The Local Langlands Conjectures for $n = 1, 2$

Chris Nicholls

December 12, 2014

1 Introduction

These notes are based heavily on Kevin Buzzard's excellent notes on the Langlands Correspondence. The aim is to give an overview of the following theorem (which is stated before explaining any of the terms!).

Theorem (Local Langlands Correspondence). *Let K be a non-Archimedean local field of characteristic 0, and $n > 0$ an integer. There is a canonical bijection between admissible irreducible complex representations of $GL_n(K)$ and n-dimensional complex F-semi-simple Weil–Deligne representations.*

We will only talk about the cases $n = 1$ and 2, and essentially nothing will be proved. The case $n = 1$ of the correspondence will be motivated from local class field theory, and the case $n = 2$ will be illustrated through examples. Later we will introduce parameters on both sides of this correspondence, and then by canonical it will be meant that these match up.

1.1 Notation

We will normally let K be a non-Archimedean local field of characteristic 0 (i.e. a finite extension of $\mathbb{Q}_p$), with ring of integers $\mathcal{O}$, unique maximal ideal $\mathfrak{p}$, and residue field $k = \mathcal{O}/\mathfrak{p}$.

2 Admissible irreducible representations for $\mathbf{GL}_n(K)$

Definition. *A complex* admissible representation *of $GL_n(K)$ is a complex vector space V equipped with an action of $GL_n(K)$, i.e. a group homomorphism $\pi : GL_n(K) \to Aut_{\mathbb{C}}(V)$, such that*

1. *if $U \subseteq GL_n(K)$ is an open subgroup, then V^U, the set of vectors of V fixed by every $u \in U$ is a finite-dimensional vector space. This is admissibility.*
2. *if $v \in V$ then the stabiliser of v in $GL_n(K)$ is open. This is normally called smoothness.*

We further call the admissible representation irreducible *if V is nonzero, but the only stable subspaces are* 0 *and V.*

2.1 $n = 1$

If $n = 1$, then an irreducible admissible representation (π, V) of $\mathrm{GL}_1(K) = K^\times$ is one-dimensional, by Schur's lemma (since $K^\times$ is commutative). We therefore have a character

$$\pi : K^\times \to \mathbb{C}^\times.$$

The admissibility condition translates into having open kernel, which means that $1 + \varpi^n \mathcal{O}$ is contained in the kernel for some $n > 0$.

2.2 $n = 2$

When $n = 2$ we have spent a lot of time building up a theory of all admissible irreducible representations of $\mathrm{GL}_2(K)$ (away from residue characteristic 2). Here is what we know:

- *Principal series (infinite-dimensional case).* If χ_1, χ_2 are two admissible complex characters of $K^\times$ (i.e. admissible as representations), such that χ_1/χ_2 is not equal to the norm character or its inverse, then we can define an admissible irreducible infinite-dimensional representation

$$PS(\chi_1, \chi_2) : \mathrm{GL}_2(K) \to \mathrm{GL}(V).$$

- *Principal series (one-dimensional case).* In some sense these correspond to the case above where χ_1/χ_2 is the norm character or its inverse. Let χ be an admissible complex character of $K^\times$, then we get an admissible irreducible representation of $\mathrm{GL}_2(K)$ by composing with the determinant map

$$\chi \circ \det : \mathrm{GL}_2(K) \to \mathbb{C}^\times.$$

- *Special representations (or Steinberg representations).* If χ is an admissible character of $K^\times$, then we can define an admissible irreducible character

$$S(\chi) : \mathrm{GL}_2(K) \to \mathrm{GL}(V).$$

- *Supercuspidal representations.* If we avoid $p = 2$, then these can be described as follows. Choose L/K a quadratic field extension and let τ be the nontrivial element in the Galois group. Then let χ an admissible character of $L^\times$ with $\chi \neq \chi^\tau$. We can define a supercuspidal ("base change") representation of $\mathrm{GL}_2(K)$ from χ. Two such $\mathrm{BC}(L/K, \chi)$ and $\mathrm{BC}(L'/K, \chi')$ are isomorphic if and only if the induced representations $\mathrm{Ind}_{W_L}^{W_K}(\chi)$ and $\mathrm{Ind}_{W'_L}^{W_K}(\chi')$ are isomorphic.

If $p = 2$ then there are more – called extraordinary. We don't talk about them here.

Our aim is now to discuss the Weil group side of the Langlands Correspondence.

3 Weil–Deligne representations

We start with a brief overview of the Weil group.

3.1 The Weil Group W_K

As before, let K be a non-Archimedean local field of characteristic 0 (so a finite extension of $\mathbb{Q}_p$). Let $\mathcal{O}$ denote the ring of integers and $\mathfrak{p}$ a maximal ideal. Then the residue field is a finite extension of $\mathbb{F}_p$, of order $q = p^n$ for some n. We would like to relate the absolute Galois group of K to the absolute Galois group of its residue field. There is a natural map $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ defined as follows. Given an automorphism $\sigma \in \mathrm{Gal}(\overline{K}/K)$, we see that $\sigma(\mathfrak{p}) = \mathfrak{p}$, since σ has to send a prime ideal to a prime ideal. Thus σ descends to an automorphism of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. Let I_K be the kernel of this map, called the *inertia group*; then there is a short exact sequence

$$1 \to I_K \to \mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q).$$

By definition, the absolute Galois group of the residue field is an inverse limit

$$\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \varprojlim_{n \geq 1} \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q).$$

Here the limit is over all n with $\mathbb{F}_{q^n}$ a Galois extension of $\mathbb{F}_q$. Moreover, $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is cyclic of order n generated by the Frobenius automorphism $a \mapsto a^q$. So we see that the residue field has absolute Galois group

$$\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \varprojlim_{n \geq 1} \mathbb{Z}/n\mathbb{Z} =: \hat{\mathbb{Z}}.$$

We can finally introduce the Weil group W_K as the subgroup of $\mathrm{Gal}(\overline{K}/K)$ of elements mapping into the copy of $\mathbb{Z}$ in $\hat{\mathbb{Z}}$. Such elements are powers of the Frobenius automorphism. This clearly contains I_K (which is the set of all elements mapping to 0), so we get the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_K & \longrightarrow & W_K & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I_K & \longrightarrow & \mathrm{Gal}(\overline{K}/K) & \longrightarrow & \hat{\mathbb{Z}} & \longrightarrow & 1 \end{array}$$

3.2 Local class field theory

Many of the references seem to just suggest that certain results hold "by local class field theory", so we will content ourselves with just a sketch here. The following theorem is from [JH06] Section 29.

Theorem (Local class field theory)**.** *There is a canonical continuous group homomorphism*

$$\theta_K : W_K \to K^\times$$

such that

1. *The map θ_K induces a topological isomorphism $W_K^{ab} \cong K^\times$.*

2. *An element $x \in W_K$ is a geometric Frobenius if and only if $\theta_K(x)$ is a uniformiser of K.*

3. *We have that $\theta_K(I_K) = \mathcal{O}^\times$.*

The homomorphism θ_K is called the *Artin reciprocity map*. Note that a geometric Frobenius element is one that is sent to the inverse of the Frobenius element in the absolute Galois group of the residue field. An important consequence is that we have an isomorphism

$$\chi \mapsto \chi \circ \theta_K,$$

of the group of characters of $K^\times$ with that of W_K. This statement is precisely the Local Langlands Correspondence for $n = 1$. But we haven't yet defined Weil–Deligne representations, so this will become clear in the next section!

In fact the last statement in the theorem says that it maps unramified characters of $K^\times$ to unramified characters of W_K.

3.3 Weil–Deligne representations

Now that we have a description of the Weil group, let us see what an n-dimensional complex F-semi-simple Weil–Deligne representation is. Let K be a finite extension of $\mathbb{Q}_p$. We give the Weil group W_K the topology such that the subgroup I_K of W_K is open with its profinite topology.

Definition. *A Weil–Deligne representation of K is a pair (ρ, N) where*

1. *$\rho : W_K \to GL_n(\mathbb{C})$ is continuous with respect to the discrete topology on $GL_n(\mathbb{C})$;*

2. *$\rho(\sigma)N\rho(\sigma)^{-1} = |\sigma|^{-1}N$, where N is a nilpotent n by n matrix, and $|\sigma| := q^{-v(\sigma)}$. Here q is the order of the residue field of K and v is the valuation map.*

4 Langlands Correspondence for $n = 2$

The Local Langlands Correspondence says for $n = 2$ that there is a bijection between isomorphism classes of admissible irreducible complex representations of $\mathrm{GL}_2(K)$, and isomorphism classes of 2-dimensional complex F-semisimple Weil–Deligne representations of W_K.

Let $|\cdot|^{1/2}$ denote the character which is the root of the norm character, i.e. $|\sigma|^{1/2} = q^{-v(\sigma)/2}$, where q is the order of the residue field of K and v is the valuation on K.

- For the 1-dimensional principal series, we have the associated Weil–Deligne representation

$$\chi \circ \det \longleftrightarrow \chi|\cdot|^{1/2} \oplus \chi|\cdot|^{-1/2},$$

  with $N = 0$. Recall that χ is an admissible character of $K^\times$.

- To the infinite-dimensional principal series, we have $N = 0$ also and

$$PS(\chi_1, \chi_2) \longleftrightarrow \chi_1 \oplus \chi_2.$$

  Here χ_1, χ_2 are two admissible representations with χ_1/χ_2 not the norm character, or its inverse. Note that the principal series have similar looking Weil–Deligne representations.

- To the special representation we have the association

$$S(\chi) \longleftrightarrow \chi \oplus \chi|\cdot|,$$

  with $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. This is the only case with N nonzero.

- To the "base change" representation we associate

$$BC(L/K, \chi) \leftrightarrow \mathrm{Ind}_{W_L}^{W_K}(\chi),$$

  which is induced from the 1-dimensional representation of W_L which corresponds to χ, and $N = 0$.

- The extraordinary supercuspidal admissible representations that we didn't talk about much before correspond to a Weil–Deligne representation with $N = 0$, but with more complicated image than before. These do not arise if $p \neq 2$.

Note that principal series correspond to semi-simple but reducible Weil–Deligne representations with $N = 0$, special representations correspond to semi-simple reducible Weil–Deligne representations with $N \neq 0$, and supercuspidal representations correspond to irreducible Weil–Deligne representations.

Chapter 4.9 of [Bum98] essentially talks about the correspondence between the irreducible representations of the Weil group W_K and the supercuspidal $\mathrm{GL}_2(K)$ admissible irreducible representations.

5 Canonical Bijection

We would now like to say something about what the word 'canonical' means in this bijection.

5.1 Conductors

5.1.1 Weil–Deligne representations

For Weil–Deligne representations we mainly follow [Ulm13].

Ramification groups Let E/K be a finite extension of fields, where K is a non-Archimedean local field of characteristic zero. To define the conductor of a representation $\rho : \mathrm{Gal}(E/K) \to \mathrm{GL}_n(\mathbb{C})$, it is necessary to introduce *ramification groups* of $G = \mathrm{Gal}(E/K)$. These are defined as follows.

The ramification groups $G_{-1}, G_0, G_1, \ldots$ are defined by

$$\sigma \in G_i \iff v_E(\sigma(x) - x) \geq i + 1, \forall x \in \mathcal{O}_K.$$

We have that $G_{-1} = G$, G_0 is the inertia subgroup of G, and for all sufficiently large i we have $G_i = 0$.

For representations of Gal(E/K) Given a representation $\rho : \mathrm{Gal}(E/K) \to \mathrm{GL}_n(\mathbb{C})$, we define the conductor of ρ as

$$a(\rho) := \sum_{i=0}^{\infty} \frac{\mathrm{codim} V^{G_i}}{[G_0 : G_i]}.$$

Here, if H is a subgroup of G, then $V^H = \{v \in V \mid \rho(h)(v) = v, \forall v \in H\}$ is the H-fixed subspace of V.

For Weil–Deligne representations We notice that this definition of conductor does not depend on N. Suppose (ρ, N) is a Weil–Deligne representation, then we define its conductor to be

$$a(\rho, N) := a(\rho) + \dim V^{I_K} - \dim dimV_N^{I_K}.$$

Here V_N is the kernel of N on V, so that

$$V_N^{I_K} = \{v \in V \mid N(v) = 0, \rho(w)(v) = v, \forall w \in I_K\}.$$

5.1.2 Representations of $\mathbf{GL}_2(K)$

In [Buz98], there is an ad hoc definition of conductor for admissible representations of $\mathrm{GL}_n(K)$ for $n = 1, 2$.

$n = 1$ In this case, we define a decreasing sequence of subgroups of $K^\times$ by

$$V(O) = \mathcal{O}^\times, \qquad\qquad V(t) = 1 + (\varpi)^t$$

for each positive integer t.

Definition. *Let χ be an admissible character of $K^\times$. Then the* conductor *of χ, denoted $a(\chi)$ in these notes, is the smallest integer t such that χ is trivial on $V(t)$.*

$n = 2$ We deal with everything case by case here.

- *1-dimensional principal series.* Here the representation π factors through the determinant as $\pi = \chi \circ \det$, for some admissible character χ of $K^\times$. Define

$$a(\pi) := 2a(\chi).$$

- *Infinite-dimensional principal series.* Then $\pi = PS(\chi_1, \chi_2)$ for some admissible characters χ_1, χ_2. Define

$$a(\pi) := a(\chi_1) + a(\chi_2).$$

- *Special representation.* If $\pi = S(\chi)$, then define

$$a(\pi) := \begin{cases} 1, & \text{if} \chi \text{is unramified} \\ 2a(\chi), & \text{if} \chi \text{is ramified.} \end{cases}$$

- *Supercuspidal.* If $\pi = BC(\chi)$, where χ is an admissible character of a quadratic extension L/K of K, then

$$a(\pi) := \begin{cases} 2a(\chi), & \text{if} L/K \text{is unramified} \\ a(\chi) + a(\rho), & \text{if} \chi \text{is unramified,} \end{cases}$$

where ρ is the quadratic character of the absolute Galois group of K corresponding to L/K.

Again, things are more complicated if $p = 2$, so we avoid this case.

We could also have defined the conductor as follows, for any infinite-dimensional irreducible admissible representation of $\mathrm{GL}_2(K)$. For any t a non-negative integer, let

$$U_1(t) = \{g \in \mathrm{GL}_2(\mathcal{O}) \mid g \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \mod \varpi^t\}.$$

Theorem. *There is a minimal non-negative integer t such that $V^{U_1(t)}$ is non-zero.*

Define the conductor of the representation to be this minimal t.

5.2 The automorphic representations associated to newforms

Given an eigenform f of weight k on $S_k(\Gamma_1(N))$, we can associate to f an admissible irreducible representation $V_{f,p}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ for all primes p. This was done last week. An important thing to keep in mind is that f has a character, and we will use this throughout the next section, denoted by χ.

Proposition. *If f is a newform of level N, then the conductor of $V_{f,p}$ is t, where p^t exactly defines N.*

Here, Local Langlands gives an $F - semi - simple$ complex Weil–Deligne representation of $W_{\mathbb{Q}_p}$, and one can show that this is the same as the restriction to the local Galois group of the global representation constructed by Deligne. But we haven't talked about any of that here.

We now investigate $V_{f,p}$ using the conductor.

- If $a(V_{f,p}) = 0$, so that p does not divide the level of f, then $V_{f,p}$ is principal series of infinite dimension, and both of its characters are unramified. They can be specified by their value on a uniformiser, and these values are the two roots of $X^2 - a_p X + \chi(p)p^{k-1}$, where k is the weight and χ is the character of f.

- If $a(V_{f,p}) = 1$, then there are two cases. (a) If $a(\chi)$ is coprime to p, then $V_{f,p}$ is special; (b) otherwise $V_{f,p}$ is a principal series representation associated to two characters; an unramified one χ_1 such that $\chi_1(\varpi) = a_p$, and a tamely ramified one.

- If $a(V_{f,p}) > 1$, then things are more complicated.

5.3 Buzzard's recipe for elliptic curves

Now that we have defined the conductor, let us see what it gives us for elliptic curves.

Suppose we are given a modular elliptic curve E coming from a modular form f. We can ask how the possible local behaviours of E at a prime p correspond to the admissible representation $V_{f,p}$.

- $V_{f,p}$ is unramified principal series if and only if E has good reduction at p. This is because these are the only ones with conductor 0.

- $V_{f,p}$ is special if and only if E has potentially multiplicative reduction at p. This is because these are the only cases having image of inertia in the associated Galois representation infinite.

- A subcase of the former: $V_{f,p}$ is special associated to an unramified character if and only if E has multiplicative reduction. This is because these are the only ones with conductor 1.

- $V_{f,p}$ is ramified principal series or supercuspidal if and only if E has bad, but potentially good, reduction. These are the only cases left.

- To distinguish them, $V_{f,p}$ is ramified principal series if and only if E attains good reduction over an abelian extension of $\mathbb{Q}_p$.

5.4 L-functions and ϵ-factors

We now introduce the L-functions and ϵ-factors which are also part of the canonical bijection. They are a pair of invariants which will help to characterise the representation. It seems that the L-functions are much more easily described than the ϵ-factors, so we do that first.

It will turn out that $L(s,\pi)$ is an elementary function of the form $f(q^{-s})^{-1}$, where $f(t)$ is a complex polynomial of degree at most two. The *local constant* $\epsilon(s,\pi,\psi)$ is of the form cq^{-ms} for a nonzero constant c and integer m. Defining these properly would involve deriving a functional equation involving the L-function, which is all done in [JH06] Ch. 6.

The Converse Theorem tells us that an irreducible admissible representation π of G is determined, up to equivalence, by $\chi \mapsto L(s,\chi\pi)$ and $\chi \mapsto \epsilon(s,\chi\pi,\psi)$ where χ ranges over all characters of $F^\times$.

For $n = 1$ In [JH06] we have the following definition

Definition. *If $\chi : K^\times \to \mathbb{C}^\times$ is a character of $K^\times$, then we define*

$$L(s,\chi) = \begin{cases} (1-\chi(\varpi)q^{-s})^{-1}, & \text{if} \chi \text{is unramified,} \\ 1, & \text{otherwise.} \end{cases}$$

Note that this says nothing about the case when χ is ramified.

For $n = 2$ Here we assume that we are working with $K = \mathbb{Q}_p$, for simplicity, so the q from the $n = 1$ case will simply be p here. We make the following definition.

Definition. *For irreducible admissible representations π of $GL_2(\mathbb{Q}_p)$, we define the* local L-factor *$L(s,\pi)$ as follows.*

1. *For irreducible principal series $\pi(\chi_1,\chi_2)$, we set*

$$L(s,\pi) = \frac{1}{(1-\alpha_1 p^{-s})(1-\alpha_2 p^{-s})},$$

where $\alpha_i = \chi_i(p)$ if χ_i is unramified and $\alpha_i = 0$ otherwise.

2. *For a special representation, written* $\pi(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2}) = St \otimes \chi$*, we set*

$$L(s, \pi) = \frac{1}{1 - \alpha p^{-s}},$$

where $\alpha = \chi(p)|p|_p^{1/2} = p^{-1/2}\chi(p)$ *if* $\chi|\cdot|^{1/2}$ *is unramified, and* $\alpha = 0$ *otherwise.*

3. *For* π *supercuspidal, we set*

$$L(s, \pi) = 1.$$

Note that, as expected, L plays no role in the cuspidal representations.

Following [Mar11], we only define the ϵ-factors on $\mathrm{PGL}_2(\mathbb{Q}_p)$. Let ψ be the standard additive character of $\mathbb{Q}_p$. That is, $\psi(x) = e^{2\pi i\{x\}_p}$, where $\{x\}_p$ is the fractional part of x, defined as all terms with p-adic valuation negative. Note that given any nonconstant additive character χ of $\mathbb{Q}_p$, then any other additive character is $\chi(ax)$ for some $a \in \mathbb{Q}_p$, so we only need to fix one such character.

Definition. *Let* π *be an irreducible admissible representation of* $PGL_2(\mathbb{Q}_p)$*. The* local ϵ-factor $\epsilon(s, \pi, \psi)$ *attached to* π *is*

$$\epsilon(s, \pi, \psi) = \epsilon p^{c(\pi)(1/2-s)},$$

where $\epsilon = \pm 1$.

In [JH06], we have the following theorem.

Theorem (Converse Theorem)**.** *Let* ψ *be an nonconstant additive character of* $K^\times$*, and let* π_1, π_2 *be irreducible admissible representations of* $G = GL_2(K)$*. Suppose that*

$$L(\chi\pi_1, s) = L(\chi\pi_2, s), \qquad \text{and} \qquad \epsilon(s, \chi\pi_1, \psi) = \epsilon(s, \chi\pi_2, \psi)$$

for all characters χ *of* $K^\times$*. Then* $\pi_1 \cong \pi_2$*.*

For Weil–Deligne representations We have so far only defined L and ϵ for representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ for $n = 1, 2$. To define them for Weil–Deligne representations the hard part is $n \geq 2$, since by the local class field theory theorem we have an isomorphism $\chi \mapsto \chi \circ \theta_K$, and we simply define

$$L(s, \chi \circ \theta_K) = L(s, \chi)$$
$$\epsilon(s, \chi \circ \theta_K, \psi) = \epsilon(s, \chi, \psi).$$

It is unsurprising that these match up correctly.

Extending to $n \geq 2$ For the L-function, we simply make the definition

$$L(s, \sigma) = 1,$$

for an irreducible admissible Weil–Deligne representation σ of dimension $n \geq 2$. For a semi-simple σ, we define

$$L(s, \sigma_1 \oplus \sigma_2) = L(s, \sigma_1) L(s, \sigma_2).$$

The local constant is more complicated though, and we don't go into it here.

6 The Langlands Correspondence

We now quote the theorem from [JH06]. Let $\mathcal{G}_2(K)$ denote the set of isomorphism classes of 2-dimensional, F-semi-simple Weil–Deligne representations, and let $\mathcal{A}_2(K)$ denote the set of isomorphism classes of irreducible admissible representations of $G = \mathrm{GL}_2(K)$.

Theorem (The Langlands Correspondence)**.** *Let ψ be a nonconstant additive character of K. There is a unique map*

$$\pi : \mathcal{G}_2(K) \to \mathcal{A}_2(K),$$

such that

$$L(s, \chi\pi(\rho)) = L(s, \chi \otimes \rho)$$
$$\epsilon(s, \chi\pi(\rho), \psi) = \epsilon(s, \chi \otimes \rho, \psi)$$

for all $\rho \in \mathcal{G}_2(K)$ and all characters χ of $K^\times$. The map is a bijection.

References

[Bum98] Daniel Bump. *Automorphic Forms and Representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1998.

[Buz98] Kevin Buzzard. Some introductory notes on the local langlands correspondence. 1998.

[JH06] Bushnell Colin J. and Guy Henniart. *The Local Langlands Conjecture for GL(2)*. A Series of Comprehensive Studies in Mathematics. Springer, 2006.

[Mar11] Kimball Martin. Automorphic representations course notes. 2011.

[Ulm13] Douglas Ulmer. Conductors of l-adic representations. 2013.