

# OVERCONVERGENT MODULAR FORMS

AND THEIR EXPLICIT ARITHMETIC

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ABSTRACT. These are the notes for a mini-course taught in Bordeaux, at the end of June 2019. They introduce the theory of  $p$ -adic overconvergent modular forms, with an emphasis on their explicit computation, and discuss some arithmetic applications, including the computation of  $p$ -adic L-functions of real quadratic fields.

## INTRODUCTION

The  $p$ -adic theory of modular forms goes back at least to the work of Serre [Ser73] and Katz [Kat73], and has since taken up a central role in algebraic number theory, becoming an indispensable item in the toolbox of many a mathematician working in the field. In these notes, we attempt to give a quick overview of the basics of the theory, favouring explicit examples and computations over detailed proofs of the foundations of the theory. As such, there will inevitably be some discussion that remains rather informal, particularly surrounding the notion of eigencurves, but we hope that nonetheless these notes can provide some of the intuition and ideas that underly the subject. For the reader interested in learning the subject thoroughly, there is no substitute for the original papers that shaped the field, many of which are extremely well written, such as for instance [Ser73, Kat73, Col96, Col97b, CM98, Buz03, Pil13].

The subject of overconvergent forms is not an unpopular one for mathematical events like this one, and as a byproduct there are many good expository sources available, which I would be foolish to ignore. Most notably, I borrowed several expository ideas and examples from the wonderful notes of Calegari [Cal13].

The *first* lecture will start from a historical perspective, discussing some of the early work on congruences between modular forms in the spirit of the approach towards the subject due to Serre [Ser73]. As is also done in *loc. cit.*, our excursion will be motivated by a desire to construct the  $p$ -adic L-function of Kubota–Leopoldt by means of the prototypical example of a  $p$ -adic family provided by Eisenstein series. This idea of Serre, whereby information on the constant coefficient of the Fourier expansion of a modular form is inferred from the higher coefficients, will be the recurrent theme around which we structure our discussion, and will reappear many times in later lectures.

The success of the approach of Serre in constructing  $p$ -adic L-functions, as well as the shortcomings inherent to the central role played by  $q$ -expansions, motivate us to reinterpret the constructions geometrically, which we will do in the *second lecture* following Katz [Kat73]. We explore several important aspects of the basic theory on a concrete example where the modular curve  $X_0(p)$  has genus zero, and an explicit orthonormal basis can be constructed, leading to a hands-on encounter with some spaces of overconvergent modular forms and their properties. The discussion is presented in a way which I hope, perhaps naively, will encourage (or at least enable) students to experiment with their own implementations, both to recover the numerical data in the text, as well as to explore some slightly different settings independently.

After our preliminary computations with specific examples, the *third lecture* briefly discusses the idea of  $p$ -adic families of modular forms, and the eigencurve. As a reprise of Serre’s idea discussed in the previous lecture, we use  $p$ -adic families of Eisenstein series to prove Leopoldt’s formula for the value  $L_p(\chi, 0)$ . The rest of the lecture is focussed on the computation of spaces of overconvergent forms in general, using the notion of Katz expansions, following Lauder’s algorithm [Lau11]. We use these algorithms to explore some folklore conjectural properties about slopes of modular forms, and present some data related to the Gouvêa–Mazur conjecture and Chow–Heegner points on elliptic curves.

In the *fourth* and final lecture, we come back to Serre’s idea from the first lecture, and discuss diagonal restrictions of Hilbert Eisenstein series, and their relevance to both the theoretical construction, and the practical computation, of  $p$ -adic L-functions of real quadratic fields.

The **prerequisites** are a good knowledge of the classical theory of modular forms, and some familiarity with their algebro-geometric definitions. In particular, the discussion on algebraic modular forms, the line bundles  $\omega^{\otimes k}$  on modular curves, Hecke correspondences, Tate curves, etc., will be very brief, and previous exposure to these ideas would be helpful. An excellent treatment can be found in Loeffler [Loe14]. That said, these notes, and a fortiori the even more concise contents of the lectures, will not spend much time developing the basic geometric theory, in favour of explicit computations. The uninitiated reader may therefore opt to focus on the explicit aspects, and start experimenting before reading up on the more formal aspects of the theory. We also use some very basic concepts from  $p$ -adic geometry, but certainly not enough to merit a section on the topic. A casual reading of any basic summary of the subject will more than cover it.

Finally, since these notes still have many rough corners and certainly contain many errors, I would be grateful for any corrections or suggestions, which I warmly invite at `vonk@maths.ox.ac.uk`.

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1. CONGRUENCES BETWEEN MODULAR FORMS

Recall that the Ramanujan  $\Delta$ -function is the unique normalised cusp form of weight 12 for the group  $\Gamma = \text{SL}_2(\mathbf{Z})$ . Its  $q$ -expansion is given by the following infinite product expansion due to Jacobi:

$$(1) \quad \Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

We tabulate its first few Fourier coefficients  $a_n$  for future reference:

$n$	1	2	3	4	5	6
$a_n$	1	-24	252	-1472	4830	-6048
$n$	7	8	9	10	11	12
$a_n$	-16744	84480	-113643	-115920	534612	-370944

The explicit product expansion of the Ramanujan  $\Delta$ -function, as well as basic facts about the dimensions of spaces of cusp forms, allow us to easily establish a number of congruences between these Fourier coefficients and the Fourier coefficients of various other modular forms. Classically, some of the most famous congruences satisfied by  $\Delta$  are the following:

**Example 1.** The following congruence is due to Ramanujan [Ram16]. For any even  $k \geq 4$ , consider the weight  $k$  normalised Eisenstein series

$$(3) \quad \mathbf{G}_k(q) = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \quad \text{where} \quad \sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$$

When  $k = 12$ , the constant term is equal to  $\frac{691}{65520}$ , whereas for  $k = 6$  the constant term is  $\frac{-1}{504}$ . Since the space  $M_{12}(\text{SL}_2(\mathbf{Z}))$  is two-dimensional, spanned by  $\mathbf{G}_{12}$  and  $\Delta$ , the form  $\mathbf{G}_6^2$  must be a linear combination of the two. Computing the first two terms of all three  $q$ -expansions, we find that

$$(4) \quad \frac{691}{65520} \cdot 504^2 \cdot \mathbf{G}_6(q)^2 = \mathbf{G}_{12}(q) - \frac{756}{65} \Delta(q)$$

and since all three modular forms involved have 691-integral  $q$ -expansions, we obtain as a consequence the congruence  $\mathbf{G}_{12}(q) \equiv \Delta(q) \pmod{691}$ . In particular, we see that for any prime  $p$ , we get the celebrated *Ramanujan congruence*

$$(5) \quad \tau(p) \equiv 1 + p^{11} \pmod{691}.$$

**Example 2.** The following example is due to Wilton [Wil30], and establishes a congruence modulo 23 between  $\Delta$  and a certain form of weight 1. We have the obvious congruence

$$(6) \quad q \prod_{n=1}^{\infty} (1 - q^n)^{24} \equiv \left( q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \right) \cdot \left( q^{23/24} \prod_{n=1}^{\infty} (1 - q^{23n}) \right) \pmod{23}$$

and we recognise the right hand side as  $\eta(q)\eta(q^{23})$ , where  $\eta(q)$  is the  $q$ -expansion of the Dedekind  $\eta$ -function, which is a modular form of weight  $1/2$  for some character  $\chi_{24}$  of the metaplectic double cover of  $\mathrm{SL}_2(\mathbf{Z})$ , of order 24. Using the *Euler identity*

$$(7) \quad \prod_{n=1}^{\infty} (1 - q^n) = \sum_{n \in \mathbf{Z}} (-1)^n q^{\frac{3n^2+n}{2}}$$

one can prove (see exercises) that we have the congruence

$$(8) \quad q \prod_{n=1}^{\infty} (1 - q^n)^{24} \equiv \frac{1}{2} \sum_{u,v \in \mathbf{Z}} \left( q^{u^2+uv+6v^2} - q^{2u^2+uv+3v^2} \right) \pmod{23}.$$

We deduce from this identity (see exercises) that for any prime  $p \neq 23$  we get the following congruences

$$(9) \quad \begin{cases} \tau(p) \equiv 0 \pmod{23} & \text{if } \left( \frac{-23}{p} \right) = -1, \\ \tau(p) \equiv 2 \pmod{23} & \text{if } \left( \frac{-23}{p} \right) = 1 \text{ and } p = u^2 + 23v^2, \\ \tau(p) \equiv -1 \pmod{23} & \text{if } \left( \frac{-23}{p} \right) = 1 \text{ and } p \neq u^2 + 23v^2, \end{cases}$$

Note that the right hand side of (8) is a modular form of weight 1. It is in fact a Hecke eigenform, with an associated Artin representation that we can identify easily: Consider the quadratic field

$$(10) \quad K = \mathbf{Q}(\sqrt{-23}),$$

which has class number 3. Its Hilbert class field  $H$  is obtained by adjoining a root of the polynomial

$$(11) \quad f(x) = x^3 - x - 1$$

which has discriminant  $-23$ . The representation  $\rho_H$  of  $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  obtained from the unique 2-dimensional irreducible representation of its quotient  $\mathrm{Gal}(H/\mathbf{Q}) \simeq S_3$  gives a 2-dimensional Artin representation. The traces of Frobenius match up with the Fourier coefficients of the right hand side of (8) (see exercises).

**Example 3.** Finally, we note that using a similar argument to the one used in the previous example, we obtain the congruence

$$(12) \quad q \prod_{n=1}^{\infty} (1 - q^n)^{24} \equiv q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 \pmod{11}$$

The right hand side is the weight 2 newform of level  $\Gamma_0(11)$ , associated to the elliptic curve

$$(13) \quad E : y^2 + y = x^3 - x^2 - 10x - 20$$

The three examples of congruences above are of very different flavours, and illustrate different but related phenomena that arise in the  $p$ -adic theory of modular forms:

- The first is a congruence between a *cuspidal form* and an *Eisenstein series*, of the *same weight*. Such congruences are central to the subject of Iwasawa theory, and directly related to the notion of the *Eisenstein ideal*. We will not discuss this theme much during these lectures, though it will no doubt

appear in many of the other lectures.

- The second is a congruence between two *cuspidal forms*, of *different weights*. This fits in the framework of  $p$ -adic families of modular forms, as developed by [Hid86b, Hid86a, Col97b, CM98] and many others. This example has special significance, in the sense that it exhibits congruences between a modular form of weight 1 and a modular form of higher weight. The existence of such congruences are a crucial ingredient in the proof of Deligne–Serre [DS74] of the existence of Artin representations attached to modular forms of weight 1.
- The third is again a congruence between two cuspidal forms of different weights, one of them being associated to an elliptic curve. Unlike the previous example there is no simply phrased criterion on the prime (for instance, a congruence condition) that predicts the congruence class of the  $p$ -th coefficient modulo 11. The reason is that this rule is governed by the traces of the Galois representation associated to the 11-adic representation attached to the elliptic curve  $E : y^2 + y = x^3 - x^2 - 10x - 20$ . It can be shown that its mod 11 reduction, which is the Galois representation on the 11-torsion points of this curve, has image equal to  $\mathrm{GL}_2(\mathbf{F}_{11})$ . Since this group is not solvable, the law governing the traces of Frobenius of various primes is not given by a simple expression. In contrast, in the previous example this law was governed by the splitting behaviour in a generalised dihedral (and hence solvable) extension with Galois group  $S_3$ .

**1.1. The  $p$ -adic family of Eisenstein series.** In these notes, we will focus primarily on the theme of congruences between modular forms of different weights, and  $p$ -adic families. Traditionally, the theory was built around the prototypical example of the *Eisenstein family*, as in Coleman [Col97b], until more recent advances due to Pilloni [Pil13] and Andreatta–Iovita–Stevens [AIS14] on the geometric interpolation of line bundles, which allows us to develop the theory abstractly, without building it around the Eisenstein family. From a practical and computational point of view, this family remains of primordial importance, so we will quickly review it, motivated by the strategy of Serre to show the existence of the Kubota–Leopoldt  $p$ -adic L-function.

Recall that the Riemann zeta function  $\zeta(s)$  may be analytically continued to the entire complex plane, except for a simple pole with residue 1 at the point  $s = 1$ . It satisfies the functional equation

$$(14) \quad \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Of special importance are its values at negative odd integers (or equivalently, by the functional equation, at positive even integers), which were computed first by Euler in 1734, and read on 5 December 1735 in the St. Petersburg Academy of Sciences. The starting point for Euler was the easily verified identity

$$(15) \quad \sin(\pi z) = \pi z \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2}\right)$$

By taking the logarithmic derivative, we obtain the following identities

$$(16) \quad \pi z \cot(\pi z) = \sum_{n \in \mathbf{Z}} \frac{z^2}{z^2 - n^2}$$

$$(17) \quad = 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{z^{2k}}{n^{2k}}$$

$$(18) \quad = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k}$$

On the other hand, the Bernoulli numbers are defined via the generating series

$$(19) \quad \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$$

and hence we can formally extract the even part of this series as

$$(20) \quad \frac{1}{2} \left( \frac{t}{e^t - 1} - \frac{-t}{e^{-t} - 1} \right) = \frac{t}{2} \cdot \frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}}$$

$$(21) \quad = \frac{t}{2} \cdot \coth\left(\frac{t}{2}\right)$$

Bearing in mind that  $i \coth(iz) = \cot(z)$ , we obtain the identity

$$(22) \quad \cot(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} z^{2k-1}.$$

It now follows formally from (18) and (22) that

$$(23) \quad \zeta(2k) = \frac{(-1)^{k-1} (2\pi)^{2k}}{2(2k)!} B_{2k}$$

and hence by the functional equation

$$(24) \quad \zeta(1 - 2k) = \frac{-B_{2k}}{2k}.$$

The fact that the values of the zeta function at negative odd integers is a *rational number* is remarkable. Moreover, the Bernoulli numbers have many interesting  $p$ -adic properties, notably two results established in the mid-19<sup>th</sup> century: The Clausen–von Staudt theorem [Cla40, vS40] and the *Kummer congruences* [Kum51], which tell us the following:

**Lemma 1.1.** *Suppose  $k, k'$  are two positive even integers such that  $k \equiv k' \pmod{(p-1)p^n}$ , then*

$$(25) \quad \begin{array}{ll} \text{If } (p-1) \nmid k & : (1-p^{k-1})B_k/k \equiv (1-p^{k'-1})B_{k'}/k' \pmod{p^{n+1}} \\ \text{If } (p-1) \mid k & : v_p(B_k/k) = -1 - v_p(k) \end{array}$$

The proof requires work. The Kummer congruences especially are a striking property of Bernoulli numbers, suggesting that suitably modified values of the zeta function at negative odd integers interpolate to a  $p$ -adically continuous (or indeed, analytic) function. This was proved by Kubota–Leopoldt [KL64], and there is a good chance that their work will be discussed in the course of Ellen Eischen.

Instead of discussing the arguments of Kummer [Kum51] and Kubota–Leopoldt [KL64] we will look at a suggestion of Serre, who observed that the congruences of Bernoulli numbers given in (27) can be upgraded to congruences between  $q$ -expansions of modular forms. Notice first that we see from the above theorem

that the Bernoulli numbers need to be modified in order to interpolate nicely. Likewise, we need to adjust the Eisenstein series introduced above, by setting

$$(26) \quad \mathbf{G}_k^{(p)} = (1 - p^{k-1}U_p) \mathbf{G}_k$$

which is a modular form for  $\Gamma_0(p)$ , often referred to as a  $p$ -stabilisation of  $\mathbf{G}_k$ . Likewise, we define  $\mathbf{E}_k^{(p)}$  to be its normalised version, whose constant coefficient is 1. Observe now that elementary congruences for the non-constant (henceforth called *higher*) Fourier coefficients yield an upgraded version of the above congruences, as the statement that whenever  $k \equiv k' \pmod{(p-1)p^n}$  we have

$$(27) \quad \begin{aligned} \text{If } (p-1) \nmid k & : \mathbf{G}_k^{(p)}(q) \equiv \mathbf{G}_{k'}^{(p)}(q) \pmod{p^{n+1}} \\ \text{If } (p-1) \mid k & : \mathbf{E}_k^{(p)}(q) \equiv \mathbf{E}_{k'}^{(p)}(q) \pmod{p^{n+1}} \end{aligned}$$

The observation of Serre [Ser73] was that in establishing these congruences of Eisenstein series, there is a striking dichotomy between the congruences between the constant terms (which are the Kummer congruences, and hence somewhat deep) and the higher coefficients (which follow trivially from Fermat's little theorem, and are hence not deep). His idea was to try and obtain the Kummer congruences, and hence the construction of the Kubota–Leopoldt zeta function  $\zeta_p(s)$ , by inheriting congruences of a more elementary nature from the higher coefficients, through the notion of  $p$ -adic modular forms. This idea, whereby information on the constant coefficient is transferred from the higher coefficients, will appear several times throughout these lectures, and is very powerful and useful in a variety of contexts. We will mark the paragraphs where it comes back by a small light bulb in the margin as shown here.



**1.2.  $p$ -Adic modular forms.** We now follow Serre [Ser73] and establish some basic definitions of  $p$ -adic modular forms. We follow Serre in restricting to the case of level 1 modular forms defined over  $\mathbf{Q}_p$ , but reassure the reader who is nervous about this that these assumptions will eventually be lifted when we adopt the more geometric viewpoint due to Katz in the next lecture.

For any formal power series in the variable  $q$  given by

$$(28) \quad f(q) = a_0 + a_1q + a_2q^2 + \dots \in \mathbf{Q}_p[[q]],$$

we define  $v_p(f) = \inf_n(v_p(a_n))$ , where  $v_p$  is the usual  $p$ -adic valuation on  $\mathbf{Q}_p$ . We define the space of  $p$ -adic modular forms to be the collection of  $f(q) \in \mathbf{Q}_p[[q]]$  such that there is a sequence  $f_i$  satisfying

$$(29) \quad v_p(f(q) - f_i(q)) \rightarrow \infty, \quad f_i \in M_{k_i}(\mathrm{SL}_2(\mathbf{Z}), \mathbf{Q}).$$

A  $p$ -adic modular form  $f(q)$  therefore is obtained as a limit of  $q$ -expansions of classical modular forms. The following important proposition of Serre [Ser73, § 1.3 Théorème 1] states that the sequence of their weights  $k_i$  must tend to a limit  $p$ -adically. Its proof, which lies much deeper than the rest of the contents of [Ser73] which are otherwise largely established by elementary means, is merely sketched here, since we have not yet introduced the necessary objects (such as the Hasse invariant).

**Proposition 1.2.** *Let  $f, g$  be two classical modular forms of weights  $k, \ell$  on  $\mathrm{SL}_2(\mathbf{Z})$ , both nonzero and normalised such that  $v_p(f) = 0$ . Suppose that we have*

$$(30) \quad v_p(f - g) \geq m$$

for some positive integer  $m$ , then it must be true that

$$(31) \quad \begin{aligned} k &\equiv \ell \pmod{(p-1)p^{m-1}} && \text{if } p \geq 3 \\ k &\equiv \ell \pmod{2^{m-2}} && \text{if } p = 2 \end{aligned}$$

**Proof.** Let us briefly sketch the proof, for  $p \geq 5$ . Let  $\overline{M}$  be the ring of modular forms modulo  $p$ , obtained by reduction from classical modular forms with  $p$ -integral  $q$ -expansion. Its structure can be determined following Swinnerton–Dyer: Since we are in level 1, the algebra of modular forms in characteristic zero is generated by the Eisenstein series  $\mathbf{E}_4$  and  $\mathbf{E}_6$ , and hence

$$(32) \quad \overline{M} \simeq \mathbf{F}_p[x, y]/\mathfrak{a}$$

for some ideal  $\mathfrak{a}$ . If  $A$  is the Hasse invariant, then clearly  $(A - 1) \subseteq \mathfrak{a}$ . Since  $A$  has simple zeroes, the ideal  $(A - 1)$  must be prime, and therefore  $(A - 1) = \mathfrak{a}$  since  $\mathfrak{a}$  is clearly not maximal, and the Krull dimension of  $\mathbf{F}_p[x, y]$  is 2. As an immediate consequence, we see that  $\overline{M}$  is the direct sum of the reductions of modular forms whose weight is in a fixed residue class modulo  $(p - 1)$ . The statement of the theorem then immediately follows in the case  $m = 1$ .

For general  $m$ , set  $h = k' - k$  and suppose that  $v_p(h) < m - 1$ . First one calculates that

$$(33) \quad f\phi = g \pmod{p}, \quad \phi = \sum_{n \geq 1} \sigma_{h-1}(n)q^n$$

where  $g$  is a multiple (by a  $p$ -unit) of the modular form  $p^{-v_p(h)-1}(f \mathbf{E}_h - f')$ . Since  $f$  and  $g$  have the same weight modulo  $(p - 1)$ , this shows that  $\phi$  is in the field of fractions of  $\overline{M}_0$ , the summand of  $\overline{M}$  of reductions of forms whose weight is divisible by  $(p - 1)$ , which is isomorphic to the ring of functions on the ordinary locus of the modular curve modulo  $p$ . On the other hand, we have

$$(34) \quad \phi - \phi^p \equiv \psi := \frac{-1}{24} \left( \frac{qd}{dq} \right)^{h-1} \pmod{\mathbf{E}_2}$$

which is an element of  $\overline{M}_0$ , and since  $\overline{M}_0$  is integrally closed (since the ordinary modular curve is irreducible) it follows that  $\overline{\phi} \in \overline{M}_0$ . On the other hand, Serre uses a weight filtration argument to show directly that  $\overline{\phi} \notin \overline{M}_0$ , which is a contradiction.  $\square$

As a consequence of this proposition, every  $p$ -adic modular form  $f$  has a well defined *weight*

$$(35) \quad k := \varprojlim_i k_i \in \mathbf{Z}_p \times \mathbf{Z} / (p - 1) \mathbf{Z} = \varprojlim_m \mathbf{Z} / (p - 1)p^m \mathbf{Z}.$$

This is the point where Serre is able to realise the idea of “inheriting” congruences for the constant terms of Eisenstein series, from the much more elementary congruences between their higher coefficients.

**Theorem 1.3** (Serre). *Suppose we have a sequence of  $p$ -adic modular forms of weights  $k_i$ :*

$$(36) \quad f_i(q) = a_0^{(i)} + a_1^{(i)}q + a_2^{(i)}q^2 + \dots$$

which satisfy the following two properties:

- The sequences  $a_n^{(i)}$  tend uniformly to a limit  $a_n \in \mathbf{Q}_p$ ,
- The weights  $k_i$  tend to a limit  $k \neq 0$ .

Then the constant terms  $a_0^{(i)}$  tend to a limit  $a_0$ , and the  $q$ -series

$$(37) \quad f(q) = a_0 + a_1q + a_2q^2 + \dots \in \mathbf{Q}_p[[q]]$$

is a  $p$ -adic modular form.





**Proof.** Assume for simplicity that  $p \geq 3$ , the proof of the case  $p = 2$  being nearly identical. Let  $m$  be such that the image of  $k_i$  in  $\mathbf{Z}/p^m(p-1)\mathbf{Z}$  is eventually equal to a fixed non-zero class for all  $i$  large enough, which is possible by the second assumption. By Proposition 1.2, applied to the forms  $f_i(q)/a_0^{(i)}$  (of weight  $k_i$ ) and the constant 1 (of weight 0) we obtain

$$(38) \quad \inf_{n \geq 1} v_p(a_n^{(i)}) = v_p(f_i(q) - a_0^{(i)}) \leq v_p(a_0^{(i)}) + m$$

By the first assumption, the quantity on the left is bounded from below. It follows that  $v_p(a_0^{(i)})$  is bounded from below as  $i$  grows, and hence there is a subsequence that converges. For any other convergent subsequence, the difference of the limits is a  $p$ -adic modular form of weight  $k$ , but it is also equal to the difference of the limits of the constant terms, and hence of weight 0. Since  $k \neq 0$ , this is only possible if the two limits of constant terms are equal. The theorem follows.  $\square$

Notice that we may use the above theorem to show the existence of a continuous function interpolating the constant terms of the Eisenstein family! This therefore gives a construction of the Kubota–Leopoldt  $p$ -adic L-function function. The rest of the paper of Serre pushes this idea even further, and strengthens this significantly by deducing also its analytic properties. In particular, the above arguments may be strengthened to give an *effective* version of the claimed convergence, whose rate may be controlled to truly recover the Kummer congruences for Bernoulli numbers from elementary congruences between the higher coefficients.

**Hecke operators.** We mention that the space of  $p$ -adic modular forms is also equipped with actions of Hecke operators, as was shown by Serre [Ser73, §2] (see exercises). The action of these Hecke operators may be given explicitly on a  $p$ -adic modular form  $f(q) = a_0 + a_1q + a_2q^2 + \dots$  of weight  $k$  by

$$(39) \quad T_\ell f(q) = \sum_{n \geq 0} a_{\ell n} q^n + \ell^{k-1} \sum_{n \geq 0} a_n q^{\ell n}$$

when  $\ell \neq p$  is prime, and by

$$(40) \quad U_p f(q) = \sum_{n \geq 0} a_{np} q^n, \quad V_p f(q) = \sum_{n \geq 0} a_n q^{np}.$$

Despite the great success of establishing the existence of the Kubota–Leopoldt  $p$ -adic L-function, this is a point where the theory of  $p$ -adic modular forms starts lacking a bit. Indeed, it was defined in a rather ad-hoc fashion, and its definition based solely on  $q$ -expansions lacks any kind of rigidity, and we have captured a tremendous amount of power series in the space of  $p$ -adic modular forms. One way in which this became apparent is that the spectrum of the Hecke operators is far from interesting. For example, every element of  $\mathbf{C}_p$  is an eigenvalue of  $U_p$  on the space of  $p$ -adic modular forms, base changed to  $\overline{\mathbf{Q}}_p$ !

**1.3. Exercises.** We now collect some exercises related to the material discussed in this lecture, mostly filling in the gaps of some arguments that were merely sketched above.

1. Prove the Euler identity (7), and use it to deduce (8). Finally, show that the resulting weight 1 modular form congruent to  $\Delta$  modulo 23 satisfies, for almost all  $p$ , that

$$a_p = \text{Tr}(\rho_H(\text{Frob}_p)).$$

2. Prove that the operators  $T_\ell, U_p, V_p$  on the space of  $q$ -expansions preserve the subspace of  $p$ -adic modular forms of weight  $k$ , as claimed in the text.

3. Prove that the  $q$ -expansion

$$\mathbf{E}_2(q) = 1 + 24 \sum_{n \geq 1} \left( \sum_{d|n} d \right) q^n$$

is a  $p$ -adic modular form, for all primes  $p$ . (Hint: Use the Hecke operators  $U_p$  and  $V_p$ )

4. Unlike what we know for classical modular forms, there is no obstruction to a  $p$ -adic modular form having a negative weight. Show that the  $q$ -expansion

$$\begin{aligned} \mathbf{E}_4(q)^{-1} &= (1 + 240q + 2160q^2 + 6720q^3 + \dots)^{-1} \\ &= 1 - 240q + 55440q^2 - 12793920q^3 + 2952385680q^4 + \dots \end{aligned}$$

is a 2-adic, 3-adic, and 5-adic modular form of weight  $-4$ . Finally, prove that for any  $p$ , there exists a  $p$ -adic modular form of negative weight, by generalising your argument.

5. Prove that the weight of a  $p$ -adic modular form  $f$  is well-defined. In other words, if there exist two sequences of modular forms  $f_i$  and  $g_i$ , such that

$$\begin{aligned} v_p(f - f_i) &\longrightarrow \infty, \\ v_p(f - g_i) &\longrightarrow \infty, \end{aligned}$$

then the weights of the sequences  $f_i$  and  $g_i$  have the same limit in  $\mathbf{Z}_p$ .

6. Let  $f$  be any  $p$ -adic modular form, and  $\lambda \in p\mathbf{Z}_p$ . Show that

$$f_\lambda = (1 - \lambda V_p)^{-1} f^{[p]}, \quad \text{where } f^{[p]} = (1 - V_p U_p) f$$

exists as a  $p$ -adic modular form, has the same weight as  $f$ , and  $U_p f_\lambda = \lambda f_\lambda$ .<sup>1</sup>

7. ( $\star$ ) Prove that if  $f \in M_k(\Gamma_0(p))$ , then  $f(q)$  is a  $p$ -adic modular form of weight  $k$ .

## 2. OVERCONVERGENT MODULAR FORMS

Last time, we encountered the Kubota–Leopoldt  $p$ -adic zeta function, and explored an idea of Serre that uses the  $p$ -adic Eisenstein family to construct it. This culminated in the notion of  $p$ -adic modular forms, which served a great purpose, but otherwise seems lacking in good structural properties, as evidenced by the absence of an interesting spectrum of Hecke operators (see the exercises in the previous section). In today’s lecture, we will follow Katz in reinterpreting the viewpoint of Serre geometrically, and identifying much smaller (though still infinite-dimensional) subspaces of the space of  $p$ -adic modular forms.

We will assume some familiarity with the algebro-geometric theory modular forms. Excellent expositions can be found for instance in Katz [Kat73], Calegari [Cal13], and Loeffler [Loe14].

**2.1. The Hasse invariant.** Suppose  $S$  is a scheme over  $\mathbf{F}_p$ , then there is an *absolute Frobenius* morphism

$$(41) \quad F_{\text{abs}} : S \longrightarrow S$$

given on affine opens by the map on functions  $f \mapsto f^p$ . If  $X/S$  is an  $S$ -scheme, we define the scheme  $X^{(p)} = X \times_S S$  where the fibre product is taken over  $S$ , viewed as an  $S$ -scheme via  $F_{\text{abs}}$ . The *relative*

<sup>1</sup>Extending scalars to  $\mathbf{C}_p$ , the same argument shows that any  $\lambda \in \mathbf{C}_p$  of positive valuation is an eigenvalue of  $U_p$ .

Frobenius morphism  $F = F_{X/S}$  is defined by the following commutative diagram, where the square is Cartesian:

$$(42) \quad \begin{array}{ccccc} & & & & F_{\text{abs}} \\ & & & & \curvearrowright \\ & & & & X \\ & & \swarrow & & \searrow \\ X & \xrightarrow{F_{X/S}} & X^{(p)} & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & S & \xrightarrow{F_{\text{abs}}} & S \end{array}$$

Notice that the relative Frobenius is an  $S$ -linear morphism, whereas the absolute Frobenius is not! Also, the scheme  $X^{(p)}$  is hardly a mysterious thing: Suppose  $X$  is of finite type over  $\mathbf{F}_q/\mathbf{F}_p$ , then  $X^{(p)}$  is given by the same equations as  $X$ , but where all the coefficients are raised to the  $p$ -th power. Note that if  $q = p$ , then we have  $X^{(p)} = X$ .

Now suppose that  $E/S$  is an elliptic curve, then the relative Frobenius  $F = F_{E/S}$  is an isogeny, and hence has a dual isogeny  $V$ :

$$(43) \quad \begin{array}{lcl} F : E & \longrightarrow & E^{(p)} \quad \text{“Frobenius”} \\ V : E^{(p)} & \longrightarrow & E \quad \text{“Verschiebung”} \end{array}$$

Suppose now that  $S = \text{Spec}(\overline{\mathbf{F}}_p)$ , then we define

$$(44) \quad \begin{cases} E \text{ is ordinary} & \text{if } E[p](\overline{\mathbf{F}}_p) \neq 1 \\ E \text{ is supersingular} & \text{if } E[p](\overline{\mathbf{F}}_p) = 1 \end{cases}$$

In general, we say  $E/S$  is ordinary/supersingular if all its geometric fibres are.

**Proposition 2.1.** *Suppose  $E/S$  is an elliptic curve, and  $S$  is an  $\mathbf{F}_p$ -scheme. Then we have:*

- $E/S$  is ordinary if and only if  $V : E^{(p)} \rightarrow E$  is étale.
- $E/\overline{\mathbf{F}}_p$  is supersingular, only if  $E$  is defined over  $\mathbf{F}_{p^2}$ .

**Proof.** We can factor the multiplication by  $p$  map as

$$(45) \quad [p] : E \xrightarrow{F} E^{(p)} \xrightarrow{V} E.$$

This implies that  $V$  is separable if and only if  $\text{Ker}(V)(\overline{\mathbf{F}}_p) \neq 1$  on all geometric fibres. Since the kernel of Frobenius only has the trivial geometric point, this is equivalent to  $\text{Ker}([p])(\overline{\mathbf{F}}_p) \neq 1$ . This proves the first statement. For the second statement, we have that  $E/S$  is supersingular if and only if  $V$  is inseparable, which means it must factor through Frobenius:

$$(46) \quad V : E^{(p)} \xrightarrow{F} E^{(p^2)} \longrightarrow E$$

The latter map must be finite of degree 1, and hence an isomorphism. Thus  $E$  is defined over  $\mathbf{F}_{p^2}$ .  $\square$

Finally, we define the *Hasse invariant* of an elliptic curve  $E/R$  where  $R$  is a ring of characteristic  $p$ . First, choose  $\omega \in H^0(E, \Omega_{E/R}^1)$  to be an  $R$ -basis, and let

$$(47) \quad \eta \in H^1(E, \mathcal{O}_E)$$

be the  $R$ -basis defined via Serre duality. The Hasse invariant  $A(E, \omega) \in R$  is defined by

$$(48) \quad F_{\text{abs}}^*(\eta) = A(E, \omega) \cdot \eta$$

Note that by the previous proposition,  $E/\overline{\mathbf{F}}_p$  is ordinary if and only if  $A(E, \omega) \neq 0$  for any choice of  $\omega$ .

The Hasse invariant is a modular form of weight  $p - 1$ , in the following sense. Recall (see for instance Katz [Kat73, Chapter 1]) that a *weakly holomorphic modular form* of weight  $k \in \mathbf{Z}$  over a ring  $A$  is a rule which assigns to any isomorphism class of pairs

$$(49) \quad (E/R, \omega)$$

where  $E/R$  is an elliptic curve over an  $A$ -algebra  $R$ , and  $\omega$  is a global section of  $\Omega_{E/R}^1$ , an element  $f(E/R, \omega) \in R$  such that the following two properties are satisfied:

- (Base change) If  $\phi : R \rightarrow R'$  is a morphism of  $A$ -algebras, then

$$(50) \quad f((E/R, \omega) \otimes_{\phi} R') = \phi(f(E, \omega)).$$

- (Weight  $k$  homogeneity) For all  $\lambda \in R^{\times}$  we have

$$(51) \quad f(E, \lambda\omega) = \lambda^{-k} f(E, \omega).$$

The  $q$ -expansion of a weakly holomorphic modular form  $f$  is defined as

$$(52) \quad f(q) := f(\text{Tate}(q)_{\mathbf{Z}((q))}, \omega_{\text{can}}) \otimes R \in R((q)),$$

where  $\text{Tate}(q)$  is the Tate elliptic curve over  $\mathbf{Z}((q))$  defined by

$$(53) \quad y^2 + xy = x^3 + B_q x + C_q, \quad \omega_{\text{can}} = \frac{dx}{2y + x}$$

with coefficients defined by the explicit  $q$ -series in  $\mathbf{Z}[[q]]$

$$(54) \quad \begin{aligned} B_q &= \sum_{n \geq 1} -5\sigma_3(n)q^n \\ C_q &= \sum_{n \geq 1} \frac{-5\sigma_3(n) - 7\sigma_5(n)}{12} q^n \end{aligned}$$

We say a weakly holomorphic modular form is an *algebraic* (or *holomorphic*) modular form if its  $q$ -expansion, which a priori is an element of  $R((q))$ , is in fact in  $R[[q]]$ .

With these definition, the Hasse invariant is an algebraic modular form of weight  $p - 1$ . Indeed, the functoriality is clear by definition, and for any  $\lambda \in R^{\times}$  we do the following formal calculation:

$$(55) \quad A(E, \lambda\omega) \cdot \lambda^{-1}\eta = F_{\text{abs}}^*(\lambda^{-1}\eta)$$

$$(56) \quad = \lambda^{-p} F_{\text{abs}}^*(\eta)$$

$$(57) \quad = \lambda^{1-p} \cdot A(E, \omega) \cdot \lambda^{-1}\eta$$

We conclude that the Hasse invariant  $A$  defines an (algebraic) weakly holomorphic modular form of weight  $p - 1$  (and level one). It also has the following important properties, whose proof we omit, as they would lead us too far afield:

- The  $q$ -expansion of the Hasse invariant was computed in [Kat73, KM85] and is simply given by  $A(q) = 1$ . The proof is a beautiful argument using the Cartier operator.
- We already know that for  $E/k$  over  $k = \overline{\mathbf{F}}_p$ , the Hasse invariant vanishes if and only if  $E$  is supersingular. In fact, it has simple zeroes in the sense that if  $R$  is a local Artinian  $k$ -algebra, and  $E/R$  is such that  $V : E^{(p)} \rightarrow E$  induces the zero map on tangent spaces, then it must be true that there is a supersingular elliptic curve  $E_0/k$  such that

$$(58) \quad E_0 \times_k R \simeq E.$$

**2.2. Overconvergent modular forms.** We now come to Katz' geometric reinterpretation of Serre's space of  $p$ -adic modular forms, and introduce the notion of overconvergent modular forms. To phrase everything as geometrically as possible, we will choose some auxiliary level structure to rigidify the moduli problem of elliptic curves into something representable by a modular curve. The reader should keep in mind that this is done just for simplicity, and in principle one may work directly on the moduli stack of level one, by working with algebraic modular forms as we did in our discussion of the Hasse invariant.

The Hasse invariant  $A$  is a beautiful example of a modular form of weight  $p - 1$  and level one. Suppose we have a lift  $\tilde{A}$  of the Hasse invariant, meaning a modular form of weight  $p - 1$  over  $\mathbf{Z}_p$  whose  $q$ -expansion is congruent to 1 modulo  $p$ . In this case, we have the elementary observation

$$(59) \quad v_p \left( \tilde{A}(q)^{p^n-1} - \tilde{A}(q)^{-1} \right) \rightarrow \infty$$

so that  $\tilde{A}(q)^{-1}$  is a  $p$ -adic modular form in the sense of Serre. See also the exercises in the previous section, where this argument already appeared. This observation lies at the basis of the work of Katz [Kat73], who showed that the space of  $p$ -adic modular forms of weight  $k \in \mathbf{Z}$  is the set of sections of the line bundle  $\omega^{\otimes k}$  on the ordinary locus of the modular curve, where any lift  $\tilde{A}$  is invertible.

Let  $N \geq 5$  and  $p \nmid N$  be a prime. We let  $\mathcal{X} / \mathbf{Z}_p$  be the moduli space of generalised elliptic curves with  $\Gamma_1(N)$ -level structure, universal curve  $\pi : \mathcal{E} \rightarrow \mathcal{X}$ , and closed subscheme of cusps  $\mathcal{I}_C$ . We frequently denote its generic and special fibres by  $X$  and  $\mathcal{X}_s$  respectively. Furthermore, we set

$$(60) \quad \omega := \pi_* \Omega_{\mathcal{E}/\mathcal{X}}^1(\log \pi^{-1} \mathcal{I}_C),$$

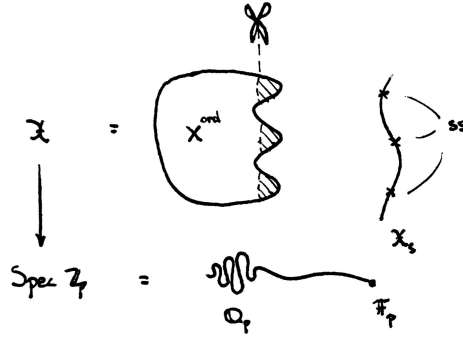
which is a line bundle on  $\mathcal{X}$ . The Hasse invariant is the unique section

$$(61) \quad A \in H^0(\mathcal{X}_s, \omega^{\otimes p-1})$$

with  $q$ -expansion 1, and indeed more generally: any modular form with a given  $q$ -expansion and weight is uniquely characterised by this data. Since the relative curve  $\mathcal{X} / \mathbf{Z}_p$  is proper, every  $\mathbf{C}_p$ -point extends uniquely to an  $\mathcal{O}_{\mathbf{C}_p}$ -point, and we obtain a reduction map

$$(62) \quad \text{red} : \mathcal{X}(\mathbf{C}_p) \rightarrow \mathcal{X}_s(\overline{\mathbf{F}}_p).$$

The inverse image  $\text{red}^{-1}(x)$  of a closed point of the special fibre is isomorphic to a rigid analytic open disk. We saw previously that the vanishing locus of the Hasse invariant is precisely the supersingular locus of  $\mathcal{X}_s$ , which consists of a finite set of closed points. Therefore, any lift of the Hasse invariant is invertible on the *ordinary locus*  $X^{\text{ord}}$ , which is the affinoid whose set of  $\mathbf{C}_p$ -points correspond to elliptic curves with ordinary reduction. It is the complement of a finite number of rigid analytic open disks.



As a consequence, the space of  $p$ -adic modular forms of weight  $k \in \mathbf{Z}$  is given by  $H^0(X^{\text{ord}}, \omega^{\otimes k})$ . We saw previously that this space is too large to have nice structural properties, prompting Katz to consider the subspace of sections that extend to sections on affinoids strictly containing  $X^{\text{ord}}$ .

More precisely, let  $0 \leq r \leq 1$ , and define  $X^{\text{ord}} \subset X_r \subset X^{\text{rig}}$  by

$$(63) \quad X_r(\mathbf{C}_p) := \{x \in X(\mathbf{C}_p) : v_p(\tilde{A}_x) \leq r\},$$

where  $\tilde{A}_x$  is a local lift of the Hasse invariant  $A$  at  $x$ . Note we do not require a global lift of the Hasse invariant to exist, which may fail in general when  $p \leq 3$ . We define the space of  $r$ -overconvergent modular forms of integer weight  $k$  on  $\Gamma_1(N)$  to be

$$(64) \quad M_k^\dagger(r) := H^0(X_r, \omega^{\otimes k}).$$

These spaces come with a collection of Hecke operators  $T_\ell$  for  $\ell \nmid Np$ , and  $U_\ell$  for  $\ell \mid N$ , which can be defined by restricting the Hecke correspondences on  $\mathcal{X}$  and have the usual effect on  $q$ -expansions.

In addition, the operators  $U_p$  and  $V_p$  defined on  $p$ -adic modular forms may be defined geometrically, and preserve the subspace of overconvergent modular forms. More precisely, they are defined for every  $r < 1/(p+1)$  and have the following effect on the rate of overconvergence:

$$(65) \quad \begin{aligned} U_p &: M_k^\dagger(r) &\longrightarrow & M_k^\dagger(pr) \\ V_p &: M_k^\dagger(pr) &\longrightarrow & M_k^\dagger(r) \end{aligned}$$

In particular, the operator  $U_p$  improves the rate of overconvergence. The reason for the existence of the operators  $U_p$  and  $V_p$  is the *canonical subgroup section*  $s$  of the natural forgetful map of modular curves, which exists for any  $r < p/(p+1)$ :

$$(66) \quad \begin{array}{ccc} X(\Gamma_1(N) \cap \Gamma_0(p))^{\text{rig}} & \longrightarrow & X^{\text{rig}} \\ & & \cup \\ & \xleftarrow{s} & X_r \end{array}$$

This yields two ways to view spaces of overconvergent modular forms: On affinoid opens of  $X$  (no level at  $p$ ) or as affinoid opens of the modular curve with additional  $\Gamma_0(p)$ -structure. For theoretical questions, the latter is frequently more convenient, whereas for computational purposes, the former has advantages.

**Extended example.** Let us explore these rather abstract definitions in a particular case, to get a feeling for the various objects involved. Consider the case where  $p = 2$  and  $k = 0$ , in level one. In this case, we

can be very explicit about the spaces of  $p$ -adic and  $r$ -overconvergent modular forms, both from the “tame” viewpoint (in level one) or via the canonical subgroup section (on  $X_0(2)$ ).

2.2.1. *The “tame” viewpoint.* Consider the moduli stack  $X$  of elliptic curves. Of the 4 values in  $\mathbf{F}_4$  for the  $j$ -invariant, only  $j = 0$  is supersingular, so that its special fibre at  $p = 2$  has a unique supersingular point corresponding to the vanishing locus of  $j$ . It follows that the ordinary locus on  $X$  is described by  $|j^{-1}| \leq 1$ , and hence the space of 2-adic modular forms of weight 0 is isomorphic to

$$(67) \quad \mathbf{C}_2\langle j^{-1} \rangle = \{a_0 + a_1j^{-1} + a_2j^{-2} + \dots \mid a_n \rightarrow 0\}$$

For any  $r$ , the space  $M_0^\dagger(r)$  defines a Banach space contained inside of this Tate algebra, which we can explicitly identify through growth conditions on the coefficients  $a_n$ . Precisely, we use the observation that

$$(68) \quad j = \frac{E_4^3}{\Delta}$$

and  $E_4 = 1 + 240(\dots)$  is the Eisenstein series of weight 4, which is a lift of the fourth power of the Hasse invariant  $A^4$ . In particular, we find that on the supersingular disk (where  $\Delta$  is invertible, and hence  $v_2(\Delta) = 0$ , we have that

$$(69) \quad v_2(A) \leq r \iff v_2(j) \leq 12r.$$

and as a consequence, we get that the subspace of  $r$ -overconvergent forms is given by

$$(70) \quad M_0^\dagger(r) = \{a_0 + a_1j^{-1} + a_2j^{-2} + \dots : |a_n|p^{12nr} \rightarrow 0\}$$

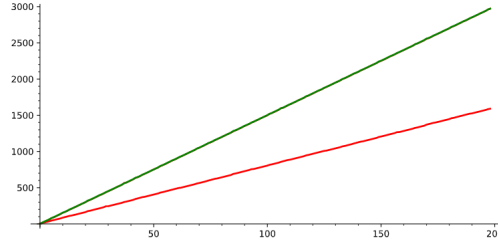
Finally, let us compute some Hecke operators, and see whether the obtained results make sense with what is said above. First, note that we can compute very rapidly (most serious computer algebra packages like Magma, PARI/GP, or Sage will already have a function implemented) the  $q$ -expansion of  $j^{-1}$ . Given any 2-adic modular form of weight 0, we can then compute its  $j^{-1}$ -expansion very rapidly by the simple observation that  $j^{-1}$  vanishes to order 1 at the cusp infinity, and hence we can inductively subtract powers of  $j^{-1}$  until we are left with zero. Carrying out this procedure in Magma [BCP97], we obtain that

$$\begin{aligned} U_2j^{-1} &= -744 j^{-1} \\ &= -140914688 j^{-2} \\ &= -16324041375744 j^{-3} \\ &= -1528926232501026816 j^{-4} \\ &\quad + \dots \\ T_3j^{-1} &= 356652 j^{-1} \\ &\quad -16114360320000 j^{-2} \\ &\quad +1298216343568384000000/3 j^{-3} \\ &\quad + \dots \\ T_5j^{-1} &= 49336682190 j^{-1} \\ &\quad -122566701099729715200000 j^{-2} \\ &\quad +177278377115100363578123747328000000 j^{-3} \\ &\quad + \dots \end{aligned}$$

where we calculated in reality hundreds of terms, which look rather unappetising. Things become very interesting when we look at the 2-adic valuations of the coefficients  $a_1, a_2, a_3, \dots$  of  $U_2j^{-1}$  and  $T_\ell j^{-1}$  tabulated above, which give us the following sequences:

$$(71) \quad \begin{aligned} U_2j^{-1} &: v_2(a_n) = 3, 12, 20, 28, 35, 46, 52, 60, 67, 76, 86, 94, \dots \\ T_3j^{-1} &: v_2(a_n) = 2, 16, 32, 45, 60, 79, 91, 105, 120, 136, 154, 165, \dots \\ T_5j^{-1} &: v_2(a_n) = 1, 18, 33, 47, 61, 80, 92, 107, 121, 138, 155, 167, \dots \end{aligned}$$

We see very clearly that the latter two sequences grow roughly at the same rate, whereas the first one grows significantly more slowly! In fact, if we plot these three sequences in red, green, and blue respectively, for the first two hundred terms, we obtain the following picture:



They all look like linear functions! The green and blue plots are virtually indistinguishable at this scale, and look roughly like a linear function of slope 15. On the other hand, at this scale the red plot looks roughly like a linear function of slope 8. This is precisely what we expected from the general theory, since  $j^{-1}$  is  $r$ -overconvergent for any  $r$  (indeed, it converges on the entire modular curve  $X$  except for a simple pole at the cusp  $0!$ ) and its image under the  $U_2$ -operator is therefore only guaranteed to be  $r$ -overconvergent for any  $r < p/(p+1) = 2/3$ . With respect to the identification (70), this shows that the valuation of the coefficients should grow at least like a linear function of slope  $8 = (2/3) \cdot 12$ .

*2.2.2. The “canonical subgroup” viewpoint.* Even though we can compute things to our heart’s desire, it is hard to get any more specific information in the tame description of this space. Following Buzzard–Calegari [BC05], we will now see that we can get a lot of mileage from working on  $X_0(2)$  instead, which we know we can by the theory of the canonical subgroup. Define the *Hauptmodul*

$$(72) \quad h = \Delta(2z)/\Delta(z) = q \prod_{n \geq 1} (1 + q^n)^{24}$$

which is a meromorphic function on  $X_0(2)$  with a simple zero at the cusp  $\infty$ , and a pole at the cusp  $0$ . It is related to the  $j$ -function by

$$(73) \quad \frac{h}{(1 + 2^8 h)^3} = j^{-1}$$

Using a Newton polygon argument, we see that we can find a canonical section of the forgetful map whenever  $v_p(j^{-1}) > -8$  exactly as predicted by the theory of canonical subgroups. Note also that in this case, we see that this section does not extend to any larger region, so the result was optimal! This means that we get an alternative description for (70) of the form

$$(74) \quad M_0^\dagger(r) = \{a_0 + a_1 h + a_2 h^2 + \dots : |a_n| p^{12nr} \rightarrow 0\}$$

The advantage is the following: The Hecke operators are defined as correspondences on  $X_0(2)$ , and hence we know that  $U_2(h)$  and  $T_\ell(h)$  are *polynomials* in  $h$ ! This is in stark contrast with the tame situation, where we got a rather mysterious set of power series, which we could compute to any accuracy, but never exactly. In contrast, on  $X_0(2)$  we can do the computation exactly, and we obtain:

$$\begin{aligned} U_2(h) &= 24h + 2048h^2 \\ T_3(h) &= 300h + 98304h^2 + 16777216/3h^3 \\ T_5(h) &= 18126h + 40239104h^2 + 14696841216h^3 + 1649267441664h^4 + 281474976710656/5h^5 \end{aligned}$$



Together with (74), this can be seen as a complete description of the Hecke module  $M_0^\dagger(r)$ . This is what is used by Buzzard–Calegari [BC04] to determine the valuations of all the eigenvalues of  $U_2$  on this space, see also the exercises where you will be guided towards this result in several steps.

A number of years ago, explicit computations of the sort we did above, and are about to do in the exercises, were a popular theme in the literature, since they are often the only fruitful way to try and obtain information about the eigenvalues of  $U_p$  on spaces of overconvergent modular forms. The spectrum of  $U_p$  remains to this day an enigma, and is the subject of several celebrated conjectures in the literature, notably Buzzard’s slope conjectures, and the Gouvêa–Mazur conjecture. We will review these conjectures in the next lecture. For more about these questions, see for instance the works [Buz03, Buz04, BC04, BC05, BC06, BK05, Roe14] and the references contained in them.

**2.3. Spectral theory of  $U_p$ .** We finish this section with some brief comments about a very important part of the subject, which is the spectral theory of the Hecke operator  $U_p$ . Spaces of  $r$ -overconvergent forms may naturally be endowed with the structure of a  $\mathbf{C}_p$ -Banach space, and we will see that this structure allows us to develop a meaningful spectral theory for  $U_p$ .

We begin by defining a norm  $\|\cdot\|_r$  on  $M_k^\dagger(r)$ . Pick a point  $x \in X_r$ , let  $K$  be a finite extension of the residue field of  $x$ , and let  $\text{Spec}(K) \rightarrow \mathcal{X}_{\mathbf{Q}_p}$  be a point whose image corresponds to  $x$ . The properness of  $\mathcal{X}$  implies that this extends uniquely to a point  $\varphi : \text{Spec}(\mathcal{O}_K) \rightarrow \mathcal{X}$ . Now let  $f \in M_k^\dagger(r)$ , then  $\varphi^* f = a_f s$  for some section  $s$  generating the trivial line bundle  $\varphi^* \omega^{\otimes k}$  and some  $a \in \mathcal{O}_K$ . We set

$$(75) \quad |f(x)| := |a_f|,$$

which is independent of the choice of  $s$ . The norm

$$(76) \quad \|f\|_r := \sup\{|f(x)| : x \in X_r\}$$

makes  $M_k^\dagger(r)$  into a  $p$ -adic Banach space. This induces the structure of a  $p$ -adic Fréchet space on

$$(77) \quad M_k^\dagger := \varinjlim_{r>0} M_k^\dagger(r)$$

which we call the space of *overconvergent modular forms*. The Banach spaces  $M_k^\dagger(r)$  are infinite-dimensional, and there is a priori no meaningful way to talk about the spectrum of an operator, unless we know more.

Suppose we have a continuous bounded operator  $T$  on a separable  $\mathbf{C}_p$ -Banach space  $B$ , then we say that  $T$  is *compact* if it is the limit of operators of finite rank. Equivalently,  $T$  is compact if and only if the image of the unit ball is relatively compact. There is a well-developed spectral theory for compact operators, see [Dwo62, Ser62, Col97b], which has the following pleasant consequences for compact operators:

- $T$  has a discrete spectrum of non-zero eigenvalues

$$(78) \quad |\lambda_1| \geq |\lambda_2| \geq \dots$$

where  $|\lambda_i| \rightarrow 0$  as  $i \rightarrow \infty$ , whose inverses are the roots of a well-defined characteristic series

$$\begin{aligned} P(t) &= \text{“det}(1 - Tt)\text{”} \\ &= a_0 + a_1 t + a_2 t^2 + \dots, \quad \text{where } a_i \rightarrow 0 \text{ as } i \rightarrow 0. \end{aligned}$$

- For every  $v \in B$  there are constants  $c_i$  and generalised eigenvectors  $v_i$  with eigenvalue  $\lambda_i$  such that for any  $\varepsilon > 0$  we have (asymptotically in  $n$ ) that

$$(79) \quad \varepsilon^{-n} \left\| T^n v - T^n \sum_{|\lambda_i| \geq \varepsilon} c_i v_i \right\| \rightarrow 0.$$

The constants  $c_i$  are often called the coefficients of the *asymptotic expansion* of  $v$ .

Now we turn to the specific case of Hecke operators acting on the Banach spaces  $M_k^\dagger(r)$ . Note that we established that the operator  $U_p$  exhibits a *contractive* nature, which is the underlying reason it improves overconvergence as described by (65). Using this property, it can be shown that this implies that the operator  $U_p$  is compact, and hence it possesses a well-defined characteristic series. Here is one concrete way to think about this series (and indeed, to compute it in examples!) as explained by Serre [Ser62] and Coleman [Col97b, Theorem A2.1]: Suppose we have an orthonormal basis

$$(80) \quad \{f_1, f_2, f_3 \dots\} \quad \text{for } M_k^\dagger(r),$$

then we obtain an infinite matrix representation of  $U_p$ . In the example above, where  $p = 2$  and  $k = 0$ , we already noted that we have an algorithm to compute this matrix *exactly*, or at least any finite submatrix of it. To see what compactness really means in practice, we compute the first  $10 \times 10$  submatrix with respect to the basis  $f_i = (2^8 h)^i$  of the cuspidal subspace, and look at the 2-adic valuations of its entries:

$$(81) \quad v_2(U_2(i, j))_{i, j} = \begin{pmatrix} 3 & 8 & & & & & & & & & \\ 3 & 7 & 11 & 16 & & & & & & & \\ & 8 & 12 & 17 & 19 & 24 & & & & & \\ & & 7 & 11 & 15 & 21 & 23 & 27 & 32 & & \\ & & & 11 & 19 & 20 & 25 & 27 & 35 & 35 & \dots \\ & & & & 11 & 16 & 20 & 24 & 27 & 33 & 35 \\ & & & & & 17 & 19 & 24 & 29 & 34 & 35 \\ & & & & & & 15 & 20 & 23 & 27 & 31 & 38 \\ & & & & & & & & 19 & 24 & 27 & 37 & 36 \\ & & & & & & & & & \vdots & & & \ddots \end{pmatrix}$$

Here, we omitted the entries of  $U_2$  that were equal to zero. The compactness of  $U_p$  in orthonormalisable situations like this one is *equivalent* to the statement that the column vectors converge uniformly to 0 in the infinite matrix representation. In the above example, that certainly looks plausible, as the entries of the columns seem to have valuation which grows roughly at the same rate. To contrast this with what happens in general, let us compute with respect to the same basis the first  $10 \times 10$  submatrix for  $T_3$ :

$$(82) \quad v_3(T_3(i, j))_{i, j} = \begin{pmatrix} 2 & 12 & 16 & & & & & & & & \\ 7 & 2 & 11 & 20 & 27 & 32 & & & & & \\ 8 & 8 & 2 & 14 & 17 & 28 & 34 & 46 & 48 & & \\ & 11 & 8 & 2 & 12 & 19 & 29 & 36 & 43 & & \\ & 16 & 9 & 10 & 2 & 12 & 16 & 32 & 34 & \dots & \\ & 16 & 15 & 12 & 7 & 2 & 11 & 22 & 28 & & \\ & & 18 & 19 & 8 & 8 & 2 & 16 & 18 & & \\ & & & 23 & 19 & 17 & 12 & 9 & 2 & 13 & \\ & & & & 24 & 25 & 18 & 17 & 10 & 12 & 2 \\ & & & & & & & \vdots & & & \ddots \end{pmatrix}$$

Notice the stark contrast with the matrix of  $U_2$ . Whereas the general entry of every column seems like it tends to zero (as it should, since  $T_3$  still defines an operator on the Banach space  $M_0^\dagger(2/3)$  after all) it does

not look like the general column tends uniformly to zero. Most strikingly, the diagonal entries all seem to have valuation 2, suggesting this operator is not compact.

For the operator  $U_2$  we can also compute an approximation for its characteristic series  $P(t)$ , using the above matrix. One can easily analyse to which precision the given answer is correct, but we will ignore such issues here. We truncate the matrix for  $U_2$  as above, and obtain a polynomial whose coefficients are 2-adically close to those of  $P(t)$ . Using a Newton polygon argument, we check that the valuations of the eigenvalues of  $U_2$  on the full space  $M_0^\dagger(r)$  for any  $r$  are as follows:

$$(83) \quad \mathbf{0}_1, \mathbf{3}_1, \mathbf{7}_1, \mathbf{13}_1, \mathbf{15}_1, \mathbf{17}_1, \dots$$

Here, we denote the valuations of the eigenvalues by bold type, and the multiplicity of that valuation by a subscript. It is striking that these are all integers, since there is no a priori reason that they should be! In this particular example, there is an explicit expression for the general term in this sequence (see the exercises) proving in particular that every valuation in this infinite sequence is an integer, and occurs with multiplicity one.

In general, the valuations of the eigenvalues of  $U_p$  are usually referred to as the *slopes* of this Hecke operator, and they are the subject of many results and conjectures in the literature. There are few cases where things can be proved explicitly, and in general there is a conjectural recipe to find the sequence of slopes, known as *Buzzard's slope conjectures*. They remain today completely mysterious. See [Buz05, BC05, BK05, Roe14, BP16, LWX17] and the references contained therein.

**2.4. Exercises.** We collect a handful of exercises related to the material in this section.

1. Prove that if  $f \in M_k(\Gamma_0(p))$ , then  $f(q)$  is a  $p$ -adic modular form of weight  $k$ .
2. Consider the modular function  $U_2j - 744$ , where  $j$  is the usual Klein invariant

$$j(q) = \frac{1}{q} + 744 + 196884q + \dots = \sum_{n \geq -1} a_n q^n$$

and show that it is in the cuspidal subspace of  $M_0^\dagger(2/3)$  of overconvergent 2-adic modular forms. Deduce from our computation of 2-adic slopes (83) the congruence proved by Lehner [Leh49] which states that for all  $n > 0$  we have

$$a_n \equiv \pmod{2^{3m+8}} \quad \text{whenever} \quad n \equiv 0 \pmod{2^m}.$$

3. Let  $h$  be the Hauptmodul defined in (72). Prove that for  $n \geq 2$  it satisfies the recursion

$$U_p(h^n) = (48h + 4096h^2)U_p(h^{n-1}) + hU_p(h^{n-2}).$$

We know from (74) that the powers of  $2^6h$  form an orthonormal basis of the Banach space  $M_0^\dagger(1/2)$ . Prove that the  $(i, j)$ -th entry in the matrix for  $U_p$  with respect to this basis is given by

$$\frac{3j(i+j-1)!2^{2i+2j-1}}{(2i-j)!(2j-i)!}$$

4. Assume without proof<sup>2</sup> that there exist matrices  $A, B$  with entries in  $\mathbf{Z}_2$  which are both congruent to the identity matrix modulo 2, and such that  $ADB$  equals the matrix of  $U_p$  computed in the

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<sup>2</sup>This was shown by Buzzard–Calegari [BC05, Lemma 4] via an explicit construction, followed by a really intriguing direct computation using a hypergeometric summation formula.

previous exercise, where  $D$  is the diagonal matrix with  $(i, i)$ -th entry given by

$$\frac{2^{4i+1}(3i)!^2 i!^2}{3(2i)!^4}.$$

Deduce that the matrix of  $U_p$  has a characteristic series whose Newton polygon is the same as that for the matrix  $D$ . Conclude that the slope sequence (83) is none other than the sequence

$$1 + 2v_2 \left( \frac{(3n)!}{n!} \right).$$

### 3. FAMILIES OF MODULAR FORMS

In this lecture, we make some final remarks on families of modular forms, before we discuss how to compute explicitly with overconvergent modular forms in general.

**3.1. The eigencurve.** The above constructions may be extended to incorporate families of elliptic curves, culminating in the existence of the *eigencurve*, which is a mysterious geometric object that provides a very helpful mental picture to have in mind when thinking about families of overconvergent modular forms. The theory is due mainly to Coleman [Col96, Col97b] and Coleman–Mazur [CM98] and was revisited more recently by Pilloni [Pil13] and Andreatta–Iovita–Stevens [AIS14], who provided an extremely satisfactory and flexible framework. We content ourselves with a very brief discussion in these notes.

We start by noting that the geometric theory of overconvergent forms due to Katz has some crucial drawbacks. Most notably, it is restricted to the setting of integral weights  $k \in \mathbf{Z}$ , whereas already Serre’s theory of  $p$ -adic modular forms allows for more general weights in  $\mathbf{Z}_p$ . As a consequence, it is difficult to get a theory of continuous (analytic) families of modular forms in different weights, if we cannot interpolate  $p$ -adically between weights. To overcome the lack of a sheaf  $\omega^\kappa$  for any  $p$ -adic weight other than  $\kappa \in \mathbf{Z}$ , the idea of Coleman was to turn once more to the Eisenstein family, where the analytic variation of all the Fourier coefficients is known. In fact, in those cases we can define the coefficients for any *weight-character*

$$(84) \quad \kappa \in \mathcal{W} := \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$$

where we can view a pair  $(k, \chi)$  consisting of  $k \in \mathbf{Z}$  and  $\chi : (\mathbf{Z}/p^n \mathbf{Z})^\times \rightarrow \mathbf{C}_p^\times$  as a subset via the embedding defined by the continuous homomorphism

$$(85) \quad (k, \chi) : \mathbf{Z}_p^\times \longrightarrow \mathbf{C}_p^\times, \quad a \longmapsto \chi(a)a^k.$$

where  $\chi$  is now thought of as a character of  $\mathbf{Z}_p^\times$  by composing with reduction modulo  $p^n$ . The subset of weight characters for which  $\kappa$  induces the trivial character on  $(\mathbf{Z}/p \mathbf{Z})^\times$  is denoted by  $\mathcal{W}_0$ .

The coefficients of Eisenstein series are naturally functions of  $(k, \chi)$ , and one can easily show that they extend to functions of  $\mathcal{W}$ . The only part that needs clarification is how to view the Kubota–Leopoldt zeta function  $\zeta_p$  as a function of  $\kappa \in \mathcal{W}$ . Denote  $\Delta$  for the torsion subgroup of  $\mathbf{Z}_p^\times$ , which is cyclic of order  $\phi(q)$ , where  $q = 4$  if  $p = 2$ , and  $q = p$  otherwise. There is an isomorphism

$$(86) \quad \mathbf{Z}_p^\times \xrightarrow{\sim} \Delta \times (1 + q \mathbf{Z}_p), \quad a \longmapsto (\omega(a), \langle a \rangle).$$

The character  $\omega$  is called the *Teichmüller character*. Let  $\Lambda = \mathbf{Z}_p[[\mathbf{Z}_p^\times]]$  be the Iwasawa algebra, which is the ring of functions on  $\mathcal{W}$ , then we have an isomorphism

$$(87) \quad \Lambda \simeq \mathbf{Z}_p[\Delta][[T]], \quad 1 + q \longmapsto 1 + T.$$

This way, the Kubota–Leopoldt zeta function  $\zeta_p$  can be viewed as a function on  $\mathcal{W}$ , satisfying  $\zeta_p((1+q)^k - 1) = (1 - p^{k-1})\zeta(1 - k)$ , giving us the Eisenstein family

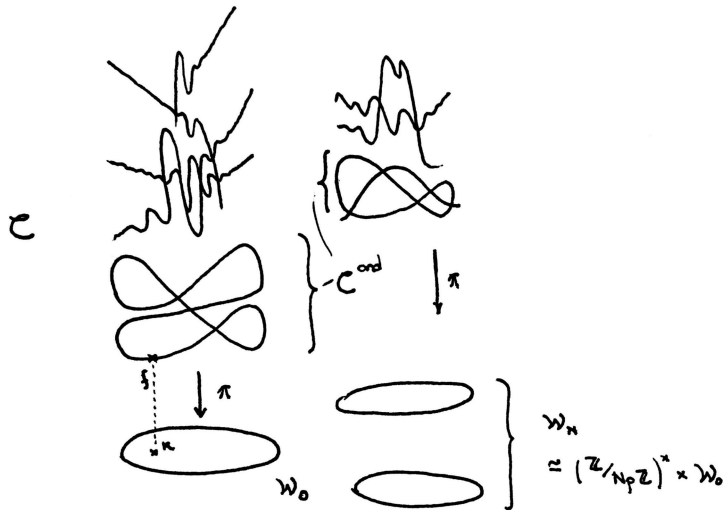
$$(88) \quad \begin{aligned} \mathbf{G}_\kappa(q) &= \frac{\zeta_p(\kappa)}{2} + \sum_{n \geq 1} \left( \sum_{p^{\ell} d | n} \kappa(d)/d \right) q^n & \kappa \notin \mathcal{W}_0 \\ \mathbf{E}_\kappa(q) &= 1 + \frac{2}{\zeta_p(\kappa)} \sum_{n \geq 1} \left( \sum_{p^{\ell} d | n} \kappa(d)/d \right) q^n & \kappa \in \mathcal{W}_0 \end{aligned}$$

The idea of Coleman was to define an overconvergent modular form of weight  $\kappa$  to be any  $q$ -expansion with the property that its quotient by the Eisenstein series of weight  $\kappa$  is an overconvergent modular function. Since then, a more satisfactory definition has been given by Pilloni, who gave a geometric construction of line bundles  $\omega^\kappa$  on the affinoids  $X_r$  for some  $r$  that depends on  $\kappa$ . He shows that the Eisenstein series of weight  $\kappa$  is a section of his line bundle, therefore giving a completely geometric definition of the space of  $r$ -overconvergent forms  $M_\kappa^\dagger(r)$  for any weight-character  $\kappa$ , as long as  $r$  is sufficiently small.

This work culminated in the construction, due to Coleman–Mazur [CM98], of the *eigencurve* for any level  $N$  coprime to  $p$ . This is a rigid analytic curve  $\mathcal{C}_N$ , whose  $\mathbf{C}_p$ -points classify overconvergent eigenforms  $f$  of the Hecke operators  $U_p$  and  $T_\ell$  for  $\ell \nmid Np$ , which are not in the kernel of  $U_p$  (in this case, we say  $f$  is of *finite slope*). We define  $\mathcal{W}_N$ , the weight space of level  $N$ , as a rigid analytic variety, via

$$(89) \quad \mathcal{W}_N = (\mathrm{Spf} \Lambda_N)^{\mathrm{rig}}, \quad \text{where } \Lambda_N = \mathbf{Z}_p[[\mathbf{Z}/N\mathbf{Z}]^\times \times \mathbf{Z}_p^\times]$$

and there is a natural map  $\pi : \mathcal{C}_N \rightarrow \mathcal{W}_N$  associating to every overconvergent eigenform  $f$  its weight character  $\kappa$ . The geometric properties of  $\mathcal{C}_N$  therefore dictate all the possible  $p$ -adic variations of modular forms of finite slope in families. Very little is known about its geometry, but one important fact is that it is a curve, justifying its name. This means that every overconvergent eigenform of finite slope may be interpolated in a  $p$ -adic family. Should this overconvergent eigenform correspond to a singularity of  $\mathcal{C}_N$ , this may even be possible in several different ways.



**Hida theory.** There has been a lot of research on the geometric properties of the eigencurve, and though this has yielded extremely interesting results, much of its geometry remains elusive. One part

of the eigencurve that is fairly well understood is the so-called *ordinary part*. Recall that an overconvergent form is called *ordinary* if it is a  $U_p$ -eigenvector with an eigenvalue that is a  $p$ -adic unit, or, said differently, which is of slope zero. Hida considered the *ordinary projection* operator

$$(90) \quad e^{\text{ord}} = \lim_{n \rightarrow \infty} U_p^{n!}$$

whose limit exists as an operator on  $M_\kappa^\dagger(r)$  for any  $\kappa \in \mathcal{W}_N$ . Then Hida showed:

**Theorem 3.1** (Hida). *The image of  $e^{\text{ord}}$  on  $M_\kappa^\dagger(r)$  is a finite-dimensional vector space, whose dimension depends only on the connected component of  $\mathcal{W}_N$  containing  $\kappa$ .*

This statement is miraculous, and shows that even though the slopes of the spectrum of  $U_p$  can vary wildly, the dimension of the part of slope 0 is locally constant on  $\mathcal{W}_N$ . Note that the connected components of  $\mathcal{W}_N$  are indexed by the characters  $(\mathbf{Z}/Nq\mathbf{Z})^\times \rightarrow \mathbf{C}_p^\times$ , and the dimension of the ordinary subspace is constant over each component. Hida in fact proved the following statement: Suppose

$$(91) \quad \pi^{\text{ord}} : \mathcal{C}_N^{\text{ord}} \rightarrow \mathcal{W}_N$$

is the projection map from the ordinary part of the eigencurve to weight space, then  $\pi^{\text{ord}}$  is finite flat. The ordinary part of  $\mathcal{C}_N^{\text{ord}}$  is often referred to, at least locally, as the *Hida family*.

A very important fact is that specialisations of Hida families at classical weights  $k \geq 2$  are always *classical modular forms*. More precisely, and more generally, the following theorem was proved by Coleman:

**Theorem 3.2** (Coleman). *Suppose that  $k \geq 2$  is a classical weight, and  $f \in M_k^\dagger$  is a  $U_p$ -eigenform of slope strictly less than  $k - 1$ . Then  $f$  is classical, in the sense that it belongs to the finite-dimensional subspace*

$$(92) \quad f \in M_k(\Gamma_0(Np)) \subset M_k^\dagger.$$

**Leopoldt's formula.** We finish our brief discussion of  $p$ -adic families of modular forms by proving Leopoldt's formula, which is a classical result on the value at  $s = 1$  of  $p$ -adic L-functions attached to Dirichlet characters. We follow here the treatment in [BCD<sup>+</sup>]. Once more, this is an incarnation of Serre's idea of obtaining information about L-values by identifying them as the constant term in the Fourier expansion of a modular form. In the situation at hand, this provides a rigidity for the power series that allows us to identify the constant term as an explicit combination of units. A purely algebraic proof for Leopoldt's formula for  $L_p(1, \chi)$  can be found in Washington [Was97, §5.4].

Suppose that  $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \rightarrow \mathbf{C}^\times$  is a primitive even Dirichlet character with conductor  $N > 1$  coprime to  $p$ , then we have the  $p$ -adic Eisenstein family of overconvergent forms

$$(93) \quad E_{k,\chi}^{(p)}(q) = L_p(1-k, \chi) + 2 \sum_{n \geq 1} \sigma_{k,\chi}^{(p)}(n) q^n, \quad \text{where } \sigma_{k,\chi}^{(p)}(n) = \sum_{p \nmid d | n} \chi(d) d^{k-1}.$$

This family specialises at  $k = 0$  to a rigid analytic function on  $X^{\text{ord}} = X_1(N)^{\text{ord}}$ , whose value at the cusp  $\infty$  is the value  $L_p(1, \chi)$ . Now choose a primitive  $N$ -th root of unity  $\zeta$ , then there is a collection of *Siegel units*  $g_a \in \mathcal{O}_{Y_1(N)}^\times$  whose  $q$ -expansions are given by

$$(94) \quad g_a(q) = q^{1/12} (1 - \zeta^a) \prod_{n \geq 1} (1 - q^n \zeta^a) (1 - q^n \zeta^{-a}), \quad 1 \leq a \leq N - 1.$$

These Siegel units are the building blocks of Kato's Euler system, and will no doubt make several appearances at this conference. We have seen that the operator  $V_p$  decreases overconvergence, as dictated by



(65), but it does define an operator on the space of  $p$ -adic modular functions (i.e. the case  $r = 0$ ). We can therefore use it to define the rigid analytic function

$$(95) \quad F_\chi^{(p)} = \frac{1}{p\mathfrak{g}(\chi^{-1})} \sum_{a=1}^{N-1} \chi^{-1}(a) \log_p (V_p(g_{pa})g_a^{-1})$$

which is defined on the ordinary locus  $X^{\text{ord}}$ , where  $\mathfrak{g}$  denotes the standard Gauß sum, obtained by summing  $\chi^{-1}(a)\zeta^a$  over  $a$ . A direct computation using the expression (94) shows that the higher coefficients of its  $q$ -expansion agree with that of  $E_{0,\chi}^{(p)}$ . Therefore the modular form

$$(96) \quad E_{0,\chi}^{(p)} - F_\chi^{(p)},$$

which is a constant function, must be equal to zero, since it has nebentype  $\chi$ . We conclude that the constant terms of both series are equal, yielding Leopoldt's formula:

$$(97) \quad L_p(1, \chi) = -\frac{(1 - \chi(p)p^{-1})}{\mathfrak{g}(\chi^{-1})} \sum_{a=1}^{N-1} \chi^{-1}(a) \log_p(1 - \zeta^a).$$

**3.2. Computing overconvergent forms.** We now explain how to compute explicit bases for  $r$ -overconvergent forms, following Katz [Kat73] and Lauder [Lau11]. Note that in the example we encountered in the previous section, namely where  $(p, N) = (2, 1)$  and  $k = 0$ , we were particularly lucky in the sense that the modular curve  $X_0(2)$  had genus zero, and the overconvergent regions  $X_r$  were isomorphic to a rigid analytic disk, for which we could identify an explicit parameter. This procedure can be repeated for any prime  $p$  for which  $X_0(p)$  has genus zero (i.e. for  $p = 2, 3, 5, 7, 13$ ), where one can likewise write down a power basis for the space of overconvergent modular forms, for any weight  $k$ . See Loeffler [Loe07] for a detailed discussion of this case, as well as many interesting results and computations.

For general values of  $p$ , we are faced with a more complicated geometric picture, as the overconvergent regions  $X_r$  are isomorphic to the complement of a finite number of disks in  $\mathbf{P}^1$ :



Moreover, in cases where we also have a nontrivial tame level  $N$ , the modular curve from which we remove these finitely many disks is no longer isomorphic to  $\mathbf{P}^1$ . Therefore, finding an explicit basis for the set of sections over the overconvergent regions  $X_r$  becomes significantly more subtle. In his foundational paper on the subject, Katz [Kat73, Chapter 2] identifies an explicit basis for these spaces, such that any overconvergent form may be written as a unique linear combination of it, referred to as its *Katz expansion*.

Let  $\mathcal{X}$  be the modular curve over  $\mathbf{Z}_p$  with  $\Gamma_1(N)$ -level structure for  $p \nmid N \geq 5$ . Let  $n$  be the smallest power of  $p$  such that the  $n$ -th power of the Hasse invariant  $A^n$  lifts to a level 1 Eisenstein series  $E$  of

weight  $k_E = n(p-1)$ . Throughout this section, we assume  $nr \leq 1$ . Our notation is summarised in the following table: In practice, there is a lot of flexibility with the setup, and the computations below are

$p$	2	3	$\geq 5$
$E$	$E_4$	$E_6$	$E_{p-1}$
$n$	4	3	1
$k_E$	4	6	$p-1$

usually for  $\Gamma_0(N)$  instead of  $\Gamma_1(N)$ . To justify this, some additional analysis is required to deal with the lack of representability, see [BC05, Appendix].

We now describe an explicit basis for the  $p$ -adic Banach spaces  $M_k^\dagger(r)$ . Suppose  $r = v_p(s)$  for some  $s \in \mathbf{C}_p$ , then let  $\mathcal{I}_r$  be the sheaf of ideals in  $\mathrm{Sym}(\omega^{\otimes k_E})$  generated by  $E - s^n$ , and define the line bundle

$$(98) \quad \mathcal{L} = \mathrm{Spec}_{\mathcal{X}}(\mathrm{Sym}(\omega^{\otimes k_E})/\mathcal{I}_r) \xrightarrow{\pi_{\mathcal{L}}} \mathcal{X}.$$

Assuming that  $k \neq 1$ , we can apply the base change theorems from [Kat73, Theorem 1.7.1] to show that

$$(99) \quad \begin{aligned} M_k^\dagger(r) &= \mathrm{H}^0(\mathcal{L}^{\mathrm{rig}}, \pi_{\mathcal{L}}^* \omega^{\otimes k}) \\ (100) \quad &= \mathrm{H}^0(\mathcal{X}, \omega^{\otimes k} \otimes \mathrm{Sym}(\omega^{\otimes k_E})) / \mathrm{H}^0(\mathcal{X}, \mathcal{I}_r). \end{aligned}$$

Having this concrete description in hand, we now attempt to eliminate the relation  $E = s^n$  by investigating the map given by multiplication by  $E$  on modular forms as in [Kat73, Lemma 2.6.1].

**Lemma 3.3.** *Let  $k \neq 1$ , then the injection given by the multiplication by  $E$ -map*

$$(101) \quad - \times E : \mathrm{H}^0(\mathcal{X}, \omega^{\otimes k}) \longrightarrow \mathrm{H}^0(\mathcal{X}, \omega^{\otimes k+k_E})$$

*splits as a map of  $\mathbf{Z}_p$ -modules.*

**Proof.** The result is clear for  $k \leq 0$ . For  $k \geq 2$ , we have  $\mathrm{H}^1(\mathcal{X}, \omega^{\otimes k}) = 0$  by computing the degree of  $\omega$  as in [Kat73, Theorem 1.7.1]. We obtain the short exact sequence

$$(102) \quad 0 \rightarrow \mathrm{H}^0(\mathcal{X}, \omega^{\otimes k}) \xrightarrow{\times E} \mathrm{H}^0(\mathcal{X}, \omega^{\otimes k+k_E}) \longrightarrow \mathrm{H}^0(\mathcal{X}, \mathcal{F}) \rightarrow 0,$$

where  $\mathcal{F}$  is the quotient sheaf. This sheaf  $\mathcal{F}$  is flat over  $\mathbf{Z}_p$ , and since  $\mathcal{F}$  is a skyscraper sheaf over  $\mathbf{F}_p$  it follows that  $\mathrm{H}^1(\mathcal{X}_s, \mathcal{F}) = 0$  and hence  $\mathrm{Supp} \mathrm{R}^1 f_* \mathcal{F} = \emptyset$ , where  $f : \mathcal{X} \rightarrow \mathrm{Spec}(\mathbf{Z}_p)$  is the defining morphism for  $\mathcal{X}$ . We conclude that  $\mathrm{H}^0(\mathcal{X}, \mathcal{F})$  is a free  $\mathbf{Z}_p$ -module, from which the conclusion follows.  $\square$

For every  $i \geq 0$ , choose generators  $\{a_{i,j}\}_j$  for a complement of the submodule

$$(103) \quad \mathrm{Im}(- \times E) \subseteq \mathrm{H}^0(\mathcal{X}, \omega^{\otimes k+ik_E}).$$

This choice is not canonical, but we will fix it once and for all in what follows. As in [Kat73, Proposition 2.6.2], one obtains the following as a consequence of (100) and Lemma 3.3.

**Theorem 3.4.** *The set  $\{e_{i,j}\}_{i,j}$  is an orthonormal basis for the  $p$ -adic Banach space  $M_k^\dagger(r)$ , where*

$$(104) \quad e_{i,j} = s^{ni} \frac{a_{i,j}}{E^i}$$

Note that we have avoided the case  $k = 1$ , which we can still compute with by appropriately twisting by  $U_p$ , thereby reducing the computation to one in higher weight for which the results above hold. This



technique is often referred to as *Coleman's trick*, see [Col97b, Eqn. (3.3)], and is also frequently useful in other situations. It is based on the observation that multiplication by  $E^j$  defines an isomorphism

$$(105) \quad M_k^{\dagger, r} \rightarrow M_{k+jk_E}^{\dagger, r},$$

as well as the fact that the  $U_p$ -operator is *Frobenius linear* in the sense that

$$(106) \quad U_p(fV_p(E)) = U_p(f)E.$$

It follows from these two simple facts that  $P_{k+jk_E}(t)$  equals the characteristic series of  $U_p \circ G^j$  on  $M_k^{\dagger, r}$ , where we denote  $G = E/V_pE$ . This allows us to flexibly change the weights of the spaces of overconvergent forms we are interested in. In particular, we can compute overconvergent forms in weight 1 by reducing the computation to, say, weight  $p$ . In addition, if we would like to compute the operator  $U_p$  on  $M_k^{\dagger}$  for some extremely large weight  $k$ , we can use Coleman's trick to reduce the computation to a much smaller weight.

Now that we know, by Theorem 3.4, an explicit basis  $e_{i,j}$  for the Banach space  $M_k^{\dagger}(r)$ , we are in a position to compute approximations of the matrix of  $U_p$  on  $q$ -expansions. Since we can only compute finitely many of its entries, we need a good estimate on the valuations of its entries, so we know how many elements of the basis we need to compute before we are guaranteed that the end result is correct up to some chosen  $p$ -adic precision. To do this, let us first fix some notation for these entries. We write

$$(107) \quad U_p \circ G^j(e_{u,v}) = \sum_{w,z} A_{u,v}^{w,z}(j) e_{w,z},$$

for some  $A_{u,v}^{w,z}(j) \in \mathbf{C}_p$ . Said differently, the numbers  $A_{u,v}^{w,z}(j)$  are the entries of the infinite matrix of  $U_p \circ G^j$  with respect to our chosen orthonormal basis for  $M_k^{\dagger}(r)$ . The following lemma estimates their  $p$ -adic valuations, and is an easy extension of Wan [Wan98, Lemma 3.1], see [Von15].

**Lemma 3.5.** *We have*

$$(108) \quad v_p(A_{u,v}^{w,z}(j)) \geq wrk_E - 1 - r(n-1).$$

The reader may have wondered why in the above precision estimate, we included the parameter  $j$ , corresponding to a twist of the  $U_p$  operator by  $G^j = (E/V_pE)^j$ , rather than simply putting  $j = 0$ . The reason is that this allows us to easily move between different weights, and perform the computation of  $U_p$  in several weights at once. The example below illustrates this, by computing the  $U_p$ -operator in families.

**3.2.1. The spectral curve.** We now have two crucial active ingredients for a working algorithm to compute with spaces of overconvergent modular forms, since we have (a) an explicit basis due to Katz, provided by Theorem 3.4, and (b) a precision estimate for the concomitant entries of the matrix of  $U_p$  due to Wan, provided by (108). Lauder [Lau11] combines these two ingredients into an efficient algorithm for computing  $U_p$  on  $M_k^{\dagger}(r)$ . We note that the estimate (108) is independent of  $j$ , and hence the computation may be performed at several  $p$ -adic weights at once. In this example, we compute the resulting 2-variable series  $P(\kappa, t)$ . The curve in  $\mathcal{W}_N \times \mathbf{G}_m$  cut out by this equation is often referred to as the *spectral curve* of  $U_p$ , which yields the eigencurve after an additional modification, see [CM98].

Let  $f : \mathcal{W}_N \rightarrow \mathbf{C}_p$  be a function in the Iwasawa algebra, and  $\{\kappa_0, \kappa_1, \dots, \kappa_n\}$  a finite set of Type-I points. Then we denote  $f[\kappa_0] = f(\kappa_0)$  and we inductively define the *divided difference* of order  $n$  to be

$$f[\kappa_0, \kappa_1, \dots, \kappa_n] := \frac{f[\kappa_1, \dots, \kappa_n] - f[\kappa_0, \dots, \kappa_{n-1}]}{\kappa_n - \kappa_0}.$$

We now define the  $n$ -th Newton series to be

$$(109) \quad P_n(\kappa, t) = \sum_{i=0}^n P[\kappa_0, \kappa_1, \dots, \kappa_i](t) \times (\kappa - \kappa_0)(\kappa - \kappa_1) \cdots (\kappa - \kappa_i),$$

where  $P[\kappa_0, \dots, \kappa_n](t)$  is the power series in  $t$  obtained by taking the corresponding finite differences on the coefficients of  $P(\kappa, t)$  of  $t$ , which are elements of the Iwasawa algebra by Coleman [Col97a]. The theory of finite differences then shows that upon increasing the number of interpolation points, the  $n$ -th Newton series  $p$ -adically approaches the series  $P(\kappa, t)$ . This means that all we need to do to compute an approximation for  $P(\kappa, t)$ , is to choose our interpolation points carefully and estimate the error term.

We explicitly compute some examples, starting by revisiting the example of Buzzard–Calegari [BC05] familiar from the previous lecture, and then venturing into more unfamiliar territory relating to situations that were considered in the literature by Buzzard–Kilford [BK05], Roe [Roe14] and the work on boundary slopes and the spectral halo by Andreatta–Iovita–Pilloni [AIP18] and Bergdall–Pollack [BP16]. We note that an alternative approach using overconvergent modular symbols has been developed in [DHH<sup>+</sup>16], for Hida families. Their algorithms yield explicit  $q$ -expansions of Hida families, where the coefficients are elements of  $\Lambda$ , but is only equipped to handle the ordinary part of the spectrum of  $U_p$ .

**Example 1.** We revisit the case of  $p = 2$  and tame level  $N = 1$ , where we computed with the space for  $k = 0$  in the previous lecture. Using an interpolation as described above, we can compute the two variable power series  $P(\kappa, t)$ , whose specialisation at  $\kappa \in \mathcal{W}$  recovers the characteristic series  $P_\kappa(t)$  of  $U_2$  on the space of overconvergent modular forms  $M_\kappa^\dagger(r)$ . We obtain

$$\begin{aligned} P(\kappa, t) = & 1 + 519736167t + 413685912t^2 + 148708352t^3 + 1065353216t^4 \\ & + \kappa (36306799t + 374998993t^2 + 380696768t^3 + 281739264t^4) \\ & + \kappa^2 (43984100t + 481404364t^2 + 496002384t^3 + 387895296t^4 + 1811939328t^5) \\ & + \kappa^3 (874017364t + 890496879t^2 + 487943741t^3 + 4077568t^4 + 964689920t^5) \\ & + \kappa^4 (392124398t + 264203079t^2 + 839291211t^3 + 908503936t^4 + 817102848t^5) \\ & + O(\kappa^5, 2^{30}), \end{aligned}$$

We actually computed  $P(\kappa, t)$  to precision  $O(\kappa^{25}, 2^{70})$ , which took about 5 minutes, but I truncated the result to get output that fits in this document. Let us now investigate various specialisations:

- The computation we did in the previous section is contained in this one, and if we set  $\kappa = 5^k - 1 = 0$ , which corresponds to weight  $k = 0$ , we recover the same power series as before, up to the used precision. In particular, we can read off that first few slopes are  $\mathbf{0}_1, \mathbf{3}_1, \mathbf{7}_1, \dots$ , which agrees with the result of Buzzard–Calegari [BC05], proved in the exercises above, that in weight 0 the  $n$ -th slope is equal to

$$1 + 2v_2 \left( \frac{(3n)!}{n!} \right),$$

with multiplicity 1.

- As for the other extreme, the main result of Buzzard–Kilford [BK05] states that the slopes for  $1/8 < |\kappa| < 1$  form an arithmetic progression with  $n$ -th term  $nv_2(\kappa)$ , all with multiplicity 1. Indeed, by substituting  $\kappa = 2$  we obtain the slope sequence  $0, 1, 2, 3, 4, \dots$ , while for  $\kappa = 4$  we recover  $0, 2, 4, 6, 8, \dots$ . Our computed power series  $P(\kappa, t)$  hence combines the best of both worlds, by describing the spectral curve over the inner regions of  $\mathcal{W}$  as well as the outskirts. Notice the striking contrast between the nature of the slope sequence at  $k = 0$  and that close to the boundary! A

folklore conjecture predicts that the same phenomenon happens in general, and a result of this flavour was obtained by Liu–Wan–Xiao [LWX17].

In the above computation, we focussed on the variation of  $P_\kappa(t)$  with the weight  $\kappa$ , but we can interchange the variables  $\kappa$  and  $t$  and study instead the powers series in  $\kappa$  appearing as the coefficients of the above series in  $t$ . For instance, up to precision  $(2^{21}, \kappa^7)$  we obtain

$$\begin{aligned} P(\kappa, t) \equiv & 1 + t(1739623 + 655215\kappa + 2041060\kappa^2 + 1602132\kappa^3 + 2054126\kappa^4 + 779022\kappa^5 + 1634724\kappa^6) \\ & + t^2(546968 + 1705937\kappa + 1156556\kappa^2 + 1304431\kappa^3 + 2059079\kappa^4 + 1677821\kappa^5 + 644339\kappa^6) \\ & + t^3(1907712 + 1112256\kappa + 1074512\kappa^2 + 1404477\kappa^3 + 430411\kappa^4 + 51909\kappa^5 + 1261732\kappa^6) \\ & + t^4(720896\kappa + 2019328\kappa^2 + 1980416\kappa^3 + 437120\kappa^4 + 1161264\kappa^5 + 1648837\kappa^6) \\ & + t^5(1310720\kappa^4 + 524288\kappa^5 + 1101824\kappa^6) \\ & + O(2^{21}, \kappa^7) \end{aligned}$$

Investigating the coefficients  $a_i(\kappa)$  of  $P(\kappa, t)$  for small values, we see that their valuation on  $\kappa \in \mathbf{Z}_2$  only seems to depend on  $\kappa \pmod{2^6}$ . This can be made into a rigorous proof of this fact, by using the uniform estimates in Wan [Wan98] for the Newton polygon in  $t$  of  $P(\kappa, t)$  recalled above. After possibly redoing the computation to a higher precision, to assure that all the slopes are indeed correct, we recover the following theorem, which may be found in Emerton [Eme98, Theorem 1.1].

**Theorem 3.6** (Emerton). *The minimal non-zero slope of  $U_2$  on  $M_k^\dagger$  in tame level 1, along with its multiplicity, depends only on  $k \pmod{16}$ . In particular, the minimal  $U_2$  slope with multiplicity is  $\mathbf{3}_1$  when  $k \equiv 0 \pmod{4}$ ,  $\mathbf{4}_1$  when  $k \equiv 2 \pmod{8}$ ,  $\mathbf{5}_1$  when  $k \equiv 6 \pmod{16}$  and  $\mathbf{6}_2$  when  $k \equiv 14 \pmod{16}$ .*

We note that the calculations of Emerton [Eme98] rely crucially on the explicit uniformisations of 2-adic regions on the genus 0 modular curves  $X_0(2^n)$  for small values of  $n$ , which are hard to come by in higher levels and primes. Our algorithms do not rely on any specifics of the situation  $(p, N) = (2, 1)$ , and therefore similar arguments work in more general settings.

Looking further into the above coefficients, let  $\lambda(i)$  be the number of roots of  $a_i(\kappa)$  in the open unit disk. The following table displays the 2-adic valuations of these roots, along with their multiplicities:

Coefficient	Valuations	$\lambda$
$a_0(\kappa) = 1$	$\emptyset$	0
$a_1(\kappa)$	$\emptyset$	0
$a_2(\kappa)$	$\mathbf{3}_1$	1
$a_3(\kappa)$	$\mathbf{3}_2, \mathbf{4}_1$	3
$a_4(\kappa)$	$\mathbf{3}_4, \mathbf{4}_1, \mathbf{7}_1$	6
$a_5(\kappa)$	$\mathbf{3}_6, \mathbf{4}_2, \mathbf{5}_1, \mathbf{7}_1$	10
$a_6(\kappa)$	$\mathbf{3}_9, \mathbf{4}_3, \mathbf{5}_2, \mathbf{6}_1$	15
$a_7(\kappa)$	$\mathbf{3}_{12}, \mathbf{4}_5, \mathbf{5}_2, \mathbf{6}_1, \mathbf{8}_1$	21

By inspecting the 2-adic valuations of the coefficients we computed, we see that this output is provably correct and complete. Note that

$$\lambda(i) = \binom{i}{2},$$

which also follows from the main result of Buzzard–Kilford [BK05]. In Bergdall–Pollack [BP16], precise conjectures are made about the location of the zeroes of  $a_i$ .

**Example 2.** Let us set  $(p, N) = (3, 1)$  and compute  $P(\kappa, t)$  up to precision  $O(3^{90}, \kappa^{60})$ . The motivated reader is encouraged to try and recover this computation, for instance using an explicit basis similar to that used in the previous lecture, which is possible since  $X_0(3)$  has genus 0. There is a particularly nice basis, described by Loeffler [Loe07], which can be twisted by an Eisenstein series to obtain the computation in all weights.

With the same notation as above, we find the following slopes of the zeroes of the coefficients  $a_i(\kappa)$ :

Coefficient	Valuations	$\lambda$
$a_0(\kappa) = 1$	$\emptyset$	0
$a_1(\kappa)$	$\emptyset$	0
$a_2(\kappa)$	$\mathbf{1}_2$	2
$a_3(\kappa)$	$\mathbf{1}_5, \mathbf{3}_1$	6
$a_4(\kappa)$	$\mathbf{1}_9, \mathbf{2}_2, \mathbf{3}_1$	12
$a_5(\kappa)$	$\mathbf{1}_{15}, \mathbf{2}_4, \mathbf{3}_1$	20
$a_6(\kappa)$	$\mathbf{1}_{22}, \mathbf{2}_5, \mathbf{3}_2, \mathbf{4}_1$	30
$a_7(\kappa)$	$\mathbf{1}_{30}, \mathbf{2}_8, \mathbf{3}_2, \mathbf{4}_2$	42
$a_8(\kappa)$	$\mathbf{1}_{40}, \mathbf{2}_{11}, \mathbf{3}_2, \mathbf{4}_3$	56

Again, this output is complete and provably correct. Notice that

$$\lambda(i) = 2 \binom{i}{2},$$

which follows from the main result of Roe [Roe14], who showed that near the boundary, the slopes form an arithmetic progression with an explicit argument that depends on the valuation of  $\kappa$ . Roe tackled this more complicated situation using the same techniques as Buzzard–Kilford [BK05].

**Example 3.** We now turn to  $(p, N) = (2, 3)$  and compute  $P(\kappa, t)$  up to precision  $O(2^{60}, \kappa^{20})$ . This computation took about 90 minutes on a standard laptop. In addition to the notation above, let  $\mu(i)$  be the largest power of  $p$  that divides  $a_i(\kappa)$ . The work of Bergdall–Pollack [BP16] uses Koike’s trace formula to prove that  $\mu(i) = 0$  whenever  $N = 1$ . However, in our situation  $\mu$  appears to be larger for several  $i$ :

Coefficient	Valuations	$\lambda$	$\mu$
$a_0(\kappa) = 1$	$\emptyset$	0	0
$a_1(\kappa)$	–	–	–
$a_2(\kappa)$	$\emptyset$	0	0
$a_3(\kappa)$	$\emptyset$	0	1
$a_4(\kappa)$	$\mathbf{4}_1$	1	0
$a_5(\kappa)$	$\mathbf{3}_2$	2	1
$a_6(\kappa)$	$\mathbf{3}_2, \mathbf{4}_1$	3	0
$a_7(\kappa)$	$\mathbf{3}_2, \mathbf{4}_1, \mathbf{8}_1$	4	1
$a_8(\kappa)$	$\mathbf{3}_3, \mathbf{4}_1, \mathbf{5}_1, \mathbf{6}_1$	6	0
$a_9(\kappa)$	$\mathbf{3}_4, \mathbf{4}_3, \mathbf{6}_1$	8	1
$a_{10}(\kappa)$	$\mathbf{3}_5, \mathbf{4}_3, \mathbf{5}_1, \mathbf{8}_1$	10	0
$a_{11}(\kappa)$	$\mathbf{3}_6, \mathbf{4}_4, \mathbf{5}_2$	12	1
$a_{12}(\kappa)$	$\mathbf{3}_7, \mathbf{4}_5, \mathbf{5}_2, \mathbf{7}_1$	15	0

Computing  $P(\kappa, t)$  up to precision  $O(2, \kappa^{30})$  takes about one minute. Extracting the degrees of the  $t$ -coefficients, our data suggests the boundary slope sequence

$$0_2, 1/2_2, 1_2, 3/2_2, 2_2, 5/2_2, 3_2, 7/2_2, \dots$$

which is indeed in accordance with the Newton polygon of  $\lambda + \mu$  computed above, up to the chosen precisions. Notice the similarity with the slope sequence for  $(p, N) = (2, 1)$ .

**Example 4.** As above, set  $(p, N) = (11, 1)$  and compute  $P(\kappa, t)$  up to precision  $O(11, \kappa^{60})$ , which takes about two minutes. We compute the degrees of the  $t$ -coefficients, which suggest the boundary slope sequence:

$$0_1, 1_1, 2_1, 3_1, 4_2, 5_1, 6_1, 7_1, 9_2, \dots$$

3.2.2. *The Gouvêa–Mazur conjecture.* An enormous amount of arithmetic information is encoded in the slopes of overconvergent modular forms, which are the valuations of their  $U_p$ -eigenvalues. One of the consequences of the theory of Coleman [Col97b] is that for any  $\alpha > 0$ , there exists a smallest integer  $N_\alpha$  with the following property: If  $k_1$  and  $k_2$  are integers such that

$$(110) \quad k_1 \equiv k_2 \pmod{p^{N_\alpha}(p-1)}$$

then the collection of slopes  $\leq \alpha$  in weights  $k_1$  and  $k_2$  agree, with multiplicities. Gouvêa and Mazur conjectured in [GM92] that  $N_\alpha \leq \lfloor \alpha \rfloor$ . However, Wan [Wan98] exhibits an explicit quadratic upper bound for  $N_\alpha$ , depending on  $p$  and the level.<sup>3</sup>

The key observation for Wan is that the lower bound (108) is independent of  $j$ . After taking determinants, we obtain a lower bound on the coefficients of the characteristic series of  $U_p$  in weight  $k + jk_E$ , again independent of  $j$ . Wan then proceeds by proving a very general reciprocity lemma on Newton polygons, which allows him to transform the lower bound for those coefficients into an upper bound for  $N_\alpha$ . The analysis goes through without modifications, and using Wan's results we deduce from (108) that

**Theorem 3.7.** *There is an explicitly computable quadratic polynomial  $P \in \mathbf{Q}[x]$ , depending only on  $p$  and the level, such that  $N_\alpha \leq P(\alpha)$ .*

Since Gouvêa and Mazur conjectured in [GM92] that  $N_\alpha \leq \lfloor \alpha \rfloor$ , this is still an order of magnitude from what we expect. However, the conjecture of Gouvêa–Mazur is known to be false, and a counterexample was given in [BC04]. It should be noted that the counterexample of Buzzard–Calegari is only a very small violation of the conjecture, and on average it seems that in fact something much stronger than Gouvêa–Mazur is true! Let us illustrate this with two examples.

The case  $p = 2$  is prolific soil for finding counterexamples to the Gouvêa–Mazur conjecture. As noted above, the first counterexample was given in [BC04] for  $p = 59$  and level 1, and a further one for  $p = 79$  in

<sup>3</sup>Strictly speaking, Wan assumes that  $p \geq 5$ , but his arguments easily extend to  $p = 2, 3$  using our basis described above.

[Lau11]. For  $p = 2$ , we obtain the following slope sequences in level  $\Gamma_0(19)$ :

$$\begin{aligned}
k = -2 : & \quad \mathbf{0}_4, \mathbf{1}/\mathbf{2}_2, \mathbf{1}_3, \mathbf{2}_5, \mathbf{9}/\mathbf{4}_4, \mathbf{4}_3, \mathbf{5}_2, \mathbf{6}_{21}, \mathbf{15}/\mathbf{2}_2, \dots \\
k = 0 : & \quad \mathbf{0}_4, \mathbf{1}/\mathbf{2}_2, \mathbf{1}_5, \mathbf{3}_{11}, \mathbf{13}/\mathbf{4}_4, \mathbf{7}_{25}, \mathbf{25}/\mathbf{2}_4, \mathbf{13}_{11}, \dots \\
k = 2 : & \quad \mathbf{0}_4, \mathbf{1}/\mathbf{2}_2, \mathbf{1}_3, \mathbf{3}/\mathbf{2}_2, \mathbf{2}_5, \mathbf{4}_{11}, \mathbf{17}/\mathbf{4}_4, \mathbf{8}_{25}, \mathbf{27}/\mathbf{2}_4, \dots \\
k = 4 : & \quad \mathbf{0}_4, \mathbf{1}/\mathbf{2}_2, \mathbf{1}_5, \mathbf{5}/\mathbf{2}_2, \mathbf{3}_6, \mathbf{7}/\mathbf{2}_2, \mathbf{4}_3, \mathbf{5}_5, \mathbf{21}/\mathbf{4}_4, \dots \\
k = 6 : & \quad \mathbf{0}_4, \mathbf{1}/\mathbf{2}_2, \mathbf{1}_3, \mathbf{2}_7, \mathbf{5}/\mathbf{2}_2, \mathbf{4}_3, \mathbf{9}/\mathbf{2}_2, \mathbf{5}_6, \mathbf{11}/\mathbf{2}_2, \dots \\
k = 8 : & \quad \mathbf{0}_4, \mathbf{1}/\mathbf{2}_2, \mathbf{1}_5, \mathbf{3}_{13}, \mathbf{7}/\mathbf{2}_2, \mathbf{6}_5, \mathbf{13}/\mathbf{2}_2, \mathbf{7}_6, \mathbf{15}/\mathbf{2}_2, \dots
\end{aligned}$$

Notice the aberration in the dimensions of the slope 1 subspaces, as well as the slope 3 subspaces in weights 0 and 8. Whereas these are all *near misses*, in that the smallest slopes for which discrepancies arise are exactly equal to the valuation of the weight difference, we note a 2-dimensional slope  $3/2$  subspace in weight 2, which is completely absent in weight 6, whereas  $3/2 < v_2(6 - 2) = 2$ . Similarly, the slope  $9/4$  subspace in weight  $-2$  does not exist in weight  $6 = -2 + 2^3$ .

On the other hand, to see how Gouvêa–Mazur is frequently much weaker than the truth, consider the first few slopes of  $U_3$  acting on  $M_{278}^\dagger(\Gamma_0(41))$ , which we computed using Lauder’s algorithm to be

$$(111) \quad \mathbf{0}_{12}, \mathbf{1}_{14}, \mathbf{3}_{48}, \mathbf{6}_{14}, \mathbf{7}_{22}, \mathbf{8}_6, \mathbf{9}_{22}, \mathbf{10}_{14}, \mathbf{12}_{48}, \mathbf{14}_{14}, \mathbf{16}_{22}, \mathbf{17}_6, \mathbf{18}_{22}, \dots$$

where the subscripts denote multiplicities. Repeating the same computation in weight 8, we find the exact same slope sequence for all the terms we display here, whereas the Gouvêa–Mazur conjecture would only predict the slopes up to 3 to agree. This behaviour is somewhat typical when one computes lots of examples.

**3.3. Chow–Heegner points.** We end this lecture with an example of how the practical computation of spaces of overconvergent forms, using the above algorithms, can be used to construct arithmeto-geometric invariants. We chose to discuss the Heegner-type point construction on elliptic curves, following the theory of Darmon–Rotger [DR14], since there seems to be a very good chance that this topic will also be discussed by Víctor Rotger in his course, and if it is not, then it is still interesting anyways.

Let  $p$  be a prime and  $E/\mathbf{Q}$  an elliptic curve of conductor  $N$ , associated to a  $p$ -ordinary form  $f \in S_2^{\text{new}}(\Gamma_0(N))$ . Let  $g$  be any other weight 2 newform which is  $p$ -ordinary. It can be deduced from the work of Darmon–Rotger [DR14, Theorem 1.3] that there exists a global (rational) point  $P_g \in E(\mathbf{Q})$  that satisfies the Gross–Zagier type formula

$$(112) \quad \log(P_g) = 2d_g \cdot \frac{\mathcal{E}_0(g) \mathcal{E}_1(g)}{\mathcal{E}(g, f, g)} \cdot \mathcal{L}_p(\mathbf{g}, \mathbf{f}, \mathbf{g})(2, 2, 2),$$

where the quantities appearing in the formula are

- $\log$  is the formal  $p$ -adic logarithm on the elliptic curve  $E$ ,
- $d_g$  is an integer described in [DDL15, Remark 3.1.3],
- the  $\mathcal{E}$ -factors are quadratic numbers depending only on the  $p$ -th coefficients of  $f$  and  $g$ ,
- $\mathcal{L}_p(\mathbf{g}, \mathbf{f}, \mathbf{g})$  is the Rankin triple product  $p$ -adic L-function of the Hida families  $\mathbf{f}, \mathbf{g}$  through  $f, g$ .

The last item in this list deserves some discussion. We will not define the Rankin triple product  $p$ -adic L-function here, as that would lead us too far from the topic of these notes, and the exposition in Darmon–Rotger [DR14] is excellent. We will however explain how one computes the special value appearing in

formula (112). As before, we let  $e^{\text{ord}} = \lim_n U_p^{n!}$  be Hida's ordinary projector. Start by computing

$$(113) \quad e^{\text{ord}}(\theta^{-1} f^{[p]} \times g)$$

where  $f^{[p]}$  denotes the  $p$ -depletion  $(1 - V_p U_p) f$  of  $f$ . Here, we have used Serre's differential operator  $\theta = qd/dq$ , which is an important object in the theory of overconvergent forms, which would merit an entire lecture to do it justice. The inverse of this operator is defined by the  $p$ -adic limit

$$(114) \quad \theta^{-1} = \lim_{n \rightarrow \infty} \theta^{p^n - 1}.$$

By Coleman's criterion, we conclude that the overconvergent form (113) is classical, and hence it can be written as a finite linear combination of Hecke eigenforms of weight 2 and level  $\Gamma_0(p)$ . The special value  $\mathcal{L}_p(\mathbf{g}, \mathbf{f}, \mathbf{g})(2, 2, 2)$  is the coefficient of  $g$  in this linear combination.

**Remark.** The formula (112) has a similar flavour to the celebrated Gross–Zagier formula, but it has the advantage of giving an equality

$$(\text{Special value of L-function}) = (\text{Logarithm of a global point}).$$

In the Gross–Zagier formula, an equality between a special value of an L-function and the *height* of a global point is obtained. From the value of the height, it is not easy to reconstruct the global point, but from the value of the logarithm it is, since we can formally exponentiate! This means that the formula (112) has the additional advantage that it may be used to *construct explicitly* global points on  $E$ .

**Example 1.** Consider the elliptic curve

$$(115) \quad E : y^2 + xy = x^3 - x^2 - x + 1$$

which has rank 1 and conductor 58. Consider its associated newform  $f$ , and let  $g$  be the unique newform on  $\Gamma_0(58)$  different from  $f$ , then

$$(116) \quad \begin{aligned} f(q) &= q - q^2 - 3q^3 + q^4 - 3q^5 + 3q^6 - 2q^7 - q^8 + 6q^9 + 3q^{10} - q^{11} + \dots \\ g(q) &= q + q^2 - q^3 + q^4 + q^5 - q^6 - 2q^7 + q^8 - 2q^9 + q^{10} - 3q^{11} + \dots \end{aligned}$$

Both  $f$  and  $g$  are 2-ordinary. Letting  $P = (0, 1)$  be a generator for  $E(\mathbf{Q})$ , we compute that

$$(117) \quad \mathcal{L}_2(\mathbf{g}, \mathbf{f}, \mathbf{g})(2, 2, 2) \equiv 3 \log_E(P) \pmod{2^{200}},$$

as predicted by the theory in [DR14].

Let us end this section on a more speculative note. In the above theory, it is important that we assume ordinarity of  $f$ . Whereas it is conceivable that this may be extended to eigenforms of finite slope through the use of Coleman families, it is not clear that the Rankin triple product  $p$ -adic L-function may even be constructed in cases where  $f$  is of infinite slope. Nonetheless, the special value as computed above yields an explicit number even in those situations, and we now compute a few examples where the Tate module of  $E_{\mathbf{Q}}$  is wildly ramified at 2 or 3, and associated newform  $f$  is of infinite slope.

**Example 2a.** Consider the elliptic curve

$$(118) \quad E : y^2 + y = x^3 + 9x - 10$$

which is of conductor  $4617 = 3^5 \cdot 19$  and rank 1. Consider the newforms

$$(119) \quad \begin{aligned} f(q) &= q - 2q^2 + 2q^4 - 2q^5 - 3q^7 + 4q^{10} - 6q^{11} + \dots \\ g(q) &= q - 2q^3 - 2q^4 + 3q^5 - q^7 + q^9 + 3q^{11} + \dots \end{aligned}$$

where  $f$  is associated to  $E$ , and  $g$  is the unique cuspidal newform of weight 2 on  $\Gamma_0(19)$ . Despite  $f$  being of infinite 3-adic slope, we can run the computation and find a numerical value for  $\mathcal{L}_2(\mathbf{g}, "f", \mathbf{g})(2, 2, 2)$ . We find that

$$(120) \quad \mathcal{L}_3(\mathbf{g}, "f", \mathbf{g})(2, 2, 2) \equiv t \cdot \log_E(P) \pmod{3^{200}} \quad \text{where } 2t^2 + 48t + 729 = 0,$$

where  $P = (4, 9)$  is a generator of  $E(\mathbf{Q})$ . The fact that both quantities are related by a quadratic number  $t$  of small height suggests that a more general analogue of the theory for ordinary forms in [DR14], and more specifically equation (112), might exist.

**Example 2b.** Consider the elliptic curve

$$(121) \quad E : y^2 = x^3 + x^2 - 62893x - 6091893$$

which is of rank 1 and conductor  $15104 = 2^8 \cdot 59$ . Let  $f$  be its associated newform, and let  $g$  be the newform of level 118 associated to the elliptic curve with Cremona label 118.a1, then

$$(122) \quad \begin{aligned} f(q) &= q - 2q^3 - 3q^7 + q^9 + 3q^{11} - 3q^{13} + \dots \\ g(q) &= q - q^2 - q^3 + q^4 - 3q^5 + q^6 - q^7 - q^8 - 2q^9 + 3q^{10} - 2q^{11} + \dots \end{aligned}$$

Note that  $g$  is 2-ordinary. We compute that

$$(123) \quad \mathcal{L}_2(\mathbf{g}, "f", \mathbf{g})(2, 2, 2) \equiv 6 \log_E(P) \pmod{2^{100}},$$

where  $P = (20821, 3004216)$  is a generator of  $E(\mathbf{Q})$ . As in the previous example, this suggests that an analogue of (112) holds for  $f$  of infinite slope. Note that in this example already, the generator of  $E(\mathbf{Q})$  has very large height, and is therefore not trivial to find. For examples like these, there may already be some value in reversing the Gross–Zagier formula, to *construct* rational points on  $E$ .

#### 4. P-ADIC L-FUNCTIONS OF REAL QUADRATIC FIELDS

In this lecture, we will discuss one approach to the computation of  $p$ -adic L-functions of real quadratic fields, which once more is an incarnation of Serre’s idea, whereby this function is computed through a similar ‘rigidification’ procedure to what we discussed before. First, we realise it as the constant coefficient of a modular form, and subsequently we compute its higher Fourier coefficients instead.

We note that the problems of defining, and computing, both theoretically and computationally,  $p$ -adic L-functions of totally real fields go back a long time, and are rooted in an idea of Siegel. Recent work of Roblot [Rob15] also describes an efficient algorithm for computing  $p$ -adic L-functions of totally real fields, based on a different construction due to Cassou–Noguès [CN79].

(This material was presented with slides)

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