

# ANALYTIC MODULAR FORMS OVER $\mathbf{R}$ AND $\mathbf{Q}_p$

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ABSTRACT. These are the notes for a seminar talk which took place virtually on 7 April 2020 during the coronavirus lockdown. They are undoubtedly full of mistakes, and I greatly welcome comments and corrections.

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## 1. INTRODUCTION

In this talk, we will give a very brief overview of some compelling analogies that exist between the spectral theory of the hyperbolic Laplacian on Maaß forms, and the operator  $U_p$  on  $p$ -adic overconvergent modular forms. We will mostly focus on the conjectures made by Blasius, which were substantially developed and popularised by Calegari, exhibiting precise instances of a more vague analogy:

$$\left\{ \begin{array}{l} \text{Maaß forms } \mathcal{M}_{\infty}^{\text{an}} \\ \text{Laplacian } \Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Overconvergent forms } \mathcal{M}_p^{\text{an},r} \\ \text{“Laplacian” } \Delta = \log |U_p| \end{array} \right\}$$

Since the left side has been extensively studied for about a century, a much clearer picture is available, and we shall therefore focus on the more speculative right side of this analogy. Nonetheless, perhaps mainly for my own edification, we give some of the basic definitions of Maaß forms, and compute some explicit examples. Overconvergent forms were defined by Koji last week, and we recall this notion by treating a simple example in level 1, for  $p = 2$ , which is due to Buzzard–Calegari [BC05]. The subject of this talk is the spectral questions on the right side which are informed by known statements and conjectures on the left side, such as the Weyl law and spectral gap, and some related topics. We present some evidence for overconvergent forms in § 3, building on some simple extensions of the methods of Lauder [Lau11].

## 2. BASIC DEFINITIONS AND EXAMPLES

We begin by illustrating the basic definitions of Maaß forms and overconvergent forms with some explicit examples, before we dive into the spectral theory of the Laplacians in section § 3.

**2.1. Maaß forms.** This will be familiar to most people in the audience. For the purposes of this talk, we will use the phrase *Maaß form* of level  $N$  and character  $\chi$  to mean any complex valued smooth function on the upper half plane  $\mathcal{H}$  that satisfies

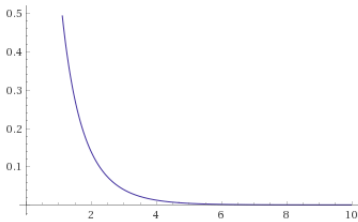
- We have  $f(\gamma z) = \chi(d)f(z)$ , for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ ,
- We have  $\Delta f = \lambda f$ , for some  $\lambda \in \mathbf{C}$ ,
- At the cusps,  $f$  has at most polynomial growth.

If  $f$  vanishes at every cusp then we say  $f$  is a Maaß cusp form. The space of cusp forms is denoted by  $\mathcal{M}_\infty^{\text{an}}$ , or  $\mathcal{M}_\infty^{\text{an}}(N, \chi)$  if we want to make the level and/or nebentypus explicit. There is a natural outer automorphism of the modular group, and its congruence subgroups, given by the reflection  $z \mapsto -\bar{z}$  along the imaginary axis. We say a Maaß form is *even* (resp. *odd*) if it is invariant (resp. negated) by this involution.

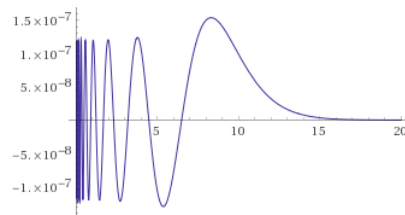
In order to obtain the analogue of the  $q$ -expansion of a holomorphic modular form, we need to introduce the Bessel function defined by

$$(1) \quad K_\nu(y) = \frac{1}{2} \int_0^\infty \exp\left(-y \cdot \frac{t^{-1} + t}{2}\right) \cdot t^\nu \cdot \frac{dt}{t},$$

where  $\nu \in \mathbf{C}$ , and  $y \in \mathbf{R}_{>0}$ . Bessel functions appear all over the literature, and there are many variants such as the modified, Hankel, and spherical incarnations one may encounter in different sources. This version above is usually referred to as the modified Bessel function of the second kind.



The function  $K_\nu$  for  $\nu \in \mathbf{R}$



The function  $K_\nu$  for  $\nu \in i\mathbf{R}$

For us, the case where  $\nu$  is purely imaginary is most relevant, with exponential decay at infinity and accelerated oscillation at zero. The Bessel function  $K_\nu$  is in the kernel of the differential operator

$$(2) \quad y^2 \frac{\partial^2}{\partial y^2} + y \frac{\partial}{\partial y} - (y^2 + \nu^2) = 0.$$

Using this differential equation, one can check directly that the function

$$(3) \quad \sqrt{y} K_\nu(2\pi y) e^{2\pi i x}$$

is an eigenfunction for the Laplacian  $\Delta$ , with eigenvalue  $1/4 - \nu^2$ . Any even Maaß cusp form may then be written as an infinite linear combination

$$(4) \quad f(z) = \sum_{i=1}^{\infty} a_n \sqrt{y} K_{\nu}(2\pi ny) \cos(2\pi nx)$$

with  $\cos(2\pi nx)$  replaced by  $\sin(2\pi nx)$  when  $f$  is odd. Here, the  $a_n$  are complex numbers, which we will refer to as the *Fourier coefficients* of  $f$  in what follows.

Maaß forms are very difficult to construct, and even where we succeed in constructing them, the arithmetic meaning of their Fourier coefficients is less apparent at first sight than is the case for holomorphic modular forms. The first examples were constructed by Maaß [Maa49] and remain today the most well-understood ones. They are associated to Hecke Grössencharaktere of real quadratic fields.

**Example 1.** Let us start with an algebraic example corresponding to an even Artin representation. Take  $K = \mathbf{Q}(\sqrt{205})$ , which has class number two, but narrow class group

$$\text{Cl}^+(K) = \mathbf{Z}/4\mathbf{Z}.$$

The narrow Hilbert class field  $H^+$  is a  $D_4$ -extension, and since it must be a CM field, complex conjugation acts as the non-trivial central element. This means the associated Maaß form is *odd*. We can easily find that  $H^+$  is the field with LMFDB label  $8.0.1766100625.1$  and defining polynomial

$$(5) \quad x^8 + 15x^6 + 48x^4 + 15x^2 + 1.$$

This field contains the biquadratic field  $\mathbf{Q}(\sqrt{5}, \sqrt{41})$ , so it is quite easy to compute the Fourier coefficients of the associated Maaß form, which are just the traces of Frobenius for the associated even Artin representation, which has determinant character  $\chi = \left(\frac{205}{\bullet}\right)$  and is determined by the rules

- Computing inertia invariants, we find  $a_5 = 1$  and  $a_{41} = -1$ .
- If  $p$  is inert in either  $\mathbf{Q}(\sqrt{5})$  or  $\mathbf{Q}(\sqrt{41})$ , then  $a_p = 0$ .
- If  $p$  splits completely in  $\mathbf{Q}(\sqrt{5}, \sqrt{41})$ , then

$$(6) \quad \begin{aligned} a_p &= 2 && \text{if } p \text{ splits completely in } H^+, \\ a_p &= -2 && \text{otherwise.} \end{aligned}$$

Given  $p$ , one easily decides which is the case from the factorisation of the polynomial (5) over  $\mathbf{F}_p$ .

We see that the coefficients of this Maaß form are quite sparse, since about 3/4 of primes have vanishing coefficient  $a_p$ . Here is a table of a few small nonzero values of  $a_p$ :

$p$	31	59	61	131	139	241	251	269	271	349
$a_p$	-2	2	-2	2	2	2	-2	2	2	-2
$p$	359	379	389	401	409	419	431	449	461	491
$a_p$	2	-2	-2	-2	2	-2	-2	-2	-2	-2

Using the above description, in reality we computed this table for  $10^6$  coefficients in a matter of a few seconds, and extend by the usual multiplicativity relations to get all the coefficients  $a_n$  up to that bound. As was made explicit in the original construction by Maaß, the spectral parameter  $\nu$  is determined by the infinity type, which is trivial here, giving us an explicit odd Maaß form of level 205 and character  $\chi = \left(\frac{205}{\bullet}\right)$ ,

with eigenvalue  $\lambda = 1/4$  and Fourier expansion

$$(7) \quad f_{205}(x + iy) = \sum_{n=1}^{\infty} a_n \sqrt{y} K_0(2\pi ny) \sin(2\pi nx).$$

So how do we come up with a useful check, to assure that we did these computations correctly? My first inclination was to look it up in the LMFDB, which at the time of writing (April 2020) contains 15659 examples of Maaß forms. Unfortunately, it's not there! In fact, I couldn't seem to find a single  $\lambda = 1/4$  example in the entire database. The range of composite levels considered there only goes up to about 100, which is not enough for us to find such an example there.

We know by the results of Maaß that this function must be invariant under the action of  $\Gamma_1(205)$ , and with the above we can hope to get a convincing sanity check out of this. With about a million coefficients at our disposal, we get decent approximations of the values, as long as  $y$  is not too small. We compute that

$$(8) \quad \begin{aligned} f_{205}(z) &\approx -0.0329258531808939434846607, & \text{where } z &= -\frac{1}{159} + \frac{i}{147} \\ f_{205}\left(\frac{z}{205z+1}\right) &\approx -0.0329258531808939434846607. \end{aligned}$$

The point  $z$  here is chosen randomly, but I fiddled around with it a bit so that both  $z$  and  $z/(205z+1)$  had comparable imaginary parts – which are then necessarily quite small – for precision reasons, as well as running times. Other random points and matrices in  $\Gamma_1(205)$  also showed agreement. We can also check decay at the cusps. It seems that the exponential decay of the Bessel function implies this at  $\infty$ , but at most of the other cusps some non-trivial cancellation should happen, since the Bessel function has a singularity at  $y = 0$ . Let us test this at the cusps  $1/5$  and  $1/41$ :

$z$	$f_{205}\left(\left(\begin{smallmatrix} 1 & -1 \\ 5 & -4 \end{smallmatrix}\right) \cdot z\right)$	$f_{205}\left(\left(\begin{smallmatrix} 1 & -1 \\ 41 & -40 \end{smallmatrix}\right) \cdot z\right)$
$i$	0.28938791323581092313	-0.0000000013831183226
$2i$	0.27394178958472227820	-0.00000001076504097306
$3i$	0.25081145098182262108	0.00000028030710513947
$4i$	0.22489469378987509144	-0.00000414269572531373
$5i$	0.19905036102642492449	-0.00001567413089581437
$6i$	0.17465665398778523978	-0.00000574090965257855
$7i$	0.15234632535756018371	0.00006881303091392039
$8i$	0.13233932568466237909	0.00024080092127293898
$9i$	0.11462694863937518081	0.00052466858846924810
$10i$	0.09908042495654999596	0.00091944836621727120

We see convincing decay at the cusp  $1/5$ , whereas at the cusp  $1/41$  we observe a discouraging growth. Continuing this table to higher and higher multiples of  $i$ , we see it turn around in the numbers (of which I will spare the reader), and at  $z = 30i$  the value is down to about  $-0.0936$  and starts decaying slowly.

**Remark.** Note that the invariance and decay the cusps are really the only content to the statement that the above example is a Maaß cusp form. Indeed, by construction the infinite sum (7) is already an eigenform for the Hecke algebra, and every individual term is an eigenvector for  $\Delta$  of eigenvalue  $1/4$ .

**Remark.** More generally, Maaß [Maa49] shows how to attach such an automorphic form to Hecke characters of real quadratic fields  $K$  with suitable non-trivial infinity types. Since we failed to construct even one of the examples contained in LMFDB, let us do one example of this form to see how this works. The advantage here is that we can drop the level down, at the cost of raising the spectral parameter  $\nu$ . Recall

that a *Hecke Grössencharakter* is a character

$$(9) \quad \psi : \mathbf{A}_K^\times / K^\times \rightarrow \mathbf{C}^\times$$

of the idèle class group. The following example involves Hecke characters which are *unitary* in the sense that their image is contained in the unit circle. When  $K$  has narrow class number one, such characters admit an especially simple description, since  $\mathbf{A}_K^\times = K^\times \widehat{\mathcal{O}}_K^\times \mathbf{R}_{>0}^2$ . Thus a Grössencharakter is just

$$(10) \quad \psi : \mathbf{R}_{>0}^2 / (\mathcal{O}_K^+)^\times \rightarrow \mathbf{C}^\times.$$

**Example 2.** Take the real quadratic field  $K = \mathbf{Q}(\sqrt{5})$  which has the smallest possible discriminant, namely 5. Its unit group (resp. totally positive unit group) is given by

$$(11) \quad \begin{aligned} \mathcal{O}_K^\times &= \pm \left( \frac{1+\sqrt{5}}{2} \right)^{\mathbf{Z}}, \\ (\mathcal{O}_K^+)^\times &= \left( \frac{3+\sqrt{5}}{2} \right)^{\mathbf{Z}}. \end{aligned}$$

By the preceding remark, we see that for any  $m \neq 0$ , the following defines the Grössencharakter:

$$(12) \quad \psi_m : (a, b) \mapsto a^{i\pi m / \log\left(\frac{1+\sqrt{5}}{2}\right)} \cdot b^{-i\pi m / \log\left(\frac{1+\sqrt{5}}{2}\right)}$$

The Maaß form with the same associated L-function as this character has Laplacian eigenvalue

$$(13) \quad \lambda = \frac{1}{4} + \left( \frac{\pi m}{\log\left(\frac{1+\sqrt{5}}{2}\right)} \right)^2$$

whereas its Fourier coefficients  $a_p$  are determined by the rules

- For  $p = 5$ , we compute that  $a_5 = 1$ ,
- If the prime  $p$  is inert in  $K$ , then  $a_p = 0$ ,
- If the prime  $p = \alpha\alpha'$  splits in  $K$ , then

$$(14) \quad \begin{aligned} a_p &= 2 \operatorname{Re} \left| \frac{\alpha}{\alpha'} \right|^{i\pi m / \log\left(\frac{1+\sqrt{5}}{2}\right)} \\ &= 2 \cos \left( \pi m \frac{\log(\alpha) - \log(\alpha')}{\log\left(\frac{1+\sqrt{5}}{2}\right)} \right). \end{aligned}$$

Let us choose  $m = 1$ , then we get an explicit odd Maaß form of level 5 and Fourier expansion

$$(15) \quad f_5(x + iy) = \sum_{n=1}^{\infty} a_n \sqrt{y} K_\nu(2\pi n y) \cos(2\pi n x).$$

where we list here a table of a few small nonzero values of  $a_p$ , where in reality we computed  $10^5$  coefficients to 100 digits of precision, which only takes a few seconds:

$p$	$a_p$	$p$	$a_p$
11	-0.760821475828...	61	1.341882064886...
19	-1.970563138731...	71	-0.312009656935...
29	-1.800993877371...	79	0.209316308437...
31	0.778492130648...	89	0.053978048745...
41	-1.284802072271...	101	1.592962957055...
59	-1.641914084178...	109	0.348617043298...

We get the very convincing sanity check

$$(16) \quad \begin{aligned} f_5(z) &\approx -2.2094847821266402728339875, & \text{where } z = -\frac{1}{19} + \frac{i}{17} \\ f_5\left(\frac{z}{5z+1}\right) &\approx -2.2094847821266402728339875. \end{aligned}$$

Decay at the cusps seems less strong as a check, since unlike the growth in the previous example, the Bessel function is now bounded as we approach zero. As we approach the cusps 0 and  $1/2$ , we get

$z$	$f_5\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot z\right)$	$f_5\left(\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \cdot z\right)$
$i$	$2.985861 \cdot 10^{-5}$	$8.091818 \cdot 10^{-85}$
$2i$	$-1.812735 \cdot 10^{-5}$	$-1.492931 \cdot 10^{-5}$
$3i$	$6.936199 \cdot 10^{-6}$	$5.610642 \cdot 10^{-6}$
$4i$	$3.742521 \cdot 10^{-5}$	$3.027763 \cdot 10^{-5}$
$5i$	$2.985862 \cdot 10^{-5}$	$2.415613 \cdot 10^{-5}$
$6i$	$1.573027 \cdot 10^{-5}$	$1.272606 \cdot 10^{-5}$
$7i$	$6.813508 \cdot 10^{-6}$	$5.512244 \cdot 10^{-6}$
$8i$	$2.637609 \cdot 10^{-6}$	$2.133871 \cdot 10^{-6}$
$9i$	$9.505706 \cdot 10^{-7}$	$7.690278 \cdot 10^{-7}$
$10i$	$3.262883 \cdot 10^{-7}$	$2.639728 \cdot 10^{-7}$
$11i$	$1.081678 \cdot 10^{-7}$	$8.750959 \cdot 10^{-8}$
$12i$	$3.494493 \cdot 10^{-8}$	$2.827104 \cdot 10^{-8}$

It would be nice to double check our example in the LMFDB database. This example has eigenvalue  $\lambda \approx 42.871346267056 \dots$  and since the level is 5, we are now well in the range of levels that appear in the database at the time of writing. But alas, in spite of our efforts, this form does not appear to be there yet!

**Remark 1.** For an alternative example of the second form, see Gelbart [Gel75, § C.]. He constructs one for  $\mathbf{Q}(\sqrt{2})$  and claims it has level 2, but as Buzzard points out, the level is really equal to 8. Following also the computation there, we obtain a Maaß form whose eigenvalue for the Laplacian is given by

$$(17) \quad \lambda = \frac{1}{4} + \left( \frac{\pi}{\log(1 + \sqrt{2})} \right)^2 \approx 12.9551466562$$

which once more does not appear in the LMFDB. With such a small level and spectral parameter, this surprises me a little. The database does seem to contain many forms with non-trivial nebentype, in the range  $1 \leq N \leq 10$ , and with spectral parameter bounded by 10 (which includes the above examples). Also, I found that the website often says ‘Eigenvalue’ in the table header, where really the spectral parameter is what is displayed, so be very careful with that. Nonetheless, I looked around a bit and did some more examples of the above form (which all passed very convincing invariance checks), but could not recover a single example in LMFDB. Maybe the way their data was computed is biased in some way not to pick up examples that come from Größencharakteren. The website currently says the following

‘‘It is believed that each database entry corresponds to an actual Maass form and the given decimal numbers are reasonable approximations to the true value. However, some eigenvalues may be missing. Using current methods it is not feasible to prove that the data for an individual Maass form is correct, nor that the list of Maass eigenvalues in a given range is complete.’’

It is extremely difficult to construct examples beyond the ones coming from Größencharakteren. Note that such examples are never of level 1, even though the space  $\mathcal{M}_\infty^{\text{an}}$  is already very rich there. It is my understanding that this problem was the main motivation for Selberg [Sel56] to develop his trace formula. This can also be turned into practical algorithms, see Booker–Strömbergsson–Venkatesh [BSV06, BS07].

**Remark 2.** Notice the striking contrast between the Fourier coefficients in the first example, which were all integers, and those in the second example, which are transcendental. It was shown by Sarnak [Sar02] that any Maaß cusp form with integer coefficients must be of the first type, and be attached to an even Artin representation which is conjugate to one with projective image contained in  $C_2 \times C_2, S_3$ , or  $A_4$ .

**2.2. Overconvergent forms.** Last week, Koji gave us a beautiful overview of the theory of overconvergent modular forms. To avoid repetition, I will simply recall the relevant objects from last week on one simple example. As we did for Maaß forms above, I will only discuss the case of weight zero in these notes.

Let us take  $p = 2$ , and work in level  $N = 1$ . In this case, we can be very explicit about the spaces of  $p$ -adic and  $r$ -overconvergent modular forms. We will use the Klein  $j$ -invariant

$$(18) \quad j(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

which is a level one modular form of weight zero with a simple pole at the cusp. Let  $X$  be the moduli stack of elliptic curves. Koji defined overconvergent modular forms to be the analytic functions on  $X$ , where we remove certain supersingular disks. Of the 4 values in  $\mathbf{F}_4$  for the  $j$ -invariant, only  $j = 0$  is supersingular, corresponding to the supersingular elliptic curve

$$(19) \quad y^2 + y = x^3 \quad \text{over } \mathbf{F}_2.$$

In other words, the special fibre of  $X$  at  $p = 2$  has a unique supersingular point corresponding to the vanishing locus of  $j$ . It follows that the ordinary locus on  $X$  is described by  $|j^{-1}| \leq 1$ , and hence the space of 2-adic modular forms of weight 0 which Koji introduced last week<sup>1</sup> is isomorphic to

$$(20) \quad \mathbf{C}_2\langle j^{-1} \rangle = \{a_0 + a_1j^{-1} + a_2j^{-2} + \dots \mid a_n \rightarrow 0\}.$$

**Remark.** We will be concerned mainly with spectral theory, and therefore we must pass immediately to the more sophisticated – but still explicit – notion of overconvergent forms due to Katz [Kat73] and Coleman [Col97]. However, let us not dismiss the much bigger space of  $p$ -adic modular forms as useless, and recall also that last week, Koji explained that the notion of  $p$ -adic modular forms can be put to great use, and was, by Serre [Ser72]. An old idea of Hecke [Hec24] that was developed by Klingen [Kli62] and Siegel [Sie68] aims to prove the rationality of special values of zeta functions for totally real fields  $F$  by viewing them as the constant term of a Hilbert Eisenstein series, and restricting it to the diagonal. The higher Fourier coefficients are rational, and therefore this diagonal restriction must be a rational combination of a rational basis for the space of classical modular forms containing it. The rationality of  $\zeta_F(1 - k)$  then follows, and for small values it even leads to an explicit formula for it. Serre pushed this idea much further, and deduced the  $p$ -adic analytic nature of the variation in the weight  $k$  of this constant term, from on the (elementary)  $p$ -adic analytic variation of the higher Fourier coefficients.

Recall that Koji showed us that the space of  $p$ -adic modular forms comes equipped with an action of the Hecke algebra, but that their spectral theory is not particularly interesting. Concretely, Koji showed that

<sup>1</sup>Recall that this is the set of 2-adic limits of  $q$ -expansions of classical modular forms, whose weights tend to 0.

the Hecke operator  $U_p$  defined by

$$(21) \quad U_p : \sum_n a_n q^n \mapsto \sum_n a_{np} q^n$$

has almost any eigenvalue we like, and we produced hosts of ‘random’ eigenforms for them. He resolved this (or rather, Katz did) by considering for any  $r$ , the space  $\mathcal{M}_p^{\text{an},r}$  of  $r$ -overconvergent modular forms consisting of functions on the space

$$(22) \quad X_r = X \setminus \{\text{slightly smaller supersingular disks}\}$$

where  $r$  is a measure for how small these disks are. Concretely, in this example, we can identify this space through growth conditions on the coefficients  $a_n$  appearing in (20). Note that  $j = E_4^3/\Delta$ , where  $E_4 = 1 + 240q + \dots$  is the normalised Eisenstein series of weight 4, which is a lift of the fourth power of the Hasse invariant  $A^4$  mentioned by Koji, and  $\Delta$  is the Ramanujan form of weight 12. This is important, since the Hasse invariant is a modular form that vanishes with simple zeroes at supersingular points, and therefore tells us how to parametrise the supersingular disks correctly. In particular, we find that on the supersingular disk (where  $\Delta$  is invertible, and hence  $v_2(\Delta) = 0$ , we have that

$$(23) \quad v_2(A) \leq r \iff v_2(j) \leq 12r.$$

and as a consequence, we get that the subspace of  $r$ -overconvergent forms is given by

$$(24) \quad \mathcal{M}_2^{\text{an},r} = \{a_0 + a_1 j^{-1} + a_2 j^{-2} + \dots : |a_n| p^{12nr} \rightarrow 0\}.$$

**Remark.** Whereas we can happily work with the space  $\mathcal{M}_2^{\text{an},r}$  via the description (24), a much better description is given by power series in the Hauptmodul  $h = \Delta(2z)/\Delta(z) = q \prod_{n \geq 1} (1 + q^n)^{24}$  which is a modular unit on  $X_0(2)$ . The justification for this lies in the theory of the *canonical subgroup* due to Katz [Kat73, § 3], which shows that there is a section of the forgetful map  $X_0(p) \rightarrow X$  over the region  $X_r$ . One huge advantage of doing this is apparent in the observation

$$(25) \quad \begin{array}{lll} U_2 j^{-1} & = & -744 j^{-1} - 140914688 j^{-2} + \dots = \text{Horrible power series} \\ U_2 h & = & 24h + 2048h^2 = \text{Simple polynomial} \end{array}$$

which is not surprising, since  $U_2$  is defined as a correspondence on  $X_0(2)$ . This lies at the basis of the work of Buzzard–Calegari [BC05], who find an explicit recursion for the matrix entries of  $U_2$ , and find the exact values of the 2-adic valuations of its eigenvalues. This brings us to the main topic of today’s lecture.

### 3. SPECTRAL PROPERTIES OF LAPLACIANS OVER $\mathbf{R}$ AND $\mathbf{Q}_p$

Now that we have some examples under the belt, we turn to spectral questions about Laplacians. These have been the subject of a huge amount of intensive study by generations of brilliant researchers for the spaces  $\mathcal{M}_\infty^{\text{an}}(N)$ . For the spaces  $\mathcal{M}_p^{\text{an},r}(N)$  we have only just begun our investigations. The study of *slopes* was very popular in the early 21<sup>st</sup> century, see for instance [Buz05, BC04, BC05, BP16, BG16] and the references contained therein. This domain has in recent years shifted its fashions towards the *boundary* of weight space, such as the works [BK05, Roe14, LWX17, AIP18] and many others. This is a notion that falls outside the narrative we take here, and whereas all indications are that it started off as a bit of a curiosity, it found a spectacular recent application in the proof by Newton–Thorne [NT19] of modularity of  $\text{Sym}^n(f)$  when  $f$  is a cuspidal eigenform satisfying certain conditions, including all forms of level 1.

In this talk however, we stick with weight zero, and discuss some of the analogies suggested by Blasius and Calegari. These suggest very original avenues of investigation, most of which are, as far as I am aware,



not yet explicitly addressed in any of the research literature. Therefore this is still virgin soil, which perhaps forms a good justification for our choice to illustrate these phenomena with lots of explicit data.

**Goal.** In what follows, we attempt to exhibit various analogies between the set of eigenvalues of  $\Delta$  in the discrete spectrum of  $\mathcal{M}_\infty^{\text{an}}(N)$ , and the set of  $p$ -adic valuations of the eigenvalues of  $U_p$  on  $\mathcal{M}_p^{\text{an},r}(N)$ . We also discuss some conjectural properties of the eigenfunctions in both situations.

**3.1. Spectral gap and Weyl's law.** We start by a discussion of two important aspects of the discrete spectrum of the Laplacian operator  $\Delta$  on  $\mathcal{M}_\infty^{\text{an}}$ : The Selberg 1/4 conjecture on the spectral gap, and the statistical distribution of eigenvalues inside  $\mathbf{R}_{>0}$  as made precise by Weyl's law. We then investigate the analogous statements for the spectrum of the Atkin operator  $U_p$  on  $\mathcal{M}_p^{\text{an},r}$ .

**The  $\infty$ -adic case.** The central actor here is the hyperbolic Laplacian  $\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ , acting on the space of smooth automorphic functions for a congruence subgroup  $\Gamma$ , which will henceforth usually be taken to be  $\Gamma_0(N)$  for some level  $N$ . On the subspace of bounded and smooth functions with bounded image under  $\Delta$  inside the Hilbert space  $L^2(\Gamma \backslash \mathcal{H})$ , we have

$$(26) \quad \begin{aligned} \langle \Delta f, g \rangle &= \langle f, \Delta g \rangle \\ \langle \Delta f, f \rangle &\geq 0 \end{aligned}$$

and the density of such functions implies that  $\Delta$  has a unique self-adjoint extension to  $L^2(\Gamma \backslash \mathcal{H})$ . It follows that every eigenvalue of  $\Delta$  on the space  $\mathcal{M}_\infty^{\text{an}}$  of Maaß cusp forms is of the form

$$\lambda = \frac{1}{4} - \nu^2$$

for some  $\nu$  which is either purely imaginary, or contained in the real interval  $[-1/2, 1/2]$ . The latter case is called an *exceptional* eigenvalue. One of the most celebrated conjectures in this area is the following.

**Conjecture 1** (Selberg). *Any positive eigenvalue  $\lambda$  of  $\Delta$  on  $\mathcal{M}_\infty^{\text{an}}$  is at least  $1/4$ .*

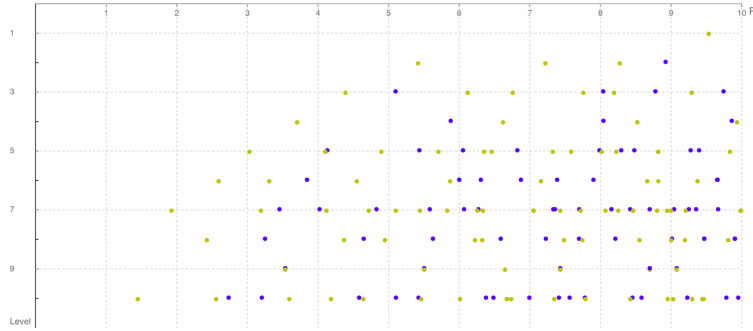
For small levels  $N$ , the smallest positive eigenvalue (sometimes called the spectral gap) is in fact much larger than that. For instance, in level one it is roughly  $\lambda \approx 91.1413$ . With so much space, it is perhaps not surprising that one can find relatively elementary estimates that are strong enough in small levels to prove Selberg's conjecture. The following can be proved elementarily, via a nice argument that uses only things we have discussed so far, incorporating ideas of Roelke [Roe56]. See Iwaniec [Iwa95, Theorem 11.4].

**Theorem 1.** *The cuspidal eigenvalues  $\lambda$  for  $\Gamma_0(N)$  satisfy the lower bound*

$$(27) \quad \lambda \geq \frac{3}{2} \left( \frac{\pi}{N} \right)^2.$$

**Remark.** Selberg's conjecture is therefore true in level at most 7. The above argument readily generalises to subgroups of finite index, where the denominator  $N$  is replaced by the maximal width of a cusp. In general, the Selberg conjecture is open, but very good bounds have been established. Initially, Selberg [Sel65] proved a lower bound of  $3/16$ , and Luo–Rudnick–Sarnak [LRS95] later improved this to  $21/100$ .

Given how difficult it seems in general to give good bounds, or to even determine the spectrum of  $\Delta$  computationally, it is perhaps more fruitful to approach the spectrum from a statistical point of view. From the LMFDB database, which is extensive (in spite of our experiences with Hecke Grössencharakteren), we obtain the following visualisation for the distribution of the spectrum in the level range  $1 \leq N \leq 10$ .



Here every dot corresponds (conjecturally) to a Maaß form, and the plot shows the spectral parameter  $R$ , which in the language we have been using is just  $R = |\nu|$ . The colour coding is blue for even Maaß forms, and yellow for odd ones. Note the gigantic spectral gap in level one, which is quite substantially larger than in higher levels. Moreover, if we count the number of Maaß forms in a very large range, and amazing regularity emerges from the set of eigenvalues, which sadly is not really reflected in the small range represented in the above picture. Selberg showed the following.

**Theorem 2** (Weyl's law). *Suppose that  $N(T) = |\{\lambda < T\}|$  is the counting function for the eigenvalues in the discrete spectrum of  $\Delta$ , then as  $T$  tends to infinity we have*

$$(28) \quad N(T) = \frac{\text{Vol } X_0(N)}{4\pi} T + O(\sqrt{T} \log T).$$

This is a landmark theorem. It has been hugely improved upon, and has analogues in various settings.

**The  $p$ -adic case.** When one attempts to parallel the above story, an immediate obstacle is the Hilbert spaces are a hard notion to come by in the non-Archimedean world. On the other hand, we may endow  $\mathcal{M}_p^{\text{an}, r}$  with the structure of a  $p$ -adic Banach space with respect to the supremum norm. A remarkable fact is that the operator  $U_p$  is the limit of operators of finite rank, which is a property that goes by the name *compactness*, see [Dwo62, Ser62, Col97]. It implies that there is a good spectral theory available, and in particular  $U_p$  has a discrete spectrum of non-zero eigenvalues

$$(29) \quad |\lambda_1| \geq |\lambda_2| \geq \dots$$

where  $|\lambda_i| \rightarrow 0$  as  $i \rightarrow \infty$ , and it gives us license to work with  $U_p$  through its infinite matrix representation with respect to some suitable basis. For instance, in the example above, where  $p = 2$  and the level is one, we may show very concretely what compactness really means in practice. If we compute the first  $10 \times 10$  submatrix with respect to the basis of Buzzard–Calegari [BC05] mentioned at the end of the last section, the 2-adic valuations of its entries are as follows:

$$(30) \quad v_2(U_2(i, j))_{i, j} = \begin{pmatrix} 3 & 8 & & & & & & & & & \\ 3 & 7 & 11 & 16 & & & & & & & \\ & 8 & 12 & 17 & 19 & 24 & & & & & \\ & & 7 & 11 & 15 & 21 & 23 & 27 & 32 & & \\ & & & 11 & 19 & 20 & 25 & 27 & 35 & 35 & \dots \\ & & & & 11 & 16 & 20 & 24 & 27 & 33 & 35 \\ & & & & & 17 & 19 & 24 & 29 & 34 & 35 \\ & & & & & & 15 & 20 & 23 & 27 & 31 & 38 \\ & & & & & & & 19 & 24 & 27 & 37 & 36 \\ & & & & & & & & \vdots & & & \ddots \end{pmatrix}$$

Here, we omitted the entries of  $U_2$  that were equal to zero. The compactness of  $U_p$  in orthonormalisable situations like this one is *equivalent* to the statement that the column vectors converge uniformly to 0 in the infinite matrix representation. In the above example, that certainly looks plausible, as the entries of the columns seem to have valuation which grows roughly at the same rate. To contrast this with an example that fails to have this property, let us compute with respect to the same basis the first  $10 \times 10$  submatrix for  $T_3$ :

$$(31) \quad v_2(T_3(i, j))_{i, j} = \begin{pmatrix} 2 & 12 & 16 & & & & & & & & \\ 7 & 2 & 11 & 20 & 27 & 32 & & & & & \\ 8 & 8 & 2 & 14 & 17 & 28 & 34 & 46 & 48 & & \\ & 11 & 8 & 2 & 12 & 19 & 29 & 36 & 43 & & \\ & 16 & 9 & 10 & 2 & 12 & 16 & 32 & 34 & \cdots & \\ & 16 & 15 & 12 & 7 & 2 & 11 & 22 & 28 & & \\ & & 18 & 19 & 8 & 8 & 2 & 16 & 18 & & \\ & & 23 & 19 & 17 & 12 & 9 & 2 & 13 & & \\ & & 24 & 25 & 18 & 17 & 10 & 12 & 2 & & \\ & & & & & \vdots & & & & & \ddots \end{pmatrix}$$

Notice the stark contrast with the matrix of  $U_2$ . Whereas the general entry of every column seems like it tends to zero (as it should, since  $T_3$  still defines an operator on  $\mathcal{M}_2^{\text{an}, r}$  after all) it does not look like the general column tends uniformly to zero. Most strikingly, the diagonal entries all seem to have valuation 2, suggesting this operator may not have a convergent “trace”.

Continuing this example, we may now easily compute an approximation for the characteristic series of  $U_2$ . One can easily analyse to which precision the given answer is correct, but we will ignore such issues here. We truncate the matrix for  $U_2$  as above, and obtain a polynomial whose coefficients are 2-adically close to those of its characteristic series. Looking at the Newton polygon, we see that the valuations of the eigenvalues of  $U_2$  on the full space  $\mathcal{M}_2^{\text{an}, r}$  (for any small value of  $r$ ) are as follows:

$$(32) \quad \mathbf{0}_1, \mathbf{3}_1, \mathbf{7}_1, \mathbf{13}_1, \mathbf{15}_1, \mathbf{17}_1, \dots$$

Here, we denote the valuations of the eigenvalues by bold type, and the multiplicity of that valuation by a subscript. In this particular example, Buzzard–Calegari [BC05] show that the  $n$ -th term in this sequence is given by the formula

$$(33) \quad 1 + 2v_2 \left( \frac{(3n)!}{n!} \right).$$

Note that every valuation in this infinite sequence is an integer, and occurs with multiplicity one. With a complete knowledge of the set of valuations of the eigenvalues of  $U_p$  – quantities referred to as *slopes* – we are in a good position to investigate the analogues of the spectral properties discussed for Maaß forms:

- The spectral gap: Note that the initial term  $\mathbf{0}_1$  in the slope sequence corresponds to the constant function, and the others are cuspidal. The spectral gap, which is the smallest positive slope, is equal to 3. This has very concrete consequences, for instance we can recover some very old congruences of Lehner. Start by observing that  $U_2 j - 744$  is holomorphic at  $\infty$ , and defines a cuspidal element in  $\mathcal{M}_2^{\text{an}, r}$  for any  $r < 2/3$ . If we expand it in terms of eigenforms<sup>2</sup>, it follows from the fact that the spectral gap is 3 that  $U_2^n(U_2 j - 744)$  is divisible by  $2^{3n+C}$  for some implicit constant, which we

<sup>2</sup>This is a subtle point in general, but in this case it is justified by Loeffler [Loe07].

may easily determine to be 11. This results in the classical congruence of Lehmer [Leh49] for the Fourier coefficients  $a_n$  of  $j(q)$ , which states that

$$a_n \equiv 0 \pmod{2^{3n+8}} \quad \text{whenever} \quad n \equiv 0 \pmod{2^n}.$$

Clearly, Lehmer did not use the same methods to establish these congruences, but they follow very nicely from the spectral gap computation we did above. This argument furthermore generalises to various other settings, and gives a systematic way to obtain such congruences.

- Weyl’s law: With such a complete description (33) of the set of slopes for this example, it now becomes possible to analyse the asymptotic behaviour of slopes very precisely. This yields the following analogue of Weyl’s law for this example

$$(34) \quad |\{\lambda : v_2(\lambda) \leq T\}| = \frac{\text{Vol } X_0(2)}{4\pi} T + O(\log(T)).$$

Whereas this example is certainly encouraging, we wonder what can be said for general primes  $p$  and levels  $N$ . The explicit determination (32) of the slope sequence by Buzzard–Calegari relies on a large number of numerical coincidences, and even then necessarily combined with a great amount of stamina and vision, all of which seem impossible to reproduce in other settings. It remains essentially the only complete description of a slope sequence in weight zero, and not for a lack of trying, see Loeffler [Loe07] for references.

One aspect in which Buzzard–Calegari were particularly lucky is that the modular curve  $X_0(2)$  has genus zero, and the overconvergent regions  $X_r$  are isomorphic to a rigid analytic disk, for which we could identify an explicit parameter. This procedure can be repeated for any prime  $p$  for which  $X_0(p)$  has genus zero (i.e. for  $p = 2, 3, 5, 7, 13$ ), see Loeffler [Loe07]. For general values of  $p$ , we are faced with a more complicated geometric picture, as the overconvergent regions  $X_r$  are isomorphic to the complement of a finite number of analytic open disks in  $\mathbf{P}^1$ , functions on which are therefore much more challenging to describe:



Moreover, in cases where we also have a nontrivial tame level  $N$ , the modular curve from which we remove these finitely many disks is no longer isomorphic to  $\mathbf{P}^1$ . Therefore, finding an explicit basis for the set of sections over the overconvergent regions  $X_r$  becomes significantly more subtle.

In his foundational paper on the subject, Katz [Kat73, Chapter 2] identifies an explicit basis for these spaces, such that any overconvergent form may be written as a unique linear combination of it, referred to as its *Katz expansion*. This, together with a precision analysis of Wan [Wan98], resulted in an explicit algorithm by Lauder [Lau11]. This algorithm assumes  $p \geq 5$ , but it is not so difficult to extend in general,

which we did. This gives us the ability to experiment in great generality. As a taste, we tabulate here the start of the slope sequences for  $p = 2$  and all levels up to 100.

$N$	$\lambda$							$N$	$\lambda$							
1	<b>0</b> <sub>1</sub>	<b>3</b> <sub>1</sub>	<b>7</b> <sub>1</sub>	<b>13</b> <sub>1</sub>	<b>15</b> <sub>1</sub>	<b>17</b> <sub>1</sub>	...	47	<b>0</b> <sub>9</sub>	<b>1</b> <sub>16</sub>	<b>2</b> <sub>4</sub>	<b>3</b> <sub>28</sub>	<b>5</b> <sub>4</sub>	<b>7</b> <sub>60</sub>	...	
3	<b>0</b> <sub>2</sub>	<b>1</b> <sub>1</sub>	<b>3</b> <sub>3</sub>	<b>7</b> <sub>5</sub>	<b>13</b> <sub>3</sub>	<b>15</b> <sub>5</sub>	...	49	<b>0</b> <sub>14</sub>	<b>1</b> <sub>16</sub>	<b>3</b> <sub>2</sub> <sub>4</sub>	<b>2</b> <sub>2</sub>	<b>3</b> <sub>34</sub>	<b>5</b> <sub>2</sub>	...	
5	<b>0</b> <sub>2</sub>	<b>1</b> <sub>1</sub>	<b>3</b> <sub>2</sub> <sub>2</sub>	<b>3</b> <sub>3</sub>	<b>6</b> <sub>2</sub>	<b>7</b> <sub>5</sub>	...	51	<b>0</b> <sub>12</sub>	<b>1</b> <sub>2</sub> <sub>4</sub>	<b>1</b> <sub>18</sub>	<b>3</b> <sub>2</sub> <sub>8</sub>	<b>3</b> <sub>38</sub>	<b>13</b> <sub>4</sub> <sub>8</sub>	...	
7	<b>0</b> <sub>3</sub>	<b>1</b> <sub>2</sub>	<b>3</b> <sub>6</sub>	<b>7</b> <sub>10</sub>	<b>13</b> <sub>6</sub>	<b>15</b> <sub>10</sub>	...	53	<b>0</b> <sub>10</sub>	<b>1</b> <sub>13</sub>	<b>3</b> <sub>2</sub> <sub>10</sub>	<b>3</b> <sub>31</sub>	<b>17</b> <sub>3</sub> <sub>3</sub>	<b>6</b> <sub>4</sub>	...	
9	<b>0</b> <sub>4</sub>	<b>1</b> <sub>3</sub>	<b>3</b> <sub>2</sub> <sub>2</sub>	<b>3</b> <sub>7</sub>	<b>6</b> <sub>2</sub>	<b>7</b> <sub>13</sub>	...	55	<b>0</b> <sub>12</sub>	<b>1</b> <sub>2</sub> <sub>4</sub>	<b>1</b> <sub>18</sub>	<b>3</b> <sub>2</sub> <sub>8</sub>	<b>3</b> <sub>44</sub>	<b>4</b> <sub>2</sub>	...	
11	<b>0</b> <sub>2</sub>	<b>1</b> <sub>2</sub> <sub>2</sub>	<b>1</b> <sub>3</sub>	<b>3</b> <sub>8</sub>	<b>4</b> <sub>1</sub>	<b>7</b> <sub>15</sub>	...	57	<b>0</b> <sub>12</sub>	<b>1</b> <sub>2</sub> <sub>8</sub>	<b>1</b> <sub>20</sub>	<b>3</b> <sub>2</sub> <sub>4</sub>	<b>3</b> <sub>46</sub>	<b>13</b> <sub>4</sub> <sub>8</sub>	...	
13	<b>0</b> <sub>4</sub>	<b>1</b> <sub>3</sub>	<b>3</b> <sub>2</sub> <sub>2</sub>	<b>3</b> <sub>9</sub>	<b>6</b> <sub>2</sub>	<b>7</b> <sub>15</sub>	...	59	<b>0</b> <sub>8</sub>	<b>1</b> <sub>2</sub> <sub>6</sub>	<b>1</b> <sub>15</sub>	<b>3</b> <sub>2</sub> <sub>4</sub>	<b>3</b> <sub>38</sub>	<b>4</b> <sub>3</sub>	...	
15	<b>0</b> <sub>6</sub>	<b>1</b> <sub>6</sub>	<b>3</b> <sub>2</sub> <sub>4</sub>	<b>3</b> <sub>14</sub>	<b>6</b> <sub>4</sub>	<b>7</b> <sub>26</sub>	...	61	<b>0</b> <sub>12</sub>	<b>1</b> <sub>15</sub>	<b>3</b> <sub>2</sub> <sub>10</sub>	<b>3</b> <sub>37</sub>	<b>17</b> <sub>3</sub> <sub>3</sub>	<b>6</b> <sub>4</sub>	...	
17	<b>0</b> <sub>4</sub>	<b>1</b> <sub>4</sub>	<b>3</b> <sub>2</sub> <sub>4</sub>	<b>3</b> <sub>10</sub>	<b>6</b> <sub>4</sub>	<b>7</b> <sub>18</sub>	...	63	<b>0</b> <sub>20</sub>	<b>1</b> <sub>24</sub>	<b>3</b> <sub>2</sub> <sub>16</sub>	<b>3</b> <sub>56</sub>	<b>6</b> <sub>16</sub>	<b>7</b> <sub>104</sub>	...	
19	<b>0</b> <sub>4</sub>	<b>1</b> <sub>2</sub> <sub>2</sub>	<b>1</b> <sub>5</sub>	<b>3</b> <sub>11</sub>	<b>13</b> <sub>4</sub> <sub>4</sub>	<b>7</b> <sub>25</sub>	...	65	<b>0</b> <sub>16</sub>	<b>1</b> <sub>20</sub>	<b>3</b> <sub>2</sub> <sub>16</sub>	<b>3</b> <sub>48</sub>	<b>6</b> <sub>16</sub>	<b>7</b> <sub>88</sub>	...	
21	<b>0</b> <sub>8</sub>	<b>1</b> <sub>8</sub>	<b>3</b> <sub>2</sub> <sub>4</sub>	<b>3</b> <sub>20</sub>	<b>6</b> <sub>4</sub>	<b>7</b> <sub>36</sub>	...	67	<b>0</b> <sub>12</sub>	<b>1</b> <sub>2</sub> <sub>2</sub>	<b>1</b> <sub>21</sub>	<b>2</b> <sub>4</sub>	<b>3</b> <sub>42</sub>	<b>4</b> <sub>1</sub>	...	
23	<b>0</b> <sub>5</sub>	<b>1</b> <sub>8</sub>	<b>2</b> <sub>2</sub>	<b>3</b> <sub>14</sub>	<b>5</b> <sub>2</sub>	<b>7</b> <sub>30</sub>	...	69	<b>0</b> <sub>18</sub>	<b>1</b> <sub>28</sub>	<b>3</b> <sub>2</sub> <sub>8</sub>	<b>2</b> <sub>4</sub>	<b>3</b> <sub>56</sub>	<b>5</b> <sub>4</sub>	...	
25	<b>0</b> <sub>8</sub>	<b>1</b> <sub>7</sub>	<b>3</b> <sub>2</sub> <sub>6</sub>	<b>3</b> <sub>17</sub>	<b>6</b> <sub>6</sub>	<b>7</b> <sub>31</sub>	...	71	<b>0</b> <sub>13</sub>	<b>1</b> <sub>24</sub>	<b>2</b> <sub>6</sub>	<b>3</b> <sub>42</sub>	<b>5</b> <sub>6</sub>	<b>7</b> <sub>90</sub>	...	
27	<b>0</b> <sub>8</sub>	<b>1</b> <sub>2</sub> <sub>2</sub>	<b>1</b> <sub>9</sub>	<b>3</b> <sub>2</sub> <sub>4</sub>	<b>3</b> <sub>19</sub>	<b>13</b> <sub>4</sub> <sub>4</sub>	...	73	<b>0</b> <sub>14</sub>	<b>1</b> <sub>22</sub>	<b>3</b> <sub>2</sub> <sub>4</sub>	<b>2</b> <sub>4</sub>	<b>3</b> <sub>44</sub>	<b>5</b> <sub>4</sub>	...	
29	<b>0</b> <sub>6</sub>	<b>1</b> <sub>7</sub>	<b>3</b> <sub>2</sub> <sub>6</sub>	<b>3</b> <sub>17</sub>	<b>6</b> <sub>6</sub>	<b>7</b> <sub>31</sub>	...	75	<b>0</b> <sub>24</sub>	<b>1</b> <sub>2</sub> <sub>4</sub>	<b>1</b> <sub>30</sub>	<b>3</b> <sub>2</sub> <sub>16</sub>	<b>3</b> <sub>72</sub>	<b>4</b> <sub>2</sub>	...	
31	<b>0</b> <sub>7</sub>	<b>1</b> <sub>10</sub>	<b>2</b> <sub>2</sub>	<b>3</b> <sub>20</sub>	<b>5</b> <sub>2</sub>	<b>7</b> <sub>40</sub>	...	77	<b>0</b> <sub>14</sub>	<b>1</b> <sub>2</sub> <sub>8</sub>	<b>1</b> <sub>24</sub>	<b>3</b> <sub>2</sub> <sub>8</sub>	<b>3</b> <sub>54</sub>	<b>13</b> <sub>4</sub> <sub>8</sub>	...	
33	<b>0</b> <sub>8</sub>	<b>1</b> <sub>2</sub> <sub>4</sub>	<b>1</b> <sub>12</sub>	<b>3</b> <sub>2</sub> <sub>4</sub>	<b>3</b> <sub>30</sub>	<b>4</b> <sub>2</sub>	...	79	<b>0</b> <sub>15</sub>	<b>1</b> <sub>24</sub>	<b>3</b> <sub>2</sub> <sub>4</sub>	<b>2</b> <sub>4</sub>	<b>3</b> <sub>48</sub>	<b>5</b> <sub>4</sub>	...	
35	<b>0</b> <sub>8</sub>	<b>1</b> <sub>2</sub> <sub>4</sub>	<b>1</b> <sub>12</sub>	<b>3</b> <sub>2</sub> <sub>4</sub>	<b>3</b> <sub>24</sub>	<b>13</b> <sub>4</sub> <sub>8</sub>	...	81	<b>0</b> <sub>22</sub>	<b>1</b> <sub>2</sub> <sub>4</sub>	<b>1</b> <sub>29</sub>	<b>3</b> <sub>2</sub> <sub>10</sub>	<b>2</b> <sub>2</sub>	<b>3</b> <sub>59</sub>	...	
37	<b>0</b> <sub>6</sub>	<b>1</b> <sub>2</sub> <sub>4</sub>	<b>1</b> <sub>9</sub>	<b>3</b> <sub>2</sub> <sub>2</sub>	<b>3</b> <sub>22</sub>	<b>13</b> <sub>4</sub> <sub>4</sub>	...	83	<b>0</b> <sub>12</sub>	<b>1</b> <sub>2</sub> <sub>6</sub>	<b>1</b> <sub>21</sub>	<b>3</b> <sub>2</sub> <sub>8</sub>	<b>3</b> <sub>52</sub>	<b>4</b> <sub>3</sub>	...	
39	<b>0</b> <sub>12</sub>	<b>1</b> <sub>14</sub>	<b>3</b> <sub>2</sub> <sub>8</sub>	<b>3</b> <sub>34</sub>	<b>6</b> <sub>8</sub>	<b>7</b> <sub>62</sub>	...	85	<b>0</b> <sub>20</sub>	<b>1</b> <sub>26</sub>	<b>3</b> <sub>2</sub> <sub>20</sub>	<b>3</b> <sub>62</sub>	<b>6</b> <sub>20</sub>	<b>7</b> <sub>114</sub>	...	
41	<b>0</b> <sub>8</sub>	<b>1</b> <sub>10</sub>	<b>3</b> <sub>2</sub> <sub>8</sub>	<b>3</b> <sub>24</sub>	<b>6</b> <sub>8</sub>	<b>7</b> <sub>44</sub>	...	87	<b>0</b> <sub>22</sub>	<b>1</b> <sub>34</sub>	<b>3</b> <sub>2</sub> <sub>12</sub>	<b>2</b> <sub>4</sub>	<b>3</b> <sub>70</sub>	<b>5</b> <sub>4</sub>	...	
43	<b>0</b> <sub>6</sub>	<b>1</b> <sub>2</sub> <sub>6</sub>	<b>1</b> <sub>11</sub>	<b>5</b> <sub>2</sub> <sub>2</sub>	<b>3</b> <sub>28</sub>	<b>4</b> <sub>1</sub>	...	89	<b>0</b> <sub>16</sub>	<b>1</b> <sub>22</sub>	<b>3</b> <sub>2</sub> <sub>16</sub>	<b>3</b> <sub>52</sub>	<b>29</b> <sub>5</sub> <sub>5</sub>	<b>6</b> <sub>6</sub>	...	
45	<b>0</b> <sub>16</sub>	<b>1</b> <sub>18</sub>	<b>3</b> <sub>2</sub> <sub>12</sub>	<b>3</b> <sub>42</sub>	<b>6</b> <sub>12</sub>	<b>7</b> <sub>78</sub>	...	91	<b>0</b> <sub>16</sub>	<b>1</b> <sub>2</sub> <sub>12</sub>	<b>1</b> <sub>28</sub>	<b>3</b> <sub>2</sub> <sub>4</sub>	<b>5</b> <sub>2</sub> <sub>4</sub>	<b>3</b> <sub>67</sub>	...	

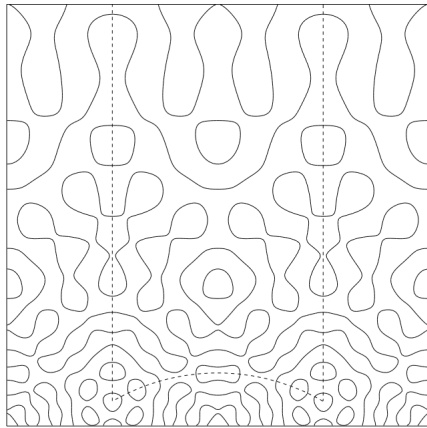
Note that the example from Buzzard–Calegari above, which corresponds to the first sequence in this table, exhibits an amusing analogy with the situation for Maaß forms, whereby there is an unseasonably large spectral gap in level one, as compared to higher levels. We conclude with a few comments about spectral properties of  $U_p$  on the spaces  $\mathcal{M}_p^{\text{an},r}(N)$ .

- Spectral gap: The smallest spectral gap we observe in this table is  $1/2$ , which remains the smallest in all experiments with higher levels or primes I have done. Note that slopes are very frequently integers. See Buzzard–Gee [BG16] for an overview of related conjectures and partial progress.
- Weyl’s law: The precise estimates of the previous example are unfortunately not known in general, though the aforementioned precision analysis of Wan [Wan98] comes very close, and implies

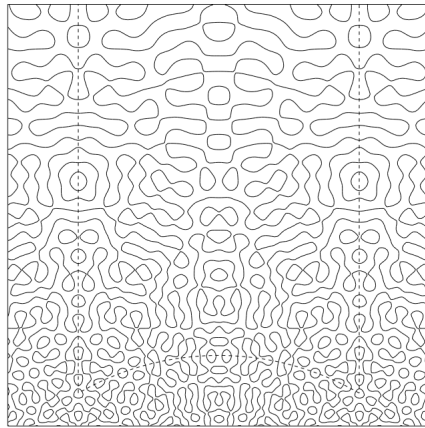
$$(35) \quad |\{\lambda : v_p(\lambda) \leq T\}| \leq \frac{\text{Vol } X_0(Np)}{4\pi} T + o(T).$$

**3.2. Nodal domains of eigenforms.** We finish with a few brief words about the zero sets of the eigenforms, specifically as the eigenvalue grows. In the case of Maaß forms, this has been intensely studied for decades, and leads to very intriguing pictures of their nodal domains, and estimates of the number of them. In the  $p$ -adic case, there are no known general results, so we focus on the toy example of Buzzard–Calegari, and do numerical experiments with other cases.

**The  $\infty$ -adic case.** Let us henceforth focus on the level one case, i.e.  $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ . Hejhal–Rackner [HR92] have produced a fascinating set of data and visualisations of the zero locus of Maaß cusp forms on this group. We include two of them here, and note that the notation  $R$  is used in *loc. cit.* to denote the spectral parameter  $|\nu|$  in our notation.



$R = 47.926558$



$R = 125.313840$

We see that on the quotient  $\mathrm{SL}_2(\mathbf{Z}) \backslash \mathcal{H}$  the zero set consists of a finite union of real analytic curves. The complement of this zero set naturally breaks up into a disjoint union of connected components, which are called *nodal domains*. This is a subject that has been very intensely studied since the appearance of Hejhal–Rackner [HR92]. For the state of the art, see Ghosh–Reznikov–Sarnak [GRS13, GRS17].

We wish to highlight here two of its features, namely the asymptotic behaviour of the *number* of nodal domains, and the equidistribution of the zero locus. The eigenvalues in the discrete spectrum may be arranged according to increasing size, and if any multiplicities should arise<sup>3</sup>, we sort them in an arbitrary way. For any eigenfunction  $\phi$ , we denote  $n_\phi$  for its numbering in this list.

- The first question is simply to determine, as  $n_\phi$  grows, the number of nodal domains. This is an open question, and it is expected that the asymptotics should be predicted by the Bogomolny–Schmit conjecture, which states that

$$(36) \quad |\{\text{Nodal domains for } \phi\}| \sim \frac{3\sqrt{3}-5}{\pi} n_\phi, \quad \text{as } n_\phi \rightarrow \infty.$$

A very general bound of Courant [CH53] shows that the number of nodal domains is at most  $n_\phi$ . Establishing a lower bound that still goes to infinity with  $n_\phi$  is difficult (but possible), and one needs a good method to produce nodal domains, see Ghosh–Reznikov–Sarnak [GRS13].

<sup>3</sup>It is presumed that the multiplicity is always one, but to the best of my limited knowledge this is an open question.

- The second question is how the zero set distributes as  $n_\phi \rightarrow \infty$ . Looking at the above picture, we may guess that it equidistributes on the modular curve. There is substantial numerical evidence for this, but it remains an open problem. For holomorphic modular forms the statement is known [Rud05], and the mass of cuspidal Maaß eigenforms is also known to equidistribute [HS10].

**The  $p$ -adic case.** We now turn to the case of overconvergent modular forms, and consider the eigenfunctions in the space  $\mathcal{M}_p^{\text{an},r}$  for the full Hecke algebra. A fair question is what the analogue should be of the notion of nodal domain. Let us go out on a limb, and work under the assumption that the *number of zeroes* of an eigenfunction  $\phi$  is the right analogue of its number of nodal domains. Do we still expect a Courant bound, where  $\phi$  has at most  $n_\phi$  zeroes?

First, let us explore the paradise that is the Buzzard–Calegari example, where we can be completely explicit. We know that the cuspidal slope sequence is

$$\mathbf{3}_1, \mathbf{7}_1, \mathbf{13}_1, \mathbf{15}_1, \mathbf{17}_1, \dots = \left\{ 1 + v_2 \left( \frac{(3n)!}{n!} \right) \right\}_{n=1,2,\dots}$$

all with multiplicity one, and therefore there is no ambiguity on how to sort the corresponding eigenfunctions  $\phi_i$ . In this special case, not only the slopes are known, but also a completely explicit pair of matrices that give us the analogue of a singular value decomposition for the matrix of  $U_p$ . These mysterious matrices were found by Buzzard–Calegari for  $r = 1/2$ , and Loeffler [Loe07] for  $5/12 < r < 7/12$ , and their properties imply that the eigenfunction  $\phi$  has *exactly*  $n_\phi$  zeroes in  $X_r$ . In other words, the Courant bound does hold, and becomes an equality! Such an exact description is of course much more than we could ever hope for in the Maaß form case. This answers the question on the number of zeroes, but it does not tell us where these zeroes are. Any cuspidal eigenfunction vanishes simply at the cusp, where  $r = 0$ . We computed the values of  $r$  at which the other zeroes occur:

$n_\phi$	$r(\text{zeroes})$	$n_\phi$	$r(\text{zeroes})$
1	$\mathbf{0}_1$	9	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{1}/\mathbf{3}_5, \mathbf{5}/\mathbf{12}_2$
2	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1$	10	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{7}/\mathbf{24}_2, \mathbf{1}/\mathbf{3}_2, \mathbf{3}/\mathbf{8}_4$
3	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{1}/\mathbf{3}_1$	11	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{31}/\mathbf{96}_8, \mathbf{1}/\mathbf{3}_1$
4	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{7}/\mathbf{24}_2$	12	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{7}/\mathbf{24}_2, \mathbf{5}/\mathbf{16}_4, \mathbf{1}/\mathbf{3}_4$
5	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{1}/\mathbf{3}_1, \mathbf{5}/\mathbf{12}_2$	13	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{31}/\mathbf{96}_8, \mathbf{1}/\mathbf{3}_1, \mathbf{5}/\mathbf{12}_2$
6	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{7}/\mathbf{24}_2, \mathbf{1}/\mathbf{3}_2$	14	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{7}/\mathbf{24}_2, \mathbf{1}/\mathbf{3}_{10}$
7	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{1}/\mathbf{3}_5$	15	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{1}/\mathbf{3}_{13}$
8	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{7}/\mathbf{24}_2, \mathbf{5}/\mathbf{16}_4$	16	$\mathbf{0}_1, \mathbf{1}/\mathbf{4}_1, \mathbf{7}/\mathbf{24}_2, \mathbf{5}/\mathbf{16}_4, \mathbf{31}/\mathbf{96}_8$

Here are a few tantalising observations about this Buzzard–Calegari case.

- The individual eigenfunctions exhibit a great deal of congruences with each other, in such a way that suggests very strongly that the function

$$\mathbf{N} \longrightarrow \mathbf{Z}_2[[q]] : n \longmapsto \phi_n$$

is continuous. A reflection of this can be seen in the above picture, in the relative similarity of sequences as the index numbers are 2-adically close. A really striking observation by Calegari is that experimentally, it seems that

$$\phi_{2^i} \longrightarrow \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \left( \sum_{d|n} d^{-1} \right) q^n,$$

which is an overconvergent modular form of *infinite slope* on the regions  $X_r$  for any  $r < 1/3$ , but for no larger  $r$ . Since it is completely explicit, one can show directly via a short calculation that it has infinitely many zeroes, all of which are located on the “boundary”  $r = 1/3$ .

- In reality, we computed the first 70 eigenfunctions, and determined the values of  $r$  at which its zeroes occur. This gives us very strong evidence for the statement that the zeroes accumulate around the  $p$ -adic unit circle  $r = 1/3$  in the sense that

$$\frac{1}{n} \sum_{\phi_n(x)=0} r(x) = \frac{1}{3} \quad \text{as } n \rightarrow \infty$$

This is also in line with the previous observation.

After these observations, it would seem interesting to leave the comforts of the Buzzard–Calegari example, and see what remains true in general. We have done computations of the above sort for any genus zero prime  $p = 2, 3, 5, 7, 13$  and observed similar phenomena. To give a flavour, here is a table for  $p = 7$ :

$n_\phi$	$r(\text{zeroes})$	$n_\phi$	$r(\text{zeroes})$	$n_\phi$	$r(\text{zeroes})$
1	$\mathbf{0}_1$	12	$\mathbf{0}_4, 1/8_8$	23	$\mathbf{0}_1, 1/8_{20}, 1/4_2$
2	$\mathbf{0}_2$	13	$\mathbf{0}_1, 1/12_6, 1/8_4, 1/4_2$	24	$\mathbf{0}_4, 3/20_{20}$
3	$\mathbf{0}_1, 1/4_2$	14	$\mathbf{0}_2, 1/12_{12}$	25	$\mathbf{0}_1, 1/10_{20}, 1/8_4$
4	$\mathbf{0}_4$	15	$\mathbf{0}_1, 1/12_6, 1/8_4, 1/4_4$	26	$\mathbf{0}_2, 1/12_{12}, 1/8_{12}$
5	$\mathbf{0}_1, 1/8_4$	16	$\mathbf{0}_4, 1/8_8, 1/4_4$	27	$\mathbf{0}_1, 1/12_6, 3/28_{14}, 1/8_4, 1/4_2$
6	$\mathbf{0}_2, 1/8_4$	17	$\mathbf{0}_1, 1/12_6, 1/8_4, 1/4_6$	28	$\mathbf{0}_4, 5/48_{24}$
7	$\mathbf{0}_1, 1/12_6$	18	$\mathbf{0}_2, 1/8_{16}$	29	$\mathbf{0}_1, 12_6, 3/28_{14}, 1/8_4, 1/4_4$
8	$\mathbf{0}_4, 1/8_4$	19	$\mathbf{0}_1, 1/12_6, 1/6_{12}$	30	$\mathbf{0}_2, 1/12_{12}, 1/8_{12}, 1/4_4$
9	$\mathbf{0}_1, 1/12_6, 1/4_2$	20	$\mathbf{0}_4, 1/8_4, 1/6_{12}$	31	$\mathbf{0}_1, 1/10_{20}, 1/8_4, 1/4_6$
10	$\mathbf{0}_2, 1/8_8$	21	$\mathbf{0}_1, 12_6, 1/7_{14}$	32	$\mathbf{0}_4, 1/8_{28}$
11	$\mathbf{0}_1, 1/12_6, 1/8_4$	22	$\mathbf{0}_2, 1/8_{20}$	33	$\mathbf{0}_1, 1/8_{32}$

I computed as far as the 49-th eigenfunction, which had zeroes with  $r$ -values  $\mathbf{0}_1, 1/12_6, 5/42_{42}$ . Then I also computed lots of examples for other primes, and hope to do more if I don’t lose my motivation after this talk. If anyone wants to join, please get in touch. An interesting case is  $p = 11$ , with two supersingular  $j$ -invariants 0 and 1728, so the domains  $X_r$  are isomorphic to annuli.

## REFERENCES

- [AIP18] F. Andreatta, A. Iovita, and V. Pilloni. Le halo spectral. *Ann. Sci. ENS*, 4(51):603–655, 2018. ↑8.
- [BC04] K. Buzzard and F. Calegari. A counterexample to the Gouvêa–Mazur conjecture. *C. R. Math. Acad. Sci. Paris*, 338:751–753, 2004. ↑8.
- [BC05] K. Buzzard and F. Calegari. Slopes of overconvergent 2-adic modular forms. *Compositio Math.*, 141(3):591–604, 2005. ↑1, 8, 10, 11.
- [BG16] K. Buzzard and T. Gee. Slopes of modular forms. In *Families of automorphic forms and the trace formula*, Simons Symp., pages 93–109. Springer-Verlag, 2016. ↑8, 13.
- [BK05] K. Buzzard and L. J. P. Kilford. The 2-adic eigencurve at the boundary of weight space. *Compositio Math.*, 141(3):605–619, 2005. ↑8.
- [BP16] J. Bergdall and R. Pollack. Arithmetic properties of Fredholm series for  $p$ -adic modular forms. *Proc. London Math. Soc.*, 113(4):419–444, 2016. ↑8.
- [BS07] A. Booker and A. Strömbergsson. Numerical computations with the trace formula and the selberg conjecture. *J. Reine Angew. Math.*, 607:113–161, 2007. ↑7.



- [BSV06] A. Booker, A. Strömbergsson, and A. Venkatesh. Effective computation of Maass cusp forms. *Int. Math. Res. Not.*, page 34 pp., 2006. ↑7.
- [Buz05] K. Buzzard. Questions about slopes of modular forms. *Astérisque*, 298, 2005. ↑8.
- [CH53] R. Courant and D. Hilbert. *Methods of mathematical physics*. Interscience Publishers, Inc., New York, 1953. ↑14.
- [Col97] R. Coleman.  $p$ -Adic Banach spaces and families of modular forms. *Invent. Math.*, 127:417–479, 1997. ↑7, 10.
- [Dwo62] B. Dwork. On the zeta function of a hypersurface. *Publ. Math. IHÉS*, (12):5–68, 1962. ↑10.
- [Gel75] S. Gelbart. *Automorphic forms on adèle groups*. Number 83 in Annals of Math. Studies. Princeton University Press, 1975. ↑6.
- [GRS13] A. Ghosh, A. Reznikov, and P. Sarnak. Nodal domains of Maass forms I. *Geom. Funct. Anal.*, 23(5):1515–1568, 2013. ↑14.
- [GRS17] A. Ghosh, A. Reznikov, and P. Sarnak. Nodal domains of Maass forms II. *Amer. J. Math.*, 139(5):1395–1447, 2017. ↑14.
- [Hec24] E. Hecke. Analytische Funktionen und Algebraische Zahlen. Zweiter Teil. *Abhandlungen aus dem Mathematischen Universitäts*, 3:213–236, 1924. ↑7.
- [HR92] D. Hejhal and B. Rackner. On the topography of Maass waveforms for  $\mathrm{PSL}(2, \mathbf{Z})$ . *Experiment. Math.*, 1(4):275–305, 1992. ↑14.
- [HS10] R. Holowinsky and K. Soundararajan. Mass equidistribution for Hecke eigenforms. *Ann. of Math. (2)*, 172(2):1517–1528, 2010. ↑15.
- [Iwa95] H. Iwaniec. *Introduction to the spectral theory of automorphic forms*. Biblioteca de la Revista Matemática Iberoamericana, 1995. ↑9.
- [Kat73] N. Katz.  $p$ -Adic properties of modular schemes and modular forms. In P. Deligne and W. Kuyk, editors, *Modular forms in one variable III*, volume 350 of *LNM*, pages 69–190. Springer-Verlag, 1973. ↑7, 8, 12.
- [Kli62] H. Klingens. über die Werte der Dedekindschen Zetafunktion. *Math. Ann.*, 145:265–272, 1962. ↑7.
- [Lau11] A. Laufer. Computations with classical and  $p$ -adic modular forms. *LMS J. Comp. Math.*, 14:214–231, 2011. ↑1, 12.
- [Leh49] J. Lehner. Further congruence properties of the fourier coefficients of the modular invariant  $j(\tau)$ . *Amer. J. Math.*, 71:373–386, 1949. ↑12.
- [Loe07] D. Loeffler. Spectral expansions of overconvergent modular functions. *Int. Math. Res. Not.*, 16, 2007. ↑11, 12, 15.
- [LRS95] W. Luo, Z. Rudnick, and P. Sarnak. On Selberg’s eigenvalue conjecture. *Geom. Funct. Anal.*, 5(2):387–401, 1995. ↑9.
- [LWX17] R. Liu, D. Wan, and L. Xiao. The eigencurve over the boundary of weight space. *Duke Math. J.*, 166(9):1739–1787, 2017. ↑8.
- [Maa49] H. Maass. Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. *Math. Ann.*, 121:141–183, 1949. ↑3, 4.
- [NT19] J. Newton and J. Thorne. Symmetric power functoriality for holomorphic modular forms. *ArXiv preprint*, 2019. ↑8.
- [Roe56] W. Roelcke. Über die Wellengleichung bei Grenzkreisgruppen erster Art. *S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl.*, pages 159–267, 1956. ↑9.
- [Roe14] D. Roe. The 3-adic eigencurve at the boundary of weight space. *Int. J. Number Theory*, 10(7):1791–1806, 2014. ↑8.
- [Rud05] Z. Rudnick. On the asymptotic distribution of zeros of modular forms. *Int. Math. Res. Not.*, (34):2059–2074, 2005. ↑15.
- [Sar02] P. Sarnak. Maass cusp forms with integer coefficients. In *A panorama of number theory from Baker’s garden*, pages 121–127. Cambridge University Press, 2002. ↑7.
- [Sel56] A. Selberg. Harmonic analysis and discontinuous groups in weakly symmetric riemannian spaces with applications to dirichlet series. *J. Indian Math. Soc.*, 20:47–87, 1956. ↑7.
- [Sel65] A. Selberg. On the estimation of fourier coefficients of modular forms. *Proc. Sympos. Pure Math.*, VIII:1–15, 1965. ↑9.
- [Ser62] J.-P. Serre. Endomorphismes complètement continus des espaces Banach  $p$ -adiques. *Publ. Math. IHÉS*, 12:69–85, 1962. ↑10.
- [Ser72] Jean-Pierre Serre. Propriétés galoisiennes des points d’ordre fini des courbes elliptiques. *Invent. Math.*, 15(4):259–331, 1972. ↑7.
- [Sie68] C.-L. Siegel. Berechnung von Zetafunktionen an ganzzahligen Stellen. *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Klasse*, 2:7–38, 1968. ↑7.
- [Wan98] D. Wan. Dimension variation of classical and  $p$ -adic modular forms. *Invent. Math.*, 133:449–463, 1998. ↑12, 13.

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