1}$ and $B = P \left(\begin{smallmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{smallmatrix}\right) P^{-1}$, and where S_2 acts via $(P, \lambda, \mu) \mapsto \left(P \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \lambda^{-1}, \mu^{-1}\right)$. To compute the virtual class of this quotient, we apply Section 3.6 with the finite (cyclic) group $S_2 = \mathbb{Z}/2\mathbb{Z}$. Using notation as in (3.12), we find

    $$\begin{aligned} [\mathbb{A}^1_k \setminus \{0, \pm 1\}]^{S_2} &= (\mathbb{L}-2) \otimes T - 1 \otimes N, \\ [\mathbb{A}^1_k \setminus \{0\}]^{S_2} &= \mathbb{L} \otimes T - 1 \otimes N, \\ [\mathrm{GL}_2/D]^{S_2} &= \mathbb{L}^2 \otimes T + \mathbb{L} \otimes N. \end{aligned}$$

    Therefore, we obtain $\mathbb{L}(\mathbb{L}-2)(\mathbb{L}-1)(\mathbb{L}+1)I_+$.

Together, these cases add up to $\mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)(\mathbb{L}+4)I_+$.

- Suppose $[A,B]=-1$. From the equivalent expressions $ABA^{-1}=-B$ and $B^{-1}AB=-A$ follows that $\operatorname{tr}A=\operatorname{tr}B=0$. In particular, we can conjugate A to $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, after which B must be of the form $\left(\begin{smallmatrix} 0 & y \\ -1/y & 0 \end{smallmatrix}\right)$. Hence, we obtain $\mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1)I_-$.
- Suppose $[A,B]\in J_+$. Conjugate $[A,B]$ to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. From $AB=\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)BA$ follows that $\operatorname{tr}B=\operatorname{tr}(ABA^{-1})=\operatorname{tr}\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)B\right)$ and hence $z=0$. Similarly, $\operatorname{tr}A=\operatorname{tr}\left(BAB^{-1}\right)=\operatorname{tr}\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^{-1}A\right)$ implies $c=0$. Now $\det A=\det B=1$ yields $d=a^{-1}$ and $w=x^{-1}$, and the only remaining equation is $y(a-a^{-1})-b(x-x^{-1})=1/ax$. Consider the following cases.
  - If $a\neq\pm1$, we can solve for $y=(1/ax+b(x-x^{-1}))/(a-a^{-1})$, and obtain $\mathbb{L}(\mathbb{L}-3)(\mathbb{L}-1)J_+$.
  - If $a=\pm1$, we must have $x\neq\pm1$ and can solve for $b=a/(1-x^2)$. We obtain $2\mathbb{L}(\mathbb{L}-3)J_+$.

  Together, these cases add up to $\mathbb{L}(\mathbb{L}-3)(\mathbb{L}+1)J_+$.
- Suppose $[A,B]\in J_-$. Conjugate $[A,B]$ to $\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$. From $\operatorname{tr}B=\operatorname{tr}(ABA^{-1})=\operatorname{tr}\left(\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)B\right)$ follows that $z=2(x+w)$, and from $\operatorname{tr}A=\operatorname{tr}(BAB^{-1})=\operatorname{tr}\left(\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)^{-1}A\right)$ follows that $c=-2(a+d)$. The only remaining equation is $ay-bw-bx-dw+dy=0$. Consider the following cases.
  - Case $w=-x$. From $\det B=1$ follows that $x=\pm i$. The action of conjugation by $\{\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)\}\cong\mathbb{G}_a$ turns this stratum into a $\mathbb{G}_a$-torsor over the stratum with $y=0$. On this stratum with $y=0$, we can solve for $d=0$, and $\det A=1$ implies $a\neq0$ and $b=1/2a$. Hence, we obtain $2\mathbb{L}(\mathbb{L}-1)J_-$.
  - Case $w\neq-x$. The action of conjugation by $\{\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)\}\cong\mathbb{G}_a$ turns this stratum into a $\mathbb{G}_a$-torsor over the stratum with $w=0$. On this stratum with $w=0$, it follows from $\det B=1$ that $x\neq0$ and $y=-1/2x$. We can solve for $b=-(a+d)/2x^2$. Finally, $\det A=1$ translates to $ad-(a+d)^2/x^2=1$.
    * Case $d=-a$. Solve for $a=\pm i$ to obtain $2\mathbb{L}(\mathbb{L}-1)J_-$.
    * Case $d\neq-a$. Make a substitution $x'=(a+d)/x$ to rewrite the equation as $ad-(x')^2=1$. This is easily seen to give $\mathbb{L}(\mathbb{L}^2-\mathbb{L}+4)J_-$.

  Together, these cases add up to $\mathbb{L}^2(\mathbb{L}+3)J_-$.
- Suppose $[A,B]\in M$. Diagonalizing $[A,B]$, this stratum can be expressed as

$$\left\{(P,A,B,\lambda)\in\mathrm{GL}_2/D\times G^2\times(\mathbb{A}^1_k\setminus\{0,\pm1\})\;\middle|\;[A,B]=\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)\right\}/\!\!/\,S_2$$

where S_2 acts via $\lambda \mapsto \lambda^{-1}$ and $P \mapsto P\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and on A and B via conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. From $\operatorname{tr} A = \operatorname{tr}(BAB^{-1}) = \operatorname{tr}\left(\left(\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{smallmatrix}\right) A\right)$ follows that $d = a/\lambda$, and from $\operatorname{tr} B = \operatorname{tr}(ABA^{-1}) = \operatorname{tr}\left(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B\right)$ that $w = \lambda x$. The relevant equations are now $ax + bz - \lambda(ax + cy) = 0$ and $\det A = a^2\lambda^{-1} - bc = 1$ and $\det B = \lambda x^2 - yz = 1$. Consider the following cases.

– Case $b = c = 0$. It follows that $a^2 = \lambda$, $x = 0$ and $z = -y^{-1}$. Note that S_2 acts via $a \mapsto d = a/\lambda = a^{-1}$ and $y \mapsto z = -y^{-1}$. Therefore, we obtain the following S_2-virtual classes

$$[\{y \neq 0\}]^{S_2} = \mathbb{L} \otimes T - 1 \otimes N$$
$$[\mathrm{GL}_2/D \times \{a^2 = \lambda\}]^{S_2}_M = X_{-2} \otimes T + (Y - X_{-2}) \otimes N.$$

Multiplying these and taking the quotient by S_2, we obtain $(\mathbb{L}+1)X_{-2} - Y$.

– Case $y = z = 0$. Similarly, we obtain $(\mathbb{L} + 1)X_{-2} - Y$.

– Case $b = 0$ or $c = 0$, but not both. The action of S_2 swaps b and c, so we can identify the S_2-quotient with the stratum where $b = 0$ and $c \neq 0$. The action of conjugation by $\left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\right\} \cong \mathbb{G}_m$ turns this stratum into a $\mathbb{G}_m$-torsor over the stratum with $c = 1$. On this stratum with $c = 1$, we find that $a^2 = \lambda$, $y = ax(\lambda^{-1} - 1)$ with $x \neq 0$, and $z = (\lambda x^2 - 1)/y$. Hence, we obtain $(\mathbb{L} - 1)^2 Y$.

– Case $y = 0$ or $z = 0$, but not both. Similarly, we obtain $(\mathbb{L} - 1)^2 Y$.

– In the above cases, we have counted twice the stratum given by $b = z = 0$ or $c = y = 0$, so we need to subtract it once. Note that these conditions cannot be satisfied simultaneously, and moreover, the action of S_2 swaps them. Therefore, we can identify the S_2-quotient with the stratum where $b = z = 0$ (and $c, y \neq 0$). The action of conjugation by $\left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\right\} \cong \mathbb{G}_m$ turns this stratum into a $\mathbb{G}_m$-torsor over the stratum with $c = 1$. On this stratum with $c = 1$, we find $a^2 = \lambda$ and solve for $(x, y) = \pm\left(a^{-1}, a^{-1} - a\right)$. Hence, we obtain $-2(\mathbb{L} - 1)Y$, where the minus sign signifies this stratum must be subtracted from the total.

– Case $bcyz \neq 0$. Solve for $c = (a^2/\lambda - 1)/b$ and $z = (\lambda x^2 - 1)/y$. The conditions $c, z \neq 0$ translate to $a^2 \neq \lambda$ and $x^{-2} \neq \lambda$. The remaining equation is

$$x^2 - \frac{a'(\lambda - 1)}{(\lambda + 1)} xy' + \left(1 - \frac{(a')^2}{\lambda + \lambda^{-1} + 2}\right)(y')^2 = \lambda^{-1},$$

where we made substitutions $y' = y/b$ and $a' = a(1 + \lambda^{-1})$. The condition $a^2 \neq \lambda$ translates to $(a')^2 \neq \lambda + \lambda^{-1} + 2$. This equation describes a family of conics over the plane $\{(a', \lambda) \mid (a')^2 \neq \lambda + \lambda^{-1} + 2\}$ with discriminant $D = (a' - 2)(a' + 2)$. To compute its virtual class, the idea is to complete this family of conics to a $\mathbb{P}^1$-fibration over the (a', λ)-plane, for $D \neq 0$, so that relation (5.4) can be used. The stratum at infinity will be computed separately, and must be subtracted from the total.

Note that the variable $b \neq 0$ is independent of a', x and y', except through the action of S_2 given by $b \mapsto c = (a'\lambda/(\lambda+1)^2 - 1)/b$. Extending b to be $\mathbb{P}^1$-valued, we can regard this stratum as a $\mathbb{P}^1$-fibration minus the stratum with $b = 0$ or $b = \infty$. Note that the cases $b = 0$ and $b = \infty$ are interchanged by the action of S_2. Hence, for the sake of the computation, we can effectively act as if b is completely independent of a', x and y', with S_2-virtual class $[\{b \neq 0\}]^{S_2} = \mathbb{L} \otimes T - 1 \otimes N$.

* Case $D = 0$. Solve for $a' = \pm 2$. Suppose $a' = 2$. Then $\left(x - \frac{\lambda-1}{\lambda+1} y'\right)^2 = \lambda^{-1}$. Write $\omega = x - \frac{\lambda-1}{\lambda+1} y'$ so that $\omega^2 = \lambda^{-1}$ and note that S_2 acts via $\omega \mapsto -\omega^{-1}$. The condition $x^2 \neq \lambda^{-1}$ translates to $x \neq \pm\omega$. Substituting $x' = x/\omega$ yields $x' \neq \pm 1$ and S_2 acts via $x' \mapsto -x'$. From the S_2-virtual classes

$$\begin{aligned} [\{b \neq 0\}]^{S_2} &= \mathbb{L} \otimes T - 1 \otimes N \\ [\{x' \neq \pm 1\}]^{S_2} &= (\mathbb{L} - 1) \otimes T - 1 \otimes N \\ [\mathrm{GL}_2/D \times \{\omega^2 = \lambda^{-1}\}]^{S_2}_M &= X_2 \otimes T + (Y - X_2) \otimes N \end{aligned}$$

we obtain $\mathbb{L}(\mathbb{L}+1)X_2 - (2\mathbb{L}-1)Y$. The case $a' = -2$ is completely similar, so we double this virtual class to obtain $2\mathbb{L}(\mathbb{L}+1)X_2 - (4\mathbb{L}-2)Y$.

* Case $D \neq 0$. Complete the family of conics to a $\mathbb{P}^1$-fibration given by

$$X^2 - \frac{a'(\lambda-1)}{(\lambda+1)} XY + \left(1 - \frac{(a')^2}{\lambda + \lambda^{-1} + 2}\right) Y^2 = \lambda^{-1} Z^2, \tag{5.5}$$

over the base $B = \mathrm{GL}_2/D \times \{(a', \lambda) \mid a' \neq \pm 2 \text{ and } (a')^2 \neq \lambda + \lambda^{-1} + 2\}$. Regarding B as the open complement of $(a')^2 = \lambda + \lambda^{-1} + 2$, we compute its S_2-virtual class as

$$\begin{aligned} [B]^{S_2}_M = (\mathbb{L} - 2)(M \otimes T + (X_{2,-2} - M) \otimes N) \\ - (X_{-2} \otimes T + (Y - X_{-2}) \otimes N). \end{aligned}$$

Multiplying by $[\mathbb{P}^1_k] = \mathbb{L} + 1$ and by $[\{b \neq 0\}]^{S_2} = \mathbb{L} \otimes T - 1 \otimes N$, and taking the quotient by S_2, we obtain

$$(\mathbb{L} - 2)(\mathbb{L}+1)^2 M - (\mathbb{L}+1)^2 X_{-2} - (\mathbb{L}-2)(\mathbb{L}+1)X_{2,-2} + (\mathbb{L}+1)Y.$$

* Now we must subtract the stratum of points at infinity, that is, the points given by $Z = 0$. Since there are no solutions with $X = 0$, we can work

with the dehomogenized coordinate Y/X. In fact, writing $u = Y/X \cdot \left(1 - \frac{(a')^2\lambda}{(\lambda+1)^2}\right)$, equation (5.5) reduces to

$$\left(2u - \frac{a'(\lambda-1)}{(\lambda+1)}\right)^2 = (a'-2)(a'+2).$$

Substituting $u' = 2u - \frac{a'(\lambda-1)}{\lambda+1}$, we find that u' is invariant under S_2, and the equation simplifies to

$$(u')^2 = (a')^2 - 4.$$

Regarding this stratum as the open complement of $(a')^2 = \lambda + \lambda^{-1} + 2$, we compute its S_2-virtual class as

$$\left[\mathrm{GL}_2/D \times \left\{\begin{smallmatrix} (u')^2=(a')^2-4\neq 0 \\ (a')^2\neq\lambda+\lambda^{-1}+2 \end{smallmatrix}\right\}\right]_M^{S_2} = (\mathbb{L}-3)(M\otimes T - (X_{2,-2}-M)\otimes N) - Y\otimes(T+N).$$

Multiplying by $[\{b\neq 0\}]^{S_2} = \mathbb{L}\otimes T - 1\otimes N$ and taking the quotient by S_2, we obtain

$$-\Big((\mathbb{L}-3)(\mathbb{L}+1)M - (\mathbb{L}-3)X_{2,-2} - (\mathbb{L}-1)Y\Big),$$

where the overall minus sign signifies this stratum should be subtracted from the total.

* Finally, we must subtract the stratum where $x^{-2} = \lambda$. In this case, we solve for $y' = 0$ or $y' = \frac{a'x(\lambda-1)(\lambda+1)}{(\lambda+1)^2-(a')^2\lambda}$. When $a' = 0$, these values coincide, so from the S_2-virtual classes

$$[\{b\neq 0\}]^{S_2} = \mathbb{L}\otimes T - 1\otimes N$$
$$[\mathrm{GL}_2/D\times\{x^{-2}=\lambda\}]_M^{S_2} = X_{-2}\otimes T + (Y - X_{-2})\otimes N$$

we obtain

$$-\Big((\mathbb{L}+1)X_{-2} - Y\Big).$$

When $a' \neq 0$, the values for y' are interchanged by the action of S_2. Hence, we can identify the S_2-quotient with the stratum where $y' = 0$. The condition $(a')^2 \neq \lambda + \lambda^{-1} + 2$ translates to $a' \neq \pm(x + x^{-1})$. This gives

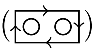
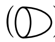
$$-\Big((\mathbb{L}-5)(\mathbb{L}-1)Y\Big).$$

Together, these cases add up to $(\mathbb{L}-1)^2(\mathbb{L}+1)M + 2\mathbb{L}(\mathbb{L}+1)X_2 - \mathbb{L}(\mathbb{L}+1)X_{-2} - (\mathbb{L}-1)^2X_{2,-2} + \mathbb{L}(\mathbb{L}-2)Y$. □

5.3 Non-orientable surfaces

Analogous to the previous section, we prove the following proposition, characterizing the second map of (5.1).

Proposition 5.3.1. *The virtual class of $G \to G$ given by $A \mapsto A^2$ in $\mathrm{K}_0(\mathbf{Var}_G)$ is equal to*

$$\left(Z_G^{\mathrm{rep}}\left(\right.\right.$$  $$\left.\right) \circ Z_G^{\mathrm{rep}}\left(\right.$$  $$\left.\left.\right)\right)(1) = 2I_+ + \mathbb{L}(\mathbb{L}+1)I_- + 2J_+ + X_{-2}.$$

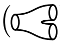
Proof. Write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and stratify based on the conjugacy class of A^2.

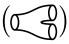
- If $A^2 = 1$, then $A = \pm 1$, so we obtain $2I_+$.
- Suppose $A^2 = -1$. If $b = 0$, then $d = a^{-1}$ with $a = \pm i$, contributing $2\mathbb{L}I_-$. If $b \neq 0$ and $c = 0$, then $d = a^{-1}$ with $a \neq \pm i$, contributing $2(\mathbb{L}-1)I_-$. If $b, c \neq 0$, then $d = -a$ and $a^2 + bc = -1$, contributing $(\mathbb{L}-2)(\mathbb{L}-1)I_-$. In total, we obtain $\mathbb{L}(\mathbb{L}+1)I_-$.
- Suppose $A^2 \in J_+$. By conjugating we can assume $A^2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. There are no solutions for $c \neq 0$, and $c = 0$ yields $a = d = b/2 = \pm 1$, so we obtain $2J_+$.
- There are no solutions with $A^2 \in J_-$.
- Suppose $A^2 \in M$. This stratum is given by

$$\left(\mathrm{GL}_2/D \times (\mathbb{A}^1_k \setminus \{0, \pm 1, \pm i\})\right) /\!\!/ S_2 \to G, \quad (P, \omega) \mapsto P \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^{-2} \end{pmatrix} P^{-1},$$

where S_2 acts on ω via $\omega \mapsto \omega^{-1}$. Hence, this is equal to X_{-2}. □

5.4 Multiplication in SL_2

In this section, we compute the images $Z_G^{\mathrm{rep}}\left(\right.$  $\left.\right)(X \otimes Y)$ for all pairs (X, Y) of generators in (5.3), in a series of lemmas. For conciseness, we will omit some cases, but those can be obtained directly from the cases we do compute. For example, the cases with $X = I_-$ are straightforward, and the cases with $X = J_-$ and $X = X_{-2}$ can be derived from those with $X = J_+$ and X_2.

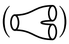
First, let us fix some notation. When computing $Z_G^{\mathrm{rep}}\left(\right.$  $\left.\right)(X \otimes Y)$ for a pair (X, Y), we write A for a point of X and $B = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ for a point of Y. When Y is of the form $(\mathrm{GL}_2/D \times \Lambda) /\!\!/ S_2$ for some S_2-variety Λ over $\mathbb{A}^1_k \setminus \{0, \pm 1\}$, we also write $B = P \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} P^{-1}$ with $P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2/D$ and $\mu \in \mathbb{A}^1_k \setminus \{0, \pm 1\}$. Recall that S_2 acts on (P, μ) via $(P, \mu) \mapsto \left(P \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \mu^{-1}\right)$. More specifically, when $\Lambda = \mathbb{A}^1_k \setminus \{0, \pm 1, \pm i\}$, we write $\mu = \omega^2$ with $\omega \in \Lambda$. Similarly, when X is of the form $(\mathrm{GL}_2/D \times \Lambda) /\!\!/ S_2$ for such Λ, we write $A = Q \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} Q^{-1}$ with $Q \in \mathrm{GL}_2/D$ and $\rho \in \mathbb{A}^1_k \setminus \{0, \pm 1\}$, and write $\rho = \nu^2$ when $\Lambda = \mathbb{A}^1_k \setminus \{0, \pm 1, \pm i\}$.

When dealing with the strata where $AB \in M$, we usually want to diagonalize AB. This can be done once we base change along the double cover $(\mathrm{GL}_2/D \times \mathbb{A}^1_k \setminus \{0, \pm 1\}) \to M$. We write λ for the coordinate on $\mathbb{A}^1_k \setminus \{0, \pm 1\}$. The group S_2 acts on this double cover via $(P, \lambda) \mapsto \left(P \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \lambda^{-1}\right)$.

Strata often admit symmetry by the action of conjugation with some subgroup of SL_2. When this happens for the subgroups $\{\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)\} \cong \mathbb{G}_a$ or $\left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\right\} \cong \mathbb{G}_m$, we will speak of $\mathbb{G}_a$*-symmetry* or $\mathbb{G}_m$*-symmetry*, respectively. In these cases, such a stratum turns into a (Zariski-locally trivial) $\mathbb{G}_a$-torsor or $\mathbb{G}_m$-torsor, so to compute its virtual class it suffices to compute that of the base.

Finally, to avoid confusion between the various S_2-actions, we write S_2^λ, S_2^μ and S_2^ρ to differentiate between them.

Lemma 5.4.1.

Z_G^{rep} ()

$$(J_+ \otimes J_+) = (\mathbb{L}+1)(\mathbb{L}-1)I_+ + (\mathbb{L}-2)J_+ + \mathbb{L}J_- + (\mathbb{L}+1)M - X_{2,-2}$$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = 1$, then $A = B^{-1}$, so we obtain $[J_+]I_+ = (\mathbb{L}+1)(\mathbb{L}-1)I_+$.
- If $AB = -1$, there are no solutions as $\operatorname{tr} A = 2 \neq -2 = -\operatorname{tr} B^{-1}$.
- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} w-z & x-y \\ -z & x \end{smallmatrix}\right)$. From $\operatorname{tr} A = \operatorname{tr} B = 2$ follows that $z = 0$ and (using $\det B = 1$) also $x = w = 1$. Furthermore, $y \neq 0, 1$ as $A, B \neq 1$, so we obtain $(\mathbb{L}-2)J_+$.
- If $AB \in J_-$, then conjugate to $AB = \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} -w-z & x+y \\ z & -x \end{smallmatrix}\right)$. From $\operatorname{tr} A = \operatorname{tr} B = 2$ follows that $z = -4$. Fix $x = 0$ using $\mathbb{G}_a$-symmetry, and solve for $w = 2$ and $y = 1/4$. We obtain $\mathbb{L}J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda w & -\lambda y \\ -z/\lambda & x/\lambda \end{smallmatrix}\right)$. From $\operatorname{tr} B = 2$ follows that $w = 2 - x$. From $\operatorname{tr} A = 2$ and $\det A = 1$ and $\lambda \neq 1$ follows that $z \neq 0$. From $\det B = 1$ follows that $y = (xw-1)/z$. From $\operatorname{tr} A = 2$ follows that $x = \frac{2\lambda}{\lambda+1}$. Make a substituting $z' = z\frac{\lambda+1}{\lambda-1}$, and note that S_2^λ acts via $z' \mapsto 1/z'$. Now, from the S_2^λ-virtual classes

$$[\{z' \neq 0\}]^{S_2^\lambda} = \mathbb{L} \otimes T - 1 \otimes N$$

$$[\mathrm{GL}_2/D \times \{\lambda \neq 0, \pm 1\}]_M^{S_2^\lambda} = M \otimes T - (X_{2,-2} - M) \otimes N$$

follows that the quotient by S_2^λ is $(\mathbb{L}+1)M - X_{2,-2}$. □

Lemma 5.4.2.

$$Z_G^{\mathrm{rep}}\left(\quad\right)(J_+ \otimes M) = \mathbb{L}(\mathbb{L}-2)(J_+ + J_-) + (\mathbb{L}-3)(\mathbb{L}+1)M + 2X_{2,-2}$$

Proof. Note that $Z_G^{\mathrm{rep}}\left(\quad\right)(X \otimes G) = [X] \cdot G$ for all $X \in \mathrm{K}_0(\mathbf{Var}_G)$. Since $G = I_+ + I_- + J_+ + J_- + M$, the result can be derived from the above lemma. $\square$

Lemma 5.4.3.

$$\begin{aligned} Z_G^{\mathrm{rep}}\left(\quad\right)(J_+ \otimes X_2) = \mathbb{L}(\mathbb{L}-3)(J_+ + J_-) + (\mathbb{L}-3)(\mathbb{L}+1)M \\ - (\mathbb{L}+1)X_2 - (\mathbb{L}-3)X_{2,-2} + \mathbb{L}Y \end{aligned}$$

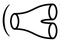
Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = \pm 1$, there are no solutions.
- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} w-z & x-y \\ -z & x \end{smallmatrix}\right)$. From $\operatorname{tr} A = 2$ follows that $z = x + w - 2$ and from $\det B = 1$ that $y = (xw-1)/z$. Furthermore, we can solve for $w = \ell^2 - x + 2$ with $\ell \neq 0, \pm 2i$. Hence, we obtain $\mathbb{L}(\mathbb{L}-3)J_+$.
- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-3)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda w & -\lambda y \\ -z/\lambda & x/\lambda \end{smallmatrix}\right)$. Consider the following cases.
  - Case $y = z = 0$. There are no solutions.
  - Case $y = 0$ or $z = 0$, but not both. Since the action of S_2^λ swaps y and z, we can identify the S_2^λ-quotient with the stratum where $z = 0$. From $\operatorname{tr} A = 2$ and $\det A = 1$ follows that $x = w^{-1} = \lambda$, so in particular $\ell^2 = \lambda + \lambda^{-1} - 2$. Since $A \neq 1$, we have $y \neq 0$, so we obtain $(\mathbb{L}-1)Y$.
  - Case $yz \neq 0$. From $\operatorname{tr} A = 2$ follows that $w = (2 - x/\lambda)/\lambda$ and from $\det B = 1$ that $y = (xw-1)/z$. We substitute $z' = z\lambda/(x-\lambda)$ so that S_2^λ acts via $z' \mapsto 1/z'$. Using $\ell^2 = \operatorname{tr} B - 2$, we can solve for $x = \lambda(\ell^2\lambda + 2\lambda - 2)/(\lambda^2 - 1)$. The conditions $y \neq 0$ and $\operatorname{tr} B \neq \pm 2$ translate to $\ell^2 \neq \lambda + \lambda^{-1} - 2$ and $\ell^2 \neq 0, -4$. From the S_2^λ-virtual classes

$$\begin{aligned} [\{z' \neq 0\}]^{S_2^\lambda} &= \mathbb{L} \otimes T - 1 \otimes N \\ \left[\mathrm{GL}_2/D \times \left\{ \begin{smallmatrix} \ell^2 \neq \lambda + \lambda^{-1} - 2 \\ \ell \neq 0, \pm 2i \end{smallmatrix} \right\}\right]_M^{S_2^\lambda} &= (\mathbb{L}-3)(M \otimes T + (X_{2,-2} - M) \otimes N) \\ &\quad - (X_2 \otimes T + (Y - X_2) \otimes N) \end{aligned}$$

  we obtain $(\mathbb{L}-3)(\mathbb{L}+1)M - (\mathbb{L}+1)X_2 - (\mathbb{L}-3)X_{2,-2} + Y$. $\square$

Lemma 5.4.4.

$$Z_G^{\mathrm{rep}}\left(\right.$$  $$\left.\right)(J_+ \otimes X_{2,-2}) = \mathbb{L}(\mathbb{L}-3)(J_+ + J_-) + (\mathbb{L}-3)(\mathbb{L}+1)M + 2X_{2,-2}$$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = \pm 1$, there are no solutions.
- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) B^{-1}$. We have $\gamma \neq 0$ since $\operatorname{tr} A = 2$ and $\mu \neq \pm 1$. Hence, we can fix $\gamma = 1$, $\alpha = 0$ and $\beta = 1$ by lifting P to GL_2 and using $\mathbb{G}_a$-symmetry. Now $\operatorname{tr} A = 2$ implies $\delta = -\frac{\mu-1}{\mu+1}$ with $\mu \neq 0, \pm 1$, so we obtain $\mathbb{L}(\mathbb{L}-3)J_+$.
- If $AB \in J_+$, then similarly we obtain $\mathbb{L}(\mathbb{L}-3)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\alpha\gamma = 0$. The action of S_2^λ swaps α and γ, so we can break the S_2^λ-action and consider only the stratum with $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. From $\operatorname{tr} A = 2$ follows that $\mu = \lambda$. Furthermore, we must have $\beta \neq 0$ to ensure $A \neq 1$, so we obtain $(\mathbb{L}-1)X_{2,-2}$.
  - Case $\alpha\gamma \neq 0$ and $\beta\delta = 0$. The action of S_2^λ swaps β and δ, so we can identify the S_2^λ-quotient with the stratum where $\delta = 0$. Fix $\beta = \gamma = 1$ by lifting P to GL_2. From $\operatorname{tr} A = 2$ follows that $\mu = \lambda^{-1}$. Furthermore, there are no conditions on α other than $\alpha \neq 0$, so we obtain another $(\mathbb{L}-1)X_{2,-2}$.
  - Case $\alpha\beta\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\mu = \lambda^{\pm 1}$, and use $\operatorname{tr} A = 2$ to solve for $\beta = \alpha\frac{(\lambda-\mu)^2}{(\lambda\mu-1)^2}$. Note that S_2^λ acts via $\alpha \mapsto \alpha^{-1}$. From the S_2^λ-virtual classes

$$\begin{aligned} [\{\alpha \neq 0\}]^{S_2^\lambda} &= \mathbb{L} \otimes T - 1 \otimes N \\ \left[\mathrm{GL}_2/D \times \left\{ \begin{smallmatrix} \lambda \neq 0, \pm 1 \\ \mu \neq 0, \pm 1, \lambda^{\pm 1} \end{smallmatrix} \right\}\right]_M^{S_2^\lambda} &= (\mathbb{L}-3)(M \otimes T + (X_{2,-2} - M) \otimes N) \\ &\quad - X_{2,-2} \otimes (T + N) \end{aligned}$$

  we obtain $(\mathbb{L}-3)(\mathbb{L}+1)M - 2(\mathbb{L}-2)X_{2,-2}$. □

Lemma 5.4.5.

$$Z_G^{\mathrm{rep}}\left(\right.$$  $$\left.\right)(J_+ \otimes Y) = \mathbb{L}(\mathbb{L}-5)(J_+ + J_-) + (\mathbb{L}-5)(\mathbb{L}+1)M - (\mathbb{L}-5)X_{2,-2} + (\mathbb{L}-1)Y$$

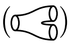
Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = \pm 1$, there are no solutions.
- If $AB \in J_+$, the computation is the same as for $A \in J_+$ and $B \in X_{2,-2}$, but with $\mu = \omega^2$ and $\omega \neq 0, \pm 1, \pm i$. Hence, we obtain $\mathbb{L}(\mathbb{L}-5)J_+$.
- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-5)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\alpha\gamma = 0$. Identify the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. From $\operatorname{tr} A = 2$ follows that $\omega^2 = \lambda$. Furthermore, we must have $\beta \neq 0$ to ensure $A \neq 1$, so we obtain $(\mathbb{L}-1)Y$.
  - Case $\alpha\gamma \neq 0$ and $\beta\delta = 0$. Identify the S_2^λ-quotient with the stratum where $\delta = 0$. Fix $\beta = \gamma = 1$ by lifting P to GL_2. From $\operatorname{tr} A = 2$ follows that $\omega^2 = \lambda^{-1}$. Furthermore, there are no conditions on α other than $\alpha \neq 0$, so we obtain another $(\mathbb{L}-1)Y$.
  - Case $\alpha\beta\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\mu = \lambda^{\pm 1}$, and use $\operatorname{tr} A = 2$ to solve for $\beta = \alpha \frac{(\lambda-\mu)^2}{(\lambda\mu-1)^2}$. Note that S_2^λ acts via $\alpha \mapsto \alpha^{-1}$. From the S_2^λ-virtual classes

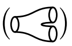
$$\begin{aligned} [\{\alpha \neq 0\}]^{S_2^\lambda} &= \mathbb{L} \otimes T - 1 \otimes N \\ \left[\mathrm{GL}_2/D \times \left\{\begin{smallmatrix} \lambda \neq 0, \pm 1 \\ \omega^2 \neq 0, \pm 1, \lambda^{\pm 1} \end{smallmatrix}\right\}\right]_M^{S_2^\lambda} &= (\mathbb{L}-5)(M \otimes T + (X_{2,-2} - M) \otimes N) \\ &\quad - Y \otimes (T + N) \end{aligned}$$

  we obtain $(\mathbb{L}-5)(\mathbb{L}+1)M - (\mathbb{L}-5)X_{2,-2} - (\mathbb{L}-1)Y$. □

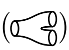
Lemma 5.4.6.

Z_G^{rep} () $(M \otimes M)$

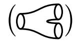
$$\begin{aligned} &= \mathbb{L}(\mathbb{L}^2 - 2\mathbb{L} - 1)(I_+ + I_-) \\ &\quad + \mathbb{L}(\mathbb{L}-3)(\mathbb{L}-1)(J_+ + J_-) \\ &\quad + (\mathbb{L}^3 - 4\mathbb{L}^2 + 3\mathbb{L} + 4)M - 4X_{2,-2} \end{aligned}$$

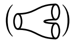
Z_G^{rep} () $(M \otimes X_2)$

$$\begin{aligned} &= \mathbb{L}(\mathbb{L}^2 - 3\mathbb{L} - 2)(I_+ + I_-) \\ &\quad + \mathbb{L}(\mathbb{L}-4)(\mathbb{L}-1)(J_+ + J_-) \\ &\quad + (\mathbb{L}^3 - 5\mathbb{L}^2 + 2\mathbb{L} + 6)M + \mathbb{L}(X_2 + X_{-2}) \\ &\quad + 2(\mathbb{L}-3)X_{2,-2} - 2\mathbb{L}Y \end{aligned}$$

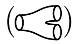
Z_G^{rep} () $(M \otimes X_{2,-2})$

$$\begin{aligned} &= \mathbb{L}(\mathbb{L}-3)(\mathbb{L}+1)(I_+ + I_-) \\ &\quad + \mathbb{L}(\mathbb{L}-3)(\mathbb{L}-1)(J_+ + J_-) \\ &\quad + (\mathbb{L}-3)(\mathbb{L}-2)(\mathbb{L}+1)M - 6X_{2,-2} \end{aligned}$$

Z_G^{rep}  $(M \otimes Y) = \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)(I_+ + I_-)$
$$+ \mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)(J_+ + J_-)$$
$$+ (\mathbb{L}-5)(\mathbb{L}-2)(\mathbb{L}+1)M + 2(\mathbb{L}-5)X_{2,-2} - 2\mathbb{L}Y$$

Proof. Note that Z_G^{rep}  $(G \otimes X) = [X] \cdot G$ for all $X \in \mathrm{K}_0(\mathbf{Var}_G)$. Since $G = I_+ + I_- + J_+ + J_- + M$, the result follows from the earlier lemmas. $\square$

Lemma 5.4.7.

Z_G^{rep}  $(X_{2,-2} \otimes X_{2,-2}) = 2\mathbb{L}(\mathbb{L}-3)(\mathbb{L}+1)(I_+ + I_-)$
$$+ \mathbb{L}(\mathbb{L}-3)(\mathbb{L}-1)(J_+ + J_-)$$
$$+ (\mathbb{L}-3)^2(\mathbb{L}+1)M + (\mathbb{L}^2 - 4\mathbb{L} - 9)X_{2,-2}$$

Proof. Stratify based on the conjugacy class of the product AB.

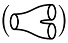
- If $AB = 1$, then solve for $A = B^{-1}$ to obtain $[X_{2,-2} \times_M X_{2,-2}]I_+ = 2[X_{2,-2}]I_+ = 2\mathbb{L}(\mathbb{L}-3)(\mathbb{L}+1)I_+$.
- If $AB = -1$, then solve for $A = -B^{-1}$ to obtain $[X_{2,-2} \times_M X_{2,-2}]I_- = 2[X_{2,-2}] = 2\mathbb{L}(\mathbb{L}-3)(\mathbb{L}+1)I_-$.
- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2, and fix $\beta = 0$ using $\mathbb{G}_a$-symmetry. Solving for $\rho = \mu^{\pm 1} \neq 0, \pm 1$, we obtain $2\mathbb{L}(\mathbb{L}-3)J_+$.
  - Case $\gamma \neq 0$. Fix $\gamma = 1$, $\alpha = 0$ and $\beta = 1$ by lifting P to GL_2 and using $\mathbb{G}_a$-symmetry. Using $\operatorname{tr} A = \rho + \rho^{-1}$, solve for $\delta = -\frac{(\mu-\rho)(\mu\rho-1)}{\rho(\mu-1)(\mu+1)}$. Since $\mu, \rho \neq 0, \pm 1$, we obtain $\mathbb{L}(\mathbb{L}-3)^2 J_+$.
- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-3)(\mathbb{L}-1)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\alpha\gamma = 0$. Identify the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\rho = (\lambda\mu^{-1})^{\pm 1}$. In both cases $\mu \neq 0, \pm 1, \pm\lambda$, so we obtain $2\mathbb{L}(\mathbb{L}-5)X_{2,-2}$.
  - Case $\alpha\gamma \neq 0$ and $\beta\delta = 0$. Identify the S_2^λ-quotient with the stratum where $\beta = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\rho = (\lambda\mu^{-1})^{\pm 1}$. In both cases $\mu \neq 0, \pm 1, \pm\lambda$, so we obtain $2(\mathbb{L}-1)(\mathbb{L}-5)X_{2,-2}$.

– Case $\alpha\beta\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Solve for $\beta = \frac{\alpha(\lambda-\mu\rho)(\lambda\rho-\mu)}{(\lambda\mu-\rho)(\lambda\mu\rho-1)}$. Note that $\alpha \neq \beta$ is automatically satisfied as there are no solutions with $\rho = \lambda^{\pm 1}\mu^{\pm 1}$. Note that S_2^λ acts via $\alpha \mapsto \alpha^{-1}$. From the S_2^λ-virtual classes

$$\begin{aligned} [\{\alpha \neq 0\}]^{S_2^\lambda} &= \mathbb{L} \otimes T - 1 \otimes N \\ \left[\mathrm{GL}_2/D \times \left\{ \begin{smallmatrix} \lambda,\mu\neq 0,\pm 1 \\ \rho \neq 0, \pm 1, \lambda^{\pm 1}\mu^{\pm 1} \end{smallmatrix} \right\}\right]_M^{S_2^\lambda} &= (\mathbb{L}-3)^2(M \otimes T + (X_{2,-2} - M) \otimes N) \\ &\quad - 2(\mathbb{L}-5)X_{2,-2} \otimes (T + N) \end{aligned}$$

we obtain $(\mathbb{L}-3)^2(\mathbb{L}+1)M - (3\mathbb{L}^2 - 18\mathbb{L} + 19)X_{2,-2}$. □

Lemma 5.4.8.



$$\begin{aligned} Z_G^{\mathrm{rep}}\left(\,\cdot\,\right)(X_{2,-2} \otimes Y) &= 2\mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)(I_+ + I_-) \\ &\quad + \mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)(J_+ + J_-) \\ &\quad + (\mathbb{L}-5)(\mathbb{L}-3)(\mathbb{L}+1)M \\ &\quad + (\mathbb{L}-5)(\mathbb{L}+3)X_{2,-2} - 4\mathbb{L}Y \end{aligned}$$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = 1$, then solve for $A = B^{-1}$ to obtain $[X_{2,-2} \times_M Y]I_+ = 2[Y]I_+ = 2\mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_+$.
- If $AB = -1$, then solve for $A = -B^{-1}$ to obtain $[X_{2,-2} \times_M Y]I_- = 2[Y]I_+ = 2\mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_-$.
- If $AB \in J_+$, the computation is the same as for $A \in X_{2,-2}$ and $B \in X_{2,-2}$, but with $\mu = \omega^2$ and $\omega \neq 0, \pm 1, \pm i$. Hence, we obtain $\mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)J_+$.
- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\alpha\gamma = 0$. Identify the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\rho = (\lambda\mu^{-1})^{\pm 1}$. In both cases $\mu = \omega^2 \neq 0, \pm 1, \pm\lambda$, so we obtain $2\mathbb{L}(\mathbb{L}-5)X_{2,-2} - 4\mathbb{L}Y$.
  - Case $\alpha\gamma \neq 0$ and $\beta\delta = 0$. Identify the S_2^λ-quotient with the stratum where $\beta = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\rho = (\lambda\mu^{-1})^{\pm 1}$. In both cases $\mu = \omega^2 \neq 0, \pm 1, \pm\lambda$, so we obtain $2(\mathbb{L}-1)(\mathbb{L}-5)X_{2,-2} - 4(\mathbb{L}-1)Y$.
  - Case $\alpha\beta\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\rho = \lambda^{\pm 1}\mu^{\pm 1}$, and solve for $\beta = \frac{\alpha(\lambda-\mu\rho)(\lambda\rho-\mu)}{(\lambda\mu-\rho)(\lambda\mu\rho-1)}$. Furthermore,

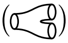
note that S_2^λ acts via $\alpha \mapsto \alpha^{-1}$. From the S_2^λ-virtual classes

$$[\{\alpha \neq 0\}]^{S_2^\lambda} = \mathbb{L} \otimes T - 1 \otimes N$$

$$\left[\mathrm{GL}_2/D \times \left\{\substack{\lambda,\omega^2,\rho\neq 0,\pm 1 \\ \rho\neq\lambda^{\pm 1}\omega^{\pm 2}}\right\}\right]_M^{S_2^\lambda} = (\mathbb{L}-5)(\mathbb{L}-3)(M \otimes T + (X_{2,-2} - M) \otimes N) - (2(\mathbb{L}-5)X_{2,-2} - 4Y) \otimes (T + N)$$

we obtain $(\mathbb{L}-5)(\mathbb{L}-3)(\mathbb{L}+1)M - (\mathbb{L}-5)(3\mathbb{L}-5)X_{2,-2} + 4(\mathbb{L}-1)Y$. □

Lemma 5.4.9.

Z_G^{rep} () $(Y \otimes Y) = 4\mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)(I_+ + I_-)$
$+ \mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)(J_+ + J_-)$
$+ (\mathbb{L}-5)^2(\mathbb{L}+1)M - (\mathbb{L}-5)^2 X_{2,-2} + 2\mathbb{L}(\mathbb{L}-9)Y$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = 1$, then solve for $A = B^{-1}$ to obtain $[Y \times_M Y]I_+ = 4[Y]I_+ = 4\mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_+$.
- If $AB = -1$, then solve for $A = -B^{-1}$ to obtain $[Y \times_M Y]I_- = 4[Y]I_- = \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_-$.
- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2, and fix $\beta = 0$ using $\mathbb{G}_a$-symmetry. Then $\nu^2 = \omega^{\pm 2}$, that is, $\nu = \pm\omega^{\pm 1} \neq 0, \pm 1, \pm i$. Hence, we obtain $4\mathbb{L}(\mathbb{L}-5)J_+$.
  - Case $\gamma \neq 0$. Fix $\gamma = 1$, $\alpha = 0$ and $\beta = 1$ by lifting P to GL_2 and using $\mathbb{G}_a$-symmetry. Solve for $\delta = -\frac{(\mu-\rho)(\mu\rho-1)}{\rho(\mu-1)(\mu+1)}$. Since $\omega, \nu \neq 0, \pm 1, \pm i$, we obtain $\mathbb{L}(\mathbb{L}-5)^2 J_+$.
- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\alpha\gamma = 0$. Identify the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\nu^2 = (\lambda\omega^{-2})^{\pm 1}$. If $\nu^2 = \lambda\omega^{-2}$, then substituting $u = \nu\omega$ yields $u^2 = \lambda$ with $\omega \neq 0, \pm 1, \pm i, \pm u, \pm iu$. The case $\nu^2 = (\lambda\omega^{-2})^{-1}$ is similar with $u = \nu/\omega$, so we obtain $2\mathbb{L}(\mathbb{L}-9)Y$.
  - Case $\alpha\gamma \neq 0$ and $\beta\delta = 0$. Identify the S_2^λ-quotient with the stratum where $\beta = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\nu^2 = (\lambda\omega^{-2})^{\pm 1}$. Again, substituting $u = \nu\omega^{\pm 1}$, respectively, we obtain $2(\mathbb{L}-9)(\mathbb{L}-1)Y$.

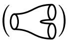
– Case $\alpha\beta\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\rho = \lambda^{\pm 1}\mu^{\pm 1}$, and solve for $\beta = \frac{\alpha(\lambda-\mu\rho)(\lambda\rho-\mu)}{(\lambda\mu-\rho)(\lambda\mu\rho-1)}$. Furthermore, note that S_2^λ acts via $\alpha \mapsto \alpha^{-1}$. From the S_2^λ-virtual classes

$$[\{\alpha \neq 0\}]^{S_2^\lambda} = \mathbb{L} \otimes T - 1 \otimes N$$

$$\left[\mathrm{GL}_2/D \times \left\{ {\lambda,\omega^2,\nu^2 \neq 0,\pm 1 \atop \nu^2 \neq \lambda^{\pm 1}\omega^{\pm 2}} \right\}\right]^{S_2^\lambda}_M = (\mathbb{L}-5)^2(M \otimes T + (X_{2,-2} - M) \otimes N) - 2(\mathbb{L}-9)Y \otimes (T+N)$$

we obtain $(\mathbb{L}-5)^2(\mathbb{L}+1)M - (\mathbb{L}-5)^2 X_{2,-2} - 2(\mathbb{L}-9)(\mathbb{L}-1)Y$. □

Lemma 5.4.10.

$$Z_G^{\mathrm{rep}}\left(\right.$$  $$\left.\right)(X_{2,-2} \otimes X_2) = \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)(I_+ + I_-) + \mathbb{L}(\mathbb{L}-4)(\mathbb{L}-1)(J_+ + J_-) + (\mathbb{L}-3)^2(\mathbb{L}+1)M + (\mathbb{L}-9)X_{2,-2} - 2\mathbb{L}Y$$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = 1$, then solve for $A = B^{-1}$ to obtain $[X_{2,-2} \times_M X_2]I_+ = [Y]I_+ = \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_+$.
- If $AB = -1$, then solve for $A = -B^{-1}$ to obtain $[X_{2,-2} \times_M X_{-2}]I_- = [Y]I_+ = \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_-$.
- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) B^{-1}$. Consider the following cases.
  - Case $\gamma\delta = 0$. Identify the S_2^μ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2, and fix $\beta = 0$ using $\mathbb{G}_a$-symmetry. Solve for $\rho = \omega^{\pm 2}$. Hence, we obtain $2\mathbb{L}(\mathbb{L}-5)J_+$.
  - Case $\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2, and fix $\alpha = 0$ using $\mathbb{G}_a$-symmetry. Note that there are no solutions with $\rho = \omega^{\pm 2}$, and solve for $\beta = -\frac{\rho(\mu-1)(\mu+1)}{(\mu-\rho)(\mu\rho-1)}$. Since

$$\left[\left\{ {\rho,\omega^2 \neq 0,\pm 1 \atop \rho \neq \omega^{\pm 2}} \right\} /\!\!/ S_2^\mu\right] = (\mathbb{L}-3)^2 - (\mathbb{L}-5) = \mathbb{L}^2 - 7\mathbb{L} + 14,$$

    we obtain $\mathbb{L}(\mathbb{L}^2 - 7\mathbb{L} + 14)J_+$.
- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-4)(\mathbb{L}-1)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.

– Case P is (anti-)diagonal. Identify the S_2^μ-quotient with the stratum where P is diagonal. Fix $\alpha = \delta = 1$ by lifting P to GL_2 and using $\mathbb{G}_m$-symmetry. Solve for $\rho = (\lambda\omega^{-2})^{\pm 1}$, and identify the S_2^λ-quotient with the stratum where $\rho = \lambda\omega^{-2}$. From the conditions $\omega \neq 0, \pm 1, \pm i$ and $\omega^2 \neq \pm\lambda$, we obtain $(\mathbb{L} - 5)X_{2,-2} - 2Y$.

– Case P has *one* zero. Identify the S_2^μ-quotient with the stratum where $\alpha\gamma = 0$, and subsequently the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\rho = (\lambda\omega^{-2})^{\pm 1}$. From the conditions $\beta \neq 0$, $\omega \neq 0, \pm 1, \pm i$ and $\omega^2 \neq \pm\lambda$, we obtain $2(\mathbb{L} - 5)(\mathbb{L} - 1)X_{2,-2} - 4(\mathbb{L} - 1)Y$.

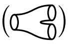
– Case P has *no* zeros. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\rho = \lambda^{\pm 1}\mu^{\pm 1}$, and solve for $\beta = \frac{\alpha(\lambda - \mu\rho)(\lambda\rho - \mu)}{(\lambda\mu - \rho)(\lambda\mu\rho - 1)}$. Substituting $\alpha' = \alpha\frac{\lambda\rho - \mu}{\lambda\mu - \rho}$, we find that S_2^λ and S_2^μ act via $\alpha' \mapsto 1/\alpha'$ and $\alpha' \mapsto \alpha'$, respectively. From the $S_2^\lambda \times S_2^\mu$-virtual classes

$$[\{\alpha' \neq 0\}]^{S_2^\lambda \times S_2^\mu} = (\mathbb{L} \otimes T^\lambda - 1 \otimes N^\lambda) \otimes T^\mu$$

$$\begin{aligned}\left[\mathrm{GL}_2/D \times \left\{\begin{smallmatrix}\rho, \omega^2 \neq 0, \pm 1 \\ \rho \neq \lambda^{\pm 1}\omega^{\pm 2}\end{smallmatrix}\right\}\right]_M^{S_2^\lambda \times S_2^\mu} &= \\ (\mathbb{L} - 3)((\mathbb{L} - 3) \otimes T^\mu - 2 \otimes N^\mu)&(M \otimes T^\lambda + (X_{2,-2} - M) \otimes N^\lambda) \\ &- ((\mathbb{L} - 5)X_{2,-2} - 2Y) \otimes (T^\lambda + N^\lambda) \otimes (T^\mu \otimes N^\mu)\end{aligned}$$

we obtain $(\mathbb{L} - 3)^2(\mathbb{L} + 1)M - 2(\mathbb{L}^2 - 6\mathbb{L} + 7)X_{2,-2} + 2(\mathbb{L} - 1)Y$. ☐

Lemma 5.4.11.

Z_G^{rep} ()

$$\begin{aligned}Z_G^{\mathrm{rep}}(\cdot)(Y \otimes X_2) = {} & 2\mathbb{L}(\mathbb{L} - 5)(\mathbb{L} + 1)(I_+ + I_-) \\ & + \mathbb{L}(\mathbb{L} - 5)(\mathbb{L} - 1)(J_+ + J_-) \\ & + (\mathbb{L} - 5)(\mathbb{L} - 3)(\mathbb{L} + 1)M \\ & - (\mathbb{L} - 5)(\mathbb{L} - 3)X_{2,-2} + \mathbb{L}(\mathbb{L} - 9)Y\end{aligned}$$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = 1$, then solve for $A = B^{-1}$ to obtain $[Y \times_M X_2]I_+ = 2[Y]I_+ = 2\mathbb{L}(\mathbb{L} - 5)(\mathbb{L} + 1)I_+$.

- If $AB = -1$, then solve for $A = -B^{-1}$ to obtain $[Y \times_M X_{-2}]I_- = 2[Y]I_- = 2\mathbb{L}(\mathbb{L} - 5)(\mathbb{L} + 1)I_-$.

- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix}1 & 1 \\ 0 & 1\end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix}1 & 1 \\ 0 & 1\end{smallmatrix}\right)B^{-1}$. Consider the following cases.

- Case $\gamma\delta = 0$. Identify the S_2^μ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2, and fix $\beta = 0$ using $\mathbb{G}_a$-symmetry. Solve for $\nu^2 = \omega^{\pm 2}$, that is, $\nu = \pm\omega^{\pm 1}$. Since $\omega \neq 0, \pm 1, \pm i$, we obtain $4\mathbb{L}(\mathbb{L}-5)J_+$.
- Case $\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2, and fix $\alpha = 0$ using $\mathbb{G}_a$-symmetry. Note that there are no solutions with $\rho = \omega^{\pm 2}$, and solve for $\beta = -\frac{\rho(\mu-1)(\mu+1)}{(\mu-\rho)(\mu\rho-1)}$. Since

$$\left[\left\{ \substack{\nu,\omega\neq 0,\pm 1,\pm i \\ \nu \neq \pm\omega^{\pm 1}} \right\} /\!\!/ S_2^\mu\right] = (\mathbb{L}-5)(\mathbb{L}-3) - 2(\mathbb{L}-5) = (\mathbb{L}-5)^2,$$

we obtain $\mathbb{L}(\mathbb{L}-5)^2 J_+$.

- If $AB \in J_-$, then similarly we obtain $\mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)J_-$.
- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) B^{-1}$. Consider the following cases.

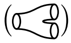
- Case P is (anti-)diagonal. Identify the S_2^μ-quotient with the stratum where P is diagonal. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\nu^2 = (\lambda\omega^{-2})^{\pm 1}$, and identify the S_2^λ-quotient with the stratum where $\nu^2 = \lambda\omega^{-2}$. Substitute $u = \nu\omega$ so that $u^2 = \lambda$. From the condition $\omega \neq 0, \pm 1, \pm i, \pm u, \pm iu$, we obtain $(\mathbb{L}-9)Y$.
- Case P has *one* zero. Identify the S_2^μ-quotient with the stratum where $\alpha\gamma = 0$, and subsequently the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Solve for $\nu^2 = (\lambda\omega^{-2})^{\pm 1}$. Again, substituting $u = \nu\omega^{\pm 1}$, respectively, we obtain $2(\mathbb{L}-9)(\mathbb{L}-1)Y$.
- Case P has *no* zeros. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\rho = \lambda^{\pm 1}\mu^{\pm 1}$, and solve for $\beta = \frac{\alpha(\lambda-\mu\rho)(\lambda\rho-\mu)}{(\lambda\mu-\rho)(\lambda\mu\rho-1)}$. Substituting $\alpha' = \alpha\frac{\lambda\rho-\mu}{\lambda\mu-\rho}$, we find that S_2^λ and S_2^μ act via $\alpha' \mapsto 1/\alpha'$ and $\alpha' \mapsto \alpha'$, respectively. From the $S_2^\lambda \times S_2^\mu$-virtual classes

$$[\{\alpha' \neq 0\}]^{S_2^\lambda \times S_2^\mu} = (\mathbb{L} \otimes T^\lambda - 1 \otimes N^\lambda) \otimes T^\mu$$

$$\begin{aligned} \left[\mathrm{GL}_2/D \times \left\{ \substack{\nu,\omega\neq 0,\pm 1,\pm i \\ \nu^2 \neq \lambda^{\pm 1}\omega^{\pm 2}} \right\}\right]_M^{S_2^\lambda \times S_2^\mu} &= \\ (\mathbb{L}-5)((\mathbb{L}-3) \otimes T^\mu - 2 \otimes N^\mu)&(M \otimes T^\lambda + (X_{2,-2} - M) \otimes N^\lambda) \\ &- (\mathbb{L}-9)Y \otimes (T^\lambda + N^\lambda) \otimes (T^\mu \otimes N^\mu) \end{aligned}$$

we obtain $(\mathbb{L}-5)(\mathbb{L}-3)(\mathbb{L}+1)M - (\mathbb{L}-5)(\mathbb{L}-3)X_{2,-2} - (\mathbb{L}-9)(\mathbb{L}-1)Y$. □

Lemma 5.4.12.

Z_G^{rep}  $(X_2 \otimes X_2)$

$$
\begin{aligned}
= {} & 2\mathbb{L}(\mathbb{L}^2 - 3\mathbb{L} - 2)I_+ + \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_- \\
& + \mathbb{L}(\mathbb{L}-5)(\mathbb{L}-1)J_+ + \mathbb{L}(\mathbb{L}-4)(\mathbb{L}-1)J_- \\
& + (\mathbb{L}-3)^2(\mathbb{L}+1)M - \mathbb{L}(\mathbb{L}-3)X_{-2} \\
& - (\mathbb{L}-3)^2 X_{2,-2} + \mathbb{L}(\mathbb{L}-6)Y
\end{aligned}
$$

Proof. Stratify based on the conjugacy class of the product AB.

- If $AB = 1$, then solve for $A = B^{-1}$. It follows that $\nu^2 = \omega^{\pm 2}$, that is, $\nu = \pm\omega^{\pm 1}$, so identify the S_2^ρ-quotient with the stratum where $\nu = \pm\omega^{-1}$. From the S_2^μ-virtual classes

$$
\begin{aligned}
[\mathrm{GL}_2/D]^{S_2^\mu} &= \mathbb{L}^2 \otimes T + \mathbb{L} \otimes N \\
\left[\left\{ \begin{smallmatrix} \omega \neq 0, \pm 1, \pm i \\ \nu \neq \pm\omega \end{smallmatrix} \right\}\right]^{S_2^\mu} &= 2((\mathbb{L}-3) \otimes T - 2 \otimes N)
\end{aligned}
$$

  we obtain $2\mathbb{L}(\mathbb{L}^2 - 3\mathbb{L} - 2)I_+$.

- If $AB = -1$, then solve for $A = -B^{-1}$. It follows that $\nu^2 = -\omega^{\pm 2}$, that is, $\nu = \pm i\omega^{\pm 1}$. Identify the S_2^ρ-quotient with the stratum where $\nu = \pm i\omega^{-1}$, and subsequently identify the S_2^μ-quotient with the stratum where $\nu = i\omega^{-1}$. We obtain $[Y]I_- = \mathbb{L}(\mathbb{L}-5)(\mathbb{L}+1)I_-$.

- If $AB \in J_+$, then conjugate to $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) B^{-1}$. Consider the following cases.

  – Case $\gamma\delta = 0$. Identify the S_2^μ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2, and fix $\beta = 0$ using $\mathbb{G}_a$-symmetry. It follows that $\nu^2 = \omega^{\pm 2}$, that is, $\nu = \pm\omega^{\pm 1}$, so identify the S_2^ρ-quotient with the stratum where $\nu = \pm\omega$. Since $\omega \neq 0, \pm 1, \pm i$, we obtain $2\mathbb{L}(\mathbb{L}-5)J_+$.

  – Case $\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2, and fix $\alpha = 0$ using $\mathbb{G}_a$-symmetry. Note that there are no solutions with $\rho = \omega^{\pm 2}$, and solve for $\beta = -\frac{\rho(\mu-1)(\mu+1)}{(\mu-\rho)(\mu\rho-1)}$. Since

$$
\left[\left\{ \begin{smallmatrix} \nu, \omega \neq 0, \pm 1, \pm i \\ \nu^2 \neq \omega^{\pm 2} \end{smallmatrix} \right\} /\!\!/ \, S_2^\mu \times S_2^\rho \right] = (\mathbb{L}-3)^2 - 2(\mathbb{L}-3) = (\mathbb{L}-5)(\mathbb{L}-3)
$$

  we obtain $\mathbb{L}(\mathbb{L}-5)(\mathbb{L}-3)J_+$.

- If $AB \in J_-$, then conjugate to $AB = \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right) B^{-1}$. Consider the following cases.

  – Case $\gamma\delta = 0$. Similarly to the above we obtain $2\mathbb{L}(\mathbb{L}-5)J_-$.

– Case $\gamma\delta \neq 0$. Fix $\gamma = \delta = 1$ by lifting P to GL_2, and fix $\alpha = 0$ using $\mathbb{G}_a$-symmetry. Note that there are no solutions with $\rho = -\mu^{\pm 1}$, and solve for $\beta = \frac{\rho(\mu-1)(\mu+1)}{(\mu+\rho)(\mu\rho+1)}$. Since

$$\left[\left\{\substack{\nu,\omega\neq 0,\pm 1,\pm i\\ \nu\neq \pm i\omega^{\pm 1}}\right\} /\!\!/ S_2^\mu \times S_2^\rho\right] = (\mathbb{L}-3)^2 - (\mathbb{L}-5) = \mathbb{L}^2 - 7\mathbb{L} + 14$$

we obtain $\mathbb{L}(\mathbb{L}^2 - 7\mathbb{L} + 14)J_-$.

- If $AB \in M$, then conjugate to $AB = \left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right)$ and solve for $A = \left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{smallmatrix}\right) B^{-1}$. Consider the following cases.

– Case P is (anti)-diagonal. Identify the S_2^μ-quotient with the stratum where P is diagonal. Fix $\alpha = \delta = 1$ by lifting P to GL_2. It follows that $\nu^2 = (\lambda\omega^{-2})^{\pm 1}$, so identify the S_2^ρ-quotient with the stratum where $\nu^2 = \lambda\omega^{-2}$. Substitute $u = \nu\omega$ so that $u^2 = \lambda$. Since $\omega \neq 0, \pm 1, \pm i, \pm u, \pm iu$, we find

$$\left[\mathrm{GL}_2/D \times \left\{\substack{u^2=\lambda\\ \omega\neq 0,\pm 1,\pm i,\pm u,\pm iu}\right\}\right]_M^{S_2^\lambda} = (X_{-2} \otimes T + (Y - X_{-2}) \otimes N)((\mathbb{L}-6) \otimes T - 3 \otimes N)$$

so we obtain $(\mathbb{L}-3)X_{-2} - 3Y$.

– Case P has *one* zero. Identify the S_2^μ-quotient with the stratum where $\alpha\gamma = 0$, and subsequently the S_2^λ-quotient with the stratum where $\gamma = 0$. Fix $\alpha = \delta = 1$ by lifting P to GL_2. Identify the S_2^ρ-quotient with the stratum where $\nu^2 = \lambda\omega^{-2}$. Substitute $u = \nu\omega$ so that $u^2 = \lambda$ and $\omega \neq 0, \pm 1, \pm i, \pm u, \pm iu$. Furthermore, $\beta \neq 0$, so we obtain $(\mathbb{L}-9)(\mathbb{L}-1)Y$.

– Case P has *no* zeros. Fix $\gamma = \delta = 1$ by lifting P to GL_2. Note that there are no solutions with $\rho = \lambda^{\pm 1}\mu^{\pm 1}$, and solve for $\beta = \frac{\alpha(\lambda-\mu\rho)(\lambda\rho-\mu)}{(\lambda\mu-\rho)(\lambda\mu\rho-1)}$. The various S_2-actions on α are given by

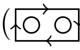
$$\alpha \overset{S_2^\lambda}{\mapsto} \alpha^{-1}, \quad \alpha \overset{S_2^\mu}{\mapsto} \beta = \frac{\alpha(\lambda-\mu\rho)(\lambda\rho-\mu)}{(\lambda\mu-\rho)(\lambda\mu\rho-1)}, \quad \alpha \overset{S_2^\rho}{\mapsto} \alpha.$$

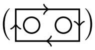
Extending α to be $\mathbb{P}^1$-valued, we can consider this stratum as a $\mathbb{P}^1$-fibration minus the stratum where $\alpha = 0$ or $\alpha = \infty$. Since the cases $\alpha = 0$ or $\alpha = \infty$ are interchanged by S_2^λ but invariant under S_2^μ, we can effectively act as if α is invariant under S_2^μ and S_2^ρ and has S_2^λ-virtual class $[\{\alpha \neq 0\}]^{S_2^\lambda} = \mathbb{L} \otimes T - 1 \otimes N$. Together with the S_2^λ-virtual class

$$\left[\mathrm{GL}_2/D \times \left\{\substack{\nu,\omega\neq 0,\pm 1,\pm i\\ \nu^2\neq\lambda^{\pm 1}\omega^{\pm 2}}\right\} /\!\!/ S_2^\mu \times S_2^\rho\right]_M^{S_2^\lambda} = (\mathbb{L}-3)^2(M \otimes T + (X_{2,-2} - M) \otimes N) - (X_{-2} \otimes T + (Y - X_{-2}) \otimes N)((\mathbb{L}-6) \otimes T - 3 \otimes N)$$

we obtain $(\mathbb{L}-3)^2(\mathbb{L}+1)M - (\mathbb{L}-3)(\mathbb{L}+1)X_{-2} - (\mathbb{L}-3)^2 X_{2,-2} + (4\mathbb{L}-6)Y$. □

5.5 Results

Using (4.13), Proposition 5.3.1 and the lemmas in Section 5.4, we obtain an expression for the matrix associated with Z_G^{rep} () with respect to the generators (5.3).

Z_G^{rep} () $=$

$$\begin{bmatrix}
2 & \mathbb{L}^2+\mathbb{L} & 2\mathbb{L}^2-2 & 0 & \mathbb{L}^3-3\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-5\mathbb{L} & 2\mathbb{L}^3-6\mathbb{L}^2-4\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-5\mathbb{L} & 2\mathbb{L}^3-8\mathbb{L}^2-10\mathbb{L} \\
\mathbb{L}^2+\mathbb{L} & 2 & 0 & 2\mathbb{L}^2-2 & \mathbb{L}^3-3\mathbb{L}^2-2\mathbb{L} & 2\mathbb{L}^3-6\mathbb{L}^2-4\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-5\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-5\mathbb{L} & 2\mathbb{L}^3-8\mathbb{L}^2-10\mathbb{L} \\
2 & 0 & \mathbb{L}^2-\mathbb{L}-2 & 2\mathbb{L}^2 & \mathbb{L}^3-3\mathbb{L}^2 & \mathbb{L}^3-3\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-\mathbb{L} & \mathbb{L}^3-3\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-5\mathbb{L} \\
0 & 2 & 2\mathbb{L}^2 & \mathbb{L}^2-\mathbb{L}-2 & \mathbb{L}^3-3\mathbb{L}^2 & \mathbb{L}^3-4\mathbb{L}^2-\mathbb{L} & \mathbb{L}^3-3\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3-3\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3-4\mathbb{L}^2-5\mathbb{L} \\
0 & 0 & \mathbb{L}^2-1 & \mathbb{L}^2-1 & \mathbb{L}^3-2\mathbb{L}^2-\mathbb{L}+2 & \mathbb{L}^3-3\mathbb{L}^2-\mathbb{L}+3 & \mathbb{L}^3-3\mathbb{L}^2-\mathbb{L}+3 & \mathbb{L}^3-3\mathbb{L}^2-\mathbb{L}+3 & \mathbb{L}^3-5\mathbb{L}^2-\mathbb{L}+5 \\
0 & 1 & 0 & -\mathbb{L}-1 & \mathbb{L} & -\mathbb{L}^2+\mathbb{L} & \mathbb{L}^2+\mathbb{L} & 0 & 0 \\
1 & 0 & -\mathbb{L}-1 & 0 & \mathbb{L} & \mathbb{L}^2+\mathbb{L} & -\mathbb{L}^2+\mathbb{L} & 0 & 0 \\
0 & 0 & 1-\mathbb{L} & 1-\mathbb{L} & 2\mathbb{L}-2 & -\mathbb{L}^2+4\mathbb{L}-3 & -\mathbb{L}^2+4\mathbb{L}-3 & \mathbb{L}^2+2\mathbb{L}-3 & -\mathbb{L}^2+6\mathbb{L}-5 \\
0 & 0 & \mathbb{L} & \mathbb{L} & -2\mathbb{L} & \mathbb{L}^2-4\mathbb{L} & \mathbb{L}^2-4\mathbb{L} & -2\mathbb{L} & 2\mathbb{L}^2-6\mathbb{L}
\end{bmatrix}$$

This matrix can be diagonalized with eigenvalues

$$0, \quad -\mathbb{L}(\mathbb{L}-1), \quad \mathbb{L}(\mathbb{L}-1), \quad \mathbb{L}(\mathbb{L}-1)(\mathbb{L}+1), \quad (\mathbb{L}-1)(\mathbb{L}+1),$$
$$-\mathbb{L}(\mathbb{L}+1), \quad 2\mathbb{L}(\mathbb{L}+1), \quad 2\mathbb{L}(\mathbb{L}-1), \quad \mathbb{L}(\mathbb{L}+1)$$

and respective eigenvectors

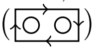
$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}
\begin{bmatrix} -\mathbb{L}-1 \\ \mathbb{L}+1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\begin{bmatrix} -\mathbb{L}^2+4\mathbb{L}+5 \\ -\mathbb{L}^2+4\mathbb{L}+5 \\ 5-\mathbb{L} \\ 5-\mathbb{L} \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\begin{bmatrix} \mathbb{L} \\ \mathbb{L} \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\begin{bmatrix} (\mathbb{L}-1)^2 \\ -(\mathbb{L}-1)^2 \\ 1-\mathbb{L} \\ \mathbb{L}-1 \\ 0 \\ -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}
\begin{bmatrix} 1-\mathbb{L} \\ \mathbb{L}-1 \\ 1 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}
\begin{bmatrix} \mathbb{L}+1 \\ \mathbb{L}+1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}
\begin{bmatrix} -(\mathbb{L}-1)^2 \\ -(\mathbb{L}-1)^2 \\ \mathbb{L}-1 \\ \mathbb{L}-1 \\ 0 \\ -2 \\ -2 \\ 0 \\ 2 \end{bmatrix}.$$

The following theorem now follows from (4.11).

Theorem 5.5.1. *For any $r \geq 0$, the virtual class of the* SL_2*-character stack of* N_r *in* $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Stck}_k)$ *is*

$$\begin{aligned}
[\mathfrak{X}_{\mathrm{SL}_2}(N_r)] &= \frac{1}{4}\mathbb{L}^{r-2}(\mathbb{L}+1)^{r-2}((\mathbb{L}-1)(1+(-1)^r)-(-2)^{r+1}) \\
&+ \frac{1}{4}\mathbb{L}^{r-2}(\mathbb{L}-1)^{r-2}((\mathbb{L}-1)(1+(-1)^r)+2^{r+1}-4) \\
&+ (\mathbb{L}^{r-2}+1)(\mathbb{L}-1)^{r-2}(\mathbb{L}+1)^{r-2}.
\end{aligned}$$

□

Remark 5.5.2. The first eigenvector, with eigenvalue 0, corresponds to the element $2M + Y - X_2 - X_{-2} - X_{2,-2} \in \mathrm{K}_0(\mathbf{Var}_G)$. We encountered this element already in Remark 5.1.2, where it was shown to be non-zero. On the other hand, the (Hodge) monodromy representation of $M + M + Y$ agrees with that of $X_2 + X_{-2} + X_{2,-2}$, so it is not surprising to encounter this element in the kernel of Z_G^{rep} ().

Similarly, for the orientable surfaces, using (4.12), Proposition 5.2.1 and the lemmas in Section 5.4, we obtain an expression for the matrix associated with Z_G^{rep} () with respect to the same set of generators.

Z_G^{rep} () =

$$\begin{bmatrix}
\mathbb{L}^4+4\mathbb{L}^3-\mathbb{L}^2-4\mathbb{L} & \mathbb{L}^3-\mathbb{L} & \mathbb{L}^5-2\mathbb{L}^4-4\mathbb{L}^3+2\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^5+3\mathbb{L}^4-\mathbb{L}^3-3\mathbb{L}^2 & \mathbb{L}^6-2\mathbb{L}^5-4\mathbb{L}^4+3\mathbb{L}^2+2\mathbb{L} & \mathbb{L}^6-11\mathbb{L}^4-3\mathbb{L}^3+10\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-3\mathbb{L}^5-8\mathbb{L}^4+7\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-2\mathbb{L}^5-9\mathbb{L}^4-\mathbb{L}^3+8\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-\mathbb{L}^5-20\mathbb{L}^4-4\mathbb{L}^3+19\mathbb{L}^2+5\mathbb{L} \\
\mathbb{L}^3-\mathbb{L} & \mathbb{L}^4+4\mathbb{L}^3-\mathbb{L}^2-4\mathbb{L} & \mathbb{L}^5+3\mathbb{L}^4-\mathbb{L}^3-3\mathbb{L}^2 & \mathbb{L}^5-2\mathbb{L}^4-4\mathbb{L}^3+2\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-2\mathbb{L}^5-4\mathbb{L}^4+3\mathbb{L}^2+2\mathbb{L} & \mathbb{L}^6-3\mathbb{L}^5-8\mathbb{L}^4+7\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-11\mathbb{L}^4-3\mathbb{L}^3+10\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-2\mathbb{L}^5-9\mathbb{L}^4-\mathbb{L}^3+8\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-\mathbb{L}^5-20\mathbb{L}^4-4\mathbb{L}^3+19\mathbb{L}^2+5\mathbb{L} \\
\mathbb{L}^3-2\mathbb{L}^2-3\mathbb{L} & \mathbb{L}^3+3\mathbb{L}^2 & \mathbb{L}^5+\mathbb{L}^4+3\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^5-3\mathbb{L}^3-6\mathbb{L}^2 & \mathbb{L}^6-2\mathbb{L}^5-3\mathbb{L}^4+\mathbb{L}^3+3\mathbb{L}^2 & \mathbb{L}^6-3\mathbb{L}^5-4\mathbb{L}^4-3\mathbb{L}^3+9\mathbb{L}^2 & \mathbb{L}^6-3\mathbb{L}^5-\mathbb{L}^4-3\mathbb{L}^3+6\mathbb{L}^2 & \mathbb{L}^6-2\mathbb{L}^5-5\mathbb{L}^4+6\mathbb{L}^2 & \mathbb{L}^6-4\mathbb{L}^5-4\mathbb{L}^4-8\mathbb{L}^3+15\mathbb{L}^2 \\
\mathbb{L}^3+3\mathbb{L}^2 & \mathbb{L}^3-2\mathbb{L}^2-3\mathbb{L} & \mathbb{L}^5-3\mathbb{L}^3-6\mathbb{L}^2 & \mathbb{L}^5+\mathbb{L}^4+3\mathbb{L}^2+3\mathbb{L} & \mathbb{L}^6-2\mathbb{L}^5-3\mathbb{L}^4+\mathbb{L}^3+3\mathbb{L}^2 & \mathbb{L}^6-3\mathbb{L}^5-\mathbb{L}^4-3\mathbb{L}^3+6\mathbb{L}^2 & \mathbb{L}^6-3\mathbb{L}^5-4\mathbb{L}^4-3\mathbb{L}^3+9\mathbb{L}^2 & \mathbb{L}^6-2\mathbb{L}^5-5\mathbb{L}^4+6\mathbb{L}^2 & \mathbb{L}^6-4\mathbb{L}^5-4\mathbb{L}^4-8\mathbb{L}^3+15\mathbb{L}^2 \\
\mathbb{L}^3-\mathbb{L}^2-\mathbb{L}+1 & \mathbb{L}^3-\mathbb{L}^2-\mathbb{L}+1 & \mathbb{L}^5-2\mathbb{L}^3+\mathbb{L} & \mathbb{L}^5-2\mathbb{L}^3+\mathbb{L} & \mathbb{L}^6-2\mathbb{L}^5-2\mathbb{L}^4+2\mathbb{L}^3+3\mathbb{L}^2-2 & \mathbb{L}^6-3\mathbb{L}^5-3\mathbb{L}^4+4\mathbb{L}^3+5\mathbb{L}^2-\mathbb{L}-3 & (\mathbb{L}-1)^2(\mathbb{L}+1)\left(\mathbb{L}^3-2\mathbb{L}^2-4\mathbb{L}-3\right) & (\mathbb{L}-3)(\mathbb{L}-1)^2(\mathbb{L}+1)^3 & (\mathbb{L}-5)(\mathbb{L}-1)^2(\mathbb{L}+1)^3 \\
2\mathbb{L}^2+2\mathbb{L} & -\mathbb{L}^2-\mathbb{L} & -2\mathbb{L}^3-4\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3+2\mathbb{L}^2+\mathbb{L} & \mathbb{L}^3+\mathbb{L}^2 & 2\mathbb{L}^4+2\mathbb{L}^3 & -2\mathbb{L}^4+2\mathbb{L}^2 & 0 & 0 \\
-\mathbb{L}^2-\mathbb{L} & 2\mathbb{L}^2+2\mathbb{L} & \mathbb{L}^3+2\mathbb{L}^2+\mathbb{L} & -2\mathbb{L}^3-4\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^3+\mathbb{L}^2 & -2\mathbb{L}^4+2\mathbb{L}^2 & 2\mathbb{L}^4+2\mathbb{L}^3 & 0 & 0 \\
-\mathbb{L}^2+2\mathbb{L}-1 & -\mathbb{L}^2+2\mathbb{L}-1 & -2\mathbb{L}^3+4\mathbb{L}^2-2\mathbb{L} & -2\mathbb{L}^3+4\mathbb{L}^2-2\mathbb{L} & 4\mathbb{L}^3-6\mathbb{L}^2+2 & -2\mathbb{L}^4+11\mathbb{L}^3-13\mathbb{L}^2+\mathbb{L}+3 & -2\mathbb{L}^4+11\mathbb{L}^3-13\mathbb{L}^2+\mathbb{L}+3 & \mathbb{L}^4+6\mathbb{L}^3-12\mathbb{L}^2+2\mathbb{L}+3 & -3\mathbb{L}^4+20\mathbb{L}^3-26\mathbb{L}^2+4\mathbb{L}+5 \\
\mathbb{L}^2-2\mathbb{L} & \mathbb{L}^2-2\mathbb{L} & 2\mathbb{L}^3-2\mathbb{L}^2+2\mathbb{L} & 2\mathbb{L}^3-2\mathbb{L}^2+2\mathbb{L} & -4\mathbb{L}^3+2\mathbb{L}^2 & 2\mathbb{L}^4-11\mathbb{L}^3+7\mathbb{L}^2 & 2\mathbb{L}^4-11\mathbb{L}^3+7\mathbb{L}^2 & -6\mathbb{L}^3+6\mathbb{L}^2 & 4\mathbb{L}^4-20\mathbb{L}^3+16\mathbb{L}^2
\end{bmatrix}$$

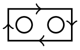
It turns out this matrix can be diagonalized using the same set of eigenvectors. The corresponding eigenvalues are

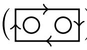
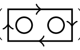
$$0, \quad \mathbb{L}^2(\mathbb{L}-1)^2, \quad \mathbb{L}^2(\mathbb{L}-1)^2, \quad \mathbb{L}^2(\mathbb{L}-1)^2(\mathbb{L}+1)^2, \quad (\mathbb{L}-1)^2(\mathbb{L}+1)^2,$$
$$\mathbb{L}^2(\mathbb{L}+1)^2, \quad 4\mathbb{L}^2(\mathbb{L}+1)^2, \quad 4\mathbb{L}^2(\mathbb{L}-1)^2, \quad \mathbb{L}^2(\mathbb{L}+1)^2.$$

The following theorem now follows from (4.10). The corresponding E-polynomials can be seen to agree with [MM16].

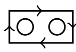
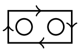
Theorem 5.5.3. *For any $g \geq 0$, the virtual class of the* SL_2*-character stack of* Σ_g *in* $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Stck}_k)$ *is*

$$\begin{aligned}[\mathfrak{X}_{\mathrm{SL}_2}(\Sigma_g)] = &\frac{1}{2}\mathbb{L}^{2g-2}(\mathbb{L}+1)^{2g-2}(2^{2g}+\mathbb{L}-1)\\ &+\frac{1}{2}\mathbb{L}^{2g-2}(\mathbb{L}-1)^{2g-2}(2^{2g}+\mathbb{L}-3)\\ &+(\mathbb{L}^{2g-2}+1)(\mathbb{L}-1)^{2g-2}(\mathbb{L}+1)^{2g-2}. \qquad \square\end{aligned}$$

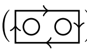
The fact that both matrices can be simultaneously diagonalized is not too surprising considering the fact that  and  commute as bordisms. Furthermore, it can be seen that

Z_G^{rep} ( $)^3 = Z_G^{\mathrm{rep}}$ () $\circ\, Z_G^{\mathrm{rep}}$ ()

which reflects the equality of bordisms

 $^3 =$  $\circ$ .

What is remarkable is that the equality

Z_G^{rep} ( $)^2 = Z_G^{\mathrm{rep}}$ ()

holds for $G = \mathrm{SL}_2$ (at least on the set of generators (5.3)), even though it does not for general G. For example, it already fails to hold for $G = \mathbb{G}_m$. Comparing Theorem 5.5.3 and Theorem 5.5.1, we find the following.

Corollary 5.5.4. $[\mathfrak{X}_{\mathrm{SL}_2}(\Sigma_g)] = [\mathfrak{X}_{\mathrm{SL}_2}(N_{2g})]$ *in* $\mathrm{K}_0^{\mathbb{P}^1}(\mathbf{Stck}_k)$ *for all* $g \geq 0$. $\square$

An explanation for this relation between the orientable and non-orientable case can be given for the corresponding E-polynomials, from the point of view of the arithmetic method.

Suppose G is a linear algebraic group over a finitely generated $\mathbb{Z}$-algebra R. Comparing Theorem 4.5.3 and Proposition 4.9.12, it follows that, for any morphism $R \to \mathbb{F}_q$, the point counts $|R_G(\Sigma_g)(\mathbb{F}_q)|$ and $|R_G(N_{2g})(\mathbb{F}_q)|$ agree whenever the Frobenius–Schur indicators ε_χ of all irreducible characters χ of $G(\mathbb{F}_q)$ are equal to ± 1. That is, if all irreducible representations of $G(\mathbb{F}_q)$ are either real or pseudoreal.

Indeed, if we take $G = \mathrm{SL}_2$ and $R = \mathbb{Z}[1/2, i]$, then a map $R \to \mathbb{F}_q$ exists if and only if $q \equiv 1 \bmod 4$. For such q, any element of $\mathrm{SL}_2(\mathbb{F}_q)$ is conjugate to its inverse, and hence

$$\chi(g) = \chi(g^{-1}) = \overline{\chi(g)}$$

for all $g \in \mathrm{SL}_2(\mathbb{F}_q)$ and irreducible characters χ of $\mathrm{SL}_2(\mathbb{F}_q)$. This shows that all irreducible characters χ of $\mathrm{SL}_2(\mathbb{F}_q)$, with $q \equiv 1 \bmod 4$, are either real or pseudoreal, that is, $\varepsilon_\chi = \pm 1$, and hence

$$|R_{\mathrm{SL}_2}(\Sigma_g)(\mathbb{F}_q)| = |R_{\mathrm{SL}_2}(N_{2g})(\mathbb{F}_q)|.$$

Since these numbers are polynomial in q, it follows from Theorem 4.6.1 (Katz' theorem) that $e(R_{\mathrm{SL}_2}(\Sigma_g)) = e(R_{\mathrm{SL}_2}(N_{2g}))$, and in turn that $e(\mathfrak{X}_{\mathrm{SL}_2}(\Sigma_g)) = e(\mathfrak{X}_{\mathrm{SL}_2}(N_{2g}))$.

Chapter 6

Upper triangular matrices

In this chapter we apply the theory of the Chapter 4 in order to study the G-character stacks of the closed orientable surfaces Σ_g, for G equal to one of the following algebraic groups, over any field k:

- the group $\mathbb{T}_n = \{A \in \mathrm{GL}_n \mid A_{ij} = 0 \text{ for } 1 \le j < i \le n\} \subseteq \mathrm{GL}_n$ of $n \times n$ upper triangular matrices, and
- its subgroup $\mathbb{U}_n = \{A \in \mathbb{T}_n \mid A_{ii} = 1 \text{ for } 1 \le i \le n\}$ of unipotent matrices.

These groups can be realized as semidirect products of copies of $\mathbb{G}_a$ and $\mathbb{G}_m$ and are therefore all special, see Proposition 3.3.16 and Example 3.3.17. In particular, the virtual class of the G-character stack of Σ_g in the Grothendieck ring of stacks is simply given by the quotient

$$[\mathfrak{X}_G(\Sigma_g)] = [R_G(\Sigma_g)]/[G] \tag{6.1}$$

as in Proposition 3.5.5. Hence, it suffices to apply the theory of Section 4.12, and work on the level of the G-representation variety.

Furthermore, these groups G are all connected. Therefore, the geometric TQFT and the arithmetic TQFT can be compared, as there is a natural transformation between them, see Corollary 4.10.5. We will consider both the geometric and the arithmetic method, and compare the results.

Note that the algebraic groups $\mathbb{T}_n$, for all $n \ge 1$, decompose as a product

$$\mathbb{T}_n = \mathbb{G}_m \times \tilde{\mathbb{T}}_n \quad \text{where} \quad \tilde{\mathbb{T}}_n = \{A \in \mathbb{T}_n \mid A_{nn} = 1\}.$$

In turn, this induces a decomposition of representation varieties

$$R_{\mathbb{T}_n}(M) \cong R_{\tilde{\mathbb{T}}_n}(M) \times R_{\mathbb{G}_m}(M). \tag{6.2}$$

As $\mathbb{G}_m$ is abelian, we have $R_{\mathbb{G}_m}(\Sigma_g) \cong \mathbb{G}_m^{2g}$, and we can focus on $\tilde{\mathbb{T}}_n$ instead. This slightly simplifies the computations as dimension is lower.

6.1 Algebraic representatives

Before doing computations, we first introduce the notion of *algebraic representatives*, which are crucial to doing computations in the later sections.

Definition 6.1.1. Let G be an algebraic group over k, and let X be a variety over k with a transitive G-action. A point $\xi \in X(k)$ is an *algebraic representative* for X if the $\text{Stab}(\xi)$-torsor

$$G \to X, \quad g \mapsto g \cdot \xi$$

is Zariski-locally trivial. Equivalently, ξ is an algebraic representative for X if every point of X has an open neighborhood U and a morphism $\gamma \colon U \to G$ such that $x = \gamma(x) \cdot \xi$ for all $x \in U$.

Remark 6.1.2.
- If there exists an algebraic representative ξ for X, then every $\xi' \in X(k)$ is an algebraic representative for X. Namely, if $\xi = g \cdot \xi'$, then one takes $\gamma'(x) = \gamma(x) g$.
- Algebraic representatives need not always exist. For example, consider the group $G = \mathbb{G}_m$ acting on $X = \mathbb{A}^1_k \setminus \{0\}$ via $t \cdot x = t^2 x$. Then X does not have an algebraic representative as $\mathbb{G}_m \to \mathbb{A}^1_k \setminus \{0\}$ given by $t \mapsto t^2$ is not Zariski-locally trivial.
- The proof of Corollary 3.3.15 shows that $\xi \in X(k)$ is an algebraic representative for X if $\text{Stab}(\xi)$ is special. However, it is possible that ξ is an algebraic representative even when $\text{Stab}(\xi)$ is not special. For example, consider any non-special group G acting trivially on a point.

For us, the main example of algebraic representatives are for any of the groups $\mathbb{U}_n, \mathbb{T}_n$ or $\tilde{\mathbb{T}}_n$ acting by conjugation on a conjugacy class.

Proposition 6.1.3. *Let G be $\mathbb{U}_n, \mathbb{T}_n$ or $\tilde{\mathbb{T}}_n$, for some $n \geq 1$, acting on itself by conjugation. Then the stabilizer* $\text{Stab}(A)$ *of any point $A \in G(k)$ is special. In particular, A is an algebraic representative for its conjugacy class.*

Proof. If $G = \mathbb{T}_n$, the stabilizer $\text{Stab}(A) \subseteq \mathbb{T}_n$ is triangularizable, and can be written as an extension

$$1 \to U \to \text{Stab}(A) \to D \to 1$$

of $D = \{B \in \text{Stab}(A) \mid B \text{ is diagonal}\}$ by the maximal normal unipotent subgroup $U = \text{Stab}(A) \cap \mathbb{U}_n$. We will show that both U and D are special, so that the result follows from Proposition 3.3.16 *(i)*. If $G = \mathbb{U}_n$, we have $\text{Stab}(A) = U$, so the result follows from the same proof. If $G = \tilde{\mathbb{T}}_n$, then the action of $\tilde{\mathbb{T}}_n$ on itself by conjugation can be extended to an action of $\mathbb{T}_n$, and the corresponding stabilizers are related by $\mathrm{Stab}_{\mathbb{T}_n}(A) \cong \mathbb{G}_m \times \mathrm{Stab}_{\tilde{\mathbb{T}}_n}(A)$. The result then follows from Proposition 3.3.16 *(ii)* or *(iii)* and the fact that $\mathrm{Stab}_{\mathbb{T}_n}(A)$ is special.

Note that $U = \{B \in \mathbb{U}_n \mid AB - BA = 0\}$ is a subgroup of $\mathbb{U}_n$ given by a linear subspace, identifying $\mathbb{U}_n \cong \mathbb{A}_k^{n(n-1)/2}$ in the usual way. From [Mil15, Example 8.46] we know that $\mathbb{U}_n$ admits a normal series

$$\mathbb{U}_n = U_n^{(0)} \supseteq \cdots \supseteq U_n^{(r)} \supseteq U_n^{(r+1)} \supseteq \cdots \supseteq U_n^{(n(n-1)/2)} = \{1\},$$

where each $U_n^{(r)} \subseteq \mathbb{U}_n$ is a normal subgroup given by a linear subspace of $\mathbb{U}_n$, whose quotients $U_n^{(r)}/U_n^{(r+1)}$ are canonically isomorphic to $\mathbb{G}_a$. Therefore, intersecting this normal series with U yields a normal series of U where each quotient is either $\mathbb{G}_a$ or 0. Hence, U is an extension of copies of $\mathbb{G}_a$, which is special by Proposition 3.3.16 *(i)*.

Furthermore, D can be identified with

$$\begin{aligned} D &= \{B \in G \mid B \text{ is diagonal and } AB - BA = 0\} \\ &= \{(B_{11}, \ldots, B_{nn}) \in \mathbb{G}_m^n \mid A_{ij}(B_{ii} - B_{jj}) = 0 \text{ for all } 1 \leq i \leq j \leq n\} \\ &= \{(B_{11}, \ldots, B_{nn}) \in \mathbb{G}_m^n \mid B_{ii} = B_{jj} \text{ whenever } A_{ij} \neq 0\}, \end{aligned}$$

which, being a product of copies of $\mathbb{G}_m$, is also special. □

The notion of algebraic representatives can be generalized to a relative setting as follows. This generalization is useful when the G-action is not transitive, and we will use it in the later sections.

Definition 6.1.4. Let G be an algebraic group over k, acting on a variety X over k, and let $\pi\colon X \to T$ be a G-invariant morphism. A *family of algebraic representatives* for X over T is a morphism $\xi\colon T \to X$ over T (that is, $\pi \circ \xi = \mathrm{id}_T$) such that the $\mathrm{Stab}(\xi)$-torsor

$$G \times T \to X, \quad (g, t) \mapsto g \cdot \xi(t)$$

of varieties over T is Zariski-locally trivial. Note that by $\mathrm{Stab}(\xi)$ we understand $(G \times T) \times_X T$ as a group over T. Equivalently, ξ is a family of algebraic representatives for X over T if every point of X has an open neighborhood U and a morphism $\gamma\colon U \to G$ such that $x = \gamma(x) \cdot \xi(\pi(x))$ for all $x \in U$.

Example 6.1.5. Consider the group $G = \mathbb{T}_2 = \left\{ \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \;\middle|\; x, z \neq 0 \right\}$ of 2×2 upper triangular matrices acting on $X = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \;\middle|\; a \neq 0, 1 \right\}$ by conjugation. Then X has a family of algebraic representatives over $T = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \;\middle|\; a \neq 0, 1 \right\}$, with π and ξ the projection and inclusion, respectively, as one can take

$$\gamma \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{1-a} \\ 0 & 1 \end{pmatrix}.$$

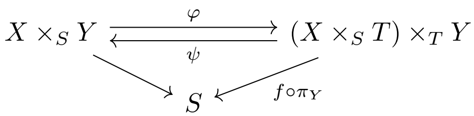
The following lemma shows why it is useful to have algebraic representatives in the context of computing virtual classes.

Proposition 6.1.6. *Let G be an algebraic group over k, acting on a variety S over k. Let $\pi\colon S \to T$ be a G-invariant morphism, and let $\xi\colon T \to S$ be a family of algebraic representatives for S over T. Then for any morphism $f\colon Y \to S$ and G-equivariant morphism $g\colon X \to S$, we have*

$$[X \times_S Y]_S = [(X \times_S T) \times_T Y]_S \in \mathrm{K}_0(\mathbf{Var}_S),$$

where $(X \times_S T) \times_T Y$ is seen as a variety over S via the composite $f \circ \pi_Y$.

Proof. Locally on X, there is a commutative diagram



where $\varphi(x, y) = (\gamma(f(y)) \cdot x, \pi(f(y)), y)$ and $\psi((x, t), y) = (\gamma(f(y)) \cdot x, y)$. One easily sees that φ and ψ are well-defined over S and inverse to each other. $\square$

In the case of algebraic representatives, that is, when T is a point, we obtain the following corollaries.

Corollary 6.1.7. *Let G be an algebraic group over k, acting on a variety S over k with algebraic representative $\xi \in S(k)$. Then for any morphism $f\colon Y \to S$ and G-equivariant morphism $g\colon X \to S$, we have*

$$[X \times_S Y]_S = [X \times_S \{\xi\}] \cdot [Y]_S \in \mathrm{K}_0(\mathbf{Var}_S). \qquad \square$$

Corollary 6.1.8. *Let G be an algebraic group over k, acting on a variety S over k with algebraic representative $\xi \in S(k)$. Then for any G-equivariant morphism $g\colon X \to S$, we have*

$$[S] \cdot [X]_S = [X] \cdot [S]_S \in \mathrm{K}_0(\mathbf{Var}_S).$$

Proof. Apply Corollary 6.1.7 with $f = \mathrm{id}_S$ to find that

$$[X]_S = [X \times_S \{\xi\}] \cdot [S]_S.$$

Applying $c_!$ to both sides of this equation, for $c\colon S \to \operatorname{Spec} k$ the final morphism, yields $[S][X \times_S \{\xi\}] = [X]$ in $\mathrm{K}_0(\mathbf{Var}_k)$. $\square$

6.2 Geometric method

The virtual classes of the G-representation varieties $R_G(\Sigma_g)$ in the Grothendieck ring of varieties can be computed, as was shown in Section 4.12, using the morphisms

Z_G^{rep} () $: \mathrm{K}_0(\mathbf{Var}_G) \to \mathrm{K}_0(\mathbf{Var}_k)$, Z_G^{rep} () $: \mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathbf{Var}_G)$

and Z_G^{rep} () $: \mathrm{K}_0(\mathbf{Var}_G) \to \mathrm{K}_0(\mathbf{Var}_G)$

of $\mathrm{K}_0(\mathbf{Var}_k)$-modules. Note that all varieties over G that we consider are naturally equipped with a G-action such that the morphism to G is G-equivariant, even though the ring $\mathrm{K}_0(\mathbf{Var}_G)$ does not remember this information. For this reason, it turns out that $\mathrm{K}_0(\mathbf{Var}_G)$ is best understood via a decomposition

$$\mathrm{K}_0(\mathbf{Var}_G) \cong \bigoplus_{i=1}^{N} \mathrm{K}_0(\mathbf{Var}_{\mathcal{C}_i}),$$

as in Proposition 3.3.6, where the $\mathcal{C}_i$ are locally closed subvarieties of G given by families of conjugacy classes. We will show that each $\mathcal{C}_i$ has a family of algebraic representatives. As a result, the submodule of $\mathrm{K}_0(\mathbf{Var}_G)$ generated by the units $\mathbf{1}_{\mathcal{C}_i} \in \mathrm{K}_0(\mathbf{Var}_{\mathcal{C}_i})$ will be invariant under Z_G^{rep} ().

Conjugacy classes of $\tilde{\mathbb{T}}_n$

Let us start by describing the conjugacy classes of $\tilde{\mathbb{T}}_n$, with a focus on the unipotent conjugacy classes. We will give algebraic representatives for the unipotent conjugacy classes, and families of algebraic representatives for (families of) non-unipotent conjugacy classes. Furthermore, we will determine equations describing the unipotent conjugacy classes, and finally, compute the virtual classes of the unipotent conjugacy classes as well as those of the stabilizers of their representatives. All of this data will be used to compute Z_G^{rep} () as a matrix with respect to the generators given by the unipotent classes.

Unipotent conjugacy classes. To find the number of unipotent conjugacy classes of $\tilde{\mathbb{T}}_n$, and a representative for each one, one can use Belitskii's algorithm as described in [Kob05]. Given a unipotent matrix $A \in \tilde{\mathbb{T}}_n$ over any field k, Belitskii's algorithm outputs a canonical representative of the conjugacy class of A. It achieves this by repeatedly conjugating A by certain elementary matrices in order to make as many entries of A as possible equal to 0 or 1, see [Kob05] for details. For $n = 1, \dots, 5$, it turns out there are only finitely many unipotent conjugacy classes $\mathcal{U}_1, \dots, \mathcal{U}_M$, and the canonical representatives $\xi_1, \dots, \xi_M$ only have entries with 0's and 1's [Kob05]. The number M of unipotent conjugacy classes is given by the following table.

n	1	2	3	4	5
M	1	2	5	16	61

We will use the convention that $\mathcal{U}_1$ is the conjugacy class of the identity. Note that the representatives will be automatically algebraic by Proposition 6.1.3.

Remark 6.2.1. The qualitative result of Belitskii's algorithm, that for $n = 1, \ldots, 5$ every unipotent matrix in $\tilde{\mathbb{T}}_n$ can be conjugated to a matrix containing only 0's and 1's, is enough to find representatives. Only finitely many such matrices exist ($2^{n(n-1)/2}$) and they are easily partitioned by whether they are conjugate. Then, one simply chooses one representative in each conjugacy class.

Non-unipotent conjugacy classes. Next, we describe the non-unipotent conjugacy classes of $\tilde{\mathbb{T}}_n$ in terms of families depending on their diagonal. Define a *diagonal pattern* to be a partition of the set $\{1, 2, \ldots, n\}$. Then, for any matrix $A \in \tilde{\mathbb{T}}_n$, the diagonal pattern δ_A of A is the partition such that i and j are equivalent if $A_{ii} = A_{jj}$. Note that two matrices A and B in $\tilde{\mathbb{T}}_n$ are conjugate only if their diagonals coincide, but not necessarily if. Now, we look at the following families of conjugacy classes:

$$\mathcal{C}_{\delta,i} = \{A \in \tilde{\mathbb{T}}_n \mid \delta_A = \delta \text{ and } A \sim \operatorname{diag}(A) + \xi_i - 1\},$$

for any diagonal pattern δ and $i = 1, \ldots, M$, where $\operatorname{diag}(A)$ denotes the diagonal part of A. We claim that any such $\mathcal{C}_{\delta,i}$ has a family of representatives over

$$C_{\delta,i} = \{A \in \tilde{\mathbb{T}}_n \mid \delta_A = \delta \text{ and } A = \operatorname{diag}(A) + \xi_i - 1\}$$

where $\pi_{\delta,i} \colon \mathcal{C}_{\delta,i} \to C_{\delta,i}$ and $\xi_{\delta,i} \colon C_{\delta,i} \to \mathcal{C}_{\delta,i}$ are given by $\pi_{\delta,i}(A) = \operatorname{diag}(A) + \xi_i - 1$ and $\xi_{\delta,i}(A) = A$. This is proved in Lemma 6.2.2 below. Of course, some $\mathcal{C}_{\delta,i}$ may be equal to $\mathcal{C}_{\delta,j}$ while $i \neq j$, but one can explicitly check whether any of the representatives are conjugate in order to remove any such duplicates. In particular, one can ensure $A_{ij} = 0$ for all $A \in C_{\delta,k}$ whenever $A_{ii} \neq A_{jj}$, after appropriate conjugation. In the end, we obtain families of conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_N$ with families of algebraic representatives over $C_1, \ldots, C_N$, where the number N is given by the following table.

n	1	2	3	4	5
N	2	3	12	61	372

We will choose our indices in such a way that the ξ_i coincide with the unipotent representatives for $i = 1, \dots, M$.

Lemma 6.2.2. *For every $i = 1, \dots, N$, the following statements hold:*

(i) the stabilizer $H_i := \mathrm{Stab}(\xi_i(t))$ is independent of $t \in C_i$,

(ii) ξ_i is a family of representatives of $\mathcal{C}_i$ over C_i,

(iii) the map $G/H_i \times C_i \to \mathcal{C}_i$ given by $(g, t) \mapsto g\xi_i(t)g^{-1}$ is an isomorphism.

Proof. *(i)* The statement can easily be verified by a computer, as there are only a finite number of cases to consider. Alternatively, write $A = \xi_i(t)$ and note that $B \in \mathrm{Stab}(A)$ if and only if for all $1 \le i \le j \le n$,

$$B_{ij}(A_{ii} - A_{jj}) + \sum_{k=i+1}^{j} B_{kj}A_{ik} - \sum_{k=i}^{j-1} B_{ik}A_{kj} = 0. \qquad (*)$$

We claim that $B_{ij} = 0$ for all $i \le j$ such that $A_{ii} \ne A_{jj}$. The result follows from this claim, because A_{ij} is independent of t for $i \ne j$ (by definition of C_i), so the solutions to $(*)$ will be independent of t. We proof the claim by induction on $j - i$, the case $j - i = 0$ being trivial. For the general case, take $i \le j$ such that $A_{ii} \ne A_{jj}$. Now, for every $k \in \{i + 1, \dots, j\}$ such that $A_{ik} \ne 0$, we have $A_{kk} = A_{ii} \ne A_{jj}$, so $B_{kj} = 0$ by the induction hypothesis. Similarly, for every $k \in \{i, \dots, j - 1\}$ such that $A_{kj} \ne 0$, we have $A_{kk} = A_{jj} \ne A_{ii}$ so $B_{ik} = 0$ by the induction hypothesis. Therefore, $(*)$ reduces to $B_{ij} = 0$.

(ii) From *(i)* follows that the map

$$G \times C_i \to \mathcal{C}_i, \quad (g, t) \mapsto g\xi_i(t)g^{-1}$$

is an H_i-torsor, which is Zariski-locally trivial because H_i is special by Proposition 6.1.3. Hence, it follows that ξ_i is a family of algebraic representatives for $\mathcal{C}_i$ over C_i. This also proves *(iii)*. □

Equations. Next, we want to find equations describing the unipotent conjugacy classes $\mathcal{U}_i$ for $i = 1, \dots, M$. For simplicity, we will compute equations for the closures $\overline{\mathcal{U}}_i$ rather than $\mathcal{U}_i$. This is sufficient using the inclusion-exclusion matrix of Section 3.3. The closure $\overline{\mathcal{U}}_i$ is the closure of the image of the morphism

$$f_i \colon G \to G, \quad g \mapsto g\xi_i g^{-1}.$$

Since $G = \tilde{\mathbb{T}}_n$ is affine, f_i can equivalently be described by the corresponding morphism on the coordinate ring of G,

$$f_i^{\#} \colon \mathcal{O}_G(G) \to \mathcal{O}_G(G).$$

In particular, the closure $\overline{\mathcal{U}}_i$ corresponds to the ideal $I_i \subseteq \mathcal{O}_G(G)$ which is the kernel of $f_i^\#$. Generators for these ideals can be computed using Gröbner basis [AL94], and this gives us the desired equations. In particular, we use [AL94, Theorem 2.4.2] in order to compute the kernel of $f_i^\#$.

Example 6.2.3. Consider the unipotent conjugacy class $\mathcal{U}$ of $\xi = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ in $G = \tilde{\mathbb{T}}_3$. The morphism $f\colon G \to G$, $g \mapsto g\xi g^{-1}$ is given by

$$f \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

that is, $f^\#(a) = f^\#(d) = 1$, $f^\#(b) = f^\#(e) = 0$ and $f^\#(c) = a$. Indeed, we find that the ideal $\ker f^\# = (a-1, b, d-1, e)$ describes the closure of the conjugacy class $\mathcal{U} = \left\{ \left(\begin{smallmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \;\middle|\; c \neq 0 \right\}$.

Orbits and stabilizers. For any $A \in \tilde{\mathbb{T}}_n$, in order to compute the virtual class of the conjugacy class of A, we can use Corollary 3.3.15. Indeed, the stabilizer $\operatorname{Stab}(A)$ of any $A \in \tilde{\mathbb{T}}_n$ is special by Proposition 6.1.3. To compute the virtual class of the stabilizer of A, we apply Algorithm 3.4.3, using the explicit description

$$\operatorname{Stab}(A) = \{B \in \tilde{\mathbb{T}}_n \mid AB - BA = 0\}.$$

Computing the TQFT

Let us return to the problem of computing the matrix associated to Z () with respect to the generators $\mathbf{1}_{\mathcal{U}_i} = [\mathcal{U}_i]_G \in \mathrm{K}_0(\mathbf{Var}_G)$. We start by computing the first column of this matrix. Recall that by convention $\mathcal{U}_1 = \{1\}$ is the conjugacy class of the identity, and that c denotes the final morphism to $\operatorname{Spec} k$. To compute the first column of this matrix, we write

$c_!\big(Z_G^{\mathrm{rep}}$ () $(\mathbf{1}_{\mathcal{U}_1})|_{\mathcal{U}_i}\big)$

$$\begin{aligned} &= [\{(A,B) \in G^2 \mid [A,B] \in \mathcal{U}_i\}] \\ &= \sum_{j=1}^N [\{(A,B) \in G \times \mathcal{C}_j \mid [A,B] \in \mathcal{U}_i\}] \\ &= \sum_{j=1}^N [\{(A,t) \in G \times C_j \mid [A,\xi_j(t)] \in \mathcal{U}_i\}] \times [\tilde{\mathbb{T}}_n / \operatorname{Stab}(\xi_j(t_0))] \\ &= \sum_{j=1}^N E_{ij}\, [\operatorname{Orbit}(\xi_j(t_0))], \end{aligned}$$

where $E_{ij} = [\{(A,t) \in G \times C_j \mid [A, \xi_j(t)] \in \mathcal{U}_i\}]$ and $t_0 \in C_j$ is any closed point. For the third equality, we used Proposition 6.1.6 with $Y = S = \mathcal{C}_j$ and $X = \{(A,B) \in G \times \mathcal{C}_j \mid [A,B] \in \mathcal{U}_i\}$, in combination with Lemma 6.2.2 *(iii)*.

Since we have computed equations describing the closures $\overline{\mathcal{U}}_i$, it is in fact easier to compute the classes $\overline{E}_{ij} = [\{(A,t) \in G \times C_j \mid [A, \xi_j(t)] \in \overline{\mathcal{U}}_i\}]$ rather than the E_{ij}. By Corollary 3.3.11, they are related through the inclusion-exclusion matrix C of the stratification by

$$E_{ij} = \sum_{k=1}^{M} C_{ik} \overline{E}_{kj}.$$

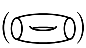
The coefficients $\overline{E}_{ij}$ can be computed using Algorithm 3.4.3. Then, using Corollary 6.1.8, we obtain

Z_G^{rep} () $(\mathbf{1}_{\mathcal{U}_1}) = \sum_{i,k=1}^{M} \sum_{j=1}^{N} C_{ik} \overline{E}_{kj} [\mathrm{Orbit}(\xi_j(t_0))]/[\mathcal{U}_i] \cdot \mathbf{1}_{\mathcal{U}_i}.$

Next, to compute the other columns of the matrix associated to Z_G^{rep} (), we will make use of the already computed first column. In particular, we have

$c_! \big(Z_G^{\mathrm{rep}}$ () $(\mathbf{1}_{\mathcal{U}_j})|_{\mathcal{U}_i} \big)$

$$
\begin{aligned}
&= [\{(g,A,B) \in \mathcal{U}_j \times G^2 \mid g[A,B] \in \mathcal{U}_i\}] \\
&= \sum_{k=1}^{M} [\{(g,A,B) \in \mathcal{U}_j \times G^2 \mid g[A,B] \in \mathcal{U}_i, [A,B] \in \mathcal{U}_k\}] \\
&= \sum_{k=1}^{M} [\{g \in \mathcal{U}_j \mid g\xi_k \in \mathcal{U}_i\}][\{(A,B) \in G^2 \mid [A,B] \in \mathcal{U}_k\}]
\end{aligned}
$$

$= \sum_{k=1}^{M} F_{ijk} \, c_! \big(Z_G^{\mathrm{rep}}$ () $(\mathbf{1}_{\mathcal{U}_1})|_{\mathcal{U}_k} \big),$

where $F_{ijk} = [\{g \in \mathcal{U}_j \mid g\xi_k \in \mathcal{U}_i\}]$. Note that the third equality follows from Corollary 6.1.7 applied to $S = \mathcal{U}_k$ and $X = \{(g,h) \in \mathcal{U}_j \times \mathcal{U}_k \mid gh \in \mathcal{U}_i\}$ and $Y = \{(A,B) \in G^2 \mid [A,B] \in \mathcal{U}_k\}$.

As for the coefficients E_{ij}, it is easier to compute $\overline{F}_{ijk} = [\{g \in \overline{\mathcal{U}}_j \mid g\xi_k \in \overline{\mathcal{U}}_i\}]$ rather than F_{ijk}, and they are related through the inclusion-exclusion matrix of the stratification by

$$F_{ijk} = \sum_{m,\ell=1}^{M} C_{im} C_{j\ell} \overline{F}_{m\ell k}.$$

The coefficients $\overline{F}_{ijk}$ can be computed using Algorithm 3.4.3. Finally, using Corollary 6.1.8 we obtain

$$Z_G^{\mathrm{rep}}\left(\bigcirc\!\!\!\!\!\!\smile\right)(\mathbf{1}_{\mathcal{U}_j}) = \sum_{i,k,\ell,m=1}^{M} C_{im} C_{j\ell} \overline{F}_{m\ell k}\, c_! \left(Z_G^{\mathrm{rep}}\left(\bigcirc\!\!\!\!\!\!\smile\right)(\mathbf{1}_{\mathcal{U}_1})|_{\mathcal{U}_k}\right) / [\mathcal{U}_i] \cdot \mathbf{1}_{\mathcal{U}_i}.$$

Remark 6.2.4. Naively computing the coefficients of the matrix representing $Z_G^{\mathrm{rep}}\left(\bigcirc\!\!\!\!\!\!\smile\right)$ would require computing the virtual class of M^2 varieties, each of which being a subvariety of G^3, with equations being mostly quadratic due to the commutator $[A, B]$. With this new setup, one needs to compute the virtual class of $MN + M^3$ varieties to obtain the coefficients $\overline{E}_{ij}$ and $\overline{F}_{ijk}$. However, the advantage of this approach is that these varieties will now be subvarieties of $G \times C_j$ and $\mathcal{U}_j$, respectively, with equations being mostly linear. In practice, the simplification of these systems of equations far outweighs the number of such systems. It is due to the (families of) algebraic representatives of the conjugacy classes $\mathcal{C}_i$ that these simplifications can be made.

Remark 6.2.5. Let us make a few computational remarks. First, to speed up the computation of the coefficients $\overline{E}_{ij}$ and $\overline{F}_{ijk}$, which are done by Algorithm 3.4.3, note that these can be performed in parallel as they are independent. Second, there are some checks one can perform to detect obvious errors. In particular, one can assert that the following equalities hold:

- $\sum_{i=1}^{M} c_! \left(Z_G^{\mathrm{rep}}\left(\bigcirc\!\!\!\!\!\!\smile\right)(\mathbf{1}_{\mathcal{U}_j})|_{\mathcal{U}_i}\right) = [G]^2\,[\mathcal{U}_j]$ for all j,
- $\sum_{i=1}^{M} F_{ijk} = [\mathcal{U}_j]$ for all j, k,
- $\sum_{j=1}^{M} F_{ijk} = [\mathcal{U}_i]$ for all i, k,
- $\sum_{i=1}^{M} E_{ij} = [G]\,[C_j]$ for all j.

Results

The code to perform the computations as described in this section can be found in [Vog22]. For every $n = 1, \ldots, 5$, the resulting matrix associated to $Z_G^{\mathrm{rep}}\left(\bigcirc\!\!\!\!\!\!\smile\right)$, with respect to the generators $\mathbf{1}_{\mathcal{U}_i}$, is a matrix whose coefficients are polynomials in the Lefschetz class $\mathbb{L}$. These matrices can be diagonalized over the field $\mathbb{Q}(\mathbb{L})$ of rational functions in $\mathbb{L}$, and the resulting eigenvalues and eigenvectors are recorded in Appendix A.

Applying equations (4.10), (6.2) and (6.1), we obtain the following theorem.

Theorem 6.2.6. *The virtual classes of the $\mathbb{T}_n$-character stacks of Σ_g in the Grothendieck ring of stacks for $n = 2, 3, 4, 5$ are given by* *(i)* $[\mathfrak{X}_{\mathbb{T}_2}(\Sigma_g)] = \mathbb{L}^{2g-2}\left(\mathbb{L}-1\right)^{2g-1} + \mathbb{L}^{2g-2}\left(\mathbb{L}-1\right)^{4g-2}$

(ii) $[\mathfrak{X}_{\mathbb{T}_3}(\Sigma_g)] = \mathbb{L}^{4g-4}\left(\mathbb{L}-1\right)^{4g-2} + \mathbb{L}^{6g-6}\left(\mathbb{L}-1\right)^{2g-1} + 2\mathbb{L}^{6g-6}\left(\mathbb{L}-1\right)^{4g-2} + \mathbb{L}^{6g-6}\left(\mathbb{L}-1\right)^{6g-3}$

(iii) $[\mathfrak{X}_{\mathbb{T}_4}(\Sigma_g)] = \mathbb{L}^{8g-8}\left(\mathbb{L}-1\right)^{4g-2} + \mathbb{L}^{8g-8}\left(\mathbb{L}-1\right)^{6g-3} + \mathbb{L}^{10g-10}\left(\mathbb{L}-1\right)^{2g-1} + 3\mathbb{L}^{10g-10}\left(\mathbb{L}-1\right)^{4g-2} + 2\mathbb{L}^{10g-10}\left(\mathbb{L}-1\right)^{6g-3} + \mathbb{L}^{12g-12}\left(\mathbb{L}-1\right)^{2g-1} + 3\mathbb{L}^{12g-12}\left(\mathbb{L}-1\right)^{4g-2} + 3\mathbb{L}^{12g-12}\left(\mathbb{L}-1\right)^{6g-3} + \mathbb{L}^{12g-12}\left(\mathbb{L}-1\right)^{8g-4}$

(iv) $[\mathfrak{X}_{\mathbb{T}_5}(\Sigma_g)] = \mathbb{L}^{12g-12}\left(\mathbb{L}-1\right)^{6g-3} + 2\mathbb{L}^{14g-14}\left(\mathbb{L}-1\right)^{4g-2} + 3\mathbb{L}^{14g-14}\left(\mathbb{L}-1\right)^{6g-3} + \mathbb{L}^{14g-14}\left(\mathbb{L}-1\right)^{8g-4} + 2\mathbb{L}^{16g-16}\left(\mathbb{L}-1\right)^{2g-1} + 7\mathbb{L}^{16g-16}\left(\mathbb{L}-1\right)^{4g-2} + 7\mathbb{L}^{16g-16}\left(\mathbb{L}-1\right)^{6g-3} + 2\mathbb{L}^{16g-16}\left(\mathbb{L}-1\right)^{8g-4} + 2\mathbb{L}^{18g-18}\left(\mathbb{L}-1\right)^{2g-1} + 7\mathbb{L}^{18g-18}\left(\mathbb{L}-1\right)^{4g-2} + 8\mathbb{L}^{18g-18}\left(\mathbb{L}-1\right)^{6g-3} + 3\mathbb{L}^{18g-18}\left(\mathbb{L}-1\right)^{8g-4} + \mathbb{L}^{20g-20}\left(\mathbb{L}-1\right)^{2g-1} + 4\mathbb{L}^{20g-20}\left(\mathbb{L}-1\right)^{4g-2} + 6\mathbb{L}^{20g-20}\left(\mathbb{L}-1\right)^{6g-3} + 4\mathbb{L}^{20g-20}\left(\mathbb{L}-1\right)^{8g-4} + \mathbb{L}^{20g-20}\left(\mathbb{L}-1\right)^{10g-5}$. □

The computation times[1] for Z_G^{rep}  for the groups $G = \tilde{\mathbb{T}}_n$ are listed in the table below. These times do not include the diagonalization of Z_G^{rep} , as this was done by hand.

n	2	3	4	5
world time	1.92s	5.17s	1m19s	1h38m
CPU time	2.11s	31.50s	28m12s	50h9m

Finally, we note that precisely the same method can be applied to the groups $G = \mathbb{U}_n$ for $n = 1, \ldots, 5$. In fact, the coefficients F_{ijk} can be reused. For these groups, the map Z_G^{rep}  is given by

Z_G^{rep}  $(\mathbf{1}_{\mathcal{U}_1}) = \sum_{i,j,k=1}^{M} C_{ik}\overline{E}_{kj}[\mathrm{Orbit}(\xi_j)]/[\mathcal{U}_i] \cdot \mathbf{1}_{\mathcal{U}_i}$

Z_G^{rep}  $(\mathbf{1}_{\mathcal{U}_j}) = \sum_{i,k,\ell,m=1}^{M} C_{im}C_{j\ell}\overline{F}_{m\ell k}\, c_! \left(Z_G^{\mathrm{rep}}\right.$  $\left.(\mathbf{1}_{\mathcal{U}_1})|_{\mathcal{U}_k}\right)/[\mathcal{U}_i] \cdot \mathbf{1}_{\mathcal{U}_i}$

where now $\overline{E}_{ij} = [\{A \in \mathbb{U}_n \mid [A, \xi_j] \in \overline{\mathcal{U}}_i\}]$, and $\overline{F}_{ijk}$ are the same as for $G = \tilde{\mathbb{T}}_n$. Importantly, we still consider the action of $\tilde{\mathbb{T}}_n$ on $\mathbb{U}_n$ by conjugation so that the orbits and stabilizers, such as $\mathrm{Orbit}(\xi_j)$, remain unchanged. This yields the following theorem.

Theorem 6.2.7. *The virtual classes of the $\mathbb{U}_n$-character stacks of Σ_g in the Grothendieck ring of stacks for $n = 2, 3, 4, 5$ are given by*

[1] As performed on an Intel®Xeon®CPU E5-4640 0 @ 2.40GHz. Since the computations were performed in parallel (64 cores), both the world time and the CPU time were recorded.

(i) $[\mathfrak{X}_{\mathbb{U}_2}(\Sigma_g)] = \mathbb{L}^{2g-1}$

(ii) $[\mathfrak{X}_{\mathbb{U}_3}(\Sigma_g)] = \mathbb{L}^{4g-4}(\mathbb{L}-1) + \mathbb{L}^{6g-4}$

(iii) $[\mathfrak{X}_{\mathbb{U}_4}(\Sigma_g)] = \mathbb{L}^{8g-7}(\mathbb{L}-1) + \mathbb{L}^{10g-9}(\mathbb{L}-1)(\mathbb{L}+1) + \mathbb{L}^{12g-9}$

(iv) $[\mathfrak{X}_{\mathbb{U}_5}(\Sigma_g)] = \mathbb{L}^{12g-12}(\mathbb{L}-1)^2 + \mathbb{L}^{14g-13}(\mathbb{L}-1)(2\mathbb{L}-1) +$
$\mathbb{L}^{16g-15}(\mathbb{L}-1)(\mathbb{L}+1)(2\mathbb{L}-1) + \mathbb{L}^{18g-16}(\mathbb{L}-1)(2\mathbb{L}+1) + \mathbb{L}^{20g-16}$. □

6.3 Arithmetic method

Let us now consider the arithmetic side of the same story, applying the theory of Section 4.5. That is, we will study the representation theory of the groups $\mathbb{T}_n$ and $\mathbb{U}_n$ over finite fields $\mathbb{F}_q$, and, in particular, we want to determine the dimensions of the irreducible representations of these finite groups G. We will encode these values in the *representation zeta function*

$$\zeta_G(s) = \sum_{\chi \in \hat{G}} \chi(1)^{-s}$$

where $\hat{G}$ denotes the set of irreducible complex characters of G. Theorem 4.5.3 shows that $\zeta_G(s)$ contains precisely enough information about the point count of the G-character groupoid of Σ_g, since the given equation can be rewritten to

$$|\mathfrak{X}_G(\Sigma_g)| = |G|^{-\chi(\Sigma_g)} \zeta_G(-\chi(\Sigma_g)), \tag{6.3}$$

where $\chi(\Sigma_g) = 2 - 2g$ denotes the Euler characteristic of Σ_g. Finally, these point counts will turn out to be polynomial in q, so that by Katz' theorem 4.6.1, these polynomials determine the E-polynomials of the character stacks.

The representation zeta functions of these finite groups will be computed algorithmically. Roughly speaking, the algorithm, which we describe below, computes representation zeta functions recursively by decomposing subgroups of $\mathbb{T}_n$ as semidirect subgroups $N \rtimes H$ with $H \subseteq \mathbb{T}_{n-1}$ and $N \subseteq \mathbb{G}_a^{n-1}$. Hence, let us recall how the representation theory of semidirect products is related to that of its factors.

Semidirect products

Consider a finite group $G = N \rtimes H$, with $N \subseteq G$ an abelian normal subgroup. Following [Ser77, Section 8.2], we describe how the representation theory of G is related to that of N and H. As N is abelian, its irreducible representations are one-dimensional and given by $X = \mathrm{Hom}(N, \mathbb{C}^\times)$. The group H acts on X via

$$(h \cdot \chi)(n) = \chi(h^{-1} n h) \quad \text{for all } \chi \in X,\ h \in H \text{ and } n \in N.$$

Let $(\chi_i)_{i\in X/H}$ be a collection of representatives for the orbits in X under H. For each $i \in X/H$, let $H_i = \{h \in H \mid h \cdot \chi_i = \chi_i\}$ denote the stabilizer of χ_i, and let $G_i = N \rtimes H_i \subseteq G$ be the corresponding subgroup of G. We can extend χ_i to G_i by setting

$$\chi_i((n,h)) = \chi_i(n) \quad \text{for all } n \in N \text{ and } h \in H_i.$$

Indeed, this defines a (1-dimensional) character of G_i as

$$\begin{aligned}\chi_i((n_1,h_1)(n_2,h_2)) &= \chi_i((n_1(h_1 n_2 h_1^{-1}), h_1 h_2)) \\ &= \chi_i(n_1 h_1 n_2 h_1^{-1}) = \chi_i(n_1)\chi_i(n_2) = \chi_i((n_1,h_1))\chi_i((n_2,h_2))\end{aligned}$$

for all $n_1, n_2 \in N$ and $h_1, h_2 \in H_i$. Now, any irreducible representation ρ of H_i induces a representation $\tilde{\rho}$ of G_i by composing with the projection $G_i \to G_i/N = H_i$, and we define

$$\theta_{i,\rho} = \mathrm{Ind}_{G_i}^{G}\left(\chi_i \otimes \tilde{\rho}\right).$$

It turns out that these are precisely all the irreducible representations of G.

Proposition 6.3.1 ([Ser77, Proposition 25]). *(i) $\theta_{i,\rho}$ is irreducible.*

(ii) If $\theta_{i,\rho}$ is isomorphic to $\theta_{i',\rho'}$, then $i = i'$ and ρ is isomorphic to ρ'.

(iii) Every irreducible representation of G is isomorphic to some $\theta_{i,\rho}$.

In terms of representation zeta functions, this proposition translates to the following corollary, using the fact that $\dim\left(\mathrm{Ind}_H^G(\rho)\right) = \dim(\rho)[G:H]$.

Corollary 6.3.2. *The representation zeta function of G is given by*

$$\zeta_G(s) = \sum_{i\in X/H} \zeta_{H_i}(s)[G:G_i]^{-s} = \sum_{i\in X/H} \zeta_{H_i}(s)[H:H_i]^{-s}. \qquad \square$$

Decomposing triangles

Consider the group $\mathbb{U}_n(\mathbb{F}_q) = \{A \in \mathrm{GL}_n(\mathbb{F}_q) \mid A_{ii} = 1 \text{ and } A_{ij} = 0 \text{ for } i > j\}$ of $n \times n$ unipotent upper triangular matrices over a finite field $\mathbb{F}_q$. Let N be the kernel of

$$\mathbb{U}_n(\mathbb{F}_q) \to \mathbb{U}_{n-1}(\mathbb{F}_q), \quad A \mapsto (A_{ij})_{i,j=1}^{n-1},$$

so that the quotient $\mathbb{U}_n(\mathbb{F}_q)/N$ is isomorphic to $\mathbb{U}_{n-1}(\mathbb{F}_q)$. Now we have a split exact sequence

$$1 \longrightarrow N \longrightarrow \mathbb{U}_n(\mathbb{F}_q) \underset{\curvearrowleft}{\longrightarrow} \mathbb{U}_{n-1}(\mathbb{F}_q) \longrightarrow 1,$$

which yields a semidirect decomposition $\mathbb{U}_n(\mathbb{F}_q) = N \rtimes \mathbb{U}_{n-1}(\mathbb{F}_q)$, where N is abelian. Moreover, for any unipotent subgroup $U \subseteq \mathbb{U}_n(\mathbb{F}_q)$, the above exact sequence can be intersected with U to obtain

$$1 \longrightarrow U \cap N \longrightarrow U \longrightarrow U \cap \mathbb{U}_{n-1}(\mathbb{F}_q) \longrightarrow 1,$$

yielding a semidirect decomposition $U = (U \cap N) \rtimes (U \cap \mathbb{U}_{n-1}(\mathbb{F}_q))$.

We identify $N \cong \mathbb{G}_a^{n-1}(\mathbb{F}_q) = \mathbb{F}_q^{n-1}$, where $\mathbb{F}_q$ as additive group is equal to $(\mathbb{Z}/p\mathbb{Z})^m$ for $q = p^m$. The irreducible characters $\chi_\alpha \in X = \mathrm{Hom}(N, \mathbb{C}^\times)$ of N can now also be identified with vectors $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{F}_q^{n-1}$, via

$$\chi_\alpha(x) = \zeta_p^{\langle \alpha, x\rangle} \qquad \text{for all } x \in N,$$

where ζ_p is a primitive p^{th} root of unity, and $\langle -, - \rangle$ denotes the trace form given by $\langle \alpha, x\rangle = \sum_{i=1}^{n-1} \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\alpha_i x_i) \in \mathbb{F}_p$, which is a non-degenerate bilinear form. Since $\mathbb{U}_{n-1}(\mathbb{F}_q)$ acts on $N \cong \mathbb{F}_q^{n-1}$ by left multiplication, it acts on $X \cong \mathbb{F}_q^{n-1}$ by right multiplication, because $\langle \alpha A, x\rangle = \langle \alpha, Ax\rangle$ for all $\alpha, x \in \mathbb{F}_q^{n-1}$ and $A \in \mathrm{GL}_{n-1}(\mathbb{F}_q)$.

From now on, to unclutter the notation, we will omit the field $\mathbb{F}_q$ from the group, simply writing G instead of $G(\mathbb{F}_q)$, and $\mathbb{U}_n$ instead of $\mathbb{U}_n(\mathbb{F}_q)$, etc.

Example 6.3.3. Consider the group $\mathbb{U}_3 \cong \mathbb{G}_a^2 \rtimes \mathbb{U}_2$. As discussed above, $H = \mathbb{U}_2$ acts on $X = \mathrm{Hom}(\mathbb{G}_a^2, \mathbb{C}^\times) \cong \mathbb{G}_a^2$ by right-multiplication, that is,

$$\begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta + a\alpha \end{pmatrix} \quad \text{for all } \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in H \text{ and } \begin{pmatrix} \alpha & \beta \end{pmatrix} \in X.$$

Hence, the orbits in X under H are given by $\left\{ \begin{pmatrix} \alpha & \beta \end{pmatrix} : \beta \in \mathbb{F}_q \right\}$ for all $\alpha \in \mathbb{F}_q^\times$ and $\left\{ \begin{pmatrix} 0 & \beta \end{pmatrix} \right\}$ for all $\beta \in \mathbb{F}_q$. We choose the following representatives:

- $\chi_\alpha = \begin{pmatrix} \alpha & 0 \end{pmatrix}$, for which $H_\alpha = \{1\}$, so the contribution to the zeta function is

$$(q-1)\, \zeta_{\{1\}}(s)\, [H : H_\alpha]^{-s} = (q-1)\, q^{-s}.$$

- $\chi_\beta = \begin{pmatrix} 0 & \beta \end{pmatrix}$, for which $H_\beta = \mathbb{U}_2$, so the contribution to the zeta function is

$$q\, \zeta_{\mathbb{U}_2}(s)\, [H : H_\beta]^{-s} = q^2.$$

Adding up the contributions, it follows from Corollary 6.3.2 that $\zeta_{\mathbb{U}_3}(s) = q^2 + (q-1)\, q^{-s}$.

Example 6.3.4. Consider $\mathbb{U}_4 \cong \mathbb{G}_a^3 \rtimes \mathbb{U}_3$, for which $X = \mathrm{Hom}(\mathbb{G}_a^3, \mathbb{C}^\times) \cong \mathbb{G}_a^3$, and $H = \mathbb{U}_3$ acts on $\begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix} \in X$ by right-multiplication, that is,

$$\begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix}\begin{pmatrix}1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}\alpha & \beta + a\alpha & \gamma + b\alpha + c\beta\end{pmatrix}.$$

Hence, the orbits in X under H are given by $\left\{\begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix} : \beta, \gamma \in \mathbb{F}_q\right\}$ for all $\alpha \in \mathbb{F}_q^\times$, $\left\{\begin{pmatrix}0 & \beta & \gamma\end{pmatrix} : \gamma \in \mathbb{F}_q\right\}$ for all $\beta \in \mathbb{F}_q^\times$, and $\left\{\begin{pmatrix}0 & 0 & \gamma\end{pmatrix}\right\}$ for all $\gamma \in \mathbb{F}_q$. We choose the following representatives:

- $\chi_\alpha = \begin{pmatrix}\alpha & 0 & 0\end{pmatrix}$ with $H_\alpha \cong \mathbb{G}_a$, contributing
$$(q-1)\,\zeta_{\mathbb{G}_a}(s)\,[H : H_\alpha]^{-s} = q^{1-2s}(q-1).$$
- $\chi_\beta = \begin{pmatrix}0 & \beta & 0\end{pmatrix}$ with $H_\beta \cong \mathbb{G}_a^2$, contributing
$$(q-1)\,\zeta_{\mathbb{G}_a^2}(s)\,[H : H_\beta]^{-s} = q^{2-s}(q-1).$$
- $\chi_\gamma = \begin{pmatrix}0 & 0 & \gamma\end{pmatrix}$ with $H_\gamma = \mathbb{U}_3$, contributing
$$q\,\zeta_{\mathbb{U}_3}(s)\,[H : H_\gamma]^{-s} = q^3 + (q-1)\,q^{1-s}.$$

In total, $\zeta_{\mathbb{U}_4}(s) = q^3 + q^{1-s}(q-1)(q+1) + q^{1-2s}(q-1)$.

The construction as described above can be applied more generally to any connected algebraic subgroup $G \subseteq \mathbb{T}_n$ as follows. Let G' be the image of the map $G \to \tilde{\mathbb{T}}_n$ given by $A \mapsto A/A_{nn}$. Then either $G \cong G'$ or $G \cong \mathbb{G}_m \times G'$, because the only connected subgroups of $\mathbb{G}_m$ are $\{1\}$ and $\mathbb{G}_m$ itself. Since $\zeta_{\mathbb{G}_m}(s) = q - 1$ is known, we may assume $G \subseteq \tilde{\mathbb{T}}_n$. The group $\tilde{\mathbb{T}}_n$ can be decomposed, similar to $\mathbb{U}_n$, as

$$1 \longrightarrow \mathbb{G}_a^{n-1} \longrightarrow \tilde{\mathbb{T}}_n \longrightarrow \mathbb{T}_{n-1} \longrightarrow 1,$$

where the map $\tilde{\mathbb{T}}_n \to \mathbb{T}_{n-1}$ is given by $A \mapsto (A_{ij})_{i,j=1}^{n-1}$. Intersecting with G, we obtain $G = N \rtimes H$ with $N = G \cap \mathbb{G}_a^{n-1}$ abelian and $H = G \cap \mathbb{T}_{n-1}$, to which we can apply Corollary 6.3.2.

Example 6.3.5. Consider $G = \mathbb{T}_2 \cong \mathbb{G}_m \times \tilde{\mathbb{T}}_2$ with $\tilde{\mathbb{T}}_2 \cong \mathbb{G}_a \rtimes \mathbb{G}_m$, for which $X = \mathrm{Hom}(\mathbb{G}_a, \mathbb{C}^\times) \cong \mathbb{G}_a$ and $H = \mathbb{G}_m$ acts on $\alpha \in X$ by multiplication. Hence, the orbits in X under H are given by $\{0\}$ and $\{\alpha : \alpha \in \mathbb{F}_q^\times\}$. We choose the following representatives:

- $\chi_0 = 0$ yields $H_0 = \mathbb{G}_m$, contributing $\zeta_{\mathbb{G}_m}(s) = q - 1$,

- $\chi_1 = 1$ yields $H_1 = \{1\}$, contributing $(q-1)^{-s}$.

In total,

$$\zeta_{\mathbb{T}_2}(s) = \zeta_{\mathbb{G}_m}(s)\,\zeta_{\tilde{\mathbb{T}}_2}(s) = (q-1)((q-1)+(q-1)^{-s}) = (q-1)^2 + (q-1)^{1-s}.$$

These examples illustrate how one computes the representation zeta function in a recursive manner using Proposition 6.3.1. Note that in all examples, the stabilizers H_α of χ_α are independent of α (and similarly for H_β, H_γ, ...). However, the following example shows that this need not always be the case: we obtain a family of stabilizers $H_{\alpha,\beta}$ which depend explicitly on the parameters α and β.

Example 6.3.6. Consider $G = \mathbb{G}_a^3 \rtimes H$ with $H = \left\{ \left(\begin{smallmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{smallmatrix} \right) \right\}$ acting naturally on $\mathbb{G}_a^3$. Then H acts on $X = \mathrm{Hom}(\mathbb{G}_a^3, \mathbb{C}^\times) \cong \mathbb{G}_a^3$ by

$$\begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma + a\alpha + b\beta \end{pmatrix}.$$

Hence, the orbits in X under H are given by $\left\{ \begin{pmatrix} 0 & 0 & \gamma \end{pmatrix} \right\}$ for all $\gamma \in \mathbb{F}_q$ and $\left\{ \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} : \gamma \in \mathbb{F}_q \right\}$ for all $\alpha, \beta \in \mathbb{F}_q$ with $(\alpha, \beta) \neq (0,0)$. We choose the following representatives:

- $\chi_\gamma = \begin{pmatrix} 0 & 0 & \gamma \end{pmatrix}$ with $H_\gamma = H$, contributing $q\,\zeta_H(s) = q^3$,
- $\chi_{\alpha,\beta} = \begin{pmatrix} \alpha & \beta & 0 \end{pmatrix}$ with $H_{\alpha,\beta} = \left\{ \left(\begin{smallmatrix} 1 & 0 & x\beta \\ 0 & 1 & -x\alpha \\ 0 & 0 & 1 \end{smallmatrix} \right) : x \in \mathbb{F}_q \right\}$. Note that $H_{\alpha,\beta}$ depends explicitly on α and β, even though $H_{\alpha,\beta} \cong \mathbb{G}_a$ for all α and β. These representatives contribute

$$(q^2-1)\,\zeta_{\mathbb{G}_a}(s)\,[H : H_{\alpha,\beta}]^{-s} = q^{1-s}(q-1)(q+1).$$

In total, $\zeta_G(s) = q^3 + q^{1-s}(q-1)(q+1)$.

Algorithmically computing $\zeta_G(s)$

Now we will describe an algorithm to compute $\zeta_G(s)$ for connected algebraic groups $G \subseteq \mathbb{T}_n$, in the style of examples 6.3.3, 6.3.4 and 6.3.6. An implementation of this algorithm can be found at [Vog22], together with the code for computing $\zeta_{\mathbb{U}_n}(s)$ and $\zeta_{\mathbb{T}_n}(s)$ for $n = 1, \ldots, 10$. The resulting zeta functions are given in Theorem 6.3.11 and Theorem 6.3.10.

Before discussing the algorithm, let us give some remarks.

The algorithm is divided into two parts. The main part, Algorithm 6.3.7, finds a semidirect decomposition $G \cong N \rtimes H$ and applies Corollary 6.3.2 in order to compute $\zeta_G(s)$. Finding representatives for the orbits in X under H is a more intricate step, and is described separately in Algorithm 6.3.8.

As highlighted in Example 6.3.6, it is possible for the stabilizers H_α to depend explicitly on the parameter α. Therefore, in order for the algorithm to work recursively, we allow the input of the algorithm to be a family G of algebraic groups $G_t \subseteq \mathbb{T}_n$ parametrized by a variety T over $\mathbb{F}_q$, that is, a subgroup $G \subseteq \mathbb{T}_n \times T$ over T. We then understand the representation zeta function of G to be

$$\zeta_G(s) = \sum_{t \in T(\mathbb{F}_q)} \zeta_{G_t}(s).$$

As we want the computations to hold over general a ground field $\mathbb{F}_q$, we will in practice work over $\mathbb{Z}$. Then $|T(\mathbb{F}_q)|$ can be computed as a polynomial in q whenever $[T] \in \mathrm{K}_0(\mathbf{Var}_{\mathbb{Z}})$ can be computed as a polynomial in $q = [\mathbb{A}^1_{\mathbb{Z}}]$ using Algorithm 3.4.3.

There are steps in the algorithm containing conditions that depend on the value of $t \in T$. At such steps, we stratify T into the stratum where the condition holds and the stratum where it does not hold, and continue the algorithm on both strata separately.

Algorithm 6.3.7. Input: A family of connected algebraic groups $G \subseteq \mathbb{T}_n \times T$ over a variety T.

Output: The representation zeta function $\zeta_G(s)$ as a polynomial in q, q^{-s} and $(q-1)^{-s}$.

1. If $n = 0$, then G is trivial, so that $\zeta_G(s) = |T(\mathbb{F}_q)|$. Hence, we can assume $n \geq 1$.

2. Since G is connected, the image of the map $G \to \mathbb{G}_m^n \times T$ given by $(A, t) \mapsto ((A_{ii})_{i=1}^n, t)$ is isomorphic to $\mathbb{G}_m^d \times T$ for some $0 \leq d \leq n$, at least locally on T, so after stratifying T we can assume this to be the case. If $d = n$, then there is an isomorphism $G \cong \mathbb{G}_m \times G'$ with $G' \subseteq \tilde{\mathbb{T}}_n \times T$ given by $(A, t) \mapsto (A_{nn}, (A/A_{nn}, t))$, so that $\zeta_G(s) = (q-1)\,\zeta_{G'}(s)$. If $d < n$, then $G \cong G' \subseteq \tilde{\mathbb{T}}_n$ via the map $A \mapsto A/A_{nn}$. Either way, we can assume $G \subseteq \tilde{\mathbb{T}}_n \times T$.

3. Write $G = N \rtimes H$ as discussed above. The group $H \subseteq \mathbb{T}_{n-1} \times T$ can be obtained as the group of minors $H = \left\{((A_{ij})_{i,j=1}^{n-1}, t) : (A, t) \in G\right\}$, and N can be obtained as the closed subgroup of G given by $A_{ij} = 0$ for $1 \leq i < j \leq n-1$ and $A_{ii} = 1$ for $1 \leq i \leq n-1$.

4. Identify $N \cong \mathbb{G}_a^r \times T$ for some $0 \leq r \leq n-1$, possibly after stratifying T. Consider induced action of H on the space of characters $X = \mathrm{Hom}_T(N, \mathbb{G}_m \times T) \cong \mathbb{G}_a^r \times T$.

5. Use Algorithm 6.3.8 to find families of representatives $\chi_i \colon T_i \to X$, parametrized by varieties T_i over T, for the orbits in X under H, together with their stabilizer $H_i \subseteq H \times_T T_i$ over T_i and index $[H \times_T T_i : H_i]$.

6. Repeat the algorithm to compute $\zeta_{H_i}(s)$ for all i, from which $\zeta_G(s)$ can be computed using Corollary 6.3.2.

Algorithm 6.3.8. Input: A family of connected algebraic groups $H \subseteq \mathbb{T}_n \times T$ over a variety T, acting linearly on a subvariety $X \subseteq \mathbb{G}_a^r \times T$ over T. Write $\alpha_1, \ldots, \alpha_r$ for the coordinates on $\mathbb{G}_a^r$.

Output: A stratification of X by H-invariant locally closed subvarieties X_i; families of representatives $\chi_i \colon T_i \to X_i$ with T_i varieties over T; the stabilizers $H_i \subseteq H \times_T T_i$ of the χ_i; such that the index $[H \times_T T_i : H_i]$ is polynomial in q.

1. Repeat steps 2 and 3 until H acts trivially on X. Then $\chi := \mathrm{id}_X \colon X \to X$ is a family of representatives, with stabilizer $H \times_T X$ and index $[H \times_T X : H \times_T X] = 1$. If both step 2 and 3 do not apply, fail.

2. If $\alpha_i \overset{H}{\mapsto} a\alpha_i$ for some coordinate a on H, then a must be a diagonal entry of H. Stratify X based on α_i:

   (i) Case $\alpha_i = 0$. Continue with the action of H restricted to the closed subvariety $X' = X \cap \{\alpha_i = 0\}$.

   (ii) Case $\alpha_i \neq 0$. By choosing representatives with $\alpha_i = 1$, we can replace X by $X' = X \cap \{\alpha_i = 1\}$ and H by $H' = H \cap \{a = 1\}$. Continue with the action of H' on X', and keep track of the index $[H : H'] = q - 1$. In the end, compose the families of representatives $\chi_i' \colon T_i \to X_i'$ with the inclusion $X_i' \to X_i = H \cdot X_i'$.

3. Write $\alpha_i \overset{H}{\mapsto} \sum_j a_j f_j$, where a_j are coordinates on H and f_j are functions on X which are not identically zero. If some f_ℓ is invariant under the action of H, then stratify X based on f_ℓ:

   (i) Case $f_\ell = 0$. Continue with the action of H restricted to the closed subvariety $X' = X \cap \{f_\ell = 0\}$.

   (ii) Case $f_\ell \neq 0$. By choosing representatives with $\alpha_i = 0$, we can replace X by $X' = X \cap \{\alpha_i = 0, f_\ell \neq 0\}$ and H by $H' = H \cap \{a_\ell = -f_\ell^{-1} \sum_{j \neq \ell} a_j f_j\}$. Continue with the action of H' on X', and keep track of the index $[H : H'] = q-1$. In the end, compose the families of representatives $\chi_i' \colon T_i \to X_i'$ with the inclusion $X_i' \to X_i = H \cdot X_i'$.

Remark 6.3.9. Unfortunately, this algorithm might possibly fail. In fact, if this algorithm were to never fail, then the representation zeta function $\zeta_G(s)$ is always a polynomial in q, q^{-s} and $(q-1)^{-s}$. Then, evaluating at $s = 0$, this would imply that the number of conjugacy classes of G is a polynomial in q. In particular, this would imply Higman's Conjecture [PS15, Conjecture 1.1]. For us, the algorithm does not fail when applied to $G = \mathbb{U}_n$ or $G = \mathbb{T}_n$ for $n = 1, \dots, 10$.

Results

The representation zeta functions of $\mathbb{U}_n$ and $\mathbb{T}_n$ were computed using Algorithm 6.3.7, and are presented in Theorem 6.3.10 and Theorem 6.3.11 below. One can evaluate these zeta functions at $s = 0$ in order to obtain the number of conjugacy classes of the groups over finite fields $\mathbb{F}_q$. For $G = \mathbb{U}_n$, the resulting polynomials in q can be seen to agree with [PS15, Appendix A], where $t = q-1$. In this sense, these zeta functions are a generalization of the polynomials $k(\mathbb{U}_n(\mathbb{F}_q))$ as in [PS15]. Furthermore, the E-polynomials of $R_{\mathbb{U}_n}(\Sigma_g)$ and $R_{\mathbb{T}_n}(\Sigma_g)$ over $k = \mathbb{C}$ can be obtained through Theorem 4.6.1 and (6.3). Indeed, one can verify that for $1 \le n \le 5$ these E-polynomials agree with the virtual classes as given by Theorem 6.2.6 and Theorem 6.2.7, via the map (3.6).

Theorem 6.3.10. *The representation zeta functions $\zeta_{\mathbb{U}_n}(s)$ for $n = 1, \dots, 10$ are given by*

(i) $\zeta_{\mathbb{U}_1}(s) = 1$

(ii) $\zeta_{\mathbb{U}_2}(s) = q$

(iii) $\zeta_{\mathbb{U}_3}(s) = q^{-s}(q-1) + q^2$

(iv) $\zeta_{\mathbb{U}_4}(s) = q^{1-s}(q-1)(q+1) + q^{1-2s}(q-1) + q^3$

(v) $\zeta_{\mathbb{U}_5}(s) = q^{1-2s}(q-1)(q+1)(2q-1) + q^{2-s}(q-1)(2q+1) + q^{1-3s}(q-1)(2q-1) + q^{-4s}(q-1)^2 + q^4$

(vi) $\zeta_{\mathbb{U}_6}(s) = q^{2-2s}(q-1)(q+2)(q^2+q-1) + q^{2-3s}(q-1)(q+1)(4q-3) + q^{-4s}(q-1)(2q^2-1)(q^2+q-1) + q^{3-s}(q-1)(3q+1) + q^{1-5s}(q-1)^2(2q+1) + q^{1-6s}(q-1)^2 + q^5$

(vii) $\zeta_{\mathbb{U}_7}(s) = q^{3-2s}(q-1)(q+1)(2q^2+3q-3) + q^{1-4s}(q-1)(2q-1)(q^4+5q^3-3q-1) + q^{4-s}(q-1)(4q+1) + q^{2-3s}(q-1)(3q^4+6q^3-2q^2-5q+1) + q^{1-5s}(q-1)(q^5+7q^4-2q^3-9q^2+3q+1) + q^{1-6s}(q-1)^2(4q^3+7q^2-3q-1) + q^{1-8s}(q-1)^2(3q-2) + q^{-7s}(q-1)^2(5q^3-3q+1) + q^{-9s}(q-1)^3 + q^6$

(viii) $\zeta_{\mathbb{U}_8}(s) = q^{4-2s}(q-1)(3q+2)(q^2+2q-2) + q^{5-s}(q-1)(5q+1) + q^{3-3s}(q-1)(q^5+5q^4+10q^3-7q^2-8q+3) + q^{3-6s}(q-1)(q^5+7q^4+16q^3-24q^2-14q+15) + q^{2-4s}(q-1)(12q^5+9q^4-16q^3-9q^2+6q+1) + q^{1-5s}(q-1)(2q^7+8q^6+13q^5-23q^4-9q^3+12q^2-1) + q^{1-7s}(q-1)^2(6q^5+18q^4+4q^3-19q^2+q+3) + q^{1-8s}(q-1)^2(q^5+13q^4+8q^3-14q^2-4q+3) + q^{1-11s}(q-1)^3(3q+1) + q^{-9s}(q-1)^2(4q^5+10q^4-7q^3-8q^2+3q+1) + q^{-10s}(q-1)^2(5q^4+q^3-6q^2+1) + q^{1-12s}(q-1)^3 + q^7$

(ix) $\zeta_{\mathbb{U}_9}(s) = q^{5-2s}(q-1)(2q+1)(2q^2+5q-5) + q^{6-s}(q-1)(6q+1) + q^{4-3s}(q-1)(2q^5+9q^4+14q^3-15q^2-11q+6) + q^{4-4s}(q-1)(4q^5+19q^4+11q^3-34q^2-10q+14) + q^{2-5s}(q-1)(q^8+5q^7+29q^6+q^5-53q^4-2q^3+27q^2-3q-2) + q^{2-6s}(q-1)(10q^7+33q^6-9q^5-68q^4+10q^3+38q^2-11q-1) + q^{1-7s}(q-1)(2q^9+8q^8+27q^7+2q^6-87q^5+20q^4+46q^3-15q^2-3q+1) + q^{1-8s}(q-1)^2(9q^7+33q^6+40q^5-45q^4-40q^3+21q^2+5q-1) + q^{1-9s}(q-1)^2(2q^7+30q^6+42q^5-44q^4-48q^3+25q^2+7q-1) + q^{1-11s}(q-1)^2(4q^6+25q^5+5q^4-48q^3+7q^2+9q+1) + q^{1-12s}(q-1)^2(10q^5+18q^4-32q^3-10q^2+18q-3) + q^{1-15s}(q-1)^3(4q-3) + q^{-10s}(q-1)^2(2q^8+13q^7+38q^6-24q^5-49q^4+20q^3+11q^2-3q-1) + q^{-13s}(q-1)^3(12q^4+10q^3-13q^2+q+1) + q^{-14s}(q-1)^3(9q^3-2q^2-5q+2) + q^{-16s}(q-1)^4 + q^8$

(x) $\zeta_{\mathbb{U}_{10}}(s) = q^{6-2s}(q-1)(5q+2)(q^2+3q-3) + q^{7-s}(q-1)(7q+1) + q^{5-3s}(q-1)(3q^5+15q^4+19q^3-28q^2-13q+10) + q^{4-4s}(q-1)(q^7+7q^6+32q^5+12q^4-65q^3-6q^2+27q-3) + q^{3-5s}(q-1)(2q^8+21q^7+42q^6-16q^5-103q^4+24q^3+50q^2-13q-3) + q^{2-6s}(q-1)(6q^9+27q^8+64q^7-73q^6-118q^5+64q^4+70q^3-39q^2+q+1) + q^{2-7s}(q-1)(2q^{10}+5q^9+39q^8+74q^7-130q^6-133q^5+128q^4+74q^3-60q^2+2q+1) + q^{2-8s}(q-1)(q^{10}+12q^9+39q^8+67q^7-137q^6-172q^5+200q^4+63q^3-80q^2+2q+6) + q^{2-9s}(q-1)^2(10q^8+65q^7+117q^6-36q^5-221q^4+18q^3+98q^2-11q-6) + q^{1-11s}(q-1)^2(6q^9+31q^8+109q^7+8q^6-240q^5-10q^4+135q^3-17q^2-8q-1) + q^{1-12s}(q-1)^2(2q^9+22q^8+77q^7+46q^6-217q^5-48q^4+156q^3-12q^2-20q+1) + q^{1-13s}(q-1)^2(10q^8+50q^7+60q^6-138q^5-110q^4+146q^3+8q^2-25q+2) + q^{1-15s}(q-1)^3(4q^6+42q^5+46q^4-51q^3-44q^2+23q+5) + q^{1-19s}(q-1)^4(4q+1) + q^{-10s}(q-1)^2(2q^{11}+8q^{10}+50q^9+112q^8-29q^7-227q^6+17q^5+123q^4-24q^3-12q^2+q+1) + q^{-14s}(q-1)^2(2q^9+24q^8+53q^7-52q^6-127q^5+84q^4+49q^3-32q^2-3q+3) + q^{-16s}(q-1)^3(10q^6+37q^5-9q^4-42q^3+6q^2+10q-1) + q^{-17s}(q-1)^3(12q^5+14q^4-21q^3-8q^2+6q+1) + q^{-18s}(q-1)^3(9q^4-q^3-9q^2+q+1) + q^{1-20s}(q-1)^4 + q^9.$ □

Theorem 6.3.11. *The representation zeta functions $\zeta_{\mathbb{T}_n}(s)$ for $n = 1, \dots, 10$ are given by*

(i) $\zeta_{\mathbb{T}_1}(s) = q - 1$

(ii) $\zeta_{\mathbb{T}_2}(s) = (q-1)^{1-s} + (q-1)^2$

(iii) $\zeta_{\mathbb{T}_3}(s) = q^{-s}(q-1)^{2-s} + 2(q-1)^{2-s} + (q-1)^{1-2s} + (q-1)^3$

(iv) $\zeta_{\mathbb{T}_4}(s) = 3q^{-s}(q-1)^{2-2s} + 2q^{-s}(q-1)^{3-s} + q^{-2s}(q-1)^{3-s} + q^{-2s}(q-1)^{2-2s} + q^{-s}(q-1)^{1-3s} + 3(q-1)^{3-s} + 3(q-1)^{2-2s} + (q-1)^{1-3s} + (q-1)^4$

(v) $\zeta_{\mathbb{T}_5}(s) = 8q^{-s}(q-1)^{3-2s} + 7q^{-2s}(q-1)^{3-2s} + 7q^{-2s}(q-1)^{2-3s} + 7q^{-s}(q-1)^{2-3s} + 3q^{-3s}(q-1)^{3-2s} + 3q^{-s}(q-1)^{4-s} + 2q^{-2s}(q-1)^{4-s} + 2q^{-2s}(q-1)^{1-4s} + 2q^{-3s}(q-1)^{2-3s} + 2q^{-s}(q-1)^{1-4s} + q^{-3s}(q-1)^{4-s} + q^{-4s}(q-1)^{3-2s} + 6(q-1)^{3-2s} + 4(q-1)^{4-s} + 4(q-1)^{2-3s} + (q-1)^{1-4s} + (q-1)^5$

(vi) $\zeta_{\mathbb{T}_6}(s) = q^{-2s}(q-1)^{1-5s}(q+7) + 29q^{-2s}(q-1)^{3-3s} + 24q^{-3s}(q-1)^{3-3s} + 23q^{-2s}(q-1)^{2-4s} + 21q^{-s}(q-1)^{3-3s} + 17q^{-3s}(q-1)^{2-4s} + 16q^{-2s}(q-1)^{4-2s} + 15q^{-4s}(q-1)^{3-3s} + 15q^{-s}(q-1)^{4-2s} + 13q^{-3s}(q-1)^{4-2s} + 13q^{-s}(q-1)^{2-4s} + 10q^{-4s}(q-1)^{2-4s} + 7q^{-4s}(q-1)^{4-2s} + 5q^{-5s}(q-1)^{3-3s} + 4q^{-3s}(q-1)^{1-5s} + 4q^{-s}(q-1)^{5-s} + 3q^{-2s}(q-1)^{5-s} + 3q^{-5s}(q-1)^{4-2s} + 3q^{-s}(q-1)^{1-5s} + 2q^{-3s}(q-1)^{5-s} + 2q^{-4s}(q-1)^{1-5s} + 2q^{-5s}(q-1)^{2-4s} + q^{-4s}(q-1)^{5-s} + q^{-6s}(q-1)^{4-2s} + q^{-6s}(q-1)^{3-3s} + 10(q-1)^{4-2s} + 10(q-1)^{3-3s} + 5(q-1)^{5-s} + 5(q-1)^{2-4s} + (q-1)^{1-5s} + (q-1)^6$

(vii) $\zeta_{\mathbb{T}_7}(s) = 2q^{-4s}(q-1)^{1-6s}(q+8) + q^{-2s}(q-1)^{2-5s}(2q+53) + q^{-2s}(q-1)^{1-6s}(2q+13) + q^{-3s}(q-1)^{2-5s}(2q+71) + q^{-3s}(q-1)^{1-6s}(3q+19) + q^{-4s}(q-1)^{2-5s}(3q+67) + q^{-5s}(q-1)^{1-6s}(q+12) + 107q^{-3s}(q-1)^{3-4s} + 104q^{-4s}(q-1)^{3-4s} + 87q^{-2s}(q-1)^{3-4s} + 79q^{-3s}(q-1)^{4-3s} + 73q^{-4s}(q-1)^{4-3s} + 73q^{-5s}(q-1)^{3-4s} + 71q^{-2s}(q-1)^{4-3s} + 49q^{-5s}(q-1)^{4-3s} + 48q^{-5s}(q-1)^{2-5s} + 46q^{-s}(q-1)^{4-3s} + 44q^{-s}(q-1)^{3-4s} + 42q^{-6s}(q-1)^{3-4s} + 30q^{-6s}(q-1)^{4-3s} + 28q^{-2s}(q-1)^{5-2s} + 27q^{-3s}(q-1)^{5-2s} + 24q^{-s}(q-1)^{5-2s} + 23q^{-6s}(q-1)^{2-5s} + 22q^{-4s}(q-1)^{5-2s} + 21q^{-s}(q-1)^{2-5s} + 15q^{-7s}(q-1)^{3-4s} + 13q^{-5s}(q-1)^{5-2s} + 12q^{-7s}(q-1)^{4-3s} + 7q^{-6s}(q-1)^{5-2s} + 5q^{-7s}(q-1)^{2-5s} + 5q^{-s}(q-1)^{6-s} + 4q^{-2s}(q-1)^{6-s} + 4q^{-6s}(q-1)^{1-6s} + 4q^{-8s}(q-1)^{4-3s} + 4q^{-s}(q-1)^{1-6s} + 3q^{-3s}(q-1)^{6-s} + 3q^{-7s}(q-1)^{5-2s} + 3q^{-8s}(q-1)^{3-4s} + 2q^{-4s}(q-1)^{6-s} + q^{-5s}(q-1)^{6-s} + q^{-8s}(q-1)^{5-2s} + q^{-9s}(q-1)^{4-3s} + 20(q-1)^{4-3s} + 15(q-1)^{5-2s} + 15(q-1)^{3-4s} + 6(q-1)^{6-s} + 6(q-1)^{2-5s} + (q-1)^{1-6s} + (q-1)^7$

(viii) $\zeta_{\mathbb{T}_8}(s) = 14q^{-3s}(q-1)^{2-6s}(q+14) + 6q^{-7s}(q-1)^{1-7s}(q+7) + 4q^{-3s}(q-1)^{3-5s}(q+87) + 2q^{-2s}(q-1)^{2-6s}(3q+50) + 2q^{-7s}(q-1)^{2-6s}(3q+89) + q^{-2s}(q-1)^{3-5s}(3q+208) + q^{-2s}(q-1)^{1-7s}(3q+20) + q^{-3s}(q-1)^{1-7s}(q^2+11q+47) + q^{-4s}(q-1)^{3-5s}(9q+457) + q^{-4s}(q-1)^{2-6s}(20q+261) + q^{-4s}(q-1)^{1-7s}(12q+61) + q^{-5s}(q-1)^{3-5s}(6q+485) + q^{-5s}(q-1)^{2-6s}(24q+305) + q^{-5s}(q-1)^{1-7s}(2q^2+20q+79) + q^{-6s}(q-1)^{3-5s}(6q+415) + q^{-6s}(q-1)^{2-6s}(19q+250) + q^{-6s}(q-1)^{1-7s}(q^2+13q+60) + q^{-8s}(q-1)^{2-6s}(3q+85) + q^{-8s}(q-1)^{1-7s}(q+15) + 410q^{-4s}(q-1)^{4-4s} + 398q^{-5s}(q-1)^{4-4s} + 340q^{-6s}(q-1)^{4-4s} + 332q^{-3s}(q-1)^{4-4s} + 297q^{-7s}(q-1)^{3-5s} + 238q^{-7s}(q-1)^{4-4s} + 229q^{-2s}(q-1)^{4-4s} + 192q^{-4s}(q-1)^{5-3s} + 174q^{-3s}(q-1)^{5-3s} + 171q^{-5s}(q-1)^{5-3s} + 168q^{-8s}(q-1)^{3-5s} + 147q^{-8s}(q-1)^{4-4s} + 139q^{-2s}(q-1)^{5-3s} + 136q^{-6s}(q-1)^{5-3s} + 110q^{-s}(q-1)^{4-4s} + 90q^{-7s}(q-1)^{5-3s} + 85q^{-s}(q-1)^{5-3s} + 80q^{-s}(q-1)^{3-5s} + 73q^{-9s}(q-1)^{3-5s} + 71q^{-9s}(q-1)^{4-4s} + 56q^{-8s}(q-1)^{5-3s} + 45q^{-3s}(q-1)^{6-2s} + 43q^{-2s}(q-1)^{6-2s} + 42q^{-4s}(q-1)^{6-2s} + 35q^{-s}(q-1)^{6-2s} + 34q^{-5s}(q-1)^{6-2s} + 31q^{-s}(q-1)^{2-6s} + 30q^{-9s}(q-1)^{2-6s} + 27q^{-10s}(q-1)^{4-4s} + 26q^{-9s}(q-1)^{5-3s} + 22q^{-6s}(q-1)^{6-2s} + 21q^{-10s}(q-1)^{3-5s} + 13q^{-7s}(q-1)^{6-2s} + 11q^{-10s}(q-1)^{5-3s} + 7q^{-8s}(q-1)^{6-2s} + 7q^{-11s}(q-1)^{4-4s} + 6q^{-s}(q-1)^{7-s} + 5q^{-2s}(q-1)^{7-s} + 5q^{-10s}(q-1)^{2-6s} + 5q^{-s}(q-1)^{1-7s} + 4q^{-3s}(q-1)^{7-s} + 4q^{-9s}(q-1)^{1-7s} + 4q^{-11s}(q-1)^{5-3s} + 3q^{-4s}(q-1)^{7-s} + 3q^{-9s}(q-1)^{6-2s} + 3q^{-11s}(q-1)^{3-5s} + 2q^{-5s}(q-1)^{7-s} + q^{-6s}(q-$

$1)^{7-s} + q^{-10s}(q-1)^{6-2s} + q^{-12s}(q-1)^{5-3s} + q^{-12s}(q-1)^{4-4s} + 35(q-1)^{5-3s} + 35(q-1)^{4-4s} + 21(q-1)^{6-2s} + 21(q-1)^{3-5s} + 7(q-1)^{7-s} + 7(q-1)^{2-6s} + (q-1)^{1-7s} + (q-1)^8$

(ix) $\zeta_{\mathbb{T}_9}(s) = 10q^{-8s}(q-1)^{4-5s}(q+186) + 10q^{-8s}(q-1)^{3-6s}(7q+204) + 6q^{-3s}(q-1)^{4-5s}(q+182) + 6q^{-11s}(q-1)^{3-6s}(q+81) + 4q^{-2s}(q-1)^{4-5s}(q+149) + 4q^{-2s}(q-1)^{1-8s}(q+7) + 4q^{-6s}(q-1)^{3-6s}(27q+598) + 4q^{-9s}(q-1)^{3-6s}(11q+377) + 3q^{-7s}(q-1)^{4-5s}(4q+739) + 2q^{-3s}(q-1)^{2-7s}(q^2+23q+212) + 2q^{-4s}(q-1)^{3-6s}(31q+754) + 2q^{-7s}(q-1)^{2-7s}(4q^2+103q+730) + 2q^{-8s}(q-1)^{2-7s}(3q^2+72q+586) + 2q^{-10s}(q-1)^{3-6s}(7q+491) + 2q^{-11s}(q-1)^{1-8s}(2q+19) + 2q^{-12s}(q-1)^{2-7s}(q+38) + q^{-2s}(q-1)^{3-6s}(12q+425) + q^{-2s}(q-1)^{2-7s}(12q+167) + q^{-3s}(q-1)^{3-6s}(31q+912) + q^{-3s}(q-1)^{1-8s}(2q^2+21q+85) + q^{-4s}(q-1)^{4-5s}(15q+1658) + q^{-4s}(q-1)^{2-7s}(2q^2+89q+758) + q^{-4s}(q-1)^{1-8s}(4q^2+45q+165) + q^{-5s}(q-1)^{4-5s}(16q+2111) + q^{-5s}(q-1)^{3-6s}(92q+2099) + q^{-5s}(q-1)^{2-7s}(9q^2+157q+1138) + q^{-5s}(q-1)^{1-8s}(q^3+12q^2+83q+262) + q^{-6s}(q-1)^{4-5s}(21q+2302) + q^{-6s}(q-1)^{2-7s}(6q^2+180q+1339) + q^{-6s}(q-1)^{1-8s}(10q^2+97q+316) + q^{-7s}(q-1)^{3-6s}(101q+2451) + q^{-7s}(q-1)^{1-8s}(2q^3+22q^2+131q+369) + q^{-8s}(q-1)^{1-8s}(9q^2+81q+277) + q^{-9s}(q-1)^{2-7s}(3q^2+80q+823) + q^{-9s}(q-1)^{1-8s}(2q^2+39q+181) + q^{-10s}(q-1)^{2-7s}(39q+536) + q^{-10s}(q-1)^{1-8s}(2q^2+25q+119) + q^{-11s}(q-1)^{2-7s}(11q+222) + 1395q^{-9s}(q-1)^{4-5s} + 1254q^{-6s}(q-1)^{5-4s} + 1224q^{-5s}(q-1)^{5-4s} + 1123q^{-7s}(q-1)^{5-4s} + 1061q^{-4s}(q-1)^{5-4s} + 923q^{-8s}(q-1)^{5-4s} + 911q^{-10s}(q-1)^{4-5s} + 776q^{-3s}(q-1)^{5-4s} + 670q^{-9s}(q-1)^{5-4s} + 505q^{-11s}(q-1)^{4-5s} + 494q^{-2s}(q-1)^{5-4s} + 436q^{-10s}(q-1)^{5-4s} + 394q^{-5s}(q-1)^{6-3s} + 381q^{-4s}(q-1)^{6-3s} + 369q^{-6s}(q-1)^{6-3s} + 319q^{-3s}(q-1)^{6-3s} + 299q^{-7s}(q-1)^{6-3s} + 251q^{-11s}(q-1)^{5-4s} + 242q^{-12s}(q-1)^{4-5s} + 239q^{-2s}(q-1)^{6-3s} + 230q^{-8s}(q-1)^{6-3s} + 230q^{-s}(q-1)^{5-4s} + 225q^{-s}(q-1)^{4-5s} + 208q^{-12s}(q-1)^{3-6s} + 154q^{-9s}(q-1)^{6-3s} + 141q^{-s}(q-1)^{6-3s} + 132q^{-s}(q-1)^{3-6s} + 126q^{-12s}(q-1)^{5-4s} + 98q^{-10s}(q-1)^{6-3s} + 89q^{-13s}(q-1)^{4-5s} + 67q^{-3s}(q-1)^{7-2s} + 67q^{-4s}(q-1)^{7-2s} + 61q^{-2s}(q-1)^{7-2s} + 61q^{-5s}(q-1)^{7-2s} + 58q^{-13s}(q-1)^{3-6s} + 53q^{-13s}(q-1)^{5-4s} + 51q^{-11s}(q-1)^{6-3s} + 50q^{-6s}(q-1)^{7-2s} + 48q^{-s}(q-1)^{7-2s} + 43q^{-s}(q-1)^{2-7s} + 34q^{-7s}(q-1)^{7-2s} + 25q^{-12s}(q-1)^{6-3s} + 25q^{-14s}(q-1)^{4-5s} + 22q^{-8s}(q-1)^{7-2s} + 18q^{-14s}(q-1)^{5-4s} + 13q^{-9s}(q-1)^{7-2s} + 12q^{-13s}(q-1)^{2-7s} + 11q^{-13s}(q-1)^{6-3s} + 9q^{-14s}(q-1)^{3-6s} + 8q^{-12s}(q-1)^{1-8s} + 7q^{-10s}(q-1)^{7-2s} + 7q^{-s}(q-1)^{8-s} + 6q^{-2s}(q-1)^{8-s} + 6q^{-s}(q-1)^{1-8s} + 5q^{-3s}(q-1)^{8-s} + 5q^{-15s}(q-1)^{5-4s} + 4q^{-4s}(q-1)^{8-s} + 4q^{-14s}(q-1)^{6-3s} + 4q^{-15s}(q-1)^{4-5s} + 3q^{-5s}(q-1)^{8-s} + 3q^{-11s}(q-1)^{7-2s} + 2q^{-6s}(q-1)^{8-s} + q^{-7s}(q-1)^{8-s} + q^{-12s}(q-1)^{7-2s} + q^{-15s}(q-1)^{6-3s} + q^{-16s}(q-1)^{5-4s} + 70(q-1)^{5-4s} + 56(q-1)^{6-3s} + 56(q-1)^{4-5s} + 28(q-1)^{7-2s} + 28(q-1)^{3-6s} + 8(q-1)^{8-s} + 8(q-1)^{2-7s} + (q-1)^{1-8s} + (q-1)^9$

(x) $\zeta_{\mathbb{T}_{10}}(s) = q^{-6s}(q-1)^{1-9s}(3q+17)(2q^2+13q+63) + 12q^{-13s}(q-1)^{3-7s}(14q+349) + 10q^{-14s}(q-1)^{3-7s}(7q+219) + 7q^{-5s}(q-1)^{4-6s}(30q+1201) + 7q^{-12s}(q-1)^{4-6s}(12q+967) + 5q^{-2s}(q-1)^{5-5s}(q+282) + 5q^{-10s}(q-1)^{5-5s}(3q+1366) + 4q^{-2s}(q-1)^{4-6s}(5q+333) + 4q^{-15s}(q-1)^{1-9s}(q+11) + 3q^{-2s}(q-1)^{3-7s}(10q+259) + 3q^{-7s}(q-1)^{4-6s}(118q+4475) + 3q^{-8s}(q-1)^{3-7s}(9q^2+385q+4636) + 3q^{-11s}(q-1)^{3-7s}(4q^2+205q+3159) + 2q^{-3s}(q-1)^{4-6s}(27q+1496) + 2q^{-4s}(q-1)^{2-8s}(8q^2+139q+889) + 2q^{-5s}(q-1)^{5-5s}(13q+$

$$3104) + 2q^{-5s}(q-1)^{2-8s}(q^3+25q^2+293q+1620) + 2q^{-8s}(q-1)^{5-5s}(19q+4337) + 2q^{-12s}(q-1)^{3-7s}(3q^2+176q+3381) + 2q^{-13s}(q-1)^{4-6s}(13q+2180) + 2q^{-14s}(q-1)^{4-6s}(5q+1223)+2q^{-15s}(q-1)^{2-8s}(11q+170)+2q^{-16s}(q-1)^{3-7s}(2q+161)+q^{-2s}(q-1)^{2-8s}(20q+257)+q^{-2s}(q-1)^{1-9s}(5q+37)+q^{-3s}(q-1)^{5-5s}(8q+2717)+q^{-3s}(q-1)^{3-7s}(3q^2+117q+2039)+q^{-3s}(q-1)^{2-8s}(6q^2+104q+791)+q^{-3s}(q-1)^{1-9s}(3q^2+33q+134)+q^{-4s}(q-1)^{5-5s}(21q+4418)+q^{-4s}(q-1)^{4-6s}(122q+5413)+q^{-4s}(q-1)^{3-7s}(4q^2+273q+4096)+q^{-4s}(q-1)^{1-9s}(q^3+15q^2+108q+342)+q^{-5s}(q-1)^{3-7s}(19q^2+545q+6943)+q^{-5s}(q-1)^{1-9s}(2q^3+35q^2+230q+659)+q^{-6s}(q-1)^{5-5s}(41q+7704)+q^{-6s}(q-1)^{4-6s}(299q+11188)+q^{-6s}(q-1)^{3-7s}(21q^2+809q+9883)+q^{-6s}(q-1)^{2-8s}(2q^3+84q^2+953q+4932)+q^{-7s}(q-1)^{5-5s}(36q+8593)+q^{-7s}(q-1)^{3-7s}(37q^2+1103q+12627)+q^{-7s}(q-1)^{2-8s}(8q^3+144q^2+1413q+6663)+q^{-7s}(q-1)^{1-9s}(2q^4+21q^3+143q^2+654q+1524)+q^{-8s}(q-1)^{4-6s}(356q+14191)+q^{-8s}(q-1)^{2-8s}(6q^3+135q^2+1587q+7639)+q^{-8s}(q-1)^{1-9s}(q^4+20q^3+165q^2+790q+1818)+q^{-9s}(q-1)^{5-5s}(20q+8073)+q^{-9s}(q-1)^{4-6s}(322q+13751)+q^{-9s}(q-1)^{3-7s}(30q^2+1125q+13721)+q^{-9s}(q-1)^{2-8s}(4q^3+125q^2+1513q+7536)+q^{-9s}(q-1)^{1-9s}(10q^3+131q^2+723q+1771)+q^{-10s}(q-1)^{4-6s}(227q+12026)+q^{-10s}(q-1)^{3-7s}(21q^2+934q+12376)+q^{-10s}(q-1)^{2-8s}(8q^3+142q^2+1407q+6977)+q^{-10s}(q-1)^{1-9s}(2q^4+22q^3+154q^2+704q+1669)+q^{-11s}(q-1)^{4-6s}(136q+9425)+q^{-11s}(q-1)^{2-8s}(61q^2+889q+5143)+q^{-11s}(q-1)^{1-9s}(6q^3+73q^2+433q+1181)+q^{-12s}(q-1)^{2-8s}(32q^2+526q+3607)+q^{-12s}(q-1)^{1-9s}(2q^3+36q^2+255q+805)+q^{-13s}(q-1)^{2-8s}(10q^2+249q+2095)+q^{-13s}(q-1)^{1-9s}(10q^2+110q+431)+q^{-14s}(q-1)^{2-8s}(5q^2+101q+983)+q^{-14s}(q-1)^{1-9s}(2q^2+33q+171)+q^{-15s}(q-1)^{3-7s}(20q+929)+q^{-16s}(q-1)^{2-8s}(2q+91)+5331q^{-11s}(q-1)^{5-5s}+3802q^{-12s}(q-1)^{5-5s}+3288q^{-7s}(q-1)^{6-4s}+3213q^{-6s}(q-1)^{6-4s}+3128q^{-8s}(q-1)^{6-4s}+2802q^{-5s}(q-1)^{6-4s}+2724q^{-9s}(q-1)^{6-4s}+2472q^{-13s}(q-1)^{5-5s}+2228q^{-4s}(q-1)^{6-4s}+2216q^{-10s}(q-1)^{6-4s}+1657q^{-11s}(q-1)^{6-4s}+1544q^{-3s}(q-1)^{6-4s}+1442q^{-14s}(q-1)^{5-5s}+1192q^{-15s}(q-1)^{4-6s}+1149q^{-12s}(q-1)^{6-4s}+937q^{-2s}(q-1)^{6-4s}+759q^{-15s}(q-1)^{5-5s}+757q^{-6s}(q-1)^{7-3s}+729q^{-5s}(q-1)^{7-3s}+725q^{-13s}(q-1)^{6-4s}+700q^{-7s}(q-1)^{7-3s}+655q^{-4s}(q-1)^{7-3s}+607q^{-8s}(q-1)^{7-3s}+525q^{-s}(q-1)^{5-5s}+524q^{-3s}(q-1)^{7-3s}+492q^{-16s}(q-1)^{4-6s}+480q^{-9s}(q-1)^{7-3s}+427q^{-s}(q-1)^{6-4s}+426q^{-14s}(q-1)^{6-4s}+413q^{-s}(q-1)^{4-6s}+377q^{-2s}(q-1)^{7-3s}+365q^{-10s}(q-1)^{7-3s}+346q^{-16s}(q-1)^{5-5s}+249q^{-11s}(q-1)^{7-3s}+225q^{-15s}(q-1)^{6-4s}+217q^{-s}(q-1)^{7-3s}+203q^{-s}(q-1)^{3-7s}+163q^{-12s}(q-1)^{7-3s}+155q^{-17s}(q-1)^{4-6s}+133q^{-17s}(q-1)^{5-5s}+105q^{-16s}(q-1)^{6-4s}+97q^{-4s}(q-1)^{8-2s}+94q^{-5s}(q-1)^{8-2s}+93q^{-3s}(q-1)^{8-2s}+92q^{-13s}(q-1)^{7-3s}+85q^{-6s}(q-1)^{8-2s}+82q^{-2s}(q-1)^{8-2s}+74q^{-17s}(q-1)^{3-7s}+70q^{-7s}(q-1)^{8-2s}+63q^{-s}(q-1)^{8-2s}+57q^{-s}(q-1)^{2-8s}+50q^{-8s}(q-1)^{8-2s}+50q^{-14s}(q-1)^{7-3s}+43q^{-17s}(q-1)^{6-4s}+42q^{-18s}(q-1)^{5-5s}+35q^{-18s}(q-1)^{4-6s}+34q^{-9s}(q-1)^{8-2s}+25q^{-15s}(q-1)^{7-3s}+22q^{-10s}(q-1)^{8-2s}+16q^{-18s}(q-1)^{6-4s}+13q^{-11s}(q-1)^{8-2s}+12q^{-17s}(q-1)^{2-8s}+11q^{-16s}(q-1)^{7-3s}+9q^{-18s}(q-1)^{3-7s}+9q^{-19s}(q-1)^{5-5s}+8q^{-16s}(q-1)^{1-9s}+8q^{-s}(q-1)^{9-s}+7q^{-2s}(q-1)^{9-s}+7q^{-12s}(q-1)^{8-2s}+7q^{-s}(q-1)^{1-9s}+6q^{-3s}(q-1)^{9-s}+5q^{-4s}(q-1)^{9-s}+5q^{-19s}(q-1)^{6-4s}+4q^{-5s}(q-1)^{9-s}+4q^{-17s}(q-1)^{7-3s}+4q^{-19s}(q-1)^{4-6s}+3q^{-6s}(q-1)^{9-s}+$$

$3q^{-13s}(q-1)^{8-2s}+2q^{-7s}(q-1)^{9-s}+q^{-8s}(q-1)^{9-s}+q^{-14s}(q-1)^{8-2s}+q^{-18s}(q-1)^{7-3s}+q^{-20s}(q-1)^{6-4s}+q^{-20s}(q-1)^{5-5s}+126(q-1)^{6-4s}+126(q-1)^{5-5s}+84(q-1)^{7-3s}+84(q-1)^{4-6s}+36(q-1)^{8-2s}+36(q-1)^{3-7s}+9(q-1)^{9-s}+9(q-1)^{2-8s}+(q-1)^{1-9s}+(q-1)^{10}$ □

The computation times[2] for the representation zeta functions $\zeta_{\mathbb{U}_n}(s)$ are given by

n	2	3	4	5	6	7	8	9	10
time	0.01s	0.08s	0.25s	0.77s	2.41s	8.18s	27.43s	1m44s	6m52s

and those for the representation zeta functions $\zeta_{\mathbb{T}_n}(s)$ are given by

n	2	3	4	5	6	7	8	9	10
time	0.03s	0.15s	0.51s	1.81s	6.32s	26.16s	1m53s	6m39s	55m46s

6.4 Arithmetic-geometric correspondence

In this section, we give some more insight into the correspondence between the arithmetic and geometric method. In one direction, Theorem 4.10.6 shows how information on the geometric side can be translated to the arithmetic side. More precisely, the eigenvalues and eigenvectors of the geometric TQFT Z_G partially describe the character tables of the finite groups $G(\mathbb{F}_q)$. Let us illustrate how, in the other direction, the arithmetic information provides geometric insight into the Z_G. In particular, we will show how the representation theory of the groups $\mathbb{U}_n(\mathbb{F}_q)$ of unipotent upper triangular matrices over $\mathbb{F}_q$ can be used to simplify the corresponding geometric TQFT Z_G. This yields a new smaller set of generators for this TQFT motivated by the arithmetic side. More precisely, we obtain this new generating set by canonically lifting the sums of equidimensional characters to the Grothendieck ring of varieties. These generators will be given by virtual classes of locally closed subvarieties of G.

Unipotent 3×3 matrices. Consider the group $\mathbb{U}_3(\mathbb{F}_q)$ of unipotent upper triangular matrices of rank 3 over a finite field $\mathbb{F}_q$,

$$\mathbb{U}_3(\mathbb{F}_q) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{F}_q \right\}.$$

The irreducible complex characters of $\mathbb{U}_3(\mathbb{F}_q)$ are of dimension 1 or q. Denote the set of 1-dimensional characters by X_1 and of the q-dimensional characters by

[2]As performed on an Intel®Xeon®CPU E5-4640 0 @ 2.40GHz.

X_q. Summing the 1-dimensional characters, we find that $v_1 = \sum_{\chi \in X_1} \chi$ is given by

$$v_1 \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{cases} q & \text{if } x = z = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Summing the q-dimensional characters, we find that $v_2 = \sum_{\chi \in X_q} \chi$ is given by

$$v_2 \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{cases} -q & \text{if } x = z = 0 \text{ and } y \neq 0, \\ q(q-1) & \text{if } x = y = z = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now, since the eigenvectors v_1 and v_2 are 'polynomial in q'-valued on locally closed subsets of $G = \mathbb{U}_3$, we can naturally lift these eigenvectors along the morphism $\mu_{[G/G]} \colon \mathrm{K}_0(\mathbf{Stck}_{[G/G]}) \to \mathbb{C}^{[G/G](\mathbb{F}_q)} = R_{\mathbb{C}}(G(\mathbb{F}_q))$ of Definition 4.10.1, replacing q by $\mathbb{L}$ to obtain

$$\mathbb{L}[\{x = z = 0\}] \quad \text{and} \quad -\mathbb{L}[\{x = z = 0,\ y \neq 0\}] + \mathbb{L}(\mathbb{L} - 1)[\{x = y = z = 0\}]$$

respectively, expressed in terms of the virtual classes of G-equivariant subvarieties of $\mathbb{U}_3$. Indeed, from the computations of Section 6.2, whose results can be found in Appendix A, it can be seen that these elements are eigenvectors of Z_G, with eigenvalues $\mathbb{L}^6$ and $\mathbb{L}^4$, respectively. In fact, the submodule of $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ generated by these two eigenvectors contains Z_G () (1) and is invariant under Z_G(), and is therefore a simplification of the $M = 5$ generators as used in Section 6.2.

Unipotent 4×4 matrices. Consider the group $\mathbb{U}_4(\mathbb{F}_q)$ of unipotent upper triangular matrices of rank 4 over a finite field $\mathbb{F}_q$,

$$\mathbb{U}_4(\mathbb{F}_q) = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} : a, b, c, d, e, f \in \mathbb{F}_q \right\}.$$

This group has three families of irreducible complex characters: the 1-dimensional characters X_1, the q-dimensional characters X_q and the q^2-dimensional characters X_{q^2}. Summing equidimensional characters, we find

$$\sum_{\chi \in X_1} \chi \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{cases} q^2 & \text{if } a = d = f = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\sum_{\chi \in X_q} \chi \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{cases} q^4 & \text{if } a = b = d = e = f = 0, \\ -q^2 & \text{if } a = d = f \\ 0 & \text{otherwise,} \end{cases}$$

$$\sum_{\chi \in X_{q^2}} \chi \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{cases} q^3(q-1) & \text{if } a = b = c = d = e = f = 0, \\ -q^3 & \text{if } a = b = d = e = f = 0 \text{ and } c \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We lift these to elements in $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$, for $G = \mathbb{U}_4$, given by

$$\begin{aligned} &\mathbb{L}^2[\{a = d = f = 0\}], \\ &\mathbb{L}^4[\{a = b = d = e = f = 0\}] - \mathbb{L}^2[\{a = d = f = 0\}], \text{ and} \\ &\mathbb{L}^3(\mathbb{L} - 1)[\{a = b = c = d = e = f = 0\}] - \mathbb{L}^3[\{a = b = d = e = f = 0,\ c \neq 0\}]. \end{aligned}$$

Again, the computations of Section 6.2, whose results can be found in Appendix A, show that these elements are eigenvectors of Z_G, with eigenvalues $\mathbb{L}^{12}$, $\mathbb{L}^{10}$ and $\mathbb{L}^8$, respectively. These three elements to generate a submodule of $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ which can replace the one from Section 6.2 with $M = 16$ generators, a significant simplification.

Remark 6.4.1. During the algorithmic computations of Section 6.3, it is, in principle, possible to keep track of the irreducible characters of $\mathbb{U}_n(\mathbb{F}_q)$ and $\mathbb{T}_n(\mathbb{F}_q)$ for $6 \leq n \leq 10$. Then, as in the above examples, the sums of equidimensional characters can be lifted to elements in $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ for $G = \mathbb{U}_n$ or $G = \mathbb{T}_n$, respectively. While we have not attempted this, these lifts would generate a submodule of $\mathrm{K}_0(\mathbf{Stck}_{[G/G]})$ which, one could hopefully show, is invariant under Z_G (). This would provide a way to extend the geometric method to groups of upper triangular matrices of rank ≥ 6, even though there are infinitely many conjugacy classes.

Chapter 7

Motivic stability

Let Γ_n be a sequence of finitely generated groups, and let G be an algebraic group over a field k. One can wonder whether the invariants of the corresponding sequence of character stacks $\mathfrak{X}_G(\Gamma_n)$ are related. We will mainly focus on the sequence $\Gamma_n = \mathbb{Z}^n$ of free abelian groups and the sequence $\Gamma_n = F_n$ of free groups, for which the character stacks parametrize (commuting) tuples of elements in G up to conjugation.

Geometric invariants of these and related spaces have been studied extensively [Bai07, AC07, PS13, FL14]. For X_n the sequence of G-representation varieties or G-character varieties of $\mathbb{Z}^n$, the homology groups $H_k(X_n)$ were computed in [RS19], and their mixed Hodge structures in [FS21]. A pattern emerged: fixing n and varying G through sequences G_r of classical groups (such as GL_r or U_r), the homology groups $H_k(X_n)$ remain constant for sufficiently large r, that is, they *stabilize*. This pattern was proved in [RS21], as well as for fixed G and increasing n, and for many related sequences X_n. Moreover, taking into account the action of the symmetric group S_n on $\mathbb{Z}^n$ by permutation, inducing an action of S_n on X_n and in turn on $H_k(X_n)$, they showed the homology groups stabilize as S_n-representations. This type of stability, called *representation stability*, was formulated in [CF13]: a sequence V_n of S_n-representations is representation stable, roughly speaking, if the multiplicities of the irreducible representations V_λ, corresponding to the partitions λ of n, stabilize. Partitions for n and $n+1$ are related by increasing the first number.

In this chapter, we combine the notion of representation stability with that of *motivic stability*. Completing the Grothendieck ring of varieties, one can study the convergence of a sequence of virtual classes. Such convergence was studied in [VW15] for sequences of symmetric powers $\mathrm{Sym}^n X$ (as an algebraic analogue of the Dold–Thom theorem) and sequences of configuration spaces $\mathrm{Conf}^n X$.

Using the theory of Section 3.6, we will generalize the notion of motivic stability, and introduce the concept of *motivic representation stability.* As an application, we will show that the sequence of GL_r-character stacks $\mathfrak{X}_{\mathrm{GL}_r}(\mathbb{Z}^n)$, with the action of S_n, is motivically representation stable.

7.1 Motivic stability

Motivic stability is a property of a sequence of varieties, which amounts to the convergence (in some sense) of their virtual classes in the topological ring $\widehat{\mathcal{M}_{\mathbb{L}}}$, which is the completion of the localization $\mathrm{K}_0(\mathbf{Var}_k)[\mathbb{L}^{-1}]$ of the Grothendieck ring of varieties. This topological ring was originally constructed by Kontsevich in the context of motivic integration [Kon95]. For more information on this object, we refer to [Bou11, Loo02, VW15].

For our applications, we adapt the standard definitions to the equivariant setting. Throughout, fix an algebraic group G over k, and denote by $\mathbb{L}$ the class $[\mathbb{A}^1_k] \in \mathrm{K}_0(\mathbf{Var}^G_k)$ of $\mathbb{A}^1_k$ on which G acts trivially.

Definition 7.1.1. Write $\mathcal{M}^G_{\mathbb{L}}$ for the localization $\mathrm{K}_0(\mathbf{Var}^G_k)[\mathbb{L}^{-1}]$. Consider the increasing filtration on $\mathcal{M}^G_{\mathbb{L}}$,

$$0 \subseteq \cdots \subseteq F_n\mathcal{M}^G_{\mathbb{L}} \subseteq F_{n+1}\mathcal{M}^G_{\mathbb{L}} \subseteq \cdots \subseteq \mathcal{M}^G_{\mathbb{L}},$$

where $F_n\mathcal{M}^G_{\mathbb{L}}$ is the subgroup of $\mathcal{M}^G_{\mathbb{L}}$ generated by all elements of the form $[X]/\mathbb{L}^m$ with $\dim X - m \le n$. Note that $\bigcup_{n\in\mathbb{Z}} F_n\widehat{\mathcal{M}^G_{\mathbb{L}}} = \mathcal{M}^G_{\mathbb{L}}$. The completion with respect to this filtration is denoted

$$\widehat{\mathcal{M}^G_{\mathbb{L}}} = \varprojlim_n \mathcal{M}^G_{\mathbb{L}}/F_n\mathcal{M}^G_{\mathbb{L}}.$$

An element $x \in \widehat{\mathcal{M}^G_{\mathbb{L}}}$ can be represented as a tuple $(x_n) \in \prod_{n\in\mathbb{Z}} \mathcal{M}^G_{\mathbb{L}}/F_n\mathcal{M}^G_{\mathbb{L}}$ such that $x_n \equiv x_m \mod F_n\mathcal{M}^G_{\mathbb{L}}$ for all $m \le n$.

The completion $\widehat{\mathcal{M}^G_{\mathbb{L}}}$ inherits, a priori, only the group structure from $\mathcal{M}^G_{\mathbb{L}}$. Multiplication is defined as follows. Let $x = (x_n)$ and $y = (y_n)$ be elements of $\widehat{\mathcal{M}^G_{\mathbb{L}}}$. Note that there exists a sufficiently large N such that $x_n = y_n = 0$ for all $n \ge N$. Now define xy by $(xy)_n = x'_{n-N}y'_{n-N} \mod F_n\mathcal{M}^G_{\mathbb{L}}$, where $x'_{n-N}, y'_{n-N} \in \mathcal{M}^G_{\mathbb{L}}$ are lifts x_{n-N} and y_{n-N}, respectively. This is independent of the choice of lift since, for any other lift x''_{n-N}, we have $x''_{n-N}y'_{n-N} - x'_{n-N}y'_{n-N} = (x''_{n-N} - x'_{n-N})y'_{n-N} \in F_{n-N}\mathcal{M}^G_{\mathbb{L}} \cdot F_N\mathcal{M}^G_{\mathbb{L}} \subseteq F_n\mathcal{M}^G_{\mathbb{L}}$. Similarly, it is independent of the choice of lift y'_{n-M}. This gives $\widehat{\mathcal{M}^G_{\mathbb{L}}}$ a ring structure.

Definition 7.1.2. Let X be a G-variety over k. For any $n \ge 0$, the *n-th G-symmetric power* of X, denoted $\mathrm{Sym}^n_G X$, is the G-variety given by the ordinary symmetric power $\operatorname{Sym}^n X = X^n /\!\!/ S_n$ with the action of G induced by the diagonal action on X^n.

Definition 7.1.3. Let X be a G-variety over k. The symmetric powers $\operatorname{Sym}^n_G X$ of X are called *motivically stable* if the limit

$$\lim_{n\to\infty} \frac{[\operatorname{Sym}^n_G X]}{\mathbb{L}^{n \dim X}}$$

exists in $\widehat{\mathcal{M}^G_{\mathbb{L}}}$. More generally, a sequence X_n of G-varieties over k is *motivically stable* if the limit

$$\lim_{n\to\infty} \frac{[X_n]}{\mathbb{L}^{\dim X_n}}$$

exists in $\widehat{\mathcal{M}^G_{\mathbb{L}}}$.

Example 7.1.4. When $G = 1$ is the trivial group, we simply write $\widehat{\mathcal{M}_{\mathbb{L}}}$ instead of $\widehat{\mathcal{M}^G_{\mathbb{L}}}$. In this case, the following sequences are motivically stable.

- From Example 3.3.5, we see that the sequence $X_n = \mathrm{GL}_n$ is motivically stable, with limit $\lim_{n\to\infty}[\mathrm{GL}_n]/\mathbb{L}^{n^2} = \prod_{i\geq 1}(1 - \mathbb{L}^{-i})$.
- Similarly, $X_n = \mathrm{SL}_n$ is motivically stable with limit $\lim_{n\to\infty}[\mathrm{SL}_n]/\mathbb{L}^{n^2-1} = \prod_{i\geq 2}(1 - \mathbb{L}^{-i})$. Since $[\mathrm{PGL}_n] = [\mathrm{SL}_n]$, the sequence $X_n = \mathrm{PGL}_n$ is also motivically stable, with the same limit.
- It is still an open conjecture [VW15, Conjecture 1.25] whether the symmetric powers of all geometrically irreducible varieties are motivically stable. However, some evidence has been presented against it [Lit14].

Example 7.1.5. Let us give some intuition for what motivic stabilization implies about the cohomology of X_n. Suppose X_n is a sequence of varieties over $k = \mathbb{C}$. Note that the E-polynomial descends to a continuous morphism

$$e \colon \widehat{\mathcal{M}^G_{\mathbb{L}}} \to \mathbb{Z}[u, v][\![(uv)^{-1}]\!]$$

where the target is equipped with the $(uv)^{-1}$-adic topology. Since

$$e([X_n]/\mathbb{L}^{\dim X_n}) = \sum_{k,p,q\in\mathbb{Z}} (-1)^k h_c^{k;p,q}(X_n)\, u^{p-\dim X_n} v^{q-\dim X_n},$$

it follows, if the sequence X_n motivically stabilizes, that, for all p and q, the numbers $h_c^{k;\dim X_n - p, \dim X_n - q}(X_n)$ are eventually constant as $n \to \infty$. If the X_n are smooth projective, then evaluating in $u = v = t$, it also follows that the dimensions $\dim_{\mathbb{C}} H_c^{\dim X_n - k}(X_n; \mathbb{C})$ are eventually constant as $n \to \infty$, as well as the dimensions $\dim_{\mathbb{C}} H_k(X_n; \mathbb{C})$ by Poincaré duality.

In the context of motivic stability, an important source of sequences of varieties are the symmetric powers of a variety X. In order to keep track of the virtual classes of these symmetric powers, we collect them as the coefficients of a power series, as first done by [Kap00].

Definition 7.1.6. Let X be a G-variety over k. The *motivic zeta function* of X is

$$Z_G(X,t) = \sum_{n\geq 0} [\operatorname{Sym}^n_G X]\, t^n \in 1 + t\cdot \mathrm{K}_0(\mathbf{Var}^G_k)[\![t]\!].$$

Lemma 7.1.7. *Let X be a G-variety over k, and $Y \subseteq X$ a G-invariant closed subvariety with open complement U. Then $Z_G(X,t) = Z_G(Y,t)\, Z_G(U,t)$, and hence $Z_G(-,t)$ descends to a group morphism*

$$Z_G(-,t)\colon \mathrm{K}_0(\mathbf{Var}^G_k) \to 1 + t\cdot \mathrm{K}_0(\mathbf{Var}^G_k)[\![t]\!]$$

with the multiplicative group structure on the right. In particular, Sym^n_G descends to a map

$$\operatorname{Sym}^n_G\colon \mathrm{K}_0(\mathbf{Var}^G_k) \to \mathrm{K}_0(\mathbf{Var}^G_k).$$

Proof. From

$$\begin{aligned}[\operatorname{Sym}^n_G X] = [X^n /\!\!/ S_n] &= \sum_{i+j=n} [(S_n\cdot (Y^i\times U^j)) /\!\!/ S_n] \\ &= \sum_{i+j=n} [(Y^i /\!\!/ S_i)\times (U^j /\!\!/ S_j)] = \sum_{i+j=n} [\operatorname{Sym}^i_G Y][\operatorname{Sym}^j_G U]\end{aligned}$$

follows that

$$Z_G(X,t) = \sum_{\substack{n\geq 0\\ i+j=n}} [\operatorname{Sym}^i_G Y][\operatorname{Sym}^j_G U]\, t^n = Z_G(Y,t)\, Z_G(U,t). \qquad \square$$

The following lemma is a variation of [Göt01, Lemma 4.4], adapted to the equivariant setting.

Proposition 7.1.8. *Let G be a finite group, and let X be a G-variety over k. For any $r\geq 0$, we have*

$$Z_G(\mathbb{L}^r[X],t) = Z_G([X],\mathbb{L}^r t).$$

Proof. It suffices to treat the case $r=1$. Denote by $\pi\colon \operatorname{Sym}^n(X\times \mathbb{A}^1_k) \to \operatorname{Sym}^n X$ the obvious projection. Note that $\operatorname{Sym}^n X$ is naturally stratified by locally closed subvarieties $(\operatorname{Sym}^n X)_\lambda$ according to the partitions λ of n. For every such partition λ, we consider the cartesian diagram

$$\begin{array}{ccc} X_*^{\ell(\lambda)} \times \prod_{i=1}^n (\mathbb{A}_k^i /\!\!/ S_i)^{a_i(\lambda)} & \longrightarrow & \pi^{-1}\left((\operatorname{Sym}^n X)_\lambda\right) \\ \downarrow & & \downarrow \pi_\lambda \\ X_*^{\ell(\lambda)} & \longrightarrow & (\operatorname{Sym}^n X)_\lambda \end{array}$$

where $a_i(\lambda)$ denotes the number of times i appears in λ, and $X_*^{\ell(\lambda)}$ the space of $\ell(\lambda) = \sum_i a_i(\lambda)$ distinct ordered points of X. Since $\prod_{i=1}^n (\mathbb{A}_k^i /\!\!/ S_i)^{a_i(\lambda)} \cong \mathbb{A}_k^n$, the diagram defines an étale trivialization of π_λ. The transition functions are given by the action of the group $S_{a_1(\lambda)} \times \cdots \times S_{a_n(\lambda)}$, which acts linearly. Hence, π_λ is a vector bundle which is étale-locally trivial, so by Hilbert's Theorem 90 [Ser58, Theorem 2] also Zariski-locally trivial. However, note that a stratification of $(\operatorname{Sym}^n X)_\lambda$ trivializing π_λ need not necessarily be G-invariant. Nevertheless, using that G is finite, any such stratification can be intersected with all of its translations by $g \in G$, in order to obtain a G-invariant stratification. Hence, we conclude that $[\operatorname{Sym}_G^n(X \times \mathbb{A}_k^1)] = \mathbb{L}^n [\operatorname{Sym}_G^n X]$. □

From the Chevalley–Shephard–Todd theorem [Che55], it is easy to see that $\operatorname{Sym}^n \mathbb{A}_k^r$ is not isomorphic to $\mathbb{A}_k^{nr}$ for $n, r > 1$. Nevertheless, the above proposition yields the following corollary.

Corollary 7.1.9. *For any $n, r \geq 0$, we have $\operatorname{Sym}^n \mathbb{L}^r = \mathbb{L}^{nr}$. In particular, $Z_G(\mathbb{L}^r, t) = 1/(1 - \mathbb{L}^r t)$.* □

Lemma 7.1.10. *Let X be a d-dimensional G-variety over k, and suppose that the symmetric powers $\operatorname{Sym}_G^n X$ are motivically stable. Then*

$$\lim_{n\to\infty} \frac{[\operatorname{Sym}_G^n X]}{\mathbb{L}^{nd}} = \Big[(1-t) Z_G(X, t/\mathbb{L}^d)\Big]_{t=1}.$$

Proof. As

$$\Big[(1-t) Z_G(X, t/\mathbb{L}^d)\Big]_{t=1} = \left[1 + \sum_{n\geq 1} \left(\frac{[\operatorname{Sym}_G^n X]}{\mathbb{L}^{nd}} - \frac{[\operatorname{Sym}_G^{n-1} X]}{\mathbb{L}^{(n-1)d}} \right) t^n \right]_{t=1}$$

evaluates to a telescoping series, it is equal to $\lim_{n\to\infty} [\operatorname{Sym}_G^n X]/\mathbb{L}^{nd}$. □

Example 7.1.11. Let X be a variety over k such that $[X] \in \mathrm{K}_0(\mathbf{Var}_k)$ is a polynomial in $\mathbb{L}$. Then the sequence of symmetric powers $X_n = \operatorname{Sym}^n X$ is motivically stable if and only if $[X]$ is monic in $\mathbb{L}$. Namely, writing $[X] = \sum_{i=0}^d a_i \mathbb{L}^i$

with $a_d \neq 0$, it follows from Lemma 7.1.7 and Corollary 7.1.9 that

$$Z_G([X],t) = \prod_{i=0}^{d} \left(\frac{1}{1-\mathbb{L}^i t} \right)^{a_i}.$$

Hence, $[\mathrm{Sym}^n X]/\mathbb{L}^{nd}$ is the n-th coefficient of

$$Z_G(X, t/\mathbb{L}^d) = \prod_{i=0}^{d} \left(\frac{1}{1-\mathbb{L}^{i-d} t} \right)^{a_i}.$$

Therefore, for $a_d = 1$, we find that

$$\lim_{n\to\infty} \frac{[\mathrm{Sym}^n X]}{\mathbb{L}^{nd}} = \left[(1-t) Z_G(X, t/\mathbb{L}^d)\right]_{t=1} = \prod_{i=0}^{d-1} \left(\frac{1}{1-\mathbb{L}^{i-d}} \right)^{a_i},$$

and for $a_d > 1$, the limit is easily seen to not exist.

Proposition 7.1.12 ([VW15, Proposition 4.2]). *Let X be a G-variety over k, and $Y \subseteq X$ a G-invariant closed subvariety of dimension $\dim Y < \dim X$, with open complement $U = X \backslash Y$. Then the symmetric powers $\mathrm{Sym}_G^n X$ are motivically stable if and only if the symmetric powers $\mathrm{Sym}_G^n U$ are motivically stable, and in this case*

$$\lim_{n\to\infty} \frac{[\mathrm{Sym}_G^n X]}{\mathbb{L}^{n \dim X}} = Z_G(Y, \mathbb{L}^{-\dim X}) \lim_{n\to\infty} \frac{[\mathrm{Sym}_G^n U]}{\mathbb{L}^{n \dim X}}.$$

Proof. Let us prove the result modulo $F_{-m}\mathcal{M}_{\mathbb{L}}^G$ for all $m \geq 0$, by induction on m. The case $m = 0$ is trivial as $[\mathrm{Sym}_G^n X]/\mathbb{L}^{n \dim X} \equiv 0 \bmod F_0\mathcal{M}_{\mathbb{L}}^G$, and similarly for U. For $m > 0$ we find, as in Lemma 7.1.7, that, for all $n \geq 1$,

$$\frac{[\mathrm{Sym}_G^n X]}{\mathbb{L}^{n \dim X}} \equiv \sum_{i=0}^{m-1} \frac{[\mathrm{Sym}_G^{n-i} U]}{\mathbb{L}^{(n-i) \dim X}} \frac{[\mathrm{Sym}_G^i Y]}{\mathbb{L}^{i \dim X}} \bmod F_{-m}\mathcal{M}_{\mathbb{L}}^G \qquad (*)$$

since $[\mathrm{Sym}_G^{n-i} U][\mathrm{Sym}_G^i Y]/\mathbb{L}^{n \dim X} \in F_{-m}\mathcal{M}_{\mathbb{L}}^G$ for $i \geq m$ as $\dim Y < \dim X$. Now, if the symmetric powers of U stabilize modulo $F_{-m}\mathcal{M}_{\mathbb{L}}^G$, say to $\ell = \lim_{n\to\infty} [\mathrm{Sym}_G^n U]/\mathbb{L}^{\dim X}$, then the right-hand side of equation $(*)$ stabilizes modulo $F_{-m}\mathcal{M}_{\mathbb{L}}^G$ to $\ell\, Z_G(Y, \mathbb{L}^{-\dim X})$. Conversely, if the symmetric powers of X stabilize modulo $F_{-m}\mathcal{M}_{\mathbb{L}}^G$, then the symmetric powers of U stabilize modulo $F_{-m+1}\mathcal{M}_{\mathbb{L}}^G$ (by the induction hypothesis), so every term on the right-hand side of $(*)$ with $i \geq 1$ stabilizes modulo $F_{-m}\mathcal{M}_{\mathbb{L}}^G$. But then also the term with $i = 0$ must stabilize, which shows that the symmetric powers of U stabilize modulo $F_{-m}\mathcal{M}_{\mathbb{L}}^G$. □

Remark 7.1.13. Suppose G is the trivial group, and write $Z(-,t)$ for $Z_G(-,t)$. The definition of $Z(-,t)$ can be extended to the Grothendieck ring of stacks

$\mathrm{K}_0(\mathbf{Stck}_k)$. Since $\mathrm{K}_0(\mathbf{Stck}_k) \cong \mathrm{K}_0(\mathbf{Var}_k)[\mathbb{L}^{-1}, (\mathbb{L}^n - 1)^{-1}]$ by Theorem 3.5.7, it suffices to recursively define $Z(x/\mathbb{L}, t)$ and $Z(x/(\mathbb{L}^n - 1), t)$ in terms of $Z(x, t)$ for all elements $x \in \mathrm{K}_0(\mathbf{Stck}_k)$, using that $Z(x, t)$ is determined for $x \in \mathrm{K}_0(\mathbf{Var}_k)$. This is done as follows.

$$Z(x/\mathbb{L}, t) = Z(x, \mathbb{L}^{-1} t)$$
$$Z(x/(\mathbb{L}^n - 1), t) = \prod_{i \geq 0} Z(x, \mathbb{L}^{in} t)^{-1}$$

Note that this gives a well-defined map

$$Z(-, t) \colon \mathrm{K}_0(\mathbf{Stck}_k) \to 1 + t \cdot \mathrm{K}_0(\mathbf{Stck}_k)[\![t]\!]$$

since

$$Z(x\mathbb{L}/\mathbb{L}, t) = Z(x, t) \quad \text{and}$$
$$Z(x(\mathbb{L}^n - 1)/(\mathbb{L}^n - 1)) = \prod_{k \geq 0} Z(x(\mathbb{L}^n - 1), \mathbb{L}^{kn} t) = \prod_{k \geq 0} \frac{Z(x, \mathbb{L}^{kn} t)}{Z(x, \mathbb{L}^{(k+1)n} t)} = Z(x, t)$$

which is easily seen to still be group morphism. In particular, looking at the n-th coefficient of $Z(-, t)$, we find that Sym^n descends to a map

$$\mathrm{Sym}^n \colon \mathrm{K}_0(\mathbf{Stck}_k) \to \mathrm{K}_0(\mathbf{Stck}_k).$$

The definition of symmetric powers does not naturally extend from varieties to stacks. However, as shown in [Eke09b], the class $\mathrm{Sym}^n[\mathfrak{X}]$ coincides with the virtual class of the stacky symmetric power $[\mathfrak{X}^n / S_n]$ for objects $\mathfrak{X}$ of $\mathbf{Stck}_k$ when $\mathrm{char}(k) = 0$ and $\mathrm{char}(k) > n$.

7.2 Equivariant stability

In this section we will show various stability results, for non-trivial algebraic groups G. Let us start by considering one of the simplest actions.

Proposition 7.2.1. *Let $G = \mathbb{G}_m$ act on $\mathbb{A}^1_k$ via $\alpha \cdot x = \alpha x$. Then*

$$\lim_{n \to \infty} \frac{\mathrm{Sym}^n_G[\mathbb{A}^1_k]}{\mathbb{L}^n} = \frac{[\mathbb{G}_m]}{\mathbb{L} - 1}$$

in $\widehat{\mathcal{M}^G_{\mathbb{L}}}$, where on the right $\mathbb{G}_m$ acts transitively on itself.

Proof. Write $X = [\mathbb{A}^1_k]$ for the described action of $\mathbb{G}_m$ on $\mathbb{A}^1_k$. Since $\mathrm{Sym}^n \mathbb{A}^1_k \cong \mathbb{A}^n_k$ has basis of coordinates given by the elementary symmetric polynomials, we have

$$\mathrm{Sym}^n_G X = \prod_{i=1}^{n} [\mathbb{A}^1_k] = \prod_{i=1}^{n} (1 + Y_i),$$

where, for any $i \geq 1$, we denote $Y_i = [\mathbb{G}_m]$ for the action $\alpha \cdot x = \alpha^i x$. Note that $Y_i Y_j = (\mathbb{L} - 1) Y_{\gcd(i,j)}$ for any $i, j \geq 1$. Indeed, there exist $a, b \in \mathbb{Z}$ such that $ai + bj = d \coloneqq \gcd(i, j)$, so the equality follows from the isomorphism

$$\begin{aligned} \mathbb{G}_m \times \mathbb{G}_m &\cong \mathbb{G}_m \times \mathbb{G}_m \\ (x, y) &\mapsto (x^a y^b, x^{j/\gcd(i,j)} y^{-i/d}) \\ (z^{i/d} w^b, z^{j/d} w^{-a}) &\leftarrow\!\shortmid (z, w), \end{aligned}$$

where $\alpha \cdot (x, y, z, w) = (\alpha^i x, \alpha^j y, \alpha^d z, w)$. Now, it follows that

$$\operatorname{Sym}_G^n X = 1 + \sum_{i \geq 1} a_{n,i} Y_i \quad \text{with} \quad a_{n,i} = \sum_S (\mathbb{L} - 1)^{|S|-1},$$

where the latter sum runs over all non-empty subsets $S \subseteq \{1, 2, \ldots, n\}$ such that $\gcd(S) = i$. Now, for any $i \geq 2$, we see that any S appearing in this sum must have $|S| \leq n/i$, so that $\deg_{\mathbb{L}}(a_{n,i}) \leq n/i - 1$. In particular,

$$\lim_{n \to \infty} \frac{a_{n,i}}{\mathbb{L}^n} = 0$$

for $i \geq 2$. Furthermore, from the equality $1 + \sum_{i=1}^n a_{n,i}(\mathbb{L} - 1) = \mathbb{L}^n$ follows that

$$\lim_{n \to \infty} \frac{a_{n,1}}{\mathbb{L}^n} = \lim_{n \to \infty} \frac{1}{\mathbb{L}^n} \left(\frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \sum_{i=2}^n a_{n,i} \right) = \frac{1}{\mathbb{L} - 1},$$

and therefore

$$\lim_{n \to \infty} \frac{\operatorname{Sym}_G^n X}{\mathbb{L}^n} = \frac{1}{\mathbb{L} - 1} Y_1. \qquad \square$$

Corollary 7.2.2. *The action of $G = \mathbb{G}_m$ on $\mathbb{A}_k^1$ given by $\alpha \cdot x = \alpha x$ extends to $\mathbb{P}_k^1$ and restricts to $\mathbb{G}_m$. The symmetric powers of $\mathbb{P}_k^1$ and $\mathbb{G}_m$ are motivically stable with limits*

$$\begin{aligned} \lim_{n \to \infty} \frac{\operatorname{Sym}_G^n [\mathbb{P}^1]}{\mathbb{L}^n} &= \frac{\mathbb{L}}{(\mathbb{L} - 1)^2} [\mathbb{G}_m] \\ \lim_{n \to \infty} \frac{\operatorname{Sym}_G^n [\mathbb{G}_m]}{\mathbb{L}^n} &= \frac{1}{\mathbb{L}} [\mathbb{G}_m]. \end{aligned}$$

Proof. This follows from Proposition 7.2.1 together with Proposition 7.1.12 and the fact that $Z_G(1, t) = 1/(1 - t)$. $\square$

Next, we will generalize this result to the groups $G = \mathrm{GL}_r$ acting on affine space. In doing so, the following definition will be useful.

Definition 7.2.3. Let X_n be a sequence of G-varieties over k. A family of G-invariant subvarieties $Y_n \subseteq X_n$ is *negligible* if $\lim_{n \to \infty} \dim X_n - \dim Y_n = \infty$. In particular, X_n is motivically stable with limit $\ell = \lim_{n \to \infty} [X_n] / \mathbb{L}^{\dim X_n}$ if and only if $Z_n = X_n \setminus Y_n$ is motivically stable with the same limit.

Proposition 7.2.4. *Let $G = \mathrm{GL}_r$ act naturally on $\mathbb{A}^r_k$ for some $r \geq 1$. Then*

$$\lim_{n\to\infty} \frac{\mathrm{Sym}^n_G[\mathbb{A}^r_k]}{\mathbb{L}^{nr}} = \frac{[\mathrm{GL}_r]}{\prod_{i=1}^r(\mathbb{L}^r - \mathbb{L}^{i-1})} = \prod_{i=1}^r \frac{[\mathbb{A}^r_k] - \mathbb{L}^{i-1}}{\mathbb{L}^r - \mathbb{L}^{i-1}}$$

with GL_r *acting transitively on itself.*

Proof. Let $X_n \subseteq \mathrm{Sym}^n_G \mathbb{A}^r_k$ be the strata where GL_r acts freely, that is, the strata of points whose stabilizer is trivial. Then $X_n \to X_n /\!\!/ \mathrm{GL}_r$ is a GL_r-torsor, so $[X_n] = [X_n /\!\!/ \mathrm{GL}_r][\mathrm{GL}_r]$ since GL_r is a special group. In particular, if we show that the complement $Y_n = (\mathrm{Sym}^n_G \mathbb{A}^r_k) \setminus X_n$ of points with non-trivial stabilizer is negligible, then the result follows as

$$\lim_{n\to\infty} \frac{\mathrm{Sym}^n_G[\mathbb{A}^r_k]}{\mathbb{L}^{nr}} = \lim_{n\to\infty} \frac{[X_n]}{\mathbb{L}^{nr}} = \lim_{n\to\infty} \frac{[X_n /\!\!/ \mathrm{GL}_r]}{\mathbb{L}^{nr}}[\mathrm{GL}_r]$$

where

$$\lim_{n\to\infty} \frac{[X_n /\!\!/ \mathrm{GL}_r]}{\mathbb{L}^{nr}} = \lim_{n\to\infty} \frac{[X_n]}{\mathbb{L}^{nr}}[\mathrm{GL}_r]^{-1} = [\mathrm{GL}_r]^{-1} = \frac{1}{\prod_{i=1}^r(\mathbb{L}^r - \mathbb{L}^{i-1})}.$$

To show that Y_n is negligible, suppose $(x_1, \ldots, x_n) \in (\mathbb{A}^r_k)^n$ is a point which (in passing to the quotient by S_n) is stabilized by some non-trivial $A \in \mathrm{GL}_r$. Then there is a permutation $\sigma \in S_n$ such that $Ax_i = x_{\sigma(i)}$ for all $i = 1, \ldots, n$. Hence, there is a surjection

$$\bigsqcup_{\sigma \in S_n} Z_\sigma \to Y_n$$

with $Z_\sigma = \left\{(A, x_1, \ldots, x_n) \in (\mathrm{GL}_r \setminus \{1\}) \times (\mathbb{A}^r_k)^n \mid Ax_i = x_{\sigma(i)}\right\}$. We claim that $\dim Z_\sigma \leq \dim \mathrm{GL}_r + nr - n$ for all $\sigma \in S_n$, from which it follows that $\dim Y_n \leq \dim \mathrm{GL}_r + nr - n$, which in turn implies Y_n is negligible. To prove this claim, fix some $\sigma \in S_n$ and write $\sigma = \tau_1\tau_2 \ldots \tau_s$ in canonical cycle notation (in particular, we do not omit 1-cycles). Then for every cycle $\tau = (i_1\ i_2\ \ldots\ i_m)$, let

$$Z_\tau = \left\{(A, x_{i_1}, \ldots, x_{i_m}) \mid Ax_{i_j} = x_{\tau(i_j)}\right\}.$$

If τ is a 1-cycle, then $\dim Z_\tau \leq \dim \mathrm{GL}_r + r - 1$ since A is non-trivial. If τ is an $(m \geq 2)$-cycle, then $\dim Z_\tau \leq \dim \mathrm{GL}_r + r$. Simple combinatorics now yields

$$\dim Z_\sigma = \dim(Z_{\tau_1} \times_{\mathrm{GL}_r \setminus \{1\}} \cdots \times_{\mathrm{GL}_r \setminus \{1\}} Z_{\tau_s}) \leq \dim \mathrm{GL}_r + nr - n. \qquad \square$$

Remark 7.2.5. Note that Proposition 7.2.1 is a special case of this proposition, but with an alternative proof.

Finally, we want to extend this result to any linear algebraic group G acting linearly on affine space. In order to relate $\widehat{\mathcal{M}^G_{\mathbb{L}}}$ for various G, consider the following lemma.

Lemma 7.2.6. *Let G be an algebraic group over k with subgroup $H \subseteq G$. The morphisms Res_H^G and Ind_H^G of Definition 3.6.5 extend to continuous morphisms*

$$\operatorname{Res}_H^G \colon \widehat{\mathcal{M}_{\mathbb{L}}^G} \to \widehat{\mathcal{M}_{\mathbb{L}}^H} \quad \text{and} \quad \operatorname{Ind}_H^G \colon \widehat{\mathcal{M}_{\mathbb{L}}^H} \to \widehat{\mathcal{M}_{\mathbb{L}}^G}.$$

In fact, Res_H^G is defined for any morphism $H \to G$ of algebraic groups over k.

Proof. Since $\operatorname{Res}_H^G(F_m\mathcal{M}_{\mathbb{L}}^G) \subseteq F_m\mathcal{M}_{\mathbb{L}}^H$ and $\operatorname{Ind}_H^G(F_m\mathcal{M}_{\mathbb{L}}^H) \subseteq F_{m'}\mathcal{M}_{\mathbb{L}}^G$, with $m' = m + \dim G - \dim H$, both Ind_H^G and Res_H^G extend to the completions. □

Corollary 7.2.7. *Let G be an algebraic group over k acting on $\mathbb{A}_k^r$ via some morphism $\rho\colon G \to \mathrm{GL}_r$ of algebraic groups. Then*

$$\lim_{n\to\infty} \frac{\operatorname{Sym}_G^n[\mathbb{A}_k^r]}{\mathbb{L}^{nr}} = \frac{[\mathrm{GL}_r]}{\prod_{i=1}^r(\mathbb{L}^r - \mathbb{L}^{i-1})}$$

where G acts on GL_r by multiplication via ρ.

Proof. Use Proposition 7.2.4 and that $\operatorname{Res}_G^{\mathrm{GL}_r} \circ \operatorname{Sym}_{\mathrm{GL}_r}^n = \operatorname{Sym}_G^n \circ \operatorname{Res}_G^{\mathrm{GL}_r}$. □

7.3 Motivic representation stability

In the context of motivic stability, it is typical to consider a sequence of symmetric powers $\operatorname{Sym}^n X = X^n /\!\!/ S_n$ of a variety X over k. However, one can more generally consider the whole X^n together with the action of S_n by permutation. One can then attempt to study the stability of the S_n-virtual class of X^n, as in Definition 3.6.12.

However, two problems arise. First of all, the group S_n depends on n, so to talk about stability, we must identify the irreducible representations of S_n for varying n. Recall that the irreducible representations of S_n are parametrized by the partitions of n [FH91]. Write V_λ for the irreducible representation of S_n corresponding to a partition λ of n. For any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ and any integer $n \geq |\lambda| + \lambda_1$, denote by $\lambda[n]$ the partition of n given by

$$\lambda[n] = (n - |\lambda|, \lambda_1, \lambda_2, \ldots).$$

Then, we think of the family $V_{\lambda[n]}$ of irreducible representations of S_n as corresponding to each other.

The second problem is that the S_n-virtual class depends on the choice of a set $\mathcal{H}$ of subgroups of S_n. One could, as in Example 3.6.15, take set of Young subgroups

$$\mathcal{H} = \{S_{\lambda_1} \times \cdots \times S_{\lambda_k} \mid \lambda \text{ is a partition of } n\}. \tag{7.1}$$

This idea will give rise to Definition 7.3.4. However, to get rid of the choice, we will first consider invariants in $\mathrm{K}_0(\mathcal{A})$ instead of $\mathrm{K}_0(\mathbf{Var}_k)$ for some suitable category $\mathcal{A}$ and functor $\mathcal{X}\colon \mathbf{Var}_k \to \mathcal{A}$. We will assume the following:

- $\mathcal{A}$ is a K-linear idempotent complete tensor triangulated category, with K a field of characteristic zero.
- The functor $\mathcal{X}$ induces a ring morphism $\mathrm{K}_0(\mathbf{Var}_k) \to \mathrm{K}_0(\mathcal{A})$. For any element $x \in \mathrm{K}_0(\mathbf{Var}_k)$, we will denote its image in $\mathrm{K}_0(\mathcal{A})$ also by x.
- For any finite group G and G-variety X over k, the coefficient of $[\mathcal{X}(X)]^G \in \mathrm{K}_0(\mathcal{A}) \otimes R_K(G)$ corresponding to the trivial representation equals $[\mathcal{X}(X /\!\!/ G)]$.

Inspired by [CF13, Definition 2.3], we introduce the following definition.

Definition 7.3.1. Let X_n be a sequence of varieties over k with an action of S_n. The sequence is *$\mathcal{A}$-representation stable* if, writing

$$[X_n]^{S_n} = \sum_{\lambda[n]} [X_n]_{\lambda[n]} \otimes [V_{\lambda[n]}] \in \mathrm{K}_0(\mathcal{A}) \otimes R_{\mathbb{Q}}(S_n),$$

the coefficients $[X_n]_{\lambda[n]}/\mathbb{L}^{\dim X_n}$ are eventually independent of n.

One way to compute the coefficients $[X_n]_\mu$, for partitions μ of n, is to look at the virtual classes of the quotients $X_n /\!\!/ S_\lambda$ with $S_\lambda \in \mathcal{H}$. This way, one inevitably encounters the Kostka numbers $K_{\mu\lambda}$. We will need the following lemma.

Lemma 7.3.2. *Let λ and μ be partitions. The Kostka number $K_{\mu[n]\lambda[n]}$ is independent of n for $n \geq |\lambda| + \mu_1$.*

Proof. Recall that $K_{\mu\lambda}$ is equal to the number of ways to fill the Young diagram of μ with λ_1 1's, λ_2 2's, etc., such that the resulting tableau is non-decreasing along rows and strictly increasing along columns [FH91]. Denote by $A_{\mu\lambda}$ the set of such tableaux. In particular, $K_{\mu\lambda} = |A_{\mu\lambda}|$.

For $|\mu| > |\lambda|$, we have $\mu[n] < \lambda[n]$, and hence $K_{\mu[n]\lambda[n]} = 0$. Now suppose $|\mu| \leq |\lambda|$. Considering $A_{\mu[n]\lambda[n]}$, note that all $(n - |\lambda|)$ 1's must be placed on the first row of the Young diagram of $\mu[n]$. Therefore, any Young tableau in $A_{\mu[n]\lambda[n]}$ is completely determined by the second through last rows and the last $|\lambda| - |\mu|$ entries of the first row. Note that, for $n \geq |\lambda| + \mu_1$, these last $|\lambda| - |\mu|$ entries do not put any restrictions on the entries of the second through last rows. Hence, we obtain a bijection between $A_{\mu[n]\lambda[n]}$ and $A_{\mu[n'],\lambda[n']}$ for all $n, n' \geq |\lambda| + \mu_1$, which shows that $K_{\mu[n]\lambda[n]} = K_{\mu[n']\lambda[n']}$. ☐

Proposition 7.3.3. *Suppose the sequences $[X_n /\!\!/ S_{\lambda[n]}]/\mathbb{L}^{\dim X_n} \in \mathrm{K}_0(\mathcal{A})$ stabilize for all partitions λ. Then, the sequence X_n is $\mathcal{A}$-representation stable.*

Proof. Write $[X_n]^{S_n} = \sum_{\lambda[n]} [X_n]_{\lambda[n]} \otimes [V_{\lambda[n]}]$. For any λ, we have, similar to Example 3.6.15,

$$\begin{aligned}[X_n /\!\!/ S_{\lambda[n]}] &= \left\langle T_{S_{\lambda[n]}}, \operatorname{Res}^{S_n}_{S_{\lambda[n]}} [X_n]^{S_n} \right\rangle \\ &= \left\langle \operatorname{Ind}^{S_n}_{S_{\lambda[n]}} T_{S_{\lambda[n]}}, [X_n]^{S_n} \right\rangle \\ &= \sum_{\mu[n] \geq \lambda[n]} K_{\mu[n]\lambda[n]} [X_n]^{\mu[n]}.\end{aligned}$$

Note that there are, independent of n, only finitely many partitions μ such that $\mu[n] \geq \lambda[n]$: those μ with $|\mu| < |\lambda|$, and those with $|\mu| = |\lambda|$ and $\mu > \lambda$.

By Lemma 7.3.2, the numbers $K_{\mu[n]\lambda[n]}$ are, for sufficiently large n, independent of n. Hence, $[X_n]_{\lambda[n]}$ can be expressed as a linear combination of $[X /\!\!/ S_{\mu[n]}]$ with $\mu[n] \geq \lambda[n]$, where the coefficients do not change for sufficiently large $n \geq 2|\lambda| \geq |\lambda| + \mu_1$. □

This motivates the following definition. Also agrees with stabilization of G-virtual class with $\mathcal{H}$ given by (7.1).

Definition 7.3.4. Let X_n be a sequence of varieties over k with an action of S_n. The sequence is said to be *motivically representation stable* if the sequences $[X_n /\!\!/ S_{\lambda[n]}]$ are motivically stable for all partitions λ. In particular, this implies X_n is $\mathcal{A}$-representation stable for all $\mathcal{A}$ and $\mathcal{X} \colon \mathbf{Var}_k \to \mathcal{A}$ as above. Also, in particular, the sequence $[X_n /\!\!/ S_n]$ is motivically stable.

More generally, a sequence X_n of $(G \times S_n)$-varieties over k is *motivically representation stable* if the sequences $[X_n /\!\!/ S_{\lambda[n]}]$ are motivically stable, as sequences of G-varieties, for all partitions λ.

Example 7.3.5. Let X be a variety over k whose sequence of symmetric powers $\operatorname{Sym}^n X$ is motivically stable, and let $X_n = X^n$ with S_n acting by permutation. Then, for any partition λ, the sequence

$$[X_n /\!\!/ S_{\lambda[n]}] = \operatorname{Sym}^{n-|\lambda|} X \times \prod_{i \geq 1} \operatorname{Sym}^{\lambda_i} X$$

is motivically stable. In particular, X_n is motivically representation stable.

7.4 $\mathbf{GL}_r$-character stacks

The goal of this section is to show the sequences of character stacks

$$\mathfrak{X}_n = \mathfrak{X}_G(\Gamma_n) = [R_G(\Gamma_n)/G]$$

of the free groups $\Gamma_n = F_n$ and the free abelian groups $\Gamma_n = \mathbb{Z}^n$ are motivic representation stable for the general linear groups $G = \mathrm{GL}_r$ of any rank $r \geq 0$ over a field k, where the action of S_n is induced from the action of S_n on Γ_n by permutation. However, since the notion of motivic representation stability is only defined for (G-)varieties, we will instead prove that the sequences of representation varieties $X_n = R_G(\Gamma_n)$ are motivically representation stable as sequences of G-varieties. Indeed, note that the action of G by conjugation and the action of S_n by permutation commute.

The case of $\Gamma_n = F_n$ turns out to be a quick consequence of the theory developed in the previous sections.

Theorem 7.4.1. *For every $r \geq 0$, the sequence of GL_r-representation varieties*

$$X_n = R_{\mathrm{GL}_r}(F_n)$$

with the action of GL_r by conjugation, and the action of S_n by permutation, is motivically representation stable.

Proof. For any $n \geq 1$, write $X_n = R_{\mathrm{GL}_r}(F_n) = (\mathrm{GL}_r)^n$. Given any partition λ, we find

$$X_n /\!\!/ S_{\lambda[n]} = \mathrm{Sym}^{n-|\lambda|}_{\mathrm{GL}_r} \mathrm{GL}_r \times \prod_{i \geq 1} \mathrm{Sym}^{\lambda_i}_{\mathrm{GL}_r} \mathrm{GL}_r,$$

where GL_r acts on itself by conjugation. Viewing GL_r as a dense open subset of $\mathbb{A}^{r^2}_k$, the action of GL_r on itself is linear, and hence the sequence $X_n /\!\!/ S_{\lambda[n]}$ is motivically stable by Corollary 7.2.7 and Proposition 7.1.12. ☐

For the remainder of this section, we will focus on the case $\Gamma_n = \mathbb{Z}^n$, and assume that k is algebraically closed.

Theorem 7.4.2. *For every $r \geq 0$, the sequence of GL_r-representation varieties*

$$X_n = R_{\mathrm{GL}_r}(\mathbb{Z}^n)$$

with the action of GL_r by conjugation, and the action of S_n by permutation, is motivically representation stable.

Notation-wise, we will use the following presentation of X_n, as the closed subvariety of $(\mathrm{GL}_r)^n$ given by commuting tuples of elements $A_i \in \mathrm{GL}_r$.

$$X_n = \left\{ (A_1, \ldots, A_n) \in (\mathrm{GL}_r)^n \;\middle|\; \text{all } A_i \text{ commute} \right\}$$

Interestingly, it turns out the cases $r \leq 3$ should be treated differently from the general case $r > 3$. We will first treat the cases $r = 2, 3$.

Proposition 7.4.3. *The* GL_2*-representation varieties* $X_n = R_{\mathrm{GL}_2}(\mathbb{Z}^n)$ *are motivically representation stable.*

Proof. Consider the possible Jordan normal forms of an element $A \in \mathrm{GL}_2$.

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

In particular, note that a matrix of the form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$, with $\lambda \neq \mu$, only commutes with diagonal matrices, and that a matrix of the form $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ only commutes with matrices of the form $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$. Therefore, X_n can be stratified by the subvarieties

$$\begin{aligned} Y_n &= \{A \in X_n \mid \text{all } A_i \text{ are scalar}\}, \\ J_n &= \left\{A \in X_n \mid \text{some } A_i \text{ is conjugate to } \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)\right\} \\ \text{and} \quad M_n &= \left\{A \in X_n \mid \text{some } A_i \text{ is conjugate to } \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)\right\}. \end{aligned}$$

Simultaneously conjugating the A_i into normal form, we can express J_n as

$$J_n = \mathrm{Ind}_H^{\mathrm{GL}_2} (J^n \setminus Y_n)$$

where $J = \{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) \mid x \neq 0\}$ and $H = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, c \neq 0\}$ the stabilizer of J. Similarly, we have

$$M_n = \mathrm{Ind}_K^{\mathrm{GL}_2} (M^n \setminus Y_n)$$

where $M = \left\{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right) \mid x, y \neq 0\right\}$ and $K = \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right) \mid a, b \neq 0\}$ the stabilizer of M. Clearly $\dim Y_n = n$ while $\dim J_n, \dim M_n \geq 2n$, implying $Y_n \subseteq X_n$ is negligible. Hence, it suffices to show that the sequences J'_n and M'_n, given by

$$J'_n = \mathrm{Ind}_H^{\mathrm{GL}_2} (J^n) \qquad \text{and} \qquad M'_n = \mathrm{Ind}_K^{\mathrm{GL}_2} (M^n)$$

are motivically representation stable.

First we consider J'_n. Note that, for any partition λ, the actions of $S_{\lambda[n]}$ and GL_2 on J^n commute, so that

$$J'_n /\!\!/ S_{\lambda[n]} = \mathrm{Ind}_H^{\mathrm{GL}_2} \Big(\mathrm{Sym}_H^{n-|\lambda|} J \times \prod_{i \geq 1} \mathrm{Sym}_H^{\lambda_i} J\Big).$$

Note that the action of H on J is linear, viewing J as an open dense subvariety of $\mathbb{A}^2_k$. Hence, the sequence $J'_n /\!\!/ S_{\lambda[n]}$ is motivically stable as a result of Corollary 7.2.7, Proposition 7.1.12 and Lemma 7.2.6.

The argument regarding M_n is analogous: for any partition λ, we have

$$M'_n /\!\!/ S_{\lambda[n]} = \mathrm{Ind}_K^{\mathrm{GL}_2} \Big(\mathrm{Sym}_K^{n-|\lambda|} M \times \prod_{i \geq 1} \mathrm{Sym}_K^{\lambda_i} M\Big).$$

Again, the action of K on $M \subseteq \mathbb{A}^2_k$ is linear, so the sequence $M'_n /\!\!/ S_{\lambda[n]}$ is also motivically stable. □

Proposition 7.4.4. *The* GL_3*-representation varieties* $X_n = R_{\mathrm{GL}_3}(\mathbb{Z}^n)$ *are motivically stable.*

Proof. The proof is very similar to that of Proposition 7.4.3. Consider the possible Jordan normal forms of an element $A \in \mathrm{GL}_3$.

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \qquad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \rho \end{pmatrix}$$

Having analyzed which matrices commute with each Jordan type, we stratify X_n by the subvarieties

$$\begin{aligned}
Y_n^0 &= \left\{ A \in X_n \mid \text{ all } A_i \text{ are conjugate to } \left(\begin{smallmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{smallmatrix}\right), \left(\begin{smallmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{smallmatrix}\right) \text{ or } \left(\begin{smallmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{smallmatrix}\right) \right\}, \\
Y_n^1 &= \left\{ A \in X_n \mid \text{ some } A_i \text{ is conjugate to } \left(\begin{smallmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{smallmatrix}\right) \right\}, \\
Y_n^2 &= \left\{ A \in X_n \mid \text{ some } A_i \text{ is conjugate to } \left(\begin{smallmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{smallmatrix}\right) \right\}, \\
Y_n^3 &= \left\{ A \in X_n \mid \text{ some } A_i \text{ is conjugate to } \left(\begin{smallmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \rho \end{smallmatrix}\right) \right\}.
\end{aligned}$$

Note that the sequences Y_n^1, Y_n^2 and Y_n^3 do not intersect since matrices that have different Jordan type in the bottom row never commute. As in the proof of Proposition 7.4.3, to show motivic representation stability of the strata Y_n^i, it suffices to show motivic representation stability of the sequences

$$Y'^i_n = \mathrm{Ind}_{H_i}^{\mathrm{GL}_3}(J_i^n) \qquad \text{for } i = 1, 2, 3,$$

where

$$\begin{aligned}
J_1 &= \left\{ \left(\begin{smallmatrix} x & y & z \\ 0 & x & y \\ 0 & 0 & x \end{smallmatrix}\right) \mid x \neq 0 \right\} & H_1 &= \left\{ \left(\begin{smallmatrix} a & b & c \\ 0 & 1 & d \\ 0 & 0 & 1/a \end{smallmatrix}\right) \mid a \neq 0 \right\} \\
J_2 &= \left\{ \left(\begin{smallmatrix} x & y & 0 \\ 0 & x & 0 \\ 0 & 0 & z \end{smallmatrix}\right) \mid x, z \neq 0 \right\} & H_2 &= \left\{ \left(\begin{smallmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{smallmatrix}\right) \mid a, c, d \neq 0 \right\} \\
J_3 &= \left\{ \left(\begin{smallmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{smallmatrix}\right) \mid x, y, z \neq 0 \right\} & H_3 &= \mathbb{G}_m^3 \rtimes S_3.
\end{aligned}$$

More precisely, $H_3 \subseteq \mathrm{GL}_r$ is the subgroup generated by the diagonal matrices and the permutation matrices.

Now, for any partition λ, we find

$$Y'^{i}_{n} /\!\!/ S_{\lambda[n]} = \mathrm{Ind}_{H_i}^{\mathrm{GL}_3}\Big(\mathrm{Sym}_{H_i}^{n-|\lambda|} J_i \times \prod_{j\geq 1} \mathrm{Sym}_{H_i}^{\lambda_j} J_i\Big).$$

For all i, the group H_i acts linearly on J_i, a dense open of $\mathbb{A}^3_k$, so it follows from Corollary 7.2.7, Proposition 7.1.12 and Lemma 7.2.6 that the limits $\lim_{n\to\infty} Y^i_n/\mathbb{L}^{3n}$ exist. Knowing that $\dim X_n \geq 3n$, we see that $Y^0_n \subseteq X_n$ is negligible, and the result follows. □

Looking at the proofs of Proposition 7.4.3 and Proposition 7.4.4, it might be tempting to think that in the general case the non-negligible strata are those containing matrices with maximal Jordan type. However, this turns out to be the case only for $r \leq 3$.

For the general case we use a result initially proved by Schur [Sch05], and later reproved by Jacobson [Jac44], about the maximum number of linearly independent commuting matrices. This leads to the idea of stratifying the representation varieties $R_{\mathrm{GL}_r}(\mathbb{Z}^n)$ by the dimension of the linear subspace inside $\mathrm{Mat}_{r\times r}$ spanned by the matrices A_i.

Proof of Theorem 7.4.2. The case $r = 0$ is obvious, and the case $r = 1$ follows from motivic representation stability of $\mathbb{G}_m^n$, see Example 7.1.11 and Example 7.3.5. The cases $r = 2$ and $r = 3$ were treated in Proposition 7.4.3 and Proposition 7.4.4, so we can assume $r > 3$.

As usual, write $X_n = R_{\mathrm{GL}_r}(\mathbb{Z}^n)$ for all $n \geq 1$. For any point $A \in X_n$ corresponding to a tuple $(A_1, \dots, A_n)$ of commuting elements in GL_r, define

$$d_A = \dim_k \langle A_1, \dots, A_n \rangle$$

to be the dimension of the linear subspace of $\mathrm{Mat}_{r\times r}(k)$ spanned by the A_i. By [Jac44, Theorem 1], we have $d_A \leq m$ with

$$m = \begin{cases} r^2/4 + 1 & \text{if } r \text{ is even,} \\ (r^2-1)/4 + 1 & \text{if } r \text{ is odd.} \end{cases}$$

Note that d_A is invariant under the actions of S_n and GL_r, so X_n can be stratified equivariantly by

$$X_{n,d} = \{A \in X_n \mid d_A = d\} \qquad \text{for } 1 \leq d \leq m.$$

Now, we will show that $X_{n,d} \subseteq X_n$ is negligible for $d < m$, so that we solely need to focus on $X_{n,m}$. Note that the dimension of X_n is at least nm, as it contains the family of commuting matrices given by

$$A_1 = \left(\begin{array}{c|c} \lambda_1 I & M_1 \\ \hline 0 & \lambda_1 I \end{array}\right), \dots, A_n = \left(\begin{array}{c|c} \lambda_n I & M_n \\ \hline 0 & \lambda_n I \end{array}\right) \qquad (*)$$

with $\lambda_i \neq 0$, $M_i \in \mathrm{Mat}_{\frac{r}{2}\times\frac{r}{2}}$ if r is even, and $M_i \in \mathrm{Mat}_{\frac{r+1}{2}\times\frac{r-1}{2}}$ if r is odd. To see why the strata $X_{n,d}$ with $d < m$ are negligible, observe that $X_{n,d}$ can be covered by a dense open of $X_d \times (\mathbb{A}^d_k)^n$, that is, there is a surjective morphism from a dense open $Y_{n,d} \subseteq X_d \times (\mathbb{A}^d_k)^n$ given by

$$Y_{n,d} \to X_{n,d}, \quad \left((A_i)_{i=1}^d, (\alpha_{ij})_{i,j=1}^{n,d}\right) \mapsto \left(\sum_{j=1}^d \alpha_{ij} A_j\right)_{i=1}^n.$$

In particular, $\dim X_{n,d} \leq \dim Y_{n,d} \leq r^2 d + nd$, and hence $\lim_{n\to\infty} \dim X_n - \dim X_{n,d} = \infty$ for $d < m$, so it follows that $X_{n,d} \subseteq X_n$ is negligible.

By [Jac44, Theorem 3], every $A \in X_{n,m}$ can be conjugated to a tuple of the form $(*)$. Hence, to show motivic representation stability of $X_{n,m}$ it suffices to show motivic representation stability of

$$X'_{n,m} = \mathrm{Ind}_H^{\mathrm{GL}_r}\left(J^n\right) \quad \text{with} \quad J = \left\{ \left(\begin{array}{c|c} \lambda I & M \\ \hline 0 & \lambda I \end{array}\right) \;\middle|\; \begin{array}{c} \lambda \neq 0 \text{ and} \\ M \in \mathrm{Mat}_{\lceil \frac{r}{2} \rceil \times \lfloor \frac{r}{2} \rfloor} \end{array} \right\},$$

where the stabilizer

$$H = \left\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) \right\} \subseteq \mathrm{GL}_r$$

acts trivially on λ, and acts on M via the linear action

$$\left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) \cdot M = AMC^{-1}.$$

Now, from Corollary 7.2.7, Proposition 7.1.12 and Lemma 7.2.6, it follows that $\lim_{n\to\infty}[X_{n,m}]/\mathbb{L}^{\dim X_{n,m}}$ exists. Moreover, as all $X_{n,d}$ with $d < m$ are negligible, this limit is equal to $\lim_{n\to\infty}[X_n]/\mathbb{L}^{\dim X_n}$. □

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Appendix A

TQFT for upper triangular matrices

The following pages describe the $\mathrm{K}_0(\mathbf{Var}_k)$-module morphism Z_G^{rep} () for the groups $G = \mathbb{U}_n$ and $G = \mathbb{T}_n$ over $k = \mathbb{C}$ for $2 \le n \le 5$. We restrict these maps to the $\mathrm{K}_0(\mathbf{Var}_k)$-submodule of $\mathrm{K}_0(\mathbf{Var}_G)$ generated by the elements $\mathbf{1}_{\mathcal{U}_1}, \ldots, \mathbf{1}_{\mathcal{U}_M} \in \mathrm{K}_0(\mathbf{Var}_G)$, corresponding to the inclusions of the unipotent conjugacy classes $\mathcal{U}_i \to G$, and express them as matrices with respect to these generators.

For every $2 \le n \le 5$, representatives for these unipotent conjugacy classes are, in order, given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&1\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0\\0&1&1\\0&0&1 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1 \end{pmatrix},$$
$$\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0\\0&1&0&1\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0\\0&1&0&0\\0&0&1&1\\0&0&0&1 \end{pmatrix},$$
$$\begin{pmatrix} 1&0&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1\\0&1&1&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&1&0\\0&1&0&0\\0&0&1&1\\0&0&0&1 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&1\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&1\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0\\0&1&0&0&1\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0\\0&1&0&0&1\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0\\0&1&0&0&1\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&0&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&1&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&1\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&1\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&1\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0\\0&1&0&0&1\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0\\0&1&0&0&1\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&1&0&0\\0&1&0&0&1\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0\\0&1&0&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&1&0&0\\0&1&0&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0\\0&1&0&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&1&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0\\0&1&1&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&1&0\\0&1&1&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&1&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0\\0&1&0&1&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0\\0&1&0&1&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&1&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&1&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&1&1&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0\\0&1&1&1&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&1&0&0\\0&1&1&1&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&1&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1 \end{pmatrix}$$

Case $G = \mathbb{U}_2$. The matrix associated to $Z^{\text{rep}}_{\mathbb{U}_2}$ () is given by

$$\begin{bmatrix} \mathbb{L}^2 & 0 \\ 0 & \mathbb{L}^2 \end{bmatrix},$$

for which both $\mathbf{1}_{\mathcal{U}_1}$ and $\mathbf{1}_{\mathcal{U}_2}$ are eigenvectors with eigenvalue $\mathbb{L}^2$.

Case $G = \mathbb{T}_2$. The matrix associated to $Z^{\text{rep}}_{\mathbb{T}_2}$ () is given by

$$\begin{bmatrix} \mathbb{L}^2(\mathbb{L}-1) & \mathbb{L}^2(\mathbb{L}-2)(\mathbb{L}-1) \\ \mathbb{L}^2(\mathbb{L}-2) & \mathbb{L}^2\left(\mathbb{L}^2-3\mathbb{L}+3\right) \end{bmatrix},$$

whose eigenvalues are $\mathbb{L}^2$ and $\mathbb{L}^2(\mathbb{L}-1)^2$ with respective eigenvectors

$$\begin{bmatrix} 1-\mathbb{L} \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Case $G = \mathbb{U}_3$**.** The matrix associated to $Z^{\mathrm{rep}}_{\mathbb{U}_3}$ () is given by

$$\begin{bmatrix} \mathbb{L}^3\left(\mathbb{L}^2+\mathbb{L}-1\right) & 0 & \mathbb{L}^3\left(\mathbb{L}-1\right)^2\left(\mathbb{L}+1\right) & 0 & 0 \\ 0 & \mathbb{L}^6 & 0 & 0 & 0 \\ \mathbb{L}^3\left(\mathbb{L}-1\right)\left(\mathbb{L}+1\right) & 0 & \mathbb{L}^3\left(\mathbb{L}^3-\mathbb{L}^2+1\right) & 0 & 0 \\ 0 & 0 & 0 & \mathbb{L}^6 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{L}^6 \end{bmatrix},$$

whose eigenvalues are $\mathbb{L}^4$ and $\mathbb{L}^6$ (with multiplicity 4), with respective eigenvectors

$$\begin{bmatrix} 1-\mathbb{L} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Case $G = \mathbb{T}_3$**.** The matrix associated to $Z^{\mathrm{rep}}_{\mathbb{T}_3}$ () is given by

$$\left[\begin{matrix} \mathbb{L}^3\left(\mathbb{L}-1\right)^2\left(\mathbb{L}^2+\mathbb{L}-1\right) & \mathbb{L}^6\left(\mathbb{L}-2\right)\left(\mathbb{L}-1\right)^2 & \mathbb{L}^3\left(\mathbb{L}-1\right)^4\left(\mathbb{L}+1\right) \\ \mathbb{L}^5\left(\mathbb{L}-2\right)\left(\mathbb{L}-1\right) & \mathbb{L}^6\left(\mathbb{L}-1\right)\left(\mathbb{L}^2-3\mathbb{L}+3\right) & \mathbb{L}^5\left(\mathbb{L}-2\right)\left(\mathbb{L}-1\right)^2 \\ \mathbb{L}^3\left(\mathbb{L}-1\right)^3\left(\mathbb{L}+1\right) & \mathbb{L}^6\left(\mathbb{L}-2\right)\left(\mathbb{L}-1\right)^2 & \mathbb{L}^3\left(\mathbb{L}-1\right)^2\left(\mathbb{L}^3-\mathbb{L}^2+1\right) \\ \mathbb{L}^5\left(\mathbb{L}-2\right)\left(\mathbb{L}-1\right) & \mathbb{L}^6\left(\mathbb{L}-2\right)^2\left(\mathbb{L}-1\right) & \mathbb{L}^5\left(\mathbb{L}-2\right)\left(\mathbb{L}-1\right)^2 \\ \mathbb{L}^5\left(\mathbb{L}-2\right)^2 & \mathbb{L}^6\left(\mathbb{L}-2\right)\left(\mathbb{L}^2-3\mathbb{L}+3\right) & \mathbb{L}^5\left(\mathbb{L}-2\right)^2\left(\mathbb{L}-1\right) \end{matrix}\right.$$

$$\left.\begin{matrix} \mathbb{L}^6\left(\mathbb{L}-2\right)\left(\mathbb{L}-1\right)^2 & \mathbb{L}^6\left(\mathbb{L}-2\right)^2\left(\mathbb{L}-1\right)^2 \\ \mathbb{L}^6\left(\mathbb{L}-2\right)^2\left(\mathbb{L}-1\right) & \mathbb{L}^6\left(\mathbb{L}-2\right)\left(\mathbb{L}-1\right)\left(\mathbb{L}^2-3\mathbb{L}+3\right) \\ \mathbb{L}^6\left(\mathbb{L}-2\right)\left(\mathbb{L}-1\right)^2 & \mathbb{L}^6\left(\mathbb{L}-2\right)^2\left(\mathbb{L}-1\right)^2 \\ \mathbb{L}^6\left(\mathbb{L}-1\right)\left(\mathbb{L}^2-3\mathbb{L}+3\right) & \mathbb{L}^6\left(\mathbb{L}-2\right)\left(\mathbb{L}-1\right)\left(\mathbb{L}^2-3\mathbb{L}+3\right) \\ \mathbb{L}^6\left(\mathbb{L}-2\right)\left(\mathbb{L}^2-3\mathbb{L}+3\right) & \mathbb{L}^6\left(\mathbb{L}^2-3\mathbb{L}+3\right)^2 \end{matrix}\right],$$

whose eigenvalues are

$$\mathbb{L}^6, \quad \mathbb{L}^4\left(\mathbb{L}-1\right)^2, \quad \mathbb{L}^6\left(\mathbb{L}-1\right)^2, \quad \mathbb{L}^6\left(\mathbb{L}-1\right)^2, \quad \mathbb{L}^6\left(\mathbb{L}-1\right)^4$$

with respective eigenvectors

$$\begin{bmatrix} \mathbb{L}^2-2\mathbb{L}+1 \\ 1-\mathbb{L} \\ \mathbb{L}^2-2\mathbb{L}+1 \\ 1-\mathbb{L} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1-\mathbb{L} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1-\mathbb{L} \\ 2-\mathbb{L} \\ 1-\mathbb{L} \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Case $G = \mathbb{U}_4$. The matrix associated to $Z^{\text{rep}}_{\mathbb{U}_4}$ () (which we do not print due to its size) has eigenvalues, with multiplicity, given by

$$\mathbb{L}^8 \text{ (mult. 2)}, \quad \mathbb{L}^{10} \text{ (mult. 6)}, \quad \mathbb{L}^{12} \text{ (mult. 8)}$$

with respective eigenvectors

- $\mathbf{1}_{\mathcal{U}_4} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_1}$
- $\mathbf{1}_{\mathcal{U}_{14}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_5}$
- $\mathbf{1}_{\mathcal{U}_3} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_4})$
- $\mathbf{1}_{\mathcal{U}_3} - \mathbf{1}_{\mathcal{U}_6}$
- $\mathbf{1}_{\mathcal{U}_9} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_2}$
- $\mathbf{1}_{\mathcal{U}_{11}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_3}$
- $\mathbf{1}_{\mathcal{U}_{12}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_7}$
- $\mathbf{1}_{\mathcal{U}_{16}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{10}}$
- $\mathbf{1}_{\mathcal{U}_8}$
- $\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_9}$
- $\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_{11}}$
- $\mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_{12}}$
- $\mathbf{1}_{\mathcal{U}_{13}}$
- $\mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{14}}$
- $\mathbf{1}_{\mathcal{U}_{15}}$
- $\mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{16}}$.

Case $G = \mathbb{T}_4$. The matrix associated to $Z^{\text{rep}}_{\mathbb{T}_4}$ () has eigenvalues, with multiplicity, given by

$$\mathbb{L}^{10}, \quad \mathbb{L}^{12}, \quad \mathbb{L}^8(\mathbb{L}-1)^2, \quad \mathbb{L}^{10}(\mathbb{L}-1)^2 \text{ (mult. 3)}, \quad \mathbb{L}^{12}(\mathbb{L}-1)^2 \text{ (mult. 3)},$$

$$\mathbb{L}^8(\mathbb{L}-1)^4, \quad \mathbb{L}^{10}(\mathbb{L}-1)^4 \text{ (mult. 2)}, \quad \mathbb{L}^{12}(\mathbb{L}-1)^4 \text{ (mult. 3)}, \quad \mathbb{L}^{12}(\mathbb{L}-1)^6$$

with respective eigenvectors

- $\mathbf{1}_{\mathcal{U}_{16}} + (\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_4}) - (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_6}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{10}} - \mathbf{1}_{\mathcal{U}_{11}})$
- $\mathbf{1}_{\mathcal{U}_{15}} - (\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_{11}}) + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{14}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{16}})$
- $\mathbf{1}_{\mathcal{U}_{14}} + \mathbb{L}(\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_1} - \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_4} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_5}$
- $\mathbf{1}_{\mathcal{U}_9} + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_{11}})$
- $\mathbf{1}_{\mathcal{U}_9} - \mathbf{1}_{\mathcal{U}_{12}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_2} - \mathbf{1}_{\mathcal{U}_7}) + \mathbb{L}(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_3} - \mathbf{1}_{\mathcal{U}_6})$
- $\mathbf{1}_{\mathcal{U}_9} + \mathbb{L}(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_4}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_{16}}) - \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_6} + (\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_{10}}$
- $\mathbf{1}_{\mathcal{U}_8} - \mathbf{1}_{\mathcal{U}_{13}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_2} - \mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_9} - \mathbf{1}_{\mathcal{U}_{12}})$
- $\mathbf{1}_{\mathcal{U}_{15}} + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_6} - \mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_{11}} - \mathbf{1}_{\mathcal{U}_{12}}) + (\mathbb{L}-2)(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{14}}) - (2\mathbb{L}-3)\mathbf{1}_{\mathcal{U}_8}$
- $\mathbf{1}_{\mathcal{U}_8} - \mathbf{1}_{\mathcal{U}_{10}} - \mathbf{1}_{\mathcal{U}_{16}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_5} - \mathbf{1}_{\mathcal{U}_7} - \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{14}})$
- $\mathbf{1}_{\mathcal{U}_{14}} - \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_1} + \mathbb{L}\mathbf{1}_{\mathcal{U}_4} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_5}$

- $\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{11}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4})$
- $\mathbf{1}_{\mathcal{U}_9} - \mathbf{1}_{\mathcal{U}_{12}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_2} - \mathbf{1}_{\mathcal{U}_7}) - \mathbb{L}(\mathbf{1}_{\mathcal{U}_3} - \mathbf{1}_{\mathcal{U}_6})$
- $\mathbf{1}_{\mathcal{U}_2} - \mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_9} - \mathbf{1}_{\mathcal{U}_{12}} - \mathbf{1}_{\mathcal{U}_{13}}$
- $\mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_{12}} - \mathbf{1}_{\mathcal{U}_{15}} + (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_{11}}) + (\mathbb{L}-2)(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{14}}) + (\mathbb{L}-3)\mathbf{1}_{\mathcal{U}_8}$
- $\mathbf{1}_{\mathcal{U}_5} - \mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_8} - \mathbf{1}_{\mathcal{U}_{10}} - \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{14}} - \mathbf{1}_{\mathcal{U}_{16}}$
- $\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{16}}$.

Case $G = \mathbb{U}_5$. The matrix associated to $Z^{\mathrm{rep}}_{\mathbb{U}_5}$ () has eigenvalues, with multiplicity, given by

$$\mathbb{L}^{12}, \quad \mathbb{L}^{14} \text{ (mult. 6)}, \quad \mathbb{L}^{16} \text{ (mult. 18)}, \quad \mathbb{L}^{18} \text{ (mult. 20)}, \quad \mathbb{L}^{20} \text{ (mult. 16)}$$

with respective eigenvectors

- $\mathbf{1}_{\mathcal{U}_{36}} + \mathbb{L}\,(\mathbb{L}-1)^2\,\mathbf{1}_{\mathcal{U}_1} - \mathbb{L}\,(\mathbb{L}-1)\,\mathbf{1}_{\mathcal{U}_5} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_7}$
- $\mathbf{1}_{\mathcal{U}_{23}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_9}$
- $\mathbf{1}_{\mathcal{U}_{28}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_6}$
- $\mathbf{1}_{\mathcal{U}_{32}} + (\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_5}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_8})$
- $\mathbf{1}_{\mathcal{U}_{36}} - \mathbb{L}\,(\mathbb{L}-1)\,\mathbf{1}_{\mathcal{U}_1} + \mathbb{L}\mathbf{1}_{\mathcal{U}_5} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_7}$
- $\mathbf{1}_{\mathcal{U}_{46}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{15}}$
- $\mathbf{1}_{\mathcal{U}_{53}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{19}}$
- $\mathbf{1}_{\mathcal{U}_4} - \mathbf{1}_{\mathcal{U}_8}$
- $\mathbf{1}_{\mathcal{U}_{17}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{11}}$
- $\mathbf{1}_{\mathcal{U}_{22}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{10}}$
- $\mathbf{1}_{\mathcal{U}_{18}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{23}})$
- $\mathbf{1}_{\mathcal{U}_{26}} + \mathbb{L}\,(\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5}) - \mathbb{L}\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_{10}})$
- $\mathbf{1}_{\mathcal{U}_{27}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_{28}})$
- $\mathbf{1}_{\mathcal{U}_{32}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_5}) - (\mathbb{L}-2)\mathbf{1}_{\mathcal{U}_4}$
- $\mathbf{1}_{\mathcal{U}_{34}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_3}$
- $\mathbf{1}_{\mathcal{U}_{35}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_2}$
- $\mathbf{1}_{\mathcal{U}_{42}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{13}}$
- $\mathbf{1}_{\mathcal{U}_{47}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{16}}$
- $\mathbf{1}_{\mathcal{U}_{48}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{20}}$
- $\mathbf{1}_{\mathcal{U}_{50}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{21}}$
- $\mathbf{1}_{\mathcal{U}_{52}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{29}}$
- $\mathbf{1}_{\mathcal{U}_{55}} + (\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_{36}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{24}} + \mathbf{1}_{\mathcal{U}_{31}})$
- $\mathbf{1}_{\mathcal{U}_{56}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{25}}$
- $\mathbf{1}_{\mathcal{U}_{60}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{41}}$
- $\mathbf{1}_{\mathcal{U}_{61}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{51}}$
- $\mathbf{1}_{\mathcal{U}_{14}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{17}})$
- $\mathbf{1}_{\mathcal{U}_{24}} - \mathbf{1}_{\mathcal{U}_{31}}$

- $\mathbf{1}_{\mathcal{U}_{26}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_{32}}) - (\mathbb{L}-2)(\mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{22}})$
- $\mathbf{1}_{\mathcal{U}_3} - \mathbf{1}_{\mathcal{U}_{10}} - \mathbf{1}_{\mathcal{U}_{22}} + \mathbf{1}_{\mathcal{U}_{34}}$
- $\mathbf{1}_{\mathcal{U}_{33}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_{35}})$
- $\mathbf{1}_{\mathcal{U}_7} - \mathbf{1}_{\mathcal{U}_{10}} - \mathbf{1}_{\mathcal{U}_{22}} + \mathbf{1}_{\mathcal{U}_{24}} - \mathbf{1}_{\mathcal{U}_{26}} + \mathbf{1}_{\mathcal{U}_{36}}$
- $\mathbf{1}_{\mathcal{U}_{38}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{33}}$
- $\mathbf{1}_{\mathcal{U}_{40}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{12}}$
- $\mathbf{1}_{\mathcal{U}_{45}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{14}}$
- $\mathbf{1}_{\mathcal{U}_{44}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{46}})$
- $\mathbf{1}_{\mathcal{U}_{14}} - \mathbf{1}_{\mathcal{U}_{16}} - \mathbf{1}_{\mathcal{U}_{47}}$
- $\mathbf{1}_{\mathcal{U}_{21}} + \mathbf{1}_{\mathcal{U}_{50}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{23}})$
- $\mathbf{1}_{\mathcal{U}_{29}} + \mathbf{1}_{\mathcal{U}_{52}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_{27}} + \mathbf{1}_{\mathcal{U}_{28}})$
- $\mathbf{1}_{\mathcal{U}_{54}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{30}}$
- $\mathbf{1}_{\mathcal{U}_{24}} + \mathbf{1}_{\mathcal{U}_{55}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{22}} + \mathbf{1}_{\mathcal{U}_{26}})$
- $\mathbf{1}_{\mathcal{U}_{25}} - \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{56}}$
- $\mathbf{1}_{\mathcal{U}_{58}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{39}}$
- $\mathbf{1}_{\mathcal{U}_{59}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{43}}$
- $\mathbf{1}_{\mathcal{U}_{41}} + \mathbf{1}_{\mathcal{U}_{60}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{42}})$
- $\mathbf{1}_{\mathcal{U}_{51}} + \mathbf{1}_{\mathcal{U}_{61}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{48}})$
- $\mathbf{1}_{\mathcal{U}_{37}}$
- $\mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{40}}$
- $\mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{44}} + \mathbf{1}_{\mathcal{U}_{46}}$
- $\mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{17}} + \mathbf{1}_{\mathcal{U}_{45}} + \mathbf{1}_{\mathcal{U}_{47}}$
- $\mathbf{1}_{\mathcal{U}_{49}}$
- $\mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{21}} + \mathbf{1}_{\mathcal{U}_{23}} + \mathbf{1}_{\mathcal{U}_{50}}$
- $\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_{27}} + \mathbf{1}_{\mathcal{U}_{28}} + \mathbf{1}_{\mathcal{U}_{29}} + \mathbf{1}_{\mathcal{U}_{52}}$
- $\mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{53}}$
- $\mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{54}}$
- $\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{22}} + \mathbf{1}_{\mathcal{U}_{24}} + \mathbf{1}_{\mathcal{U}_{26}} + \mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{32}} + \mathbf{1}_{\mathcal{U}_{34}} + \mathbf{1}_{\mathcal{U}_{36}} + \mathbf{1}_{\mathcal{U}_{55}}$
- $\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_{25}} + \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{38}} + \mathbf{1}_{\mathcal{U}_{56}}$
- $\mathbf{1}_{\mathcal{U}_{57}}$
- $\mathbf{1}_{\mathcal{U}_{39}} + \mathbf{1}_{\mathcal{U}_{58}}$
- $\mathbf{1}_{\mathcal{U}_{43}} + \mathbf{1}_{\mathcal{U}_{59}}$
- $\mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{41}} + \mathbf{1}_{\mathcal{U}_{42}} + \mathbf{1}_{\mathcal{U}_{60}}$
- $\mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{48}} + \mathbf{1}_{\mathcal{U}_{51}} + \mathbf{1}_{\mathcal{U}_{61}}$.

Case $G = \mathbb{T}_5$. The matrix associated to $Z^{\mathrm{rep}}_{\mathbb{T}_5}$ () has eigenvalues, with multiplicity, given by

$$\mathbb{L}^{12}(\mathbb{L}-1)^4,\ \mathbb{L}^{14}(\mathbb{L}-1)^2\ \text{(mult. 2)},\ \mathbb{L}^{16}\ \text{(mult. 2)},\ \mathbb{L}^{14}(\mathbb{L}-1)^4\ \text{(mult. 3)},$$

$$\mathbb{L}^{16}(\mathbb{L}-1)^2\ \text{(mult. 7)},\ \mathbb{L}^{18}\ \text{(mult. 2)},\ \mathbb{L}^{14}(\mathbb{L}-1)^6,\ \mathbb{L}^{16}(\mathbb{L}-1)^4\ \text{(mult. 7)},$$

$$\mathbb{L}^{18}(\mathbb{L}-1)^2\ \text{(mult. 7)},\ \mathbb{L}^{20},\ \mathbb{L}^{16}(\mathbb{L}-1)^6\ \text{(mult. 2)},\ \mathbb{L}^{18}(\mathbb{L}-1)^4\ \text{(mult. 8)},$$

$$\mathbb{L}^{20}(\mathbb{L}-1)^2 \text{ (mult. 4)},\ \mathbb{L}^{18}(\mathbb{L}-1)^6 \text{ (mult. 3)},\ \mathbb{L}^{20}(\mathbb{L}-1)^4 \text{ (mult. 6)},$$
$$\mathbb{L}^{20}(\mathbb{L}-1)^6 \text{ (mult. 4)},\ \mathbb{L}^{20}(\mathbb{L}-1)^8$$

with respective eigenvectors

- $\mathbf{1}_{\mathcal{U}_{13}} + \mathbb{L}(\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_1} - \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_5} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{10}}$
- $\mathbf{1}_{\mathcal{U}_{25}} - \mathbb{L}^2(\mathbb{L}-1)^3\mathbf{1}_{\mathcal{U}_1} + \mathbb{L}^2(\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_5} + \mathbb{L}(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{18}}) - \mathbb{L}(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{22}}) - \mathbb{L}(\mathbb{L}-1)^3\mathbf{1}_{\mathcal{U}_{10}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{23}}$
- $\mathbf{1}_{\mathcal{U}_{45}} + \mathbb{L}(\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_5}) - \mathbb{L}(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_{14}}) + \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{17}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{42}}$
- $\mathbf{1}_{\mathcal{U}_{29}} + \mathbb{L}(\mathbb{L}-1)^4(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5}) - \mathbb{L}(\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{17}}) + \mathbb{L}(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_{16}} - \mathbf{1}_{\mathcal{U}_{31}}) - (\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{22}}) + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{26}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{21}} + \mathbf{1}_{\mathcal{U}_{27}} + \mathbf{1}_{\mathcal{U}_{28}}) + \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{33}}$
- $\mathbf{1}_{\mathcal{U}_{55}} + \mathbb{L}(\mathbb{L}-1)^4(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{14}}) - \mathbb{L}(\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_{17}} + \mathbf{1}_{\mathcal{U}_{30}}) - (\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_9}) + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{47}}) + \mathbb{L}(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_{34}} - \mathbf{1}_{\mathcal{U}_{43}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{37}} + \mathbf{1}_{\mathcal{U}_{49}} + \mathbf{1}_{\mathcal{U}_{53}}) + \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{46}}$
- $\mathbf{1}_{\mathcal{U}_9} - \mathbf{1}_{\mathcal{U}_{22}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_6} - \mathbf{1}_{\mathcal{U}_{18}})$
- $\mathbf{1}_{\mathcal{U}_{25}} + \mathbb{L}^2(\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_1} - \mathbb{L}^2(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_5} + \mathbb{L}(\mathbb{L}-2)(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_6} - \mathbb{L}(\mathbb{L}-2)\mathbf{1}_{\mathcal{U}_9} + \mathbb{L}(\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_{10}} - \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{13}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{23}}$
- $\mathbf{1}_{\mathcal{U}_{45}} - \mathbb{L}(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_5}) + \mathbb{L}(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_{14}}) - \mathbb{L}\mathbf{1}_{\mathcal{U}_{17}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{42}}$
- $\mathbf{1}_{\mathcal{U}_{28}} - \mathbb{L}(\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_2} + \mathbb{L}^2(\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_3} + \mathbb{L}(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{19}}) - \mathbb{L}^2(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{16}} - (\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{22}}) - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{20}} + (\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_{26}} - \mathbb{L}\mathbf{1}_{\mathcal{U}_{27}}$
- $\mathbf{1}_{\mathcal{U}_{29}} - \mathbb{L}(\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{16}}) - \mathbb{L}(\mathbb{L}-2)(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{22}}) + \mathbb{L}(\mathbb{L}-1)^4\mathbf{1}_{\mathcal{U}_3} + \mathbb{L}(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{17}}) + \mathbb{L}(\mathbb{L}-2)(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{26}}) + (\mathbb{L}-1)\left(\mathbb{L}^2-\mathbb{L}-1\right)\mathbf{1}_{\mathcal{U}_{19}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{21}} - (\mathbb{L}^2-\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{27}}$
- $\mathbf{1}_{\mathcal{U}_{27}} + \mathbf{1}_{\mathcal{U}_{33}} + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{22}}) - \mathbb{L}(\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_3} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{26}} + \mathbf{1}_{\mathcal{U}_{31}}) + \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{16}}$
- $\mathbf{1}_{\mathcal{U}_{40}} - (\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{17}}) + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{34}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{32}} + \mathbf{1}_{\mathcal{U}_{38}})$
- $\mathbf{1}_{\mathcal{U}_{37}} + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_9}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_{35}}) - \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{43}} + \mathbb{L}\mathbf{1}_{\mathcal{U}_{46}}$
- $\mathbf{1}_{\mathcal{U}_{37}} - (\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_9}) + (\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_8} + \mathbb{L}^2(\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_{30}} - \mathbb{L}^2(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{34}} - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{35}} - \mathbb{L}(\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_{41}} + \mathbb{L}(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{44}} + \mathbf{1}_{\mathcal{U}_{47}}) - \mathbb{L}\mathbf{1}_{\mathcal{U}_{49}}$
- $\mathbb{L}^2(\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{14}}) - \mathbb{L}^2(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_{17}}) - (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_9}) + (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_8} + \mathbb{L}^2(\mathbb{L}-2)(\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_{30}} - \mathbb{L}^2(\mathbb{L}-2)(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{34}} - (\mathbb{L}-1)\left(\mathbb{L}^2-\mathbb{L}-1\right)\mathbf{1}_{\mathcal{U}_{35}} + (\mathbb{L}^2-\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{37}} - \mathbb{L}(\mathbb{L}-1)^3\mathbf{1}_{\mathcal{U}_{41}} + \mathbb{L}(\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_{44}} + \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{53}} - \mathbb{L}\mathbf{1}_{\mathcal{U}_{55}}$
- $\mathbf{1}_{\mathcal{U}_{54}} - (\mathbb{L}-1)^4(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{17}}) + (\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{32}} + \mathbf{1}_{\mathcal{U}_{34}}) - (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{38}} + \mathbf{1}_{\mathcal{U}_{40}} - \mathbf{1}_{\mathcal{U}_{47}} - \mathbf{1}_{\mathcal{U}_{49}}) + (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{39}} - \mathbf{1}_{\mathcal{U}_{48}} - \mathbf{1}_{\mathcal{U}_{53}} - \mathbf{1}_{\mathcal{U}_{55}})$
- $\mathbf{1}_{\mathcal{U}_{59}} - (\mathbb{L}-1)^4(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{17}} + \mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{34}}) + (\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{32}} + \mathbf{1}_{\mathcal{U}_{38}} + \mathbf{1}_{\mathcal{U}_{41}} + \mathbf{1}_{\mathcal{U}_{44}}) - (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_{12}} - \mathbf{1}_{\mathcal{U}_{19}} - \mathbf{1}_{\mathcal{U}_{27}} + \mathbf{1}_{\mathcal{U}_{40}} + \mathbf{1}_{\mathcal{U}_{43}} + \mathbf{1}_{\mathcal{U}_{46}} + \mathbf{1}_{\mathcal{U}_{50}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{21}} + \mathbf{1}_{\mathcal{U}_{29}} - \mathbf{1}_{\mathcal{U}_{52}} + \mathbf{1}_{\mathcal{U}_{57}})$

- $\mathbf{1}_{\mathcal{U}_{25}} - \mathbb{L}^2(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_1} + \mathbb{L}^2\mathbf{1}_{\mathcal{U}_5} - \mathbb{L}(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{18}}) + \mathbb{L}(\mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{22}}) - (\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{23}}$
- $\mathbf{1}_{\mathcal{U}_{26}} + \mathbf{1}_{\mathcal{U}_{28}} + \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_2} - \mathbb{L}\mathbf{1}_{\mathcal{U}_{15}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{22}})$
- $\mathbf{1}_{\mathcal{U}_{27}} + \mathbf{1}_{\mathcal{U}_{29}} + \mathbb{L}(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5}) + \mathbb{L}(\mathbb{L}-2)(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_2} - \mathbb{L}(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{17}}) - \mathbb{L}(\mathbb{L}-2)\mathbf{1}_{\mathcal{U}_{15}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{21}})$
- $\mathbf{1}_{\mathcal{U}_{26}} + \mathbf{1}_{\mathcal{U}_{27}} + \mathbf{1}_{\mathcal{U}_{33}} + (\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_2} + \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_3} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{22}} + \mathbf{1}_{\mathcal{U}_{31}}) - \mathbb{L}\mathbf{1}_{\mathcal{U}_{16}}$
- $\mathbf{1}_{\mathcal{U}_{32}} + \mathbf{1}_{\mathcal{U}_{40}} + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{17}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{34}} + \mathbf{1}_{\mathcal{U}_{38}})$
- $\mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_{37}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{35}}) + \mathbb{L}(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{41}} - \mathbb{L}\mathbf{1}_{\mathcal{U}_{44}}$
- $\mathbf{1}_{\mathcal{U}_8} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{37}}) + \mathbb{L}^2(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{30}} - \mathbb{L}^2\mathbf{1}_{\mathcal{U}_{34}} + (\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_{35}} - \mathbb{L}(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{43}} + \mathbf{1}_{\mathcal{U}_{47}}) + \mathbb{L}(\mathbf{1}_{\mathcal{U}_{46}} + \mathbf{1}_{\mathcal{U}_{49}})$
- $\mathbb{L}^2(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{14}}) - \mathbb{L}^2(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_{17}}) + (\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_9}) - (\mathbb{L}-1)^2\mathbf{1}_{\mathcal{U}_8} + \mathbb{L}^2(\mathbb{L}-2)(\mathbb{L}-1)\mathbf{1}_{\mathcal{U}_{30}} - \mathbb{L}^2(\mathbb{L}-2)\mathbf{1}_{\mathcal{U}_{34}} + (\mathbb{L}-1)\left(\mathbb{L}^2-3\mathbb{L}+1\right)\mathbf{1}_{\mathcal{U}_{35}} - (\mathbb{L}^2-3\mathbb{L}+1)\mathbf{1}_{\mathcal{U}_{37}} + \mathbb{L}(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{43}} - \mathbf{1}_{\mathcal{U}_{53}}) - \mathbb{L}(\mathbf{1}_{\mathcal{U}_{46}} - \mathbf{1}_{\mathcal{U}_{55}})$
- $\mathbf{1}_{\mathcal{U}_{36}} - (\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{17}}) + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{32}} + \mathbf{1}_{\mathcal{U}_{34}} + \mathbf{1}_{\mathcal{U}_{38}} + \mathbf{1}_{\mathcal{U}_{40}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{37}} + \mathbf{1}_{\mathcal{U}_{39}})$
- $\mathbf{1}_{\mathcal{U}_{36}} - \mathbf{1}_{\mathcal{U}_{51}} + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{27}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_{21}} + \mathbf{1}_{\mathcal{U}_{29}} + \mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{37}} - \mathbf{1}_{\mathcal{U}_{42}} - \mathbf{1}_{\mathcal{U}_{45}}) - \mathbb{L}(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{13}} - \mathbf{1}_{\mathcal{U}_{30}} - \mathbf{1}_{\mathcal{U}_{32}} - \mathbf{1}_{\mathcal{U}_{34}} + \mathbf{1}_{\mathcal{U}_{41}} + \mathbf{1}_{\mathcal{U}_{44}}) + \mathbb{L}(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{11}} - \mathbf{1}_{\mathcal{U}_{31}} - \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{50}})$
- $\mathbf{1}_{\mathcal{U}_{21}} + \mathbf{1}_{\mathcal{U}_{29}} + \mathbf{1}_{\mathcal{U}_{52}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{27}} + \mathbf{1}_{\mathcal{U}_{43}} + \mathbf{1}_{\mathcal{U}_{46}} + \mathbf{1}_{\mathcal{U}_{50}}) + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_{41}} + \mathbf{1}_{\mathcal{U}_{44}})$
- $\mathbf{1}_{\mathcal{U}_{36}} - \mathbb{L}(\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{17}}) - (\mathbb{L}-1)^2\left(\mathbb{L}^2-3\mathbb{L}+1\right)(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_{15}}) + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_9} - \mathbf{1}_{\mathcal{U}_{39}} + \mathbf{1}_{\mathcal{U}_{48}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{37}} + \mathbf{1}_{\mathcal{U}_{54}}) + (\mathbb{L}-1)^3\mathbf{1}_{\mathcal{U}_{11}} + \mathbb{L}(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{32}} + \mathbf{1}_{\mathcal{U}_{34}}) + (\mathbb{L}-1)\left(\mathbb{L}^2-3\mathbb{L}+1\right)(\mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{33}})$
- $\mathbf{1}_{\mathcal{U}_{39}} + \mathbf{1}_{\mathcal{U}_{53}} + \mathbf{1}_{\mathcal{U}_{55}} + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_{15}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{47}} + \mathbf{1}_{\mathcal{U}_{49}})$
- $\mathbf{1}_{\mathcal{U}_{36}} - \mathbf{1}_{\mathcal{U}_{58}} + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{15}} - \mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{19}} - \mathbf{1}_{\mathcal{U}_{22}} - \mathbf{1}_{\mathcal{U}_{26}} + \mathbf{1}_{\mathcal{U}_{27}}) - \mathbb{L}(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{16}} - \mathbf{1}_{\mathcal{U}_{30}} - \mathbf{1}_{\mathcal{U}_{34}} - \mathbf{1}_{\mathcal{U}_{38}} + \mathbf{1}_{\mathcal{U}_{41}} + \mathbf{1}_{\mathcal{U}_{44}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_7} - \mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{21}} - \mathbf{1}_{\mathcal{U}_{28}} + \mathbf{1}_{\mathcal{U}_{29}} + \mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{37}} + \mathbf{1}_{\mathcal{U}_{39}} - \mathbf{1}_{\mathcal{U}_{56}}) + \mathbb{L}(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{43}} + \mathbf{1}_{\mathcal{U}_{46}})$
- $\mathbf{1}_{\mathcal{U}_{36}} - \mathbb{L}(\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{17}}) + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{57}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{37}} + \mathbf{1}_{\mathcal{U}_{39}} + \mathbf{1}_{\mathcal{U}_{59}}) - \mathbb{L}(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_{12}} - \mathbf{1}_{\mathcal{U}_{32}} - \mathbf{1}_{\mathcal{U}_{38}}) - (\mathbb{L}-2)(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{27}}) + (\mathbb{L}-2)(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{21}} + \mathbf{1}_{\mathcal{U}_{29}}) - \mathbb{L}(\mathbb{L}-2)(\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{34}})$
- $\mathbf{1}_{\mathcal{U}_{61}} + (\mathbb{L}-1)^4(\mathbf{1}_{\mathcal{U}_1} + \mathbf{1}_{\mathcal{U}_3} + \mathbf{1}_{\mathcal{U}_4} + \mathbf{1}_{\mathcal{U}_5} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{17}} + \mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{32}} + \mathbf{1}_{\mathcal{U}_{34}} + \mathbf{1}_{\mathcal{U}_{38}} + \mathbf{1}_{\mathcal{U}_{40}}) - (\mathbb{L}-1)^3(\mathbf{1}_{\mathcal{U}_2} + \mathbf{1}_{\mathcal{U}_6} + \mathbf{1}_{\mathcal{U}_8} + \mathbf{1}_{\mathcal{U}_9} + \mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{22}} + \mathbf{1}_{\mathcal{U}_{26}} + \mathbf{1}_{\mathcal{U}_{28}} + \mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{37}} + \mathbf{1}_{\mathcal{U}_{39}} + \mathbf{1}_{\mathcal{U}_{41}} + \mathbf{1}_{\mathcal{U}_{43}} + \mathbf{1}_{\mathcal{U}_{44}} + \mathbf{1}_{\mathcal{U}_{46}} + \mathbf{1}_{\mathcal{U}_{50}} + \mathbf{1}_{\mathcal{U}_{52}}) + (\mathbb{L}-1)^2(\mathbf{1}_{\mathcal{U}_7} + \mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{21}} + \mathbf{1}_{\mathcal{U}_{23}} + \mathbf{1}_{\mathcal{U}_{25}} + \mathbf{1}_{\mathcal{U}_{27}} + \mathbf{1}_{\mathcal{U}_{29}} + \mathbf{1}_{\mathcal{U}_{36}} + \mathbf{1}_{\mathcal{U}_{42}} + \mathbf{1}_{\mathcal{U}_{45}} + \mathbf{1}_{\mathcal{U}_{47}} + \mathbf{1}_{\mathcal{U}_{49}} + \mathbf{1}_{\mathcal{U}_{51}} + \mathbf{1}_{\mathcal{U}_{53}} + \mathbf{1}_{\mathcal{U}_{55}} + \mathbf{1}_{\mathcal{U}_{56}} + \mathbf{1}_{\mathcal{U}_{58}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{24}} + \mathbf{1}_{\mathcal{U}_{48}} + \mathbf{1}_{\mathcal{U}_{54}} + \mathbf{1}_{\mathcal{U}_{57}} + \mathbf{1}_{\mathcal{U}_{59}} + \mathbf{1}_{\mathcal{U}_{60}})$

- $\mathbf{1}_{\mathcal{U}_{26}} + \mathbf{1}_{\mathcal{U}_{27}} + \mathbf{1}_{\mathcal{U}_{28}} + \mathbf{1}_{\mathcal{U}_{29}} - \mathbb{L}\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{1}} + \mathbf{1}_{\mathcal{U}_{2}} + \mathbf{1}_{\mathcal{U}_{3}} + \mathbf{1}_{\mathcal{U}_{4}} + \mathbf{1}_{\mathcal{U}_{5}}) + \mathbb{L}(\mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{17}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{21}} + \mathbf{1}_{\mathcal{U}_{22}})$
- $\mathbf{1}_{\mathcal{U}_{8}} + \mathbf{1}_{\mathcal{U}_{37}} + \mathbf{1}_{\mathcal{U}_{49}} + \mathbf{1}_{\mathcal{U}_{55}} - \mathbb{L}\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{1}} + \mathbf{1}_{\mathcal{U}_{5}} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{41}}) + \mathbb{L}(\mathbf{1}_{\mathcal{U}_{4}} + \mathbf{1}_{\mathcal{U}_{17}} + \mathbf{1}_{\mathcal{U}_{34}} + \mathbf{1}_{\mathcal{U}_{44}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{6}} + \mathbf{1}_{\mathcal{U}_{9}} + \mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{47}} + \mathbf{1}_{\mathcal{U}_{53}})$
- $\mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{33}} - \mathbf{1}_{\mathcal{U}_{35}} - \mathbf{1}_{\mathcal{U}_{37}} + \mathbf{1}_{\mathcal{U}_{39}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{2}} - \mathbf{1}_{\mathcal{U}_{6}} - \mathbf{1}_{\mathcal{U}_{8}} - \mathbf{1}_{\mathcal{U}_{9}} + \mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{15}})$
- $\mathbf{1}_{\mathcal{U}_{36}} + (\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_{1}} + \mathbf{1}_{\mathcal{U}_{3}} + \mathbf{1}_{\mathcal{U}_{4}} + \mathbf{1}_{\mathcal{U}_{5}} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{17}}) + (\mathbb{L}-2)\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{6}} + \mathbf{1}_{\mathcal{U}_{8}} + \mathbf{1}_{\mathcal{U}_{9}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{7}} + \mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{32}} + \mathbf{1}_{\mathcal{U}_{34}} + \mathbf{1}_{\mathcal{U}_{38}} + \mathbf{1}_{\mathcal{U}_{40}}) - (\mathbb{L}-2)(\mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{37}})$
- $\mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{28}} - \mathbf{1}_{\mathcal{U}_{35}} - \mathbf{1}_{\mathcal{U}_{37}} + (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{6}} + \mathbf{1}_{\mathcal{U}_{8}} + \mathbf{1}_{\mathcal{U}_{9}} - \mathbf{1}_{\mathcal{U}_{18}} - \mathbf{1}_{\mathcal{U}_{22}} - \mathbf{1}_{\mathcal{U}_{26}}) - \mathbb{L}(\mathbf{1}_{\mathcal{U}_{11}} - \mathbf{1}_{\mathcal{U}_{31}} - \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{43}} + \mathbf{1}_{\mathcal{U}_{46}} - \mathbf{1}_{\mathcal{U}_{50}})$
- $\mathbf{1}_{\mathcal{U}_{36}} - \mathbf{1}_{\mathcal{U}_{51}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{7}} - \mathbf{1}_{\mathcal{U}_{42}} - \mathbf{1}_{\mathcal{U}_{45}}) + \mathbb{L}\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{13}} - \mathbf{1}_{\mathcal{U}_{30}} - \mathbf{1}_{\mathcal{U}_{32}} - \mathbf{1}_{\mathcal{U}_{34}}) + \mathbb{L}\,(\mathbb{L}-2)\,(\mathbf{1}_{\mathcal{U}_{11}} - \mathbf{1}_{\mathcal{U}_{31}} - \mathbf{1}_{\mathcal{U}_{33}})$
- $\mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{28}} - (\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_{6}} + \mathbf{1}_{\mathcal{U}_{8}} + \mathbf{1}_{\mathcal{U}_{9}}) + \mathbb{L}\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{11}} - \mathbf{1}_{\mathcal{U}_{31}} - \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{41}} + \mathbf{1}_{\mathcal{U}_{44}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{22}} + \mathbf{1}_{\mathcal{U}_{26}} - \mathbf{1}_{\mathcal{U}_{35}} - \mathbf{1}_{\mathcal{U}_{37}}) + \mathbb{L}\,(\mathbb{L}-2)\,(\mathbf{1}_{\mathcal{U}_{43}} + \mathbf{1}_{\mathcal{U}_{46}}) - \mathbb{L}\mathbf{1}_{\mathcal{U}_{52}}$
- $\mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{36}} + \mathbf{1}_{\mathcal{U}_{37}} + \mathbb{L}\,(\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_{1}} + \mathbf{1}_{\mathcal{U}_{2}} + \mathbf{1}_{\mathcal{U}_{3}} + \mathbf{1}_{\mathcal{U}_{4}} + \mathbf{1}_{\mathcal{U}_{5}} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{17}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{6}} + \mathbf{1}_{\mathcal{U}_{7}} + \mathbf{1}_{\mathcal{U}_{8}} + \mathbf{1}_{\mathcal{U}_{9}} + \mathbf{1}_{\mathcal{U}_{53}} + \mathbf{1}_{\mathcal{U}_{54}} + \mathbf{1}_{\mathcal{U}_{55}}) - \mathbb{L}\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{32}} + \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{34}}) + (\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_{47}} + \mathbf{1}_{\mathcal{U}_{48}} + \mathbf{1}_{\mathcal{U}_{49}})$
- $\mathbf{1}_{\mathcal{U}_{36}} - \mathbf{1}_{\mathcal{U}_{58}} + \mathbb{L}\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{3}} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{16}} - \mathbf{1}_{\mathcal{U}_{30}} - \mathbf{1}_{\mathcal{U}_{34}} - \mathbf{1}_{\mathcal{U}_{38}}) + (\mathbb{L}-2)\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{6}} + \mathbf{1}_{\mathcal{U}_{8}} + \mathbf{1}_{\mathcal{U}_{9}} - \mathbf{1}_{\mathcal{U}_{18}} - \mathbf{1}_{\mathcal{U}_{22}} - \mathbf{1}_{\mathcal{U}_{26}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{7}} - \mathbf{1}_{\mathcal{U}_{56}}) + (\mathbb{L}-2)(\mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{28}} - \mathbf{1}_{\mathcal{U}_{35}} - \mathbf{1}_{\mathcal{U}_{37}})$
- $\mathbf{1}_{\mathcal{U}_{36}} + \mathbb{L}\,(\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_{1}} + \mathbf{1}_{\mathcal{U}_{4}} + \mathbf{1}_{\mathcal{U}_{5}} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{17}} + \mathbf{1}_{\mathcal{U}_{41}} + \mathbf{1}_{\mathcal{U}_{44}}) + (\mathbb{L}-2)\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{6}} + \mathbf{1}_{\mathcal{U}_{8}} + \mathbf{1}_{\mathcal{U}_{9}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{7}} - \mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{21}} - \mathbf{1}_{\mathcal{U}_{28}} + \mathbf{1}_{\mathcal{U}_{29}} + \mathbf{1}_{\mathcal{U}_{59}}) + \mathbb{L}\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{12}} - \mathbf{1}_{\mathcal{U}_{32}} - \mathbf{1}_{\mathcal{U}_{38}} - \mathbf{1}_{\mathcal{U}_{43}} - \mathbf{1}_{\mathcal{U}_{46}}) - (\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_{18}} - \mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{22}} + \mathbf{1}_{\mathcal{U}_{26}} - \mathbf{1}_{\mathcal{U}_{27}} - \mathbf{1}_{\mathcal{U}_{57}}) + \mathbb{L}\,(\mathbb{L}-2)\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{34}}) - (\mathbb{L}-2)(\mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{37}})$
- $\mathbf{1}_{\mathcal{U}_{24}} - \mathbf{1}_{\mathcal{U}_{48}} - \mathbf{1}_{\mathcal{U}_{54}} + (\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{22}} + \mathbf{1}_{\mathcal{U}_{26}} + \mathbf{1}_{\mathcal{U}_{28}} - \mathbf{1}_{\mathcal{U}_{41}} - \mathbf{1}_{\mathcal{U}_{43}} - \mathbf{1}_{\mathcal{U}_{44}} - \mathbf{1}_{\mathcal{U}_{46}} - \mathbf{1}_{\mathcal{U}_{50}} - \mathbf{1}_{\mathcal{U}_{52}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{21}} + \mathbf{1}_{\mathcal{U}_{23}} + \mathbf{1}_{\mathcal{U}_{25}} + \mathbf{1}_{\mathcal{U}_{27}} + \mathbf{1}_{\mathcal{U}_{29}} - \mathbf{1}_{\mathcal{U}_{42}} - \mathbf{1}_{\mathcal{U}_{45}} - \mathbf{1}_{\mathcal{U}_{47}} - \mathbf{1}_{\mathcal{U}_{49}} - \mathbf{1}_{\mathcal{U}_{51}} - \mathbf{1}_{\mathcal{U}_{53}} - \mathbf{1}_{\mathcal{U}_{55}})$
- $\mathbf{1}_{\mathcal{U}_{24}} - \mathbf{1}_{\mathcal{U}_{57}} - \mathbf{1}_{\mathcal{U}_{59}} + (\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_{6}} + \mathbf{1}_{\mathcal{U}_{8}} + \mathbf{1}_{\mathcal{U}_{9}} + \mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{37}} - \mathbf{1}_{\mathcal{U}_{41}} - \mathbf{1}_{\mathcal{U}_{43}} - \mathbf{1}_{\mathcal{U}_{44}} - \mathbf{1}_{\mathcal{U}_{46}} - \mathbf{1}_{\mathcal{U}_{50}} - \mathbf{1}_{\mathcal{U}_{52}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{7}} + \mathbf{1}_{\mathcal{U}_{23}} + \mathbf{1}_{\mathcal{U}_{25}} + \mathbf{1}_{\mathcal{U}_{36}} - \mathbf{1}_{\mathcal{U}_{42}} - \mathbf{1}_{\mathcal{U}_{45}} - \mathbf{1}_{\mathcal{U}_{51}} - \mathbf{1}_{\mathcal{U}_{56}} - \mathbf{1}_{\mathcal{U}_{58}})$
- $\mathbf{1}_{\mathcal{U}_{48}} + \mathbf{1}_{\mathcal{U}_{54}} - \mathbf{1}_{\mathcal{U}_{60}} + (\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_{2}} + \mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{15}} - \mathbf{1}_{\mathcal{U}_{18}} - \mathbf{1}_{\mathcal{U}_{20}} - \mathbf{1}_{\mathcal{U}_{22}} - \mathbf{1}_{\mathcal{U}_{26}} - \mathbf{1}_{\mathcal{U}_{28}} + \mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{39}}) - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{7}} - \mathbf{1}_{\mathcal{U}_{23}} - \mathbf{1}_{\mathcal{U}_{25}} + \mathbf{1}_{\mathcal{U}_{36}} + \mathbf{1}_{\mathcal{U}_{42}} + \mathbf{1}_{\mathcal{U}_{45}} + \mathbf{1}_{\mathcal{U}_{51}} - \mathbf{1}_{\mathcal{U}_{56}} - \mathbf{1}_{\mathcal{U}_{58}})$
- $\mathbf{1}_{\mathcal{U}_{61}} - (\mathbb{L}-1)^3\,(\mathbf{1}_{\mathcal{U}_{1}} + \mathbf{1}_{\mathcal{U}_{3}} + \mathbf{1}_{\mathcal{U}_{4}} + \mathbf{1}_{\mathcal{U}_{5}} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{17}} + \mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{32}} + \mathbf{1}_{\mathcal{U}_{34}} + \mathbf{1}_{\mathcal{U}_{38}} + \mathbf{1}_{\mathcal{U}_{40}} - \mathbf{1}_{\mathcal{U}_{41}} - \mathbf{1}_{\mathcal{U}_{43}} - \mathbf{1}_{\mathcal{U}_{44}} - \mathbf{1}_{\mathcal{U}_{46}} - \mathbf{1}_{\mathcal{U}_{50}} - \mathbf{1}_{\mathcal{U}_{52}}) - (\mathbb{L}-2)\,(\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_{2}} + \mathbf{1}_{\mathcal{U}_{6}} + \mathbf{1}_{\mathcal{U}_{8}} + \mathbf{1}_{\mathcal{U}_{9}} + \mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{37}} + \mathbf{1}_{\mathcal{U}_{39}}) + (\mathbb{L}-1)\,(2\mathbb{L}-3)\,(\mathbf{1}_{\mathcal{U}_{7}} + \mathbf{1}_{\mathcal{U}_{36}}) + (\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_{18}} + \mathbf{1}_{\mathcal{U}_{20}} + \mathbf{1}_{\mathcal{U}_{22}} + \mathbf{1}_{\mathcal{U}_{26}} + \mathbf{1}_{\mathcal{U}_{28}} - \mathbf{1}_{\mathcal{U}_{56}} - \mathbf{1}_{\mathcal{U}_{58}}) + (\mathbb{L}-2)\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{19}} + \mathbf{1}_{\mathcal{U}_{21}} + \mathbf{1}_{\mathcal{U}_{23}} + \mathbf{1}_{\mathcal{U}_{25}} + \mathbf{1}_{\mathcal{U}_{27}} + \mathbf{1}_{\mathcal{U}_{29}}) - (2\mathbb{L}-3)\mathbf{1}_{\mathcal{U}_{24}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{48}} + \mathbf{1}_{\mathcal{U}_{54}})$
- $\mathbf{1}_{\mathcal{U}_{30}} + \mathbf{1}_{\mathcal{U}_{31}} + \mathbf{1}_{\mathcal{U}_{32}} + \mathbf{1}_{\mathcal{U}_{33}} + \mathbf{1}_{\mathcal{U}_{34}} + \mathbf{1}_{\mathcal{U}_{35}} + \mathbf{1}_{\mathcal{U}_{36}} + \mathbf{1}_{\mathcal{U}_{37}} + \mathbf{1}_{\mathcal{U}_{38}} + \mathbf{1}_{\mathcal{U}_{39}} + \mathbf{1}_{\mathcal{U}_{40}} - (\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{1}} + \mathbf{1}_{\mathcal{U}_{2}} + \mathbf{1}_{\mathcal{U}_{3}} + \mathbf{1}_{\mathcal{U}_{4}} + \mathbf{1}_{\mathcal{U}_{5}} + \mathbf{1}_{\mathcal{U}_{6}} + \mathbf{1}_{\mathcal{U}_{7}} + \mathbf{1}_{\mathcal{U}_{8}} + \mathbf{1}_{\mathcal{U}_{9}} + \mathbf{1}_{\mathcal{U}_{10}} + \mathbf{1}_{\mathcal{U}_{11}} + \mathbf{1}_{\mathcal{U}_{12}} + \mathbf{1}_{\mathcal{U}_{13}} + \mathbf{1}_{\mathcal{U}_{14}} + \mathbf{1}_{\mathcal{U}_{15}} + \mathbf{1}_{\mathcal{U}_{16}} + \mathbf{1}_{\mathcal{U}_{17}})$

- $\mathbf{1}_{\mathcal{U}_{35}}+\mathbf{1}_{\mathcal{U}_{36}}+\mathbf{1}_{\mathcal{U}_{37}}-\mathbf{1}_{\mathcal{U}_{50}}-\mathbf{1}_{\mathcal{U}_{51}}-\mathbf{1}_{\mathcal{U}_{52}}-(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{6}}+\mathbf{1}_{\mathcal{U}_{7}}+\mathbf{1}_{\mathcal{U}_{8}}+\mathbf{1}_{\mathcal{U}_{9}}-\mathbf{1}_{\mathcal{U}_{41}}-\mathbf{1}_{\mathcal{U}_{42}}-\mathbf{1}_{\mathcal{U}_{43}}-\mathbf{1}_{\mathcal{U}_{44}}-\mathbf{1}_{\mathcal{U}_{45}}-\mathbf{1}_{\mathcal{U}_{46}})-\mathbb{L}(\mathbf{1}_{\mathcal{U}_{10}}+\mathbf{1}_{\mathcal{U}_{11}}+\mathbf{1}_{\mathcal{U}_{12}}+\mathbf{1}_{\mathcal{U}_{13}}-\mathbf{1}_{\mathcal{U}_{30}}-\mathbf{1}_{\mathcal{U}_{31}}-\mathbf{1}_{\mathcal{U}_{32}}-\mathbf{1}_{\mathcal{U}_{33}}-\mathbf{1}_{\mathcal{U}_{34}})$
- $\mathbf{1}_{\mathcal{U}_{11}}-\mathbf{1}_{\mathcal{U}_{20}}-\mathbf{1}_{\mathcal{U}_{28}}+\mathbf{1}_{\mathcal{U}_{39}}+\mathbf{1}_{\mathcal{U}_{51}}-\mathbf{1}_{\mathcal{U}_{58}}-(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{2}}+\mathbf{1}_{\mathcal{U}_{15}}-\mathbf{1}_{\mathcal{U}_{18}}-\mathbf{1}_{\mathcal{U}_{22}}-\mathbf{1}_{\mathcal{U}_{26}}+\mathbf{1}_{\mathcal{U}_{31}}+\mathbf{1}_{\mathcal{U}_{33}}+\mathbf{1}_{\mathcal{U}_{42}}+\mathbf{1}_{\mathcal{U}_{45}}-\mathbf{1}_{\mathcal{U}_{56}})-\mathbb{L}(\mathbf{1}_{\mathcal{U}_{3}}-\mathbf{1}_{\mathcal{U}_{10}}-\mathbf{1}_{\mathcal{U}_{13}}+\mathbf{1}_{\mathcal{U}_{16}}+\mathbf{1}_{\mathcal{U}_{32}}-\mathbf{1}_{\mathcal{U}_{38}}+\mathbf{1}_{\mathcal{U}_{43}}+\mathbf{1}_{\mathcal{U}_{46}}-\mathbf{1}_{\mathcal{U}_{50}})$
- $\mathbf{1}_{\mathcal{U}_{24}}-\mathbf{1}_{\mathcal{U}_{48}}-\mathbf{1}_{\mathcal{U}_{54}}-(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{18}}+\mathbf{1}_{\mathcal{U}_{20}}+\mathbf{1}_{\mathcal{U}_{22}}+\mathbf{1}_{\mathcal{U}_{26}}+\mathbf{1}_{\mathcal{U}_{28}}-\mathbf{1}_{\mathcal{U}_{41}}-\mathbf{1}_{\mathcal{U}_{43}}-\mathbf{1}_{\mathcal{U}_{44}}-\mathbf{1}_{\mathcal{U}_{46}}-\mathbf{1}_{\mathcal{U}_{50}}-\mathbf{1}_{\mathcal{U}_{52}})-(\mathbb{L}-2)(\mathbf{1}_{\mathcal{U}_{19}}+\mathbf{1}_{\mathcal{U}_{21}}+\mathbf{1}_{\mathcal{U}_{27}}+\mathbf{1}_{\mathcal{U}_{29}}-\mathbf{1}_{\mathcal{U}_{42}}-\mathbf{1}_{\mathcal{U}_{45}}-\mathbf{1}_{\mathcal{U}_{51}})$
- $\mathbf{1}_{\mathcal{U}_{19}}+\mathbf{1}_{\mathcal{U}_{21}}-\mathbf{1}_{\mathcal{U}_{23}}-\mathbf{1}_{\mathcal{U}_{25}}+\mathbf{1}_{\mathcal{U}_{27}}+\mathbf{1}_{\mathcal{U}_{29}}-\mathbf{1}_{\mathcal{U}_{42}}-\mathbf{1}_{\mathcal{U}_{45}}+\mathbf{1}_{\mathcal{U}_{47}}+\mathbf{1}_{\mathcal{U}_{49}}-\mathbf{1}_{\mathcal{U}_{51}}+\mathbf{1}_{\mathcal{U}_{53}}+\mathbf{1}_{\mathcal{U}_{55}}$
- $\mathbf{1}_{\mathcal{U}_{7}}-\mathbf{1}_{\mathcal{U}_{23}}-\mathbf{1}_{\mathcal{U}_{25}}+\mathbf{1}_{\mathcal{U}_{36}}-\mathbf{1}_{\mathcal{U}_{42}}-\mathbf{1}_{\mathcal{U}_{45}}-\mathbf{1}_{\mathcal{U}_{51}}+\mathbf{1}_{\mathcal{U}_{56}}+\mathbf{1}_{\mathcal{U}_{58}}$
- $\mathbf{1}_{\mathcal{U}_{24}}-\mathbf{1}_{\mathcal{U}_{57}}-\mathbf{1}_{\mathcal{U}_{59}}-(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{6}}+\mathbf{1}_{\mathcal{U}_{8}}+\mathbf{1}_{\mathcal{U}_{9}}+\mathbf{1}_{\mathcal{U}_{35}}+\mathbf{1}_{\mathcal{U}_{37}}-\mathbf{1}_{\mathcal{U}_{41}}-\mathbf{1}_{\mathcal{U}_{43}}-\mathbf{1}_{\mathcal{U}_{44}}-\mathbf{1}_{\mathcal{U}_{46}}-\mathbf{1}_{\mathcal{U}_{50}}-\mathbf{1}_{\mathcal{U}_{52}})-(\mathbb{L}-2)(\mathbf{1}_{\mathcal{U}_{7}}+\mathbf{1}_{\mathcal{U}_{36}}-\mathbf{1}_{\mathcal{U}_{42}}-\mathbf{1}_{\mathcal{U}_{45}}-\mathbf{1}_{\mathcal{U}_{51}})$
- $\mathbf{1}_{\mathcal{U}_{24}}-\mathbf{1}_{\mathcal{U}_{60}}-(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{2}}+\mathbf{1}_{\mathcal{U}_{11}}+\mathbf{1}_{\mathcal{U}_{15}}+\mathbf{1}_{\mathcal{U}_{31}}+\mathbf{1}_{\mathcal{U}_{33}}+\mathbf{1}_{\mathcal{U}_{39}}-\mathbf{1}_{\mathcal{U}_{41}}-\mathbf{1}_{\mathcal{U}_{43}}-\mathbf{1}_{\mathcal{U}_{44}}-\mathbf{1}_{\mathcal{U}_{46}}-\mathbf{1}_{\mathcal{U}_{50}}-\mathbf{1}_{\mathcal{U}_{52}})-(\mathbb{L}-2)(\mathbf{1}_{\mathcal{U}_{7}}+\mathbf{1}_{\mathcal{U}_{19}}+\mathbf{1}_{\mathcal{U}_{21}}-\mathbf{1}_{\mathcal{U}_{23}}-\mathbf{1}_{\mathcal{U}_{25}}+\mathbf{1}_{\mathcal{U}_{27}}+\mathbf{1}_{\mathcal{U}_{29}}+\mathbf{1}_{\mathcal{U}_{36}}-\mathbf{1}_{\mathcal{U}_{42}}-\mathbf{1}_{\mathcal{U}_{45}}-\mathbf{1}_{\mathcal{U}_{51}})$
- $\mathbf{1}_{\mathcal{U}_{61}}+(\mathbb{L}-1)^2\,(\mathbf{1}_{\mathcal{U}_{1}}+\mathbf{1}_{\mathcal{U}_{3}}+\mathbf{1}_{\mathcal{U}_{4}}+\mathbf{1}_{\mathcal{U}_{5}}+\mathbf{1}_{\mathcal{U}_{10}}+\mathbf{1}_{\mathcal{U}_{12}}+\mathbf{1}_{\mathcal{U}_{13}}+\mathbf{1}_{\mathcal{U}_{14}}+\mathbf{1}_{\mathcal{U}_{16}}+\mathbf{1}_{\mathcal{U}_{17}}+\mathbf{1}_{\mathcal{U}_{30}}+\mathbf{1}_{\mathcal{U}_{32}}+\mathbf{1}_{\mathcal{U}_{34}}+\mathbf{1}_{\mathcal{U}_{38}}+\mathbf{1}_{\mathcal{U}_{40}})+(\mathbb{L}-2)\,(\mathbb{L}-1)\,(\mathbf{1}_{\mathcal{U}_{2}}+\mathbf{1}_{\mathcal{U}_{6}}+\mathbf{1}_{\mathcal{U}_{8}}+\mathbf{1}_{\mathcal{U}_{9}}+\mathbf{1}_{\mathcal{U}_{11}}+\mathbf{1}_{\mathcal{U}_{15}}+\mathbf{1}_{\mathcal{U}_{18}}+\mathbf{1}_{\mathcal{U}_{20}}+\mathbf{1}_{\mathcal{U}_{22}}+\mathbf{1}_{\mathcal{U}_{26}}+\mathbf{1}_{\mathcal{U}_{28}}+\mathbf{1}_{\mathcal{U}_{31}}+\mathbf{1}_{\mathcal{U}_{33}}+\mathbf{1}_{\mathcal{U}_{35}}+\mathbf{1}_{\mathcal{U}_{37}}+\mathbf{1}_{\mathcal{U}_{39}}-\mathbf{1}_{\mathcal{U}_{41}}-\mathbf{1}_{\mathcal{U}_{43}}-\mathbf{1}_{\mathcal{U}_{44}}-\mathbf{1}_{\mathcal{U}_{46}}-\mathbf{1}_{\mathcal{U}_{50}}-\mathbf{1}_{\mathcal{U}_{52}})+(\mathbb{L}-2)^2\,(\mathbf{1}_{\mathcal{U}_{7}}+\mathbf{1}_{\mathcal{U}_{19}}+\mathbf{1}_{\mathcal{U}_{21}}+\mathbf{1}_{\mathcal{U}_{27}}+\mathbf{1}_{\mathcal{U}_{29}}+\mathbf{1}_{\mathcal{U}_{36}})-(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{23}}+\mathbf{1}_{\mathcal{U}_{25}})-2\,(\mathbb{L}-2)\,\mathbf{1}_{\mathcal{U}_{24}}-(\mathbb{L}^2-3\mathbb{L}+3)(\mathbf{1}_{\mathcal{U}_{42}}+\mathbf{1}_{\mathcal{U}_{45}}+\mathbf{1}_{\mathcal{U}_{51}})$
- $\mathbf{1}_{\mathcal{U}_{18}}+\mathbf{1}_{\mathcal{U}_{19}}+\mathbf{1}_{\mathcal{U}_{20}}+\mathbf{1}_{\mathcal{U}_{21}}+\mathbf{1}_{\mathcal{U}_{22}}+\mathbf{1}_{\mathcal{U}_{23}}+\mathbf{1}_{\mathcal{U}_{24}}+\mathbf{1}_{\mathcal{U}_{25}}+\mathbf{1}_{\mathcal{U}_{26}}+\mathbf{1}_{\mathcal{U}_{27}}+\mathbf{1}_{\mathcal{U}_{28}}+\mathbf{1}_{\mathcal{U}_{29}}-\mathbf{1}_{\mathcal{U}_{41}}-\mathbf{1}_{\mathcal{U}_{42}}-\mathbf{1}_{\mathcal{U}_{43}}-\mathbf{1}_{\mathcal{U}_{44}}-\mathbf{1}_{\mathcal{U}_{45}}-\mathbf{1}_{\mathcal{U}_{46}}-\mathbf{1}_{\mathcal{U}_{47}}-\mathbf{1}_{\mathcal{U}_{48}}-\mathbf{1}_{\mathcal{U}_{49}}-\mathbf{1}_{\mathcal{U}_{50}}-\mathbf{1}_{\mathcal{U}_{51}}-\mathbf{1}_{\mathcal{U}_{52}}-\mathbf{1}_{\mathcal{U}_{53}}-\mathbf{1}_{\mathcal{U}_{54}}-\mathbf{1}_{\mathcal{U}_{55}}$
- $\mathbf{1}_{\mathcal{U}_{6}}+\mathbf{1}_{\mathcal{U}_{7}}+\mathbf{1}_{\mathcal{U}_{8}}+\mathbf{1}_{\mathcal{U}_{9}}+\mathbf{1}_{\mathcal{U}_{23}}+\mathbf{1}_{\mathcal{U}_{24}}+\mathbf{1}_{\mathcal{U}_{25}}+\mathbf{1}_{\mathcal{U}_{35}}+\mathbf{1}_{\mathcal{U}_{36}}+\mathbf{1}_{\mathcal{U}_{37}}-\mathbf{1}_{\mathcal{U}_{41}}-\mathbf{1}_{\mathcal{U}_{42}}-\mathbf{1}_{\mathcal{U}_{43}}-\mathbf{1}_{\mathcal{U}_{44}}-\mathbf{1}_{\mathcal{U}_{45}}-\mathbf{1}_{\mathcal{U}_{46}}-\mathbf{1}_{\mathcal{U}_{50}}-\mathbf{1}_{\mathcal{U}_{51}}-\mathbf{1}_{\mathcal{U}_{52}}-\mathbf{1}_{\mathcal{U}_{56}}-\mathbf{1}_{\mathcal{U}_{57}}-\mathbf{1}_{\mathcal{U}_{58}}-\mathbf{1}_{\mathcal{U}_{59}}$
- $\mathbf{1}_{\mathcal{U}_{2}}+\mathbf{1}_{\mathcal{U}_{7}}+\mathbf{1}_{\mathcal{U}_{11}}+\mathbf{1}_{\mathcal{U}_{15}}-\mathbf{1}_{\mathcal{U}_{18}}-\mathbf{1}_{\mathcal{U}_{20}}-\mathbf{1}_{\mathcal{U}_{22}}-\mathbf{1}_{\mathcal{U}_{23}}-\mathbf{1}_{\mathcal{U}_{25}}-\mathbf{1}_{\mathcal{U}_{26}}-\mathbf{1}_{\mathcal{U}_{28}}+\mathbf{1}_{\mathcal{U}_{31}}+\mathbf{1}_{\mathcal{U}_{33}}+\mathbf{1}_{\mathcal{U}_{36}}+\mathbf{1}_{\mathcal{U}_{39}}+\mathbf{1}_{\mathcal{U}_{42}}+\mathbf{1}_{\mathcal{U}_{45}}+\mathbf{1}_{\mathcal{U}_{48}}+\mathbf{1}_{\mathcal{U}_{51}}+\mathbf{1}_{\mathcal{U}_{54}}-\mathbf{1}_{\mathcal{U}_{56}}-\mathbf{1}_{\mathcal{U}_{58}}-\mathbf{1}_{\mathcal{U}_{60}}$
- $\mathbf{1}_{\mathcal{U}_{41}}+\mathbf{1}_{\mathcal{U}_{43}}+\mathbf{1}_{\mathcal{U}_{44}}+\mathbf{1}_{\mathcal{U}_{46}}-\mathbf{1}_{\mathcal{U}_{48}}+\mathbf{1}_{\mathcal{U}_{50}}+\mathbf{1}_{\mathcal{U}_{52}}-\mathbf{1}_{\mathcal{U}_{54}}+\mathbf{1}_{\mathcal{U}_{56}}+\mathbf{1}_{\mathcal{U}_{58}}-\mathbf{1}_{\mathcal{U}_{61}}+(\mathbb{L}-1)(\mathbf{1}_{\mathcal{U}_{1}}+\mathbf{1}_{\mathcal{U}_{3}}+\mathbf{1}_{\mathcal{U}_{4}}+\mathbf{1}_{\mathcal{U}_{5}}+\mathbf{1}_{\mathcal{U}_{10}}+\mathbf{1}_{\mathcal{U}_{12}}+\mathbf{1}_{\mathcal{U}_{13}}+\mathbf{1}_{\mathcal{U}_{14}}+\mathbf{1}_{\mathcal{U}_{16}}+\mathbf{1}_{\mathcal{U}_{17}}+\mathbf{1}_{\mathcal{U}_{18}}+\mathbf{1}_{\mathcal{U}_{20}}+\mathbf{1}_{\mathcal{U}_{22}}+\mathbf{1}_{\mathcal{U}_{26}}+\mathbf{1}_{\mathcal{U}_{28}}+\mathbf{1}_{\mathcal{U}_{30}}+\mathbf{1}_{\mathcal{U}_{32}}+\mathbf{1}_{\mathcal{U}_{34}}+\mathbf{1}_{\mathcal{U}_{38}}+\mathbf{1}_{\mathcal{U}_{40}})+(\mathbb{L}-2)(\mathbf{1}_{\mathcal{U}_{2}}+\mathbf{1}_{\mathcal{U}_{6}}+\mathbf{1}_{\mathcal{U}_{8}}+\mathbf{1}_{\mathcal{U}_{9}}+\mathbf{1}_{\mathcal{U}_{11}}+\mathbf{1}_{\mathcal{U}_{15}}+\mathbf{1}_{\mathcal{U}_{19}}+\mathbf{1}_{\mathcal{U}_{21}}+\mathbf{1}_{\mathcal{U}_{23}}+\mathbf{1}_{\mathcal{U}_{25}}+\mathbf{1}_{\mathcal{U}_{27}}+\mathbf{1}_{\mathcal{U}_{29}}+\mathbf{1}_{\mathcal{U}_{31}}+\mathbf{1}_{\mathcal{U}_{33}}+\mathbf{1}_{\mathcal{U}_{35}}+\mathbf{1}_{\mathcal{U}_{37}}+\mathbf{1}_{\mathcal{U}_{39}})+(\mathbb{L}-3)(\mathbf{1}_{\mathcal{U}_{7}}+\mathbf{1}_{\mathcal{U}_{24}}+\mathbf{1}_{\mathcal{U}_{36}})$
- $\mathbf{1}_{\mathcal{U}_{1}}+\mathbf{1}_{\mathcal{U}_{2}}+\mathbf{1}_{\mathcal{U}_{3}}+\mathbf{1}_{\mathcal{U}_{4}}+\mathbf{1}_{\mathcal{U}_{5}}+\mathbf{1}_{\mathcal{U}_{6}}+\mathbf{1}_{\mathcal{U}_{7}}+\mathbf{1}_{\mathcal{U}_{8}}+\mathbf{1}_{\mathcal{U}_{9}}+\mathbf{1}_{\mathcal{U}_{10}}+\mathbf{1}_{\mathcal{U}_{11}}+\mathbf{1}_{\mathcal{U}_{12}}+\mathbf{1}_{\mathcal{U}_{13}}+\mathbf{1}_{\mathcal{U}_{14}}+\mathbf{1}_{\mathcal{U}_{15}}+\mathbf{1}_{\mathcal{U}_{16}}+\mathbf{1}_{\mathcal{U}_{17}}+\mathbf{1}_{\mathcal{U}_{18}}+\mathbf{1}_{\mathcal{U}_{19}}+\mathbf{1}_{\mathcal{U}_{20}}+\mathbf{1}_{\mathcal{U}_{21}}+\mathbf{1}_{\mathcal{U}_{22}}+\mathbf{1}_{\mathcal{U}_{23}}+\mathbf{1}_{\mathcal{U}_{24}}+\mathbf{1}_{\mathcal{U}_{25}}+\mathbf{1}_{\mathcal{U}_{26}}+\mathbf{1}_{\mathcal{U}_{27}}+\mathbf{1}_{\mathcal{U}_{28}}+\mathbf{1}_{\mathcal{U}_{29}}+\mathbf{1}_{\mathcal{U}_{30}}+\mathbf{1}_{\mathcal{U}_{31}}+\mathbf{1}_{\mathcal{U}_{32}}+\mathbf{1}_{\mathcal{U}_{33}}+\mathbf{1}_{\mathcal{U}_{34}}+\mathbf{1}_{\mathcal{U}_{35}}+\mathbf{1}_{\mathcal{U}_{36}}+\mathbf{1}_{\mathcal{U}_{37}}+\mathbf{1}_{\mathcal{U}_{38}}+\mathbf{1}_{\mathcal{U}_{39}}+\mathbf{1}_{\mathcal{U}_{40}}+\mathbf{1}_{\mathcal{U}_{41}}+\mathbf{1}_{\mathcal{U}_{42}}+\mathbf{1}_{\mathcal{U}_{43}}+\mathbf{1}_{\mathcal{U}_{44}}+\mathbf{1}_{\mathcal{U}_{45}}+\mathbf{1}_{\mathcal{U}_{46}}+\mathbf{1}_{\mathcal{U}_{47}}+\mathbf{1}_{\mathcal{U}_{48}}+\mathbf{1}_{\mathcal{U}_{49}}+\mathbf{1}_{\mathcal{U}_{50}}+\mathbf{1}_{\mathcal{U}_{51}}+\mathbf{1}_{\mathcal{U}_{52}}+\mathbf{1}_{\mathcal{U}_{53}}+\mathbf{1}_{\mathcal{U}_{54}}+\mathbf{1}_{\mathcal{U}_{55}}+\mathbf{1}_{\mathcal{U}_{56}}+\mathbf{1}_{\mathcal{U}_{57}}+\mathbf{1}_{\mathcal{U}_{58}}+\mathbf{1}_{\mathcal{U}_{59}}+\mathbf{1}_{\mathcal{U}_{60}}+\mathbf{1}_{\mathcal{U}_{61}}.$

Summary

This thesis studies the geometry of representation varieties and character stacks. These are spaces that parametrize the representations of a finitely generated group Γ into an algebraic group G. More precisely, the representation variety parametrizes all such representations, whereas the character stack parametrizes them up to isomorphism. Usually, the finitely generated group Γ is the fundamental group of a compact manifold M, in which case the representation variety and character stack equivalently parametrize G-local systems on M. This thesis contains a number of methods to study these spaces through their invariants. Besides providing theoretical descriptions, another aim of this thesis is to explicitly compute these invariants in specific cases. Motivated by these applications, we develop a number of new computational tools.

In Chapter 1, we review the necessary background on groupoids and algebraic stacks, focusing in particular on quotient stacks and stabilizers. We use this theory in Chapter 2, where we give precise definitions of representation varieties and character stacks. Furthermore, we show that these spaces admit a number of functorial properties that are crucial for the later parts of the thesis.

In Chapter 3, we study *motivic invariants*, which are invariants χ of varieties that are additive and multiplicative in the sense that $\chi(X) = \chi(Z) + \chi(X \setminus Z)$ and $\chi(X \times Y) = \chi(X)\,\chi(Y)$ for all varieties X and Y and closed subvarieties $Z \subseteq X$. We discuss various motivic invariants and their properties, with a special focus on the universal motivic invariant, called the *virtual class*, which takes values in the Grothendieck ring of varieties. This Grothendieck ring has a natural generalization to algebraic stacks, allowing us to talk about the virtual class, and other motivic invariants, of character stacks. Furthermore, we develop tools for computing motivic invariants, such as an algorithm to compute virtual classes of certain varieties, and we study how motivic invariants behave with respect to finite group actions.

In Chapter 4, we describe two known methods for computing motivic invariants of representation varieties and character stacks. We show how both the *arithmetic*

method, which studies the character stacks of compact orientable surfaces through counting points over finite fields, and the *geometric method*, which studies the same character stacks using clever stratifications, can be expressed in terms of *Topological Quantum Field Theories (TQFTs)*. Originating from physics, TQFTs are monoidal functors from the category of bordisms to the category of modules over a fixed commutative ring. The TQFTs associated to both methods can be expressed as the composite of a field theory and a quantization functor. Comparing the field theories and quantization functors of both methods, we show that the TQFTs of both methods can be related through natural transformations.

In Chapter 5, we apply the theory of Chapter 4 to explicitly compute the virtual classes of the SL_2-character stacks of orientable and non-orientable surfaces. This results in many intricate computations. Even though similar computations already exist that compute the E-polynomial (an invariant coarser than the virtual class, reflecting the mixed Hodge structure) of these character stacks, adapting these computations to the Grothendieck ring of varieties introduces many subtle problems which we deal with.

In Chapter 6, we focus on the groups G of $n \times n$ upper triangular matrices and unipotent upper triangular matrices. By means of computer-assisted calculations, we compute the virtual classes of the G-character stacks of orientable surfaces for $n \leq 5$ through the geometric method, and their E-polynomials for $n \leq 10$ through the arithmetic method. This task, which is already difficult for small n, was made possible by introducing the notion of algebraic representatives, and using the theory of special algebraic groups. Comparing the arithmetic and geometric method, we show how the geometric method can be simplified significantly using the results from the arithmetic method, that is, using the representation theory of the groups of upper triangular matrices over finite fields.

Finally, in Chapter 7, we turn our attention to the representation varieties and character stacks of the free groups F_n and free abelian groups $\mathbb{Z}^n$. These spaces parametrize tuples (resp. commuting tuples) of elements of G. It is known that the homology of these spaces, and many variations thereof, stabilize as n tends to infinity, in a well-defined sense known as *representation stability*. Inspired by this notion, we define an analogous notion of *motivic representation stability* for stability in the Grothendieck ring of varieties. As an application, we show that the character stacks of F_n and $\mathbb{Z}^n$ stabilize in this sense for the linear groups $G = \mathrm{GL}_r$.

Samenvatting

Dit proefschrift bestudeert de meetkunde van representatievariëteiten en karakterstacks. Dit zijn ruimtes die de representaties van een eindig voortgebrachte groep Γ naar een algebraïsche groep G parametriseren. Om precies te zijn, de representatievariëteit parametriseert al zulke representaties, en de karakterstack tot op isomorfie. De eindig voortgebrachte groep Γ is doorgaans de fundamentaalgroep van een compacte differentieerbare variëteit M, in welk geval de representatievariëteit en karakterstack ook wel G-lokale systemen op M parametriseren. Dit proefschrift bevat een aantal methodes om deze ruimtes te bestuderen aan de hand van hun invarianten. Naast het geven van theoretische beschrijvingen, beoogt dit proefschrift ook om deze invarianten expliciet te berekenen. Gemotiveerd door deze toepassingen, ontwikkelen we een aantal nieuwe computationele hulpmiddelen.

In Hoofdstuk 1 geven we de nodige voorkennis over groepoïden en algebraïsche stacks, waarbij de nadruk ligt op quotiëntstacks en stabilisatoren. Deze theorie gebruiken we in Hoofdstuk 2, waar we een precieze definitie geven van representatievariëteiten en karakterstacks. Verder laten we zien dat deze ruimtes een aantal functoriële eigenschappen bezitten die cruciaal zijn voor de latere delen van het proefschrift.

In Hoofdstuk 3 bestuderen we *motivische invarianten*, dat zijn invarianten χ van variëteiten die additief en multiplicatief zijn in de zin dat $\chi(X) = \chi(Z) + \chi(X \setminus Z)$ en $\chi(X \times Y) = \chi(X)\,\chi(Y)$ voor alle variëteiten X en Y en gesloten subvariëteiten $Z \subseteq X$. We bespreken verschillende motivische invarianten en hun eigenschappen, met een nadruk op de universele motivische invariant, de *virtuele klasse*, die waardes aanneemt in de Grothendieck-ring van variëteiten. Deze Grothendieck-ring heeft een natuurlijke generalisatie naar algebraïsche stacks, die ons in staat stelt te praten over de virtuele klasse, en andere motivische invarianten, van bijvoorbeeld karakterstacks. Verder ontwikkelen we hulpmiddelen om motivische invarianten te berekenen, zoals een algoritme om virtuele klassen te berekenen, en bestuderen we hoe motivische invarianten zich gedragen onder groepswerkingen van eindige groepen.

In Hoofdstuk 4 beschrijven we twee bekende methodes om motivische invarianten te berekenen van representatievariëteiten en karakterstacks. We laten zien dat zowel de *arithmetische methode*, die de karakterstacks van compacte oriënteerbare oppervlakten bestudeert door punten te tellen over eindige lichamen, en de *meetkundige methode*, die dezelfde karakterstacks bestudeert door deze slim te stratificeren, kunnen worden beschreven als *topologische kwantumveldentheorie (TQFT).* TQFTs vinden hun oorsprong in de natuurkunde, en zijn monoïdale functoren van de categorie van bordismen naar de categorie van modulen over een vaste commutatieve ring. De TQFTs van beide methodes kunnen worden uitgedrukt als samenstelling van een veldentheorie en een kwantisatiefunctor. Door de veldentheorieën en kwantisatiefunctoren te vergelijken, laten we zien dat de twee TQFTs zijn verbonden via een natuurlijke transformatie.

In Hoofdstuk 5 passen we de theorie van Hoofdstuk 4 toe om expliciet de virtuele klassen van de SL_2-karakterstacks van oriënteerbare en niet-oriënteerbare oppervlakken uit te rekenen, met complexe berekeningen tot gevolg. Ondanks dat er al vergelijkbare berekeningen bestaan die de E-polynomen bepalen (een grovere invariant dan de virtuele klasse, die de gemengde Hodgestructuur weerspiegelt) van deze karakterstacks, introduceert het tillen van deze berekeningen naar de Grothendieck-ring van variëteiten diverse subtiele problemen die we oplossen.

In Hoofdstuk 6 richten we ons op de groepen G van $n \times n$ bovendriehoeksmatrices en unipotente bovendriehoeksmatrices. Met behulp van de computer bepalen we de virtuele klassen van de karakterstacks van oriënteerbare oppervlakken voor $n \leq 5$ via de meetkundige methode, en hun E-polynomen voor $n \leq 10$ via de arithmetische methode. Deze berekeningen, al gecompliceerd voor kleine n, zijn mogelijk gemaakt door het introduceren van algebraïsche representanten en de theorie van speciale algebraïsche groepen. We vergelijken de arithmetische en meetkundige methode, en laten zien hoe de meetkundige methode significant vereenvoudigd kan worden door gebruik te maken van de resultaten van de arithmetische methode, dat wil zeggen, gebruikmakend van de representatietheorie van de groepen van bovendriehoeksmatrices over eindige lichamen.

Ten slotte, in Hoofdstuk 7, bestuderen we de representatievariëteiten en karakterstacks van de vrije groepen F_n en de vrije abelse groepen $\mathbb{Z}^n$. Deze ruimtes parametriseren tupels (resp. commuterende tupels) van elementen van G. Het is bekend dat de homologie van deze ruimtes, en variaties daarop, stabiliseert als n naar oneindig neigt, in een goed-gedefinieerde zin bekend als *representatie-stabiliteit*. Geïnspireerd door dit begrip definiëren we een analoog begrip van *motivische representatiestabiliteit* voor stabiliteit in de Grothendieck-ring van variëteiten. Als toepassing laten we zien dat de karakterstacks van F_n en $\mathbb{Z}^n$ stabiliseren in deze zin voor de algemene lineaire groepen $G = \mathrm{GL}_r$.

Acknowledgments

This thesis was created with the help and support of many people, for which I am very grateful.

First and foremost, I would like to thank my supervisor Marci. Thank you for all the interesting discussions, in which you were always happy to answer my questions. Your way of thinking has inspired me to come up with my own problems, and ways to solve them. It has been a great pleasure working with you.

I would like to thank my promotores, Bas and Ronald, for all of their help, and the doctorate committee, Gianne, David, Carlos, Vicente and Ángel, for their useful comments to improve the contents of this thesis. In particular, I would like to thank Ángel, whose research inspired my master's thesis, which evolved into this thesis. It has been a great pleasure collaborating with you.

I would like to thank my office mates, Daan, Georgios, Onno and Pim, for all the fun and interesting conversations, for thinking along on problems, and for their baked goods.

Finally, I want to thank my family and friends for their support and confidence, and for showing interest in my research. And most of all, I want to thank Joyce for her endless support and love, and for helping me to focus on things other than work.

Curriculum Vitae

Jesse Tijs Vogel was born on the 26th of January 1996 in Loppersum. From 2008 to 2014, he followed the bilingual vwo program at the Gomarus College in Groningen. He went on to study Mathematics and Physics at the University of Groningen. In 2018, he wrote his bachelor's thesis, titled "*Generalized Geometry and Double Field Theory Applications to Closed String Theory*", and obtained both his bachelor's degrees summa cum laude.

Afterwards, Jesse moved to Leiden to continue his studies. He followed the master *Algebra, Geometry and Number Theory* at Leiden University. In 2020, he defended his master's thesis, titled "*Computing Virtual Classes of Representation Varieties using TQFTs*", with which he obtained his master's degree summa cum laude.

During his studies, he has been teaching assistant for several courses, and was awarded *Teaching Assistant of the Year* in 2018.

After his studies, Jesse continued at Leiden University as a PhD candidate under the supervision of dr. Márton Hablicsek. This thesis is the result of the research he did from 2020 to 2024.