

RIGID MEROMORPHIC COCYCLES

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INTRODUCTION

These are notes from a set of lectures given in Barcelona, June 2017, about meromorphic cocycles. Many of the results are classical, and all the new constructions and results mentioned in this document are joint work with **Henri Darmon**.

Class field theory. Let K be a number field, and C_K its idèle class group. Fix a separable closure of K , and take the maximal abelian subextension K^{ab}/K . Then there exists a *global Artin map*

$$\varphi : C_K \rightarrow \text{Gal}(K^{\text{ab}}/K).$$

This map is surjective, and its kernel is the connected component of the identity. It becomes an isomorphism of topological groups when we pass to profinite completions:

$$\varphi : \widehat{C}_K \xrightarrow{\sim} \text{Gal}(K^{\text{ab}}/K), \quad \text{where } \widehat{C}_K = \varprojlim_U C_K/U,$$

with the limit taken over all finite index open subgroups. The Artin map sets up a bijection between finite index open subgroups of C_K , and finite abelian extensions of K , where a finite abelian extension L/K corresponds to the finite index open subgroup $\text{Nm}_{L/K}C_L$ of C_K . This is a powerful dictionary, as it encodes information about extensions of K into an object that makes no reference to other fields and depends only on the internal arithmetic of K . It is relatively straightforward to describe all the finite index open subgroups of C_K , but the nature of the Artin map does not allow us to easily find generators for its corresponding abelian extension of K . This is the subject of *explicit class field theory*, also known as Hilbert's 12th problem. When $K = \mathbf{Q}$, the theorem of Kronecker–Weber guarantees that K^{ab} is generated by the special values

$$\exp(2\pi iz), \quad z \in \mathbf{Q}$$

of the transcendental exponential function. The aspiration to extend this to more general number fields is known as Kronecker's Jugendtraum, and these lectures are about the case of quadratic extensions of \mathbf{Q} . The construction of such a transcendental function will take very different forms for imaginary and real quadratic fields, and remains conjectural for the latter.

CM theory. The theory of complex multiplication for elliptic curves gives a very concrete construction of all finite abelian extensions of an imaginary quadratic number field K . The method is to consider elliptic curves $E_{\mathbf{Q}}$ whose ring of endomorphisms is isomorphic to an order in K . All finite abelian extensions of K may then be obtained by adjoining combinations of two basic invariants attached to E : Its j -invariant, and the x -coordinates of its torsion points.

By means of an example, a classical result due to Weber, which uses some refined statements in CM theory and the theory of Weber functions, shows that

$$j(\sqrt{-14}) = 2^3 \left(323 + 228\sqrt{2} + (231 + 161\sqrt{2})\sqrt{2\sqrt{2}-1} \right)^3.$$

This number lies in the Hilbert class field $H = K(\sqrt{2\sqrt{2}-1})$ of $K = \mathbf{Q}(\sqrt{-14})$. Many other examples may be computed by hand, and fast algorithms are available for precisely determining explicit generators for any finite abelian extension of imaginary quadratic fields.

Meromorphic cocycles. If one naively tries to adapt the methods of CM theory to construct abelian extensions of *real* quadratic fields, one immediately runs into a number of seemingly insurmountable difficulties. One of these arises at a very basic level: Real quadratic singularities do not lie in \mathcal{H} , where the j -function is defined! The main goal of these notes is to discuss a proposal for a p -adic analogue of singular moduli, as discussed in recent work with **Henri Darmon**.

The idea is to replace the prime ∞ by a finite prime p . Unlike \mathcal{H} , the p -adic upper half plane \mathcal{H}_p contains many real quadratic points $\tau \in \mathcal{H}_p \cap K$, where $K \subset \mathbf{C}_p$ is a real quadratic field in which the prime p is either inert or ramified. We will refer to such points as RM points. They are characterised by the fact that their stabiliser in $\Gamma = \mathrm{SL}_2(\mathbf{Z}[1/p])$ is an infinite cyclic group generated by a matrix $\gamma_\tau \in \Gamma$ whose eigenvalues belong to \mathcal{O}_K^\times . The group Γ acts on holomorphic functions \mathcal{O}^\times and meromorphic functions \mathcal{M}^\times via the usual weight 0 action. Because the Γ -orbits in \mathcal{H}_p are dense for the rigid analytic topology, we have

$$\mathrm{H}^0(\Gamma, \mathcal{O}^\times) = \mathrm{H}^0(\Gamma, \mathcal{M}^\times) = \mathbf{C}_p^\times$$

We therefore look at the first cohomology group $\mathrm{H}^1(\Gamma, \mathcal{O}^\times)$ instead, consisting of one-cocycles $c : \Gamma \rightarrow \mathcal{O}^\times$ satisfying

$$c(\gamma_1\gamma_2) = c(\gamma_1) + \gamma_1 \cdot c(\gamma_2),$$

taken modulo 1-coboundaries, of the form $c(\gamma) = \gamma \cdot f - f$, for $f \in \mathcal{O}^\times$. A class in $\mathrm{H}^1(\Gamma, \mathcal{O}^\times)$ is called a (multiplicative) *rigid analytic cocycle* for Γ , and one in $\mathrm{H}^1(\Gamma, \mathcal{M}^\times)$ is called a (multiplicative) *rigid meromorphic cocycle*.

The *value* of a rigid meromorphic cocycle Φ^\times at the RM point τ is defined to be

$$\Phi^\times[\tau] := \Phi^\times(\gamma_\tau)(\tau) \in \mathbf{C}_p,$$

a numerical invariant which depends only on the class of Φ in cohomology and not on the choice of cocycle representing it. We define, for any RM point τ_1 , a multiplicative rigid meromorphic cocycle

Φ_F^\times for Γ , with values in the set of meromorphic functions $\mathcal{M}^\times/\mathbf{C}_p^\times$. Its values

$$J_p(\tau_1, \tau_2) := \Phi_{\tau_1}^\times(\tau_2)$$

conjecturally satisfy many properties analogous to those enjoyed by singular moduli. As an example, consider the golden ratio ω , which satisfies $\omega^2 = \omega + 1$. We calculate that

$$\begin{aligned} J_3(\omega, 4\sqrt{2}) &= \frac{-70-35\sqrt{5}+40\sqrt{2}-40i+16\sqrt{10}-20\sqrt{-5}-70\sqrt{-2}-28\sqrt{-10}}{65} \\ J_3(\omega, -4\sqrt{2}) &= \frac{-70-35\sqrt{5}+40\sqrt{2}+40i+16\sqrt{10}+20\sqrt{-5}+70\sqrt{-2}-28\sqrt{-10}}{65} \end{aligned}$$

Both of these are generators for the compositum H_{12} of the Hilbert class fields H_1 and H_2 of the orders $\mathbf{Z}[\sqrt{5}]$ and $\mathbf{Z}[\sqrt{2}]$. The denominator is $5 \cdot 13$, and both of these primes are divisors of a number of the form $5 \cdot 8 - x^2$.

Outline. After quickly recalling some classical results from CM theory in Section 1, we move to the theory of quadratic forms, which occupy center stage in the proposed p -adic strategy for explicit class field theory of real quadratic number fields. This topic is discussed in Section 3 using the Conway topograph, a combinatorial tool that makes many of the later constructions more natural. We then introduce techniques from p -adic analysis, and interpret some earlier constructions of Darmon from the viewpoint of *rigid analytic cocycles*, introduced in Section 2. Finally, we introduce *rigid meromorphic cocycles* in Section 4, by making a synthesis of the p -adic analytic theta functions considered in quaternionic settings with the work on rational period functions on the Riemann sphere by Knopp, Ash, Choie–Zagier, and Duke–Imamoglu–Toth.

1. CLASSICAL CM THEORY AND SINGULAR MODULI

We start by recalling some classical results from the theory of complex multiplication for elliptic curves. The algebraic values of the transcendental function $j(\tau)$ are called *singular moduli*, and enjoy many interesting arithmetic properties. We recall some of these results, due to Gross–Zagier at the end of this section.

1.1. Abelian extensions of imaginary quadratic fields. We will start by recalling the main theorem of CM theory, as presented by Shimura [Shi70, Section 5]. As the set of CM elliptic curves is countable, it is easy to see that $j(E) \in \overline{\mathbf{Q}}$ for $E_{\mathbf{Q}}$ an elliptic curve with CM by K . The main theorem of complex multiplication fine-tunes our understanding of the set $j(E)^\sigma$, where $\sigma \in \text{Aut}(\mathbf{C}/K)$.

Let K be an imaginary quadratic extension of \mathbf{Q} . The Artin map $\varphi : C_K \rightarrow \text{Gal}(K^{\text{ab}}/K)$ is surjective, and more precisely, there is an exact sequence

$$1 \longrightarrow K^\times \mathbf{A}_{K,\infty}^\times \longrightarrow \mathbf{A}_K^\times \xrightarrow{\varphi_K} \text{Gal}(K^{\text{ab}}/K) \longrightarrow 1,$$

where $\mathbf{A}_{K,\infty}^\times$ is the archimedean part of the idèles. Let E be an elliptic curve with CM by K , where we have normalised the isomorphism $\text{End}_{\mathbf{Q}}(E) \simeq K$ s.t. the endomorphism corresponding to $\lambda \in K$ acts on the cotangent space at the origin as multiplication by λ . We know that there exists an isomorphism

$$\xi : \mathbf{C}/\mathfrak{a} \xrightarrow{\sim} E$$

for some lattice \mathfrak{a} in K . Let p be a rational prime, and define $K_p = K \otimes_{\mathbf{Q}} \mathbf{Q}_p$. Then $\mathfrak{a}_p = \mathfrak{a} \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is a lattice in K_p . We now recall how to multiply a lattice by an idèle of K :

Lemma 1.1. *Let $x \in \mathbf{A}_K^\times$, then there exists a lattice $x\mathfrak{a}$ in K such that $(x\mathfrak{a})_p = x_p\mathfrak{a}_p$ for all p , and we have a commutative diagram, where the horizontal maps are isomorphisms*

$$\begin{array}{ccc} K_p/\mathfrak{a}_p & \xrightarrow{x_p} & K_p/x_p\mathfrak{a}_p \\ \downarrow & & \downarrow \\ K/\mathfrak{a} & \xrightarrow{x} & K/x\mathfrak{a} \end{array}$$

Remark. The above decomposition of K/\mathfrak{a} into local pieces is taken from Shimura [Shi70, Section 5.2], and there is a much more general version for torsion modules over a Dedekind domain. A very precise description is given in Silverman [Sil09, Section II.8].

This gives us a method to modify a lattice \mathfrak{a} in K by an idèle. The main theorem of complex multiplication says that this is precisely the modification needed to obtain the lattice corresponding to the image of E under the Galois action associated with the idèle via class field theory.

Theorem 1.2. *Let $\sigma \in \text{Aut}(\mathbf{C}/K)$, and $s \in \mathbf{A}_K^\times$ an element corresponding to $\sigma|_{K^{\text{ab}}}$ via the Artin map attached to K . There exists an isomorphism $\xi_s : \mathbf{C}/s^{-1}\mathfrak{a} \rightarrow E^\sigma$ such that the following diagram commutes:*

$$\begin{array}{ccc} K/\mathfrak{a} & \xrightarrow{\xi} & E \\ s^{-1}\downarrow & & \downarrow \sigma \\ K/s^{-1}\mathfrak{a} & \xrightarrow{\xi_s} & E^\sigma \end{array}$$

The main theorem has far-reaching consequences, and in particular it allows us to recover a number of classical results on explicit class field theory due to Kronecker, Weber, Takagi, and Hasse. From the main theorem, it follows that $j(\mathfrak{a})^\sigma = j(s^{-1}\mathfrak{a})$, so that $j(\mathfrak{a})^\sigma$ only depends on the restriction of σ to K^{ab} . It follows that $j(\mathfrak{a}) \in K^{\text{ab}}$, and the action of the element corresponding to s is via multiplication by s^{-1} on the lattice. We can in fact be much more precise about the nature of the numbers $j(\mathfrak{a})$:

Corollary 1.3. *Let \mathfrak{a} be a proper ideal of an order \mathcal{O} in K , then we have*

- $K(j(\mathfrak{a}))$ is the ring class field of \mathcal{O} over K ,
- The map $\text{Gal}(K(j(\mathfrak{a}))/K) \rightarrow \text{Cl } \mathcal{O} : \sigma \mapsto \mathfrak{b}$, where $j(\mathfrak{a})^\sigma = j(\mathfrak{b}^{-1}\mathfrak{a})$, is an isomorphism,
- We have $\deg(K(j(\mathfrak{a}))/K) = \deg(\mathbf{Q}(j(\mathfrak{a}))/\mathbf{Q})$,

All of this, and much more, is proved in Shimura [Shi70, Section 5]. We have omitted the important role of the coordinates of torsion points, which are needed to generate all ray class fields of K , together with the values of the j -invariant. These two different proofs foreshadow the determination of the non-archimedean and archimedean contributions to the celebrated height calculations of Heegner points on $X_0(N)$ in [GZ86].

1.2. Singular moduli. The main theorems of CM theory for elliptic curves tell us that whenever $\tau \in \mathcal{H}$ generates a quadratic order of class number one, the number $j(\tau)$ is an integer. This is a crucial ingredient of Heegner's proof of the celebrated *class number one problem*, which states that there are precisely 9 quadratic imaginary fields with class number one, tabulated here:

Field	$E_{\mathbf{Q}}$ with CM by maximal order	$j(E)$
$\mathbf{Q}(\sqrt{-1})$	$y^2 = x^3 + x$	$2^6 \cdot 3^3$
$\mathbf{Q}(\sqrt{-2})$	$y^2 = x^3 + x$	$2^6 \cdot 5^3$
$\mathbf{Q}(\sqrt{-3})$	$y^2 + xy = x^3 - x^2 - 2x - 1$	0
$\mathbf{Q}(\sqrt{-7})$	$y^2 = x^3 + 4x^2 + 2x$	$-3^3 \cdot 5^3$
$\mathbf{Q}(\sqrt{-11})$	$y^2 + y = x^3 - x^2 - 7x + 10$	-2^{15}
$\mathbf{Q}(\sqrt{-19})$	$y^2 + y = x^3 - 38x + 90$	$-2^{15} \cdot 3^3$
$\mathbf{Q}(\sqrt{-43})$	$y^2 + y = x^3 - 860x + 9707$	$-2^{18} \cdot 3^3 \cdot 5^3$
$\mathbf{Q}(\sqrt{-67})$	$y^2 + y = x^3 - 7370x + 243528$	$-2^{15} \cdot 3^3 \cdot 5^3 \cdot 11^3$
$\mathbf{Q}(\sqrt{-163})$	$y^2 + y = x^3 - 2174420x + 1234136692$	$-2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3$

A striking feature of the values of j at imaginary quadratic points, often called *singular moduli*, is that they are highly divisible. This observation was made precise, and proved for norms of arbitrary singular moduli, by Gross and Zagier [GZ85]. For simplicity, let us assume that Δ_1, Δ_2 are two negative fundamental discriminants which are coprime. We define the product

$$J(\Delta_1, \Delta_2) = \left(\prod_{[\tau_1], [\tau_2]} (j(\tau_1) - j(\tau_2)) \right)^{\frac{4}{w_1 w_2}},$$

where the product runs over $\mathrm{SL}_2(\mathbf{Z})$ -equivalence classes of quadratic imaginary numbers τ_1 and τ_2 of discriminant Δ_1 and Δ_2 respectively, and w_i is the number of torsion elements in the quadratic order of discriminant Δ_i . Gross and Zagier [GZ85] prove an explicit formula for the integer $J(\Delta_1, \Delta_2)^2$, which has the following consequence

Theorem 1.4 (Gross–Zagier). *If l is a prime dividing $J(\Delta_1, \Delta_2)^2$, then*

$$\left(\frac{\Delta_1}{l} \right), \left(\frac{\Delta_2}{l} \right) \neq 1, \quad \text{and} \quad l \mid \frac{\Delta_1 \Delta_2 - b^2}{4}$$

for some $b < \sqrt{\Delta_1, \Delta_2}$.

Gross and Zagier [GZ85] offer two proofs of this theorem, one using Deuring’s lifting theorem for supersingular elliptic curves, and one using an analytic calculation using the diagonal restriction of an Eisenstein series in two variables.

2. RIGID ANALYTIC COCYCLES

The group $\mathrm{PGL}_2(\mathbf{Q}_p)$ acts naturally on the p -adic upper half plane \mathcal{H}_p by Möbius transformations, and the spaces \mathcal{O} and \mathcal{M} of rigid analytic and rigid meromorphic functions on \mathcal{H}_p is equipped, for each even integer k , with a *weight k action* given by

$$(1) \quad (f|_k \gamma)(\tau) = \frac{\det(\gamma)^{k/2}}{(c\tau + d)^k} f\left(\frac{a\tau + b}{c\tau + d}\right), \quad \text{where } \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Write \mathcal{O}_k for the space \mathcal{O} endowed with this action. In this section, we will discuss rigid analytic cocycles, i.e. elements of

$$\mathrm{H}^1(\Gamma, \mathcal{O}_k), \quad \text{where } \Gamma := \mathrm{SL}_2(\mathbf{Z}[1/p]).$$

The partial lifts of certain weight two cocycles under the logarithmic derivative map

$$\mathrm{dlog} : H^1(\Gamma, \mathcal{O}^\times) \longrightarrow H^1(\Gamma, \mathcal{O}_2)$$

are introduced, and the Gross–Stark units of [DD06] and the Stark–Heegner points of [Dar01], which are conjecturally defined over ring class fields of real quadratic fields, will be interpreted as RM values of these multiplicative cocycles. These values are natural substitutes for the CM values of classical modular forms.

2.1. Modular symbols and the residue map. As it is more convenient to work with modular symbols than with parabolic cocycles, we now recall some standard definitions. If Ω is an abelian group, we say an Ω -valued modular symbol is a function

$$m : \mathbf{P}^1(\mathbf{Q}) \times \mathbf{P}^1(\mathbf{Q}) \longrightarrow \Omega,$$

satisfying

$$m\{r, s\} = -m\{s, r\}, \quad m\{r, s\} + m\{s, t\} = m\{r, t\} \quad \text{for all } r, s, t \in \mathbf{P}^1(\mathbf{Q}).$$

The space of Ω -valued modular symbols is denoted $\mathrm{MS}(\Omega)$. If Ω has an action of $\mathrm{PGL}_2(\mathbf{Q})$, then $\mathrm{MS}(\Omega)$ is naturally endowed with the $\mathrm{PGL}_2(\mathbf{Q})$ -action defined by

$$(m|\gamma)\{r, s\} := (m\{\gamma r, \gamma s\})|\gamma.$$

The space of Γ -invariant modular symbols $\mathrm{MS}^\Gamma(\Omega) := H^0(\Gamma, \mathrm{MS}(\Omega))$ is related to cohomology by the exact sequence

$$(2) \quad 0 \longrightarrow \Omega^\Gamma \longrightarrow \Omega^{\Gamma_\infty} \longrightarrow \mathrm{MS}^\Gamma(\Omega) \longrightarrow H^1(\Gamma, \Omega) \longrightarrow H^1(\Gamma_\infty, \Omega).$$

This sequence is obtained from exact sequence of $\mathbf{Z}[\Gamma]$ -modules

$$(3) \quad 0 \longrightarrow \Omega \longrightarrow \mathrm{Fct}(\mathbf{P}^1(\mathbf{Q}), \Omega) \xrightarrow{d} \mathrm{MS}(\Omega) \longrightarrow 0,$$

where $df\{r, s\} := f(s) - f(r)$, and $\mathrm{Fct}(\mathbf{P}^1(\mathbf{Q}), \Omega)$ be the Γ -module of Ω -valued functions on $\mathbf{P}^1(\mathbf{Q})$.

Now let us specialise to the case $\Omega = \mathcal{O}_2$. In that case, the above exact sequence induces an injection $\mathrm{MS}^\Gamma(\mathcal{O}_2) \hookrightarrow H^1(\Gamma, \mathcal{O}_2)$, whose image we call the parabolic cohomology group $H_{\mathrm{par}}^1(\Gamma, \mathcal{O}_2)$. After choosing a base point $r \in \mathbf{P}^1(\mathbf{Q})$, the cohomology class $\Phi := \delta(\Phi_0)$ associated to $\Phi_0 \in \mathrm{MS}^\Gamma(\mathcal{O}_2)$ is defined by $\Phi(\gamma) = \Phi_0\{r, \gamma r\}$. Parabolic cocycles are very closely related to classical modular forms, as the following classification theorem shows.

Theorem 2.1 (Darmon–V). *There is a Hecke-equivariant isomorphism, which we call the Schneider–Teitelbaum lift:*

$$L^{\mathrm{ST}} : \mathrm{MS}^{\Gamma_0(p)}(\mathbf{C}_p) \xrightarrow{\sim} \mathrm{MS}^\Gamma(\mathcal{O}_2).$$

The proof of this theorem is inspired by the proofs of analogous results in the case where Γ is replaced by an appropriate quaternionic group. It relies on the Poisson kernel function introduced by Teitelbaum, as well as the control theorem of Stevens for overconvergent modular symbols.

There are precisely two conjugacy classes of parabolic subgroups of $\Gamma_0(p)$, the group P_∞ consisting of the upper triangular matrices, which stabilise the cusp ∞ , and the group P_0 consisting of lower triangular matrices, which stabilise the cusp 0. The $\Gamma_0(p)$ -module $\mathrm{Fct}(\mathbf{P}^1(\mathbf{Q}), \mathbf{C}_p)$ therefore decomposes as a direct sum of the two induced modules

$$\mathrm{Fct}(\mathbf{P}^1(\mathbf{Q}), \mathbf{C}_p) = \mathrm{Ind}_{P_\infty}^{\Gamma_0(p)} \mathbf{C}_p \oplus \mathrm{Ind}_{P_0}^{\Gamma_0(p)} \mathbf{C}_p,$$

and the $\Gamma_0(p)$ -cohomology of the exact sequence (3) with $\Omega = \mathbf{C}_p$ leads to a short exact sequence

$$(4) \quad 0 \longrightarrow \mathbf{C}_p \longrightarrow \mathrm{MS}^{\Gamma_0(p)} \longrightarrow \mathrm{H}_{\mathrm{par}}^1(\Gamma_0(p), \mathbf{C}_p) \longrightarrow 0.$$

2.2. Multiplicative cocycles and the multiplicative Schneider-Teitelbaum lift. The logarithmic derivative gives a natural isomorphism

$$\mathrm{dlog} : \mathcal{O}^\times / \mathbf{C}_p^\times \longrightarrow \mathcal{O}_2$$

sending the local section f to f'/f , where f' denotes the derivative with respect to τ . It induces a similar map

$$(5) \quad \mathrm{dlog} : \mathrm{MS}^\Gamma(\mathcal{O}^\times / \mathbf{C}_p^\times) \longrightarrow \mathrm{MS}^\Gamma(\mathcal{O}_2)$$

on the space of Γ -invariant modular symbols. The space $\mathrm{dlog}(\mathcal{O}^\times) \subset \mathcal{O}_2$ is called the space of *rigid differentials of the third kind* on \mathcal{H}_p , and consists of differentials whose image under ∂ are \mathbf{Z} -valued harmonic functions on \mathcal{T}_1^* . The image of (5) is likewise called the space of rigid analytic modular symbols of the third kind.

Proposition 2.2. *There is a Hecke equivariant map*

$$L_{\mathrm{ST}}^\times : \mathrm{MS}^{\Gamma_0(p)}(\mathbf{Z}) \longrightarrow \mathrm{MS}^\Gamma(\mathcal{O}^\times / \mathbf{C}_p^\times)$$

for which the diagram

$$\begin{array}{ccc} \mathrm{MS}^{\Gamma_0(p)}(\mathbf{Z}) & \xrightarrow{L_{\mathrm{ST}}^\times} & \mathrm{MS}^\Gamma(\mathcal{O}^\times / \mathbf{C}_p^\times) \\ \downarrow & & \downarrow \mathrm{dlog} \\ \mathrm{MS}^{\Gamma_0(p)}(\mathbf{C}_p) & \xrightarrow{L_{\mathrm{ST}}} & \mathrm{MS}^\Gamma(\mathcal{O}_2) \end{array}$$

commutes.

The map L_{ST}^\times is called the *multiplicative Schneider-Teitelbaum lift*. It is constructed by setting, for each $m \in \mathrm{MS}^{\Gamma_0(p)}(\mathbf{Z})$,

$$(6) \quad L_{\mathrm{ST}}^\times(m)\{r, s\}(z) := \int_{\mathbf{P}^1(\mathbf{Q}_p)} (z-t) d\mu_m\{r, s\}(t) := \lim_{\{U_\alpha\}} \prod_{\alpha} (z-t_\alpha)^{m\{r, s\}(U_\alpha)},$$

where the limit of ‘‘Riemann products’’ on the right-hand side is taken over finer and finer coverings $\{U_\alpha\}$ of $\mathbf{P}^1(\mathbf{Q}_p)$ by open balls, the point t_α is any sample point in U_α , and

$$m\{r, s\}(U_\alpha) := m\{\gamma r, \gamma s\}, \text{ with } \gamma \in \Gamma, \quad \gamma U_\alpha = \mathbf{Z}_p.$$

2.3. Lifting obstructions and RM values. Given $\bar{\Phi} \in \mathrm{MS}^\Gamma(\mathcal{O}^\times / \mathbf{C}_p^\times) \subset \mathrm{H}^1(\Gamma, \mathcal{O}^\times / \mathbf{C}_p^\times)$, it is natural to ask whether it lifts to a ‘‘genuine’’ multiplicative class in $\mathrm{H}^1(\Gamma, \mathcal{O}^\times)$. The obstruction to lifting $\bar{\Phi}$ to such a class lies in $\mathrm{H}^2(\Gamma, \mathbf{C}_p^\times)$ and is the image of $\bar{\Phi}$ under the connecting homomorphism δ in the following long exact cohomology sequence:

$$\dots \longrightarrow \mathrm{H}^1(\Gamma, \mathbf{C}_p^\times) \longrightarrow \mathrm{H}^1(\Gamma, \mathcal{O}^\times) \longrightarrow \mathrm{H}^1(\Gamma, \mathcal{O}^\times / \mathbf{C}_p^\times) \xrightarrow{\delta} \mathrm{H}^2(\Gamma, \mathbf{C}_p^\times) \longrightarrow \dots$$

Definition 2.3. The class $\kappa := \delta(\bar{\Phi}) \in \mathrm{H}^2(\Gamma, \mathbf{C}_p^\times)$ is called the *lifting obstruction* attached to $\bar{\Phi}$. A subgroup $Q \subset \mathbf{C}_p^\times$ is said to *trivialise* this lifting obstruction if the natural image of κ in $\mathrm{H}^2(\Gamma, \mathbf{C}_p^\times / Q)$ is trivial.

If Q trivialises the lifting obstruction for $\bar{\Phi}$, then this class lifts to an element of $H^1(\Gamma, \mathcal{O}^\times/Q)$. This lift is unique up to elements of order 12, since the abelianisation of Γ is a quotient of $(\mathbf{Z}/12\mathbf{Z})$ and therefore

$$H^1(\Gamma, \mathbf{C}_p^\times/Q) \subset (\mathbf{C}_p^\times/Q)[12].$$

In conclusion, after replacing Q by a slightly larger group (containing Q with finite index) one can thus associate to any modular symbol $m \in \text{MS}^{\Gamma_0(p)}(\mathbf{Z})$ a canonical multiplicative cocycle

$$(7) \quad \Phi \in H^1(\Gamma, \mathcal{O}^\times/Q)$$

of weight zero “modulo Q ”. The trivialising subgroup Q is a subtle invariant of m and a careful analysis is required to identify it in each case.

The guiding philosophy of multiplicative rigid meromorphic cocycles is that their RM values are algebraic invariants in ring class fields of the associated real quadratic field. To illustrate this, let us use this classification of rigid analytic cocycles to investigate the lifting obstructions and RM values of both the universal cocycle and the

2.4. Example: The universal cocycle. The one-dimensional kernel of the short exact sequence (4) is spanned by the modular symbol m_{univ} defined by

$$(8) \quad m_{\text{univ}}\{r, s\} = \begin{cases} 1 & \text{if } r \sim 0 \text{ and } s \sim \infty, \\ -1 & \text{if } r \sim \infty \text{ and } s \sim 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $r \sim s$ means that $r, s \in \mathbf{P}^1(\mathbf{Q})$ are $\Gamma_0(p)$ -equivalent.

We now illustrate the guiding philosophy discussed above for the multiplicative universal cocycle

$$\bar{\Phi}_{\text{univ}}\{r, s\}(z) := L_{\text{ST}}^\times(m_{\text{univ}}) = \left(\frac{z-s}{z-r} \right) \pmod{\mathbf{C}_p^\times}.$$

The explicit nature of this universal cocycle renders all proofs elementary, and the following proposition is left to the reader as an exercise.

Proposition 2.4. *The lattice Q_{univ} of \mathbf{C}_p^\times generated by -1 and p trivialises the lifting obstruction for $\bar{\Phi}_{\text{univ}}$. More precisely, the class $\bar{\Phi}_{\text{univ}}$ admits a canonical lift to a class in $\text{MS}^\Gamma(\mathcal{O}^\times/Q_{\text{univ}})$.*

We now consider the RM values of the lifted cocycle Φ_{univ} . If $F(x, s) = ax^2 + bxy + cy^2$ is a binary quadratic form of discriminant $\Delta = b^2 - 4ac$, then its root is $\tau_F = (-b + \sqrt{\Delta})/2a$, while its stabiliser is generated by

$$\gamma_F = \begin{pmatrix} u - bv & -2cv \\ 2av & u + bv \end{pmatrix}, \quad u^2 - \Delta v^2 = 1,$$

where $u + v\sqrt{\Delta}$ is a fundamental solution to Pell’s equation. A straightforward calculation shows that

$$\Phi_{\text{univ}}[\tau_F] = \Phi_{\text{univ}}\{r, \gamma_\tau r\}(\tau_F) = u \pm v\sqrt{\Delta} \pmod{\mathbf{Z}[1/p]^\times},$$

for any $r \in \mathbf{P}^1(\mathbf{Q})$.

It follows that the cocycle Φ_{univ} takes algebraic values at RM points, albeit somewhat uninteresting ones, since they always belong to the field of “real multiplication” and are just a power of the

fundamental unit in this field. More precisely, $\Phi_{\text{univ}}[\tau]$ is a fundamental unit in the order associated to τ .

2.5. Example: Elliptic modular cocycles. Let E be an elliptic curve of prime conductor p , and let f_E be the modular form attached to E , then we have

$$\int_{\tau}^s (2\pi i) f_E(z) dz = m_E^+ \{r, s\} \cdot \Omega_E^+ + m_E^- \{r, s\} \cdot \Omega_E^-,$$

where Ω_E^+ and Ω_E^- are real and imaginary periods attached to E , and $m_E^{\pm} \in \text{MS}^{\Gamma_0(p)}(\mathbf{Z})$. Consider the multiplicative Schneider-Teitelbaum lifts

$$\bar{\Phi}_E^+, \bar{\Phi}_E^- \in \text{MS}^{\Gamma}(\mathcal{O}^{\times}/\mathbf{Q}_p^{\times})$$

of m_E^+ and m_E^- respectively, and let κ_E^+ and $\kappa_E^- \in \text{H}^2(\Gamma, \mathbf{Q}_p^{\times})$ denote the associated lifting obstructions. Recall the Tate p -adic period $q_E \in \mathbf{Q}_p^{\times}$ attached to E , and let

$$\Psi_{E,p} : \mathbf{C}_p^{\times}/q_E^{\mathbf{Z}} \longrightarrow E(\mathbf{C}_p)$$

denote the Tate uniformisation of E . Theorem 1 of [Dar01] can be stated as follows:

Theorem 2.5. *There are lattices Q_E^+ and $Q_E^- \subset \mathbf{Q}_p^{\times}$ which are commensurable with the Tate lattice $q_E^{\mathbf{Z}}$ and trivialise κ_E^+ and κ_E^- respectively.*

After slightly enlarging the lattices Q_E^{\pm} , the classes $\bar{\Phi}_E^{\pm}$ lift uniquely to classes $\Phi_E^{\pm} \in \text{H}^1(\Gamma, \mathcal{O}^{\times}/Q_E^{\pm})$. Let t be an integer for which $(Q_E^{\pm})^t \subset q_E^{\mathbf{Z}}$. After replacing the multiplicative cocycles $\bar{\Phi}_E^{\pm}$ by their t -th powers and reducing modulo $q_E^{\mathbf{Z}}$, we may view Φ_E^+ and Φ_E^- as elements of $\text{H}^1(\Gamma, \mathcal{O}^{\times}/q_E^{\mathbf{Z}})$, whose values at RM points $\tau \in \mathcal{H}_p$ can then be viewed as elements of $E(\mathbf{C}_p)$ by applying the Tate uniformisation $\Psi_{E,p}$. One thus obtains two p -adic variants

$$\Phi_E^+, \Phi_E^- : \Gamma \backslash \mathcal{H}_p^{\text{RM}} \longrightarrow E(\mathbf{C}_p)$$

of the classical modular parametrisation attached to E .

Conjecture 2.6. *Let E be an elliptic curve of conductor p and let K be a real quadratic field in which p is inert. For all $\tau \in \mathcal{H}_p \cap K$,*

- (1) *the point $\Phi_E^+[\tau] \in E(\mathbf{C}_p)$ is defined over the ring class field of K attached to \mathcal{O}_{τ} ;*
- (2) *the point $\Phi_E^-[\tau] \in E(\mathbf{C}_p)$ is defined over the narrow ring class field of K attached to \mathcal{O}_{τ} , and is in the (-1) -eigenspace for the action of complex conjugation.*

This conjecture suggests that the multiplicative cocycles Φ_E^{\pm} attached to E carry arithmetic information about E that is just as rich and useful as the classical modular parametrisation, allowing the construction of global points on E that cannot be obtained (as far as we know) from the more classical parametrisations of elliptic curves by modular or Shimura curves. Extensive numerical evidence for this conjecture has been gathered in [Dar01], [DG02], [DP06], and (in much more general settings) in [GM14] and [GM15].

3. QUADRATIC FORMS AND THE CONWAY TOPOGRAPH

3.1. Binary quadratic forms. A *binary quadratic form*, or simply quadratic form, is a homogenous polynomial of degree two in $\mathbf{Z}[x, y]$. We will usually write

$$F(x, y) = ax^2 + bxy + cy^2,$$

where $a, b, c \in \mathbf{Z}$ are the *coefficients* of F . The quadratic form F is said to be *primitive* when $\gcd(a, b, c) = 1$, and we will exclusively consider primitive forms in what follows. An important invariant of a quadratic form is its *discriminant* $\Delta = b^2 - 4ac$. Quadratic forms with $\Delta > 0$ are called *indefinite*, those with $\Delta < 0$ are called *definite*, and those with $\Delta = 0$ are called *parabolic*. Definite quadratic forms come in two flavours: Positive definite forms take only positive values, negative definite forms take only negative values.

The central subject of study is the following action of the group $\mathrm{SL}_2(\mathbf{Z})$ on the set of quadratic forms:

$$F(x, y) \cdot \gamma = F(qx + ry, sx + ty) \quad \text{where } \gamma = \begin{pmatrix} q & r \\ s & t \end{pmatrix}.$$

This defines a right action of $\mathrm{GL}_2(\mathbf{Z})$ on the set of quadratic forms which preserves primitivity, and leaves the discriminant invariant. Two quadratic forms in the same $\mathrm{SL}_2(\mathbf{Z})$ -orbit are called *equivalent*. The group $\mathrm{SL}_2(\mathbf{Z})$ is generated by the elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This gives rise to the equivalences $\langle a, b, c \rangle \sim \langle c, -b, a \rangle$ and $\langle a, b, c \rangle \sim \langle a, b + 2a, a + b + c \rangle$. The set of equivalence classes of primitive quadratic forms of a given discriminant Δ has a natural structure of an abelian group, called the (narrow) class group $\mathrm{Cl}^+(\Delta)$. Every class of $\mathrm{Cl}^+(\Delta)$ however does still consist of an infinitude of quadratic forms, and has an internal structure exhibited visually by the Conway topograph.

3.2. Conway's topograph. Conway [Con97] presents a convenient visual method for investigating quadratic forms. Not only is it enlightening to think of an equivalence class of quadratic form this way, it often leads to short and clear proofs. All the pictures here are taken from the excellent treatment by Hatcher [Hat17].

It is well-known that

$$\mathrm{SL}_2(\mathbf{Z}) \simeq \mathrm{C}_4 *_{\mathrm{C}_2} \mathrm{C}_6,$$

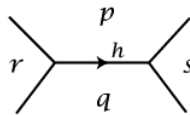
and such an amalgam corresponds to an action of $\mathrm{SL}_2(\mathbf{Z})$ on a tree by Bass–Serre theory [Ser80], which in this case is a 3-regular tree. We may interpret this tree in concrete terms as follows:

- Vertices = “Superbases”: $\{e_1, e_2, e_3\} / \pm 1$, which are vectors of \mathbf{Z}^2 such that any two of them form a basis, and $e_1 + e_2 + e_3 = 0$,
- Edges = Bases of \mathbf{Z}^2 up to sign: $\{e_1, e_2\} / \pm 1$,
- Regions = Primitive vectors of \mathbf{Z}^2 up to sign,
- Adjacency = Inclusion.

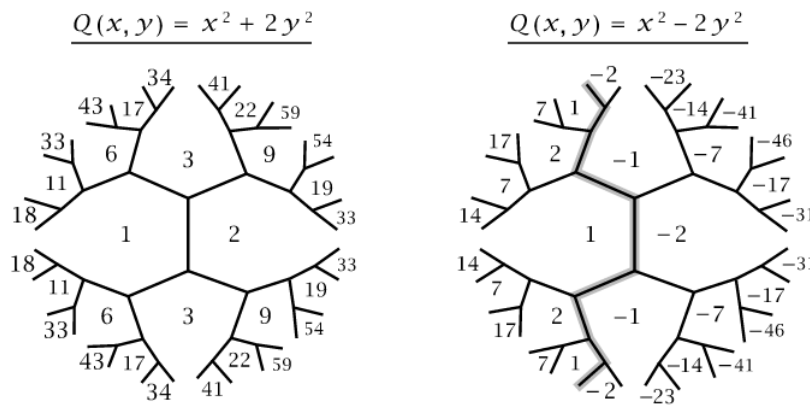
This incidence structure yields a 3-regular tree embedded in the plane, which we call the *topograph*.

Given a quadratic form F , we may visualise its $\mathrm{SL}_2(\mathbf{Z})$ -orbit \mathcal{Q}_F by assigning numbers to all regions and edges, as well as an orientation to the edges. To a region corresponding to a vector in \mathbf{Z}^2 ,

we attach the value of F on this vector. To every edge we assign the second coefficient of F in the corresponding basis. We assign an orientation to every edge to keep track of orderings of the basis, where the convention is that the value assigned to an edge is always positive, and the region to the left corresponds to the first basis vector. Starting with $F = \langle p, h, q \rangle$, we fill in the numbers of the topograph step by step as follows. Start with an arbitrary edge, and attach p and q to the two regions bordering the edge, and attaching h to the edge itself, as well as the orientation for which p is on the left, and q is on the right.



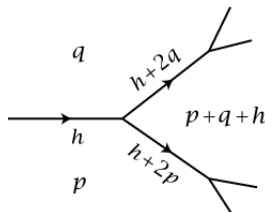
Now proceed by computing $s = p + q + h$ and $r = p + q - h$. The value of the edge between p and s is the positive square root of $\Delta + 4ps$, and its orientation is pointing towards q if $q > p + s$ and away from q otherwise. Here are two examples of topographs as shown in Hatcher [Hat17]. Complete this picture by determining the orientations of the edges:



To turn our practical computations into rigorous proofs, the following simple lemma often suffices.

Lemma 3.1 (Climbing lemma). *Suppose q, p , and h in the figure above are positive. Then the number s is also positive, and the edges adjacent to s point away from the vertex shared by p, q , and s .*

The proof of this lemma is a very easy verification, and it gives us the following schematic rendition of the topograph locally around an edge where all the numbers are positive:

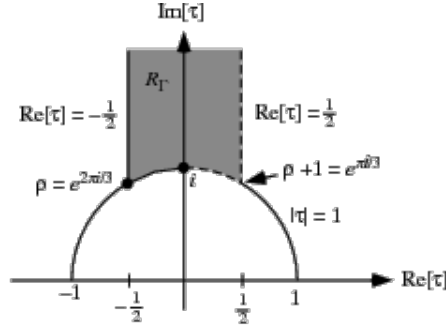


The shape of the topograph reveals much of the inner workings of an orbit of quadratic forms, and can be an extremely enlightening thing to keep in mind. We will now try and be as explicit as possible about the topograph in the various cases considered above.

3.3. Definite forms. For definite forms, every orbit consists of either positive or negative definite forms. The operation $\langle a, b, c \rangle \mapsto \langle -a, b, -c \rangle$ interchanges positive and negative definite forms, and there is no essential difference between the two theories. Let $F = \langle a, b, c \rangle$ be a primitive positive definite quadratic form. We say F is *reduced* if

$$|b| \leq a \leq c, \text{ and } b \geq 0 \text{ if either } |b| = a \text{ or } a = c.$$

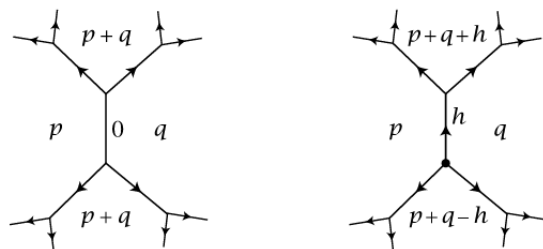
A negative definite form is reduced if its corresponding positive definite form is. Reduced forms will play the role of distinguished elements in an $\mathrm{SL}_2(\mathbf{Z})$ -orbit. This definition is equivalent to saying that one of the roots of F lies in the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ via linear fractional transformation on the upper half plane \mathfrak{H} .



Every definite form is equivalent to a unique reduced form. Since for a reduced form we have $b^2 \leq a^2$ and $a \leq c$, and hence $\Delta = b^2 - 4ac \leq a^2 - 4a^2 = -3a^2$, we obtain a practical method for finding all primitive reduced forms of a given discriminant $\Delta < 0$. Here are some examples:

Δ	Reduced forms	Δ	Reduced forms
-3	$\pm\langle 1, 1, 1 \rangle$	-19	$\pm\langle 1, 1, 5 \rangle$
-4	$\pm\langle 1, 0, 1 \rangle$	-20	$\pm\langle 1, 0, 5 \rangle, \pm\langle 2, 2, 3 \rangle$
-7	$\pm\langle 1, 1, 2 \rangle$	-23	$\pm\langle 1, 1, 6 \rangle, \pm\langle 2, \pm 1, 3 \rangle$
-8	$\pm\langle 1, 0, 2 \rangle$	-24	$\pm\langle 1, 0, 6 \rangle, \pm\langle 2, 0, 3 \rangle$
-11	$\pm\langle 1, 1, 3 \rangle$	-27	$\pm\langle 1, 1, 7 \rangle$
-12	$\pm\langle 1, 0, 3 \rangle$	-28	$\pm\langle 1, 0, 7 \rangle$
-15	$\pm\langle 1, 1, 4 \rangle, \pm\langle 2, 1, 2 \rangle$	-31	$\pm\langle 1, 1, 8 \rangle, \pm\langle 2, \pm 1, 4 \rangle$
-16	$\pm\langle 1, 0, 4 \rangle$	-32	$\pm\langle 1, 0, 8 \rangle, \pm\langle 3, 2, 3 \rangle$

There is no essential difference between positive and negative definite forms, so let us assume that we are dealing with a positive definite form F . To describe the Conway topograph of F , note that by the climbing lemma, if we follow the flow of the arrows then the values of all regions keep increasing. Retracing our steps, we see that there must be a 'source' or 'well' in the topograph, depicted in the following illustrations:



If the first case occurs, the form F must necessarily be of order at most 2 in the class group.

3.4. Indefinite forms. The theory for indefinite forms is significantly richer and more mysterious than its definite counterpart. Gauß introduced a notion of reduced form that still allows us to show that the number of orbits is finite. To simplify our presentation, we assume throughout that $\Delta > 0$ is not a square, and will briefly discuss the degenerate case of $\Delta = h^2$ at the end of this section.

Following Gauß, we say that the indefinite form $F = \langle a, b, c \rangle$ of discriminant $\Delta > 0$ is *reduced* if

$$0 < \sqrt{\Delta} - b < 2|a| < \sqrt{\Delta} + b.$$

This condition is equivalent to the following condition on the roots $\lambda^- < \lambda^+$:

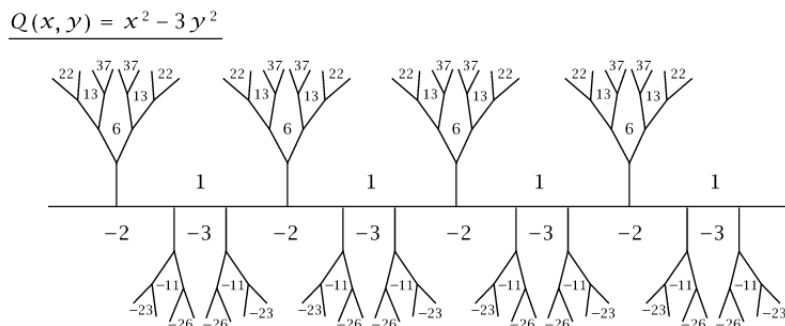
$$\begin{cases} \lambda^- < -1, \lambda^+ \in (0, 1) & \text{if } a \geq 0 \\ 1 < \lambda^+, \lambda^- \in (-1, 0) & \text{if } a < 0 \end{cases}$$

Again, reduced forms will play the role of distinguished elements in an $SL_2(\mathbf{Z})$ -orbit, with one very important difference: There can be many more than one reduced form per orbit! For instance, the two forms of discriminant $\Delta = 2021$ given by

$$\langle 5, 41, -17 \rangle \quad \text{and} \quad \langle 19, 11, -25 \rangle$$

are $SL_2(\mathbf{Z})$ -equivalent, even though they are both reduced.

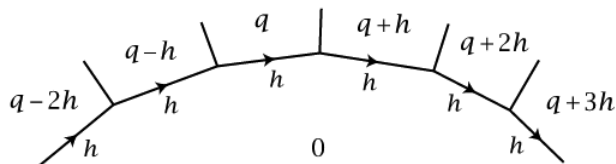
Let us now investigate the topograph attached to an indefinite form F . Assume first that $\Delta > 0$ is not a square. Then there must be an edge adjacent to regions with opposite signs, and we see that on either side of this edge, there must be another edge with the same property. The edges that separate positive and negative values must therefore be an infinite chain, which Conway calls the *river*.



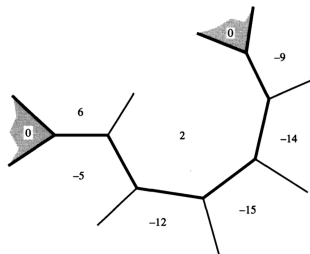
By the climbing lemma, if we move away from the river into the positive side, the values will continuously increase, whereas they will continuously decrease as we venture into the negative side. The condition $pq < 0$ means that $h^2 - 4pq = \Delta$ only has a finite number of solutions, and hence the

river must eventually become periodic! This means that the topograph has non-trivial translation symmetries, corresponding to matrices in $SL_2(\mathbf{Z})$ that fix our given quadratic form. We now **see** that the stabiliser is projectively infinite cyclic. Gauss' notion of reduced form corresponds to an edge of the river where the trees hanging off the river switch from the negative side to the positive side, or conversely. In the above picture of the form $x^2 - 3y^2$ we obtain exactly 2 reduced forms.

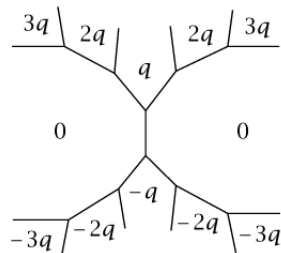
Finally, assume $\Delta = h^2$ is a square, and say h is positive. The quadratic form has rational roots and hence represents 0. There is therefore a region labelled 0, which Conway calls a *lake*:



Because we are in the case where $h \neq 0$, there are two distinct rational roots, and hence there are exactly two lakes in the topograph. As in the above picture, the regions adjacent to the lake have values that form an arithmetic progression, and hence they must change sign at some step. This shows that they must each sprout off a river, which then necessarily connect and form the following picture:



It is possible, as the example $\langle 0, 2, 0 \rangle$ shows, that the length of the river between the two lakes is zero, in which case the two lakes share an edge. See Conway [Con97] for more pictures and details.



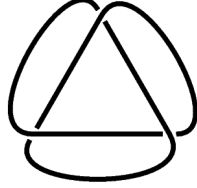
4. RIGID MEROMORPHIC COCYCLES

We now come to the definition of rigid meromorphic cocycles. The motivation will come from the work of Ghys and Duke–Imamoglu–Toth, concerning linking numbers of modular knots.

4.1. Linking numbers of modular knots. We start by discussing the rational period functions and associated cocycles that arise, notably, in the work of Knopp, Ash, Choie–Zagier, and Duke–Imamoglu–Toth. A folklore result, see for instance [?], asserts that the quotient space

$$\mathrm{SL}_2(\mathbf{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$$

has the structure of a real analytic threefold, and is isomorphic to the complement of a trefoil knot (see figure) in the 3-sphere S^3 .



Since the quotient $\mathrm{SL}_2(\mathbf{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R})$ is isomorphic to the affine modular curve $Y(1)$, we may uniquely lift a geodesic on $Y(1)$ to $\mathrm{SL}_2(\mathbf{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$. Given an indefinite integral binary quadratic form, we obtain a closed geodesic on $Y(1)$ by considering the quotient of the geodesic connecting the roots of F in the upper half plane. This means that to any indefinite form F , we can attach its *modular knot*

$$S^1 \hookrightarrow \mathrm{SL}_2(\mathbf{Z}) \backslash \mathrm{SL}_2(\mathbb{R}).$$

Since the latter space is the complement of a trefoil knot in S^3 , it is natural to ask what the linking number of the modular geodesic attached to F with the trefoil knot is. This question was answered by Ghys, who connected this topological linking number to the Dedekind–Rademacher symbol. Going one step further, Duke–Imamoglu–Toth relate the linking number of the modular geodesics attached to two indefinite forms F and G to a curious cocycle, which is the object we will find a p -adic counterpart for.

If r and s are elements of $\mathbf{P}^1(\mathbf{Q})$, let $\gamma(r, s)$ denote the geodesic on $\bar{\mathcal{H}} := \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ joining r to s and oriented in the direction from r to s . The complement of this geodesic in $\bar{\mathcal{H}}$ is partitioned into two disjoint connected subsets

$$\bar{\mathcal{H}} - \gamma(r, s) := \mathcal{H}^+[r, s] \cup \mathcal{H}^-[r, s],$$

labelled with the convention that, as one is travelling along $\gamma(r, s)$ in the positive direction from r to s , the region $\mathcal{H}^+[r, s]$ is to one's right and the region $\mathcal{H}^-[r, s]$ is to one's left. Any binary quadratic form G is said to be *linked* to $\gamma(r, s)$ if its roots belong to distinct connected components of $\bar{\mathcal{H}} - \gamma(r, s)$. In that case we write w_G^+ and w_G^- for its root in $\mathcal{H}^+[r, s]$ and $\mathcal{H}^-[r, s]$ respectively. Fix a binary quadratic form F . While the class \mathcal{Q}_F is infinite, the subset

$$\mathcal{Q}_F[r, s] := \{G \in \mathcal{Q}_F \text{ such that } G \text{ is linked to } \gamma(r, s)\}$$

is finite. The function $\omega_F : \mathbf{P}^1(\mathbf{Q}) \times \mathbf{P}^1(\mathbf{Q}) \rightarrow \mathbf{C}(z)$ defined by

$$(9) \quad \omega_F(r, s) := \sum_{G \in \mathcal{Q}_F[r, s]} \frac{1}{z - w_G^+} - \frac{1}{z - w_G^-}$$

is easily seen to be an $\mathrm{SL}_2(\mathbf{Z})$ -invariant modular symbol of weight two.

By formally integrating and exponentiating the modular symbol ω_F , or in other words applying the inverse of the isomorphism

$$\mathrm{dlog} : \mathcal{M}^\times(\mathcal{H})/\mathbf{C}^\times \longrightarrow \mathcal{M}_2(\mathcal{H}),$$

one obtains a multiplicative variant of weight zero, which is defined by

$$(10) \quad \Theta_F(r, s) := \prod_{G \in \mathcal{Q}_F[r, s]} \left(\frac{z - w_G^+}{z - w_G^-} \right).$$

Let

$$\theta_F \in \mathrm{H}^1(\mathrm{SL}_2(\mathbf{Z}), \mathcal{M}^\times(\mathcal{H})/\mathbf{C}^\times)$$

denote the image of Θ_F under the coboundary map. Although this ‘‘classical’’ modular cocycle only takes values in the quotient $\mathcal{M}^\times(\mathcal{H})/\mathbf{C}^\times$, the obstruction to lifting θ_F to a genuine modular cocycle with values in $\mathcal{M}^\times(\mathcal{H})$ lies in $\mathrm{H}^2(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}^\times)$ which is trivial, since $\mathrm{SL}_2(\mathbf{Z})$ is essentially a free group. Since $\mathrm{H}^1(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}^\times) = \mu_{12}$, and hence the lift of θ_F to $\mathrm{H}^1(\mathrm{SL}_2(\mathbf{Z}), \mathcal{M}^\times(\mathcal{H}))$ is *essentially unique*, up to 12-th roots of unity. By an abuse of notation, we will continue to denote by θ_F this essentially unique lift to $\mathrm{H}^1(\mathrm{SL}_2(\mathbf{Z}), \mathcal{M}^\times(\mathcal{H})/\mu_{12})$.

If G is any indefinite binary quadratic form which does not lie in \mathcal{Q}_F , we can attach to it a non-trivial, well-defined numerical invariant

$$J_\infty(F, G) = \frac{\theta_F(\gamma_G)(w_G)}{\theta_F(\gamma_G)(w'_G)},$$

where γ_G is a generator of the group of automorphs of G and w_G and w'_G are the two roots of G . This quantity is algebraic, but not in an interesting way, since it merely lies in the (biquadratic) field generated by the roots of F and G . The authors of [DIT] relate it to the topological linking number of the modular geodesics attached to F and G in the circle bundle $\mathrm{SL}_2(\mathbf{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$ over $\mathrm{SL}_2(\mathbf{Z}) \backslash \mathcal{H}$.

4.2. Rigid meromorphic cocycles. Let F be a primitive, integral, indefinite binary quadratic form of non-square discriminant Δ . Let $\mathcal{Q}_F^{(p)}$ denote the Γ -orbit of F . It consists of binary quadratic forms

$$ax^2 + bxy + cy^2, \quad a, b, c \in \mathbf{Z}[1/p], \quad \gcd(a, b, c) = 1, \quad b^2 - 4ac = \Delta.$$

We set

$$\mathcal{Q}_F^{(p)}[r, s] := \{G \in \mathcal{Q}_F^{(p)} \text{ such that } G \text{ is linked to } \gamma(r, s)\}.$$

For $G \in \mathcal{Q}_F^{(p)}[r, s]$, we continue to use the notation w_G^+ and w_G^- from the previous section to denote the roots of G that, as one travels along the geodesic from r to s , lie to the right and left respectively. In addition we define

$$w_G = \frac{-b + \sqrt{\Delta}}{2a}, \quad \mathrm{sgn}(G) = \begin{cases} 1 & \text{if } w_G = w_G^+ \\ -1 & \text{if } w_G = w_G^- \end{cases}$$

The set $\mathcal{Q}_F^{(p)}[r, s]$ is naturally a disjoint union $\mathcal{Q}_F^{(p)}[r, s] = \mathcal{Q}_F^{(p)}[r, s]^+ \cup \mathcal{Q}_F^{(p)}[r, s]^-$, where

$$\mathcal{Q}_F^{(p)}[r, s]^\pm = \{G \in \mathcal{Q}_F^{(p)}[r, s] : \mathrm{sgn}(G) = \pm 1\}.$$

We now define

$$(11) \quad \Theta_F(r, s)(z) := \prod_{G \in \mathcal{Q}_F^{(p)}[r, s]} [z - w_G]^{\mathrm{sgn}(G)}$$

Unlike the product in (10), this product is infinite. We will show below that it converges to a meromorphic function on \mathcal{H}_p , and defines a Γ -invariant modular symbol modulo constants.

Theorem 4.1. (a) For all $r, s \in \mathbf{P}^1(\mathbf{Q})$, the infinite product in (11) converges to a rigid-meromorphic function:

$$\Theta_F(r, s)(z) \in \mathcal{M}^\times.$$

(b) The assignment $(r, s) \mapsto \Theta_F(r, s)$ is a modular symbol with values in \mathcal{M}^\times , which means that for all $r, s, t \in \mathbf{P}^1(\mathbf{Q})$ we have the equalities

$$\begin{aligned} \Theta_F(r, s) &= \Theta_F(s, r)^{-1}, \\ \Theta_F(r, s) \times \Theta_F(s, t) &= \Theta_F(r, t), \end{aligned}$$

(c) The modular symbol Θ_F is Γ -invariant up to multiplicative constants, i.e.,

$$\Theta_F(\gamma r, \gamma s)(\gamma z) = \Theta_F(r, s)(z) \pmod{K_p^\times}.$$

Proposition 4.2. The group $p^{2-\sqrt{\varepsilon}\mathbf{Z}}$ trivialises the restriction to $\mathrm{SL}_2(\mathbf{Z})$ of the lifting obstruction κ_F , where ε is the fundamental unit of the quadratic field defined by F .

4.3. Meromorphic period functions. The aim of this section is to classify elements of $\mathrm{MS}^\Gamma(\mathcal{M}_2)$, where \mathcal{M}_2 is the space of meromorphic function on \mathcal{H}_p endowed with the weight 2 action of Γ defined by (1). As a consequence, we deduce that the modular symbols

$$\Theta_F \in \mathrm{MS}^\Gamma(\mathcal{M}^\times/\mathbf{C}_p^\times)$$

constructed above are essentially the only examples.

The key role in our classification is played by a p -adic analogue of the rational period functions introduced by Knopp [Kno78]. Any $m \in \mathrm{MS}^\Gamma(\mathcal{M}_2)$ is determined by the meromorphic function $f = m\{0, \infty\}$. This function satisfies a number of properties, which are formalised in the following definition.

Definition 4.3. A rigid meromorphic period function is a function $f \in \mathcal{M}_2$ satisfying:

- $f|(1+S) = 0$,
- $f|(1+U+U^2) = 0$,
- $f|D_p = f$.

Call Per_2 the space of all rigid meromorphic period functions. For example,

$$\Phi_F := \mathrm{dlog} \Theta_F(0, \infty) = \sum_{G \in \mathcal{Q}_F^{(p)}[0, \infty]} \frac{\mathrm{sgn}(G)}{z - w_G}$$

is easily seen to define an element in Per_2 . We will now show that any element of Per_2 is a finite linear combination of these, up to a holomorphic function. We borrow many ideas of Choie–Zagier [CZ93], who classify rational functions on $\mathbf{P}^1(\mathbf{C})$ subject to similar conditions.

Theorem 4.4. Any $f \in \mathrm{Per}_2$ may be written as a finite linear combination of period functions of the form Φ_F , up to a holomorphic period function on \mathcal{H}_p . In other words, we have a decomposition

$$\mathrm{Per}_2 = (\mathrm{Per}_2 \cap \mathcal{O}_2) \bigoplus_F \mathbf{C}_p \Phi_F,$$

where the sum runs through indefinite binary quadratic forms F with non-square discriminant.

4.4. Arithmetic intersection numbers of RM points. Let \mathcal{H}_p^{RM} denote the set of real multiplication points in \mathcal{H}_p that lie in the standard affinoid. An element $\tau \in \mathcal{H}_p^{RM}$ is the zero of a primitive integral binary quadratic form F_τ , which is unique up to sign. The discriminant of F_τ is called the *discriminant* of τ . The class group of a given discriminant D acts on the set, denoted $\mathrm{SL}_2(\mathbf{Z}) \backslash \mathcal{H}_p^D$, of $\mathrm{SL}_2(\mathbf{Z})$ -orbits $\tau \in \mathcal{H}_p^{RM}$ of discriminant D . Let C_p denote the compositum of all the quadratic extensions of \mathbf{Q}_p .

Conjecture 4.5. *There exists a function*

$$J_p : \mathcal{H}_p^{RM} \times \mathcal{H}_p^{RM} \longrightarrow C_p$$

satisfying the following properties:

- (1) (Relation with the meromorphic cocycles θ_F). For all τ_1 and τ_2 as above, which are zeroes of quadratic forms F_1 and F_2 of discriminants D_1 and D_2 ,

$$\theta_{F_1}[\tau_2] = \frac{J_p(\tau_1, \tau_2)}{J_p(\bar{\tau}_1, \tau_2)}.$$

- (2) (Algebraicity and integrality). The p -adic number $J_p(\tau_1, \tau_2)$ is an algebraic integer in the compositum $H_{12} := H_1 H_2$ of the ring class fields of discriminants D_1 and D_2 .

- (3) (Shimura reciprocity law). Suppose that Δ_1 and Δ_2 are relatively prime to each other, so that class field theory gives an identification

$$\mathrm{Gal}(H_{12}/H_1 K_2) = \mathrm{Gal}(H_2/K_2) = \mathrm{Pic}(R_{\tau_2}).$$

For all σ in this group and all $\tau_1, \tau_2 \in \mathcal{H}_p^{RM}$ of discriminant Δ_1 and Δ_2 respectively,

$$J_p(\tau_1, \tau_2)^\sigma = J_p(\tau_1, \tau_2^\sigma).$$

- (4) (Arithmetic intersections). Let $q \neq p$ be a prime that is inert or ramified in both K_1 and in K_2 , and hence splits completely in H_{12}/K_{12} . Then there exists a prime \mathfrak{q} above q for which

$$\mathrm{ord}_{\mathfrak{q}}(J_p(\tau_1, \tau_2)) = I_{pq}(F_1, F_2),$$

where $I_{pq}(F_1, F_2)$ is a certain q -weighted intersection number between the modular geodesics attached to F_1 and F_2 , in the circle bundle on the Shimura curve attached to a maximal order in the indefinite quaternion algebra of discriminant pq .

- (5) (Factorisation). The norm

$$\mathbf{J}_p(D_1, D_2) := \mathbf{J}_p(\tau_1, \tau_2) := \prod_{\sigma \in \mathrm{Gal}(H_{12}/K_{12})} J(\tau_1^{\sigma_1}, \tau_2^{\sigma_2})$$

belongs to $K_{12} \cap \mathbf{Q}_p = \mathbf{Q}(\sqrt{D_1 D_2})$ and the quantity $\mathbf{J}_p(\tau_1, \tau_2) =: \mathbf{J}_p(\tau_1, \tau_2)$ is only divisible by primes l which are inert or ramified in both $\mathbf{Q}(\sqrt{\Delta_1})$ and $\mathbf{Q}(\sqrt{\Delta_2})$, and which divide a positive integer of the form $(D_1 D_2 - b^2)/4$.

The conjectural quantity $J(\tau_1, \tau_2)$ is called the *p -adic arithmetic linking number* between the *RM* points τ_1 and τ_2 . It generalises the arithmetic intersection

$$J(\tau_1, \tau_2) = j(\tau_1) - j(\tau_2)$$

between CM points τ_1 and τ_2 that is studied in [GZ85].

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