JACQUET-LANGLANDS FOR GL_2

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Disclaimer. These are rough notes I wrote for a series of four talks I gave in Oxford in Hilary Term 2015 sketching a proof of the Jacquet-Langlands correspondence for GL_2. None of the material is my own work, my only contribution is the way the material is presented. There may be typos and mistakes in the notes so a person wishing to get a greater insight into the proof should consult the refererences at the end of the notes.

1. Background

1.1. The Global Langlands Correspondence and Langlands Functoriality.

Let G be a reductive group over $\mathbb{Q}$. The global Langlands correspodence is a conjectural correspondence between

$$\{\psi : L_{\mathbb{Q}} \to {}^L G(\mathbb{C})\} \longleftrightarrow \{\text{automorphic representations on } G(\mathbb{A})\}$$

There also needs to be compatibility with the local theory. The biggest problem is that the existence of $L_{\mathbb{Q}}$ is not known and in fact is one of the deepest problems in the theory. One of the consequences of this conjectural correspodence is Langlands functoriality; this says if we have two connected reductive groups H, G over a field $\mathbb{Q}$ with G quasi-split and an L-homomorphism between them, i.e. a map s.t. the following diagram commutes,

$$\begin{tikzcd} {}^L H \arrow[r] \arrow[d] & {}^L G \arrow[dl] \\ \Gamma_{\mathbb{Q}} & \end{tikzcd}$$

then there is a transfer map from

$$\{\text{automorphic forms on } H(\mathbb{A})\} \longrightarrow \{\text{automorphic forms on } G(\mathbb{A})\}$$

This is obvious if one has the appropriate Langlands correspondence as there is a map from

$$\{\psi : L_{\mathbb{Q}} \to {}^L H(\mathbb{C})\} \longrightarrow \{\psi : L_{\mathbb{Q}} \to {}^L G(\mathbb{C})\}$$

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We will be interested in a proving a specific instance of Langlands funcotoriality. To do this we briefly describe the theory of forms of reductive groups over a field k of characteristic zero. An algebraic group H is a k-form of G if H is defined over k and is isomorphic to G over $\overline{k}$. Γ_k acts on $\mathrm{Aut}(G_{\overline{k}}))$ (by conjugation), and this action preserves the subgroup $\mathrm{Inn}(G_{\overline{k}})$. One can show that the k-forms of G are parametrised by the cohomology set $H^1(\Gamma, \mathrm{Aut}(G_{\overline{k}}))$, H is said to be an inner form of G if the associated cocycle actually lies in the image of the natural map from $H^1(\Gamma, \mathrm{Inn}(G_{\overline{k}})) \to H^1(\Gamma, \mathrm{Aut}(G_{\overline{k}}))$. When G is split this map is injective, so inner forms can be thought of as elements of $H^1(\Gamma, \mathrm{Inn}(G_{\overline{k}}))$. Given a cocycle $c \in H^1(\Gamma, \mathrm{Aut}(G_{\overline{k}}))$, we can define $H(k) = \{h \in G(\overline{k}) : (c(\gamma) \cdot \gamma)h = h \ \forall \gamma \in \Gamma_k\}$.

A crucial fact about reductive groups G over k is the following; there exists a unique (up to isomorphism) quasi-split inner form G'. Since ${}^LG = {}^LG'$, Langlands functoriality predicts a map from automorphic representations of G to those of G'. If $G' = \mathrm{GL}_2$ then it is split, so one can take ${}^LG' = \mathrm{GL}_2$ as there is a trival Galois action.

1.2. Inner forms of GL_2.

We need to determine what the inner forms G of $G' = \mathrm{GL}_2$ are. Let k be a field. A central simple algebra A/k is a finite dimensional associative unital k-algebra in which there are no non-trivial two sided ideals (i.e. none except for $0, A$), and such that the center of A is $k = k \cdot 1$. For example, $M_n(k)$ is a CSA over k. A division algebra D/k is an associative unital k-algebra where for each non-zero $a \in D$ $\exists b \in D$ s.t. $ab = ba = 1$. One defines a quarternion algebra as a 4-dimensional central simple algebra. Wedderburn's theorem tells us that quarternion algebras are either isomorphic to $M_2(k)$ or a division algebra. Provided $\mathrm{char}(k) \neq 2$ they have the following form

$$\left(\frac{a,b}{k}\right) := \{x + iy + jz + ijw : i^2 = a, j^2 = b, ij = -ji\}$$

In particular we have the following proposition.

Proposition: Let B/k be a quaternion algebra. Then $G = B^\times$ is an inner form of G'/k (the converse also holds).

Proof. Let $B = \left(\frac{a,b}{k}\right)$. Let $E = k(\sqrt{a})$. Given $q = x + iy + jz + ijw \in B$ we can embed $B \hookrightarrow M_2(E)$

$$q = \begin{pmatrix} x + y\sqrt{a} & (z + w\sqrt{a})b \\ z - w\sqrt{a} & x - y\sqrt{a} \end{pmatrix} \in M_2(E)$$

Clearly we have $\phi : \mathrm{GL}_2(E) \cong G(E)$. As GL_2 is split the map from its inner forms are parametrised by the cocycles which are actually inner automorphisms. We therefore want to consider the cocycle $c(t) = \phi^{-1} \cdot t\phi \cdot t^{-1}$ where $t(x + y\sqrt{a}) = x - y\sqrt{a}$ and t is the non-trivial element of $\mathrm{Gal}(E/k)$. As $t^{-1} = t$ we have that

$$\begin{pmatrix} x + y\sqrt{a} & (z + w\sqrt{a})b \\ z - w\sqrt{a} & x - y\sqrt{a} \end{pmatrix} \overset{t}{\mapsto} \begin{pmatrix} \bar{x} - \bar{y}\sqrt{a} & (\bar{z} - \bar{w}\sqrt{a})b \\ \bar{z} + \bar{w}\sqrt{a} & \bar{x} + \bar{y}\sqrt{a} \end{pmatrix}$$

$$\overset{\phi}{\mapsto} \bar{x} - i\bar{y} + j\bar{z} - ij\bar{w} \overset{t}{\mapsto} x - iy + jz - ijw \xmapsto{\phi^{-1}} \begin{pmatrix} x - y\sqrt{a} & (z - w\sqrt{a})b \\ z + w\sqrt{a} & x + y\sqrt{a} \end{pmatrix}$$

In other words we have that

$$c(t)(\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}) = \begin{pmatrix} q_4 & bq_3 \\ b^{-1}q_2 & q_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ b^{-1} & 0 \end{pmatrix} \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

We set $\gamma = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$ and see that $\gamma^{-1} = \begin{pmatrix} 0 & 1 \\ b^{-1} & 0 \end{pmatrix}$. Therefore G is an inner form of G' as required. $\square$

We now state some facts about quaternion algebras we will need for later. It is not hard to show that quaternion algebras which are division algebras simply correspond to order 2 elements in the Brauer group. For local fields $k = \mathbb{Q}_p$ or $\mathbb{R}$ we see that there is a unique such quarternion algebras as $\mathrm{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}$ and $\mathrm{Br}(\mathbb{R}) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. For $k = \mathbb{Q}$, global class field theory gives us the exact sequence

$$0 \to \mathrm{Br}(\mathbb{Q}) \to \bigoplus_v \mathrm{Br}(\mathbb{Q}_v) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z} \to 0$$

Restricting to order 2 elements shows us that quaternion algebras are uniqually paramterised by an even set of places where they are non-split (i.e. not isomorphic to a matrix algebra over $\mathbb{Q}_v$).

1.3. **The Jaquet-Langlands Correspondence.**

We now have enough terminology to state the Jacquet-Langlands correspodence. Let $G = B^\times$ for some rational quaternion algebra B, and let $G' = \mathrm{GL}_2/\mathbb{Q}$. Let S be the finite set of places where $G(\mathbb{Q}_v) \ncong \mathrm{GL}_2(\mathbb{Q}_v)$.

Theorem: Let π be a cuspidal automorphic representation of G. Then $\exists$ a unique cuspidal automorphic representation π' of G' s.t. $\pi_v \cong \pi'_v$ $\forall v \notin S$.

2. The Trace Formula

In order to prove the Jacquet-Langlands correspodence we need to use the trace formula, we follow [1][§1]. Let H be a locally compact unimodualr (where left and right Haar measure agree) toplogical group, and Γ a discrete subgroup of H. Consider the unitary representation R given by right translation on $L^2(\Gamma\backslash H)$;

$$(R(y)f)(x) = f(xy) \;\; f \in L^2(\Gamma\backslash H), \;\; x, y \in H$$

If $H = G(\mathbb{A})$, $\Gamma = G(\mathbb{Q})$, one can 'roughly' define an automorphic representation π of G as an irreducible unitary representation occurring in the decomposition of R. We study R by integrating against a test function $f \in C_c(H)$, so define

$$R(f) = \int_H f(y)R(y)dy \qquad \text{on } L^2(\Gamma\backslash H)$$

We obtain the following for $\phi \in L^2(\Gamma\backslash H)$, $x \in H$

$$(R(f)\phi)(x) = \int_H (f(y)R(y)\phi)(x)dy = \int_H f(y)\phi(xy)dy = \int_H f(x^{-1}y)\phi(y)dy$$
$$= \int_{\Gamma\backslash H} \left(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y) \right) \phi(y)dy$$

Therefore $R(f)$ is an integral operator with kernel

$$K(x, y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)$$

This sum is always finite as it can be taken over in the intersection of the discrete group $\Gamma \cap (x \cdot \mathrm{supp}(f) \cdot y^{-1})$.

2.1. **The Compact Trace Formula.** We now assume that $\Gamma\backslash H$ is compact. In our situation this occurs when $G = B^\times$ where B is a non-split quaternion algebra, i.e. when $G(\mathbb{Q}) \ncong \mathrm{GL}_2(\mathbb{Q})$ and we consider the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ where

$$G(\mathbb{A})^1 = \{x \in G(\mathbb{A}) : |x\overline{x}| = 1\}$$

This assumption means that R decomposes discretely into irreducible representations π with finite multiplicty (this follows from the spectral theorem for compact operators). Moreover, $R(f)$ is of trace class with

$$\mathrm{tr}R(f) = \int_{\Gamma\backslash H} K(x, x)dx$$

Let $\{\Gamma\}$ be a set of representatives for the conjugacy classes of Γ, and for any $\gamma \in \Gamma$ let

$$\Gamma_\gamma = C_\Gamma(\gamma), H_\gamma = C_H(\gamma)$$

We find that

$$\begin{aligned}
\mathrm{tr}R(f) &= \int_{\Gamma\backslash H} K(x,x)dx = \int_{\Gamma\backslash H} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)dx \\
&= \int_{\Gamma\backslash H} \sum_{\gamma\in\{\Gamma\}} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx \\
&\left.\begin{aligned} &= \sum_{\gamma\in\{\Gamma\}} \int_{\Gamma_\gamma\backslash H} f(x^{-1}\gamma x)dx \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{H_\gamma\backslash H} \int_{\Gamma_\gamma\backslash H_\gamma} f(x^{-1}u^{-1}\gamma ux)dudx \end{aligned}\right\} = (*) \\
&= \sum_{\gamma\in\{\Gamma\}} \mathrm{vol}(\Gamma_\gamma\backslash H_\gamma) \int_{H_\gamma\backslash H} f(x^{-1}\gamma x)dx
\end{aligned}$$

Here the $(*)$ step uses Fubini's theorem and a fact concerning measures of unimodular groups. By restricting $R(f)$ to irreducible subspaces of $L^2(\Gamma\backslash H)$ we obtain the spectral expansion of $R(f)$ in terms of irreducible unitary representations π of H. Thus we get the following formula

$$\sum_{\gamma} a_\Gamma^H(\gamma) f_H(\gamma) = \sum_{\pi} a_\Gamma^H(\pi) f_H(\pi) \tag{2.1.1}$$

where the right sum is over representatives of conjugacy classes in Γ and π is summed over equivalence classes of irreducible unitary representations of H. The constant $a_\Gamma^H(\pi)$ is the multiplicty of π in the decomposisiton of R, $a_\Gamma^G(\gamma) = \mathrm{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A}))$, and the linear forms are defined as follows

$$f_H(\gamma) = \int_{H_\gamma\backslash H} f(x^{-1}\gamma x)dx$$

$$f_H(\pi) = \mathrm{tr}\pi(f) = \mathrm{tr}\left(\int_H f(y)\pi(y)dy\right)$$

Note if $H = \mathbb{R}$, $\Gamma = \mathbb{Z}$ then (2.1.1) is just the Poisson summation formula. We now apply this to the situation where $\Gamma = G(\mathbb{Q})$ and $H = G(\mathbb{A})^1$, for $G = B^\times$ where B is a non-split quaternion algebra. Let $\Gamma(G)$ denote the set of conjugacy classes of $G(\mathbb{Q})$ and let $\Pi(G)$ be the set of equivalence classes of automorphic representations of G (or more accurately their restrictions to $G(\mathbb{A})^1$). Then we have

$$\sum_{\gamma\in\Gamma(G)} a^G(\gamma) f_G(\gamma) = \sum_{\pi\in\Pi(G)} a^G(\pi) f_G(\pi) \qquad f \in C_c^\infty(G(\mathbb{A})) \tag{2.1.2}$$

where we are restricting the test functions to $G(\mathbb{A})^1$.

Note that for a general reductive group $G/\mathbb{Q}$, we have that $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is compact iff the maximal split torus in the center of G is a maximal split torus of G over $\mathbb{Q}$. In fact this is also equivalent to not having any proper parabolic subgroups defined over $\mathbb{Q}$. We would like to apply the above arguments to the case of $G' = \mathrm{GL}_2$. The problems are twofold; R no longer decomposes discretely and $R(f)$ is no longer of trace class.

2.2. **Eisenstein Series and Different Types of Spectra.**

We need to construct a trace formula for GL_2. The primary reference for the following is [3, pgs. 162-179]. Set Z_∞^+ to be the group of real scalar matrices with positive coefficicents. Set

$$X = Z_\infty^+ \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})$$

It will be easier but no less general to look at the space $L^2(X)$ instead, this contains all cuspidal automorphic forms. Let $T \subset N \subset B$ denote the subgroups of diagonal/strictly upper triangular/upper triagular matrices respectively. We define

$$T_\infty^+ = \{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_i > 0\},\ T^1(\mathbb{A}) = T_\infty^+ \backslash T(\mathbb{A})$$

$B(\mathbb{A})$ is not unimodular but we can define left and right Haar measures by

$$\int_{B(\mathbb{A})} \phi(b) d_l b = \int_{T(\mathbb{A})} \int_{N(\mathbb{A})} \phi(tn) dt dn,\ \int_{B(\mathbb{A})} \phi(b) d_r b = \int_{T(\mathbb{A})} \int_{N(\mathbb{A})} \phi(nt) dn dt$$

Recall $d_r b = \delta_B(b) d_l b$ where

$$\delta_B \left(\begin{pmatrix} a_1 & n \\ 0 & a_2 \end{pmatrix} \right) = \left| \frac{a_1}{a_2} \right|$$

One defines an injection $\mathbb{R} \hookrightarrow T_\infty^+$ by

$$r \mapsto h_r = \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix}$$

We write $x = n t h_r k z$ for $n \in N(\mathbb{A})$, $t \in T^1(\mathbb{A})$, $k \in K$ and $z \in Z_\infty^+$. r is uniquely determined by x so we set $H(x) = r$. One has that

$$H \left(\begin{pmatrix} a_1 & n \\ 0 & a_2 \end{pmatrix} \right) = \log \left| \frac{a_1}{a_2} \right|^{1/2}$$

so $\delta_B(b) = e^{2H(b)}$. We now consider certain induced representations from the subgroup $N(\mathbb{A})T(\mathbb{Q})T_\infty^+ \subset B(\mathbb{A})$, defined as follows for $z \in \mathbb{C}$

$$R(x, z) = \mathrm{Ind}_{N(\mathbb{A})T(\mathbb{Q})T_\infty^+}^{G'(\mathbb{A})} e^{zH(b)}$$

This acts by right translation on the Hilbert space $\mathbf{H}(z)$ consisting of complex-valued measurable functions ϕ on $G'(\mathbb{A})$ s.t.

$$\int_K |\phi(k)|^2 dk < \infty \tag{2.2.1}$$

$$\phi(ntg) = e^{(z+1)H(nt)}\phi(g) \tag{2.2.2}$$

One finds that if we let $\mathbf{H}$ be the Hilbert space of measurable functions ϕ on $N(\mathbb{A})T(\mathbb{Q})T_\infty^+\backslash G'(\mathbb{A})$ satisfying (2.2.1), then $R'(x, z)$ defined by

$$R'(x, z)\phi(g) = \phi(gx)e^{(z+1)H(gx)}e^{-(z+1)H(g)}$$

acting on $\mathbf{H}$ is equivalent to $R(x, z)$. One finds that $R(x, z)$ is unitary if $\Re(z) = 0$. The aim is to show that the integral

$$\int_{\Re(z)=0, \Im(z)>0} R(x, z) d|z|$$

is equivalent to a subrepresentation of $R(x)$. One uses the space $\mathbf{H}$ to define Eisenstein series, the Eisenstein series associated to ϕ is

$$E(g, \phi, z) = \sum_{\gamma \in B(\mathbb{Q})\backslash G'(\mathbb{Q})} \phi(\gamma g) e^{(z+1)H(\gamma g)}$$

If $\phi = 1$ and $g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ is real with $y > 0$, then we have a classical Eisenstein series. It turns out that the map

$$\phi \mapsto E(x, \phi, z)$$

intertwines the space of $R(x, z)$ with a subrepresentation of $R(x)$. This comes from the fact that

$$E(x, R(g, z)\phi, z) = R(g)E(x, \phi, z)$$

so the above map from $\mathbf{H}$ to the space of automorphic forms on $G'(\mathbb{A})$ commutes with $G'(\mathbb{A})$. There is a problem that $E(x, \phi, z)$ is not square integrable. However, we can use the E to define a different Hilbert space using an altered Fourier transform and obtain a map from this space to a certain subspace of $L^2(X)$. One obtains the following.

Theorem: The Hilbert space $L^2(X)$ has a decomposition

$$L^2(X) = L_0^2(X) \oplus L_0^2(E) \oplus L_1^2(E)$$

and R comes with a corresponding decomposition

$$R = R_0 \oplus R^+ \oplus R_1 = R_0^+ \oplus R_1$$

where $R_0^+ = R_0 \oplus R^+$ is the restriction of R to the discrete spectrum, and

$$R_1(x) = \int_0^\infty R(x, it)dt$$

Of course a lot of details has been left out of this theorem but the crucial point is the following, we have an explicit formula for R_1 and R.

2.3. **The Trace Formula for** GL_2.

We now follow [3, pgs. 186-226]. We have a decomposisiton $R = R_0^+ \oplus R_1$, the idea is to see if $R_0^+(f)$ is of trace class for suitable test functions. We will be interested in functions satisfying the following;

Assumption: f is a convolution of two bi-K-finite functions f' & f'' in $C_c^\infty(Z_\infty^+ \backslash G(\mathbb{A}))$.

Recall the convolution is defined as follows

$$(f' * f'')(x) = \int_X f'(xy)f''(y^{-1})dy = \int_X f'(y)f''(y^{-1}x)dy$$

The assumption is needed for the next theorem. We need to analyse the operator $R_1(f)$. If K_0^+ is the kernel of $R_0^+(f)$ then in fact $K_0^+ = K - K_1$ where K_1 is the kernel of $R_1(f)$. The reason this is helpful is that K_1 has an explicit description in terms of Eisenstein series. We pick a 'suitable' orthonormal basis $\{\phi_j\}_{j\in I}$ for $\mathbf{H}$ and let $R_{ij}(f, z)$ denote the matrix coefficient $\langle R(f, z)\phi_j, \phi_i\rangle$ of $R(f, z)$. One defines

$$K_1(x, y, f, z) = \frac{1}{4\pi} \sum_{i,j\in I} R_{ij}(f, z)E(x, \phi_i, z)\overline{E(y, \phi_j, z)}$$

We find that $R_1(f)$ is an integral operator with kernel

$$K_1(x, y) = \int_{-i\infty}^{i\infty} K_1(x, y, f, z)d|z|$$

Moreover, we have the following.

Theorem: $R_0^+(f)$ is an integral operator in $L_0^2(X)$ with kernel $K_0^+(x,x) = K(x,x) - K_1(x,x)$. Morover, it possesses a trace, K_0^+ is integrable over the diagonal and one has that

$$\mathrm{tr}(R_0^+(f)) = \int_X K_0^+(x,x)dx = \int_X (K(x,x) - K_1(x,x))dx$$

We'd now like to refine this identity. The strategy is to first break $K(x,x)$ up into elliptic and parabolic parts. One then breaks $K_1(x,x)$ into a component 'at infinity' and its complement, regroup terms and then integrate. Because our goal is the Jacquet-Langlands correspondence we will only need to describe part of this in detail. We say $\gamma \in G'(\mathbb{Q})$ is parabolic if it is $G'(\mathbb{Q})$ conjugate to some element of $B(\mathbb{Q})$, and say it is elliptic otherwise. We let G'_e denote the set of elliptic elements. Recall that $R(f)$ has kernel $K(x,y)$ where

$$K(x,y) = \sum_{\gamma \in G'(\mathbb{Q})} f(x^{-1}\gamma y)$$

We shall be interested in two parts of this sum, the elliptic part and the singular part; i.e. the sum over elements in the center. We get two series

$$I_e(x,f) = \sum_{\gamma \in G'_e} f(x^{-1}\gamma x) = \sum_{\gamma \in \{G'_e\}} \sum_{\delta \in G'_\gamma(\mathbb{Q}) \backslash G'(\mathbb{Q})} f(x^{-1}\delta^{-1}\gamma\delta x) \tag{2.3.1}$$

$$I_s(x,f) = \sum_{\mu \in Z(\mathbb{Q})} f(\mu) \tag{2.3.2}$$

We wish to prove both of these are integrable. Of course for equation (2.3.2) this follows as X has finite measure. For equation (2.3.1) it follows immediately provided $F(x) = \sum_{\gamma \in G'_e} |f(x^{-1}\gamma x)|$ is compactly supported, which is a consequence of the following lemma.

Lemma: Let C be a compact subset of $Z_\infty^+ \backslash G'(\mathbb{A})$. Then $\exists d_C > 0$ s.t. if $\gamma \in G'(\mathbb{Q})$ and $x^{-1}\gamma x \in C$ for some $x \in G'(\mathbb{A})$ with $H(x) > \log d_C$ then $\gamma \in B(\mathbb{Q})$.

Proof. If $x = ntk$ then $x^{-1}\gamma x \in C$ implies $t^{-1}n^{-1}\gamma n t \in KCK$. But if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we see that

$$\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} * & * \\ \frac{t_1 c}{t_2} & * \end{pmatrix}$$

Therefore $|ct_1t_2^{-1}|$ is bounded, so

$$|c| \leq M \left|\frac{t_1}{t_2}\right|^{-1}$$

Therefore if $\left|\frac{t_1}{t_2}\right|^{-1}$ is sufficiently small, or equivalently $H(x) = \log \left|\frac{t_1}{t_2}\right|^{1/2}$ is sufficiently large, then $c = 0$ and $\gamma \in B(\mathbb{Q})$. $\square$

Set $C = \mathrm{supp}(f)$ and we see that $F(x) = 0$ if $H(x) > \log d_C$ so it is indeed compactly supported. Consequently we can integrate (2.3.1). Moreover, we get

$$\int_X I_e(x,f)dx = \int_X \sum_{\gamma \in \{G'_e\}} \sum_{\delta \in G'_\gamma(\mathbb{Q})\backslash G'(\mathbb{Q})} f(x^{-1}\delta^{-1}\gamma\delta x)dx$$

$$= \sum_{\gamma \in \{G'_e\}} \int_{Z_\infty^+ G'_\gamma(\mathbb{Q})\backslash G'(\mathbb{A})} f(x^{-1}\gamma x)dx$$

$$= \sum_{\gamma \in \{G'_e\}} \mathrm{vol}(Z_\infty^+ G'_\gamma(\mathbb{Q})\backslash G'_\gamma(\mathbb{A})) \int_{G'_\gamma(\mathbb{A})\backslash G'(\mathbb{A})} f(x^{-1}\gamma x)dx$$

Summing this up with the integral of the singular term we get

$$\mathrm{vol}(X) \sum_{\mu \in Z(\mathbb{Q})} f(\mu) + \sum_{\gamma \in \{G'_e\}} \mathrm{vol}(Z_\infty^+ G'_\gamma(\mathbb{Q})\backslash G'_\gamma(\mathbb{A})) \int_{G'_\gamma(\mathbb{A})\backslash G'(\mathbb{A})} f(x^{-1}\gamma x)dx \tag{2.3.3}$$

This is not unlike the trace formula for compact quotient. However, we will also have contribution from parabolic conjugacy classes and Eisenstein series which make the formula more complicated. Luckily by choosing suitable test functions we can get rid of this problem.

Theorem: Let $f = \prod_v f_v$ and suppose that for at least two places v

$$\int_{N_v} \int_{K_v} f_v(k^{-1}tnk)dndk = 0 \tag{2.3.4}$$

for all $t \in T_v$. Then $\mathrm{tr} R_0^+(f)$ is equal to the expression (2.3.3).

The condition (2.3.4) happens for quite general test functions, indeed it happens if f_p is a K_p-finite matrix coefficient of a supercuspidal representation of G'_p.

3. Proof of the Correspodence

3.1. A Comparision of Two Trace Formulae.

We now want to compare the two trace formulae for $G = B^\times$ for B a non-split quaternion algebra and $G' = \mathrm{GL}_2$. Let S be the even set of places where $G(\mathbb{Q}_v) \not\cong G'(\mathbb{Q}_v)$. For G we have the trace formula

$$\sum_{\gamma \in \Gamma(G)} a^G(\gamma) f_G(\gamma) = \sum_{\pi \in \Pi(G)} a^G(\pi) f_G(\pi) \tag{3.1.1}$$

Where

$$f_G(\gamma) = \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(x^{-1}\gamma x)dx, \;\; f_G(\pi) = \mathrm{tr}\pi(f)$$

Functions $f \in C_c^\infty(G(\mathbb{A}))$ are a finite linear combination of products

$$f = \prod_v f_v \;\; f_v, \in C_c^\infty(G(\mathbb{Q}_v))$$

Suppose f is of the above form, then $f_G(\gamma)$ is just a product of the local orbital integrals $f_{v,G}(\gamma_v)$ where γ_v is the image of γ in $\Gamma(G_v)$, and $f_G(\pi)$ is a product of local characters $f_{v,G}(\pi_v)$ where π_v is the local component of π in $\Pi(G_v)$ (the equivalence classes of irreducible representations of $G(\mathbb{Q}_v)$). For $v \notin S$ the $\mathbb{Q}_v$-isomorphism is determined up to inner-automorphism so we have canonical bijections

$$\Gamma(G_v) \longrightarrow \Gamma(G'_v), \;\; \gamma_v \mapsto \gamma'_v$$

$$\Pi(G_v) \longrightarrow \Pi(G'_v), \;\; \pi_v \mapsto \pi'_v$$

For $v \notin S$ we can define a function $f'_v \in C_c^\infty(G'_v)$ s.t. $\forall \gamma_v \in \Gamma(G_v)$, $\pi_v \in \Pi(G_v)$

$$f'_{v,G'}(\gamma'_v) = f_{v,G}(\gamma_v), \;\; f'_{v,G'}(\pi'_v) = f_{v,G}(\pi_v)$$

We want to be able to define a version of this for $v \in S$.

Proposition: For $v \in S$ there is a canonical bijection $\gamma_v \mapsto \gamma'_v$ from $\Gamma(G_v)$ to the set $\Gamma_{\mathrm{ell}}(G'_v)$ of semisimple conjugacy classes in $G'(\mathbb{Q}_v)$ that are either central or don't have eigenvalues in $\mathbb{Q}_v$.

Proof. Recall that $B_v = B(\mathbb{Q}_v)$ is a division algebra. An element has the form $q = x + iy + jz + ijw$ and can be realised as a matrix by letting $B = \left(\frac{a,b}{\mathbb{Q}_v}\right)$, $E = \mathbb{Q}_v(\sqrt{a})$, and then

$$Q = \begin{pmatrix} x + y\sqrt{a} & (z + w\sqrt{a})b \\ z - w\sqrt{a} & x - y\sqrt{a} \end{pmatrix} \in M_2(E)$$

This has characteristic polynomial

$$m_Q(\lambda) = \lambda^2 - 2x\lambda + (x^2 - ay^2 - bz^2 + abw^2) = \lambda^2 - 2x\lambda + N(q)$$

Note that

$$N(q) = q\bar{q} = (x + iy + jz + ijw)(x - iy - jz - ijw)$$

Therefore q is invertible iff $N(q) \neq 0$. As $B(\mathbb{Q}_v)$ is a division algebra this occurs for all $q \neq 0$. However, looking at the discriminant of $m_Q(\lambda)$ gives

$$B^2 - 4AC = 4x^2 - 4(x^2 - ay^2 - bz^2 + abw^2) = 4(ay^2 + bz^2 - abw^2)$$

This is in $\mathbb{Q}_v^2 \iff \exists s \in \mathbb{Q}_v$ s.t. $s^2 - ay^2 - bz^2 + abw^2 = 0$ which happens iff $s = y = z = w = 0$. Hence in this case $q = x \in \mathbb{Q}_v$, so is central. Therefore the conjugacy class of Q which is described by its characteristic polynomial is either central or has characteristic polynomial with no roots in $\mathbb{Q}_v$. $\square$

This proof also shows there is a canonical bijection $\gamma \mapsto \gamma'$ from $\Gamma(G)$ onto the semisimple conjugacy classes $\gamma' \in \Gamma(G')$ s.t. $\forall v \in S$ $\gamma'_v \in \Gamma_{\text{ell}}(G'_v)$. Jacquet and Langlands used this to be able to transfer test functions for $v \in S$. For regular element $\gamma'_v \in \Gamma(G'_v)$ (those whose centraliser is a maximal torus) they defined an $f'_v \in C_c^\infty(G'(\mathbb{Q}_v))$ s.t.

$$f'_{v,G'}(\gamma'_v) = \begin{cases} f_{v,G}(\gamma_v) & \text{if } \gamma'_v \in \Gamma_{\text{ell}}(G'_v) \\ 0 & \text{otherwise} \end{cases} \tag{3.1.2}$$

This was achieved using a result of Langlands; one simply defines the function above and then shows such a function must come from an orbital integral. We now have a function

$$f' = \prod_v f'_v \in C_c^\infty(G'(\mathbb{A}))$$

s.t. for any class $\gamma' \in \Gamma(G')$

$$f'_{G'}(\gamma') = \begin{cases} f_G(\gamma) & \text{if } \gamma' \text{ is the image of some } \gamma \in \Gamma(G) \\ 0 & \text{otherwise} \end{cases} \tag{3.1.3}$$

We claim that for $v \in S$ the f'_v satisfy (2.3.4). Indeed we have for regular non-elliptic $\gamma'_v = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a \neq b$ that

$$0 = f'_{v,G'}(\gamma'_v) = \int_{G'_{\gamma'}(\mathbb{Q}_v)\backslash G'(\mathbb{Q}_v)} f'_v(x^{-1}\gamma'_v x)dx = \int_{T(\mathbb{Q}_v)\backslash G'(\mathbb{Q}_v)} f'_v(x^{-1}\gamma'_v x)dx$$

$$= \int_{N_v}\int_{K_v} f'_v(k^{-1}n^{-1}\gamma'_v nk)dndk = \int_{N_v}\int_{K_v} f'_v(k^{-1}tnk)dndk$$

The last equality comes from the following conjugacy relation for regular elements

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & (a-b)x \\ 0 & b \end{pmatrix}$$

One obtains a similar result for the non-regular non-elliptic elements. The final crucial observation is that for $\gamma \mapsto \gamma'$

$$\mathrm{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A})^1) = \mathrm{vol}(G'_{\gamma'}(\mathbb{Q})\backslash G'_{\gamma'}(\mathbb{A})^1)$$

For central γ this just comes from a fact about Tamagawa numbers. For regular γ, first consider one of the form

$$\gamma = \begin{pmatrix} x + y\sqrt{a} & 0 \\ 0 & x - y\sqrt{a} \end{pmatrix} \longleftrightarrow x + iy$$

This has characteristic polynomial $\lambda^2 - 2x\lambda + (x^2 - ay^2)$, so gets mapped to a conjugacy class in GL_2

$$\gamma' = \begin{pmatrix} x & ay \\ y & x \end{pmatrix}$$

Both elements have centraliser consisting of elements like themselves, and in fact $G_\gamma \cong_{\mathbb{Q}} G_{\gamma'}$. For a different non-central element, note we can write it in the form $x + q$ where q is a pure quaternion (one where $q^2 \in \mathbb{Q}$ but $q \notin \mathbb{Q}$). Therefore up to change of basis, we can take $q = i$ and apply the above result.

All this gives the equality

$$\mathrm{tr} R_0^+(f') = \sum_{\gamma \in \Gamma(G)} a^G(\gamma) f_G(\gamma)$$

Using the trace formulae for G and G' we get the spectral identity

$$\sum_{\pi \in \Pi(G)} a^G(\pi) \mathrm{tr}\pi(f) = \sum_{\pi' \in \Pi(G')} a^{G'}_{\mathrm{disc}}(\pi') \mathrm{tr}\pi'(f') \tag{3.1.4}$$

where $a^{G'}_{\mathrm{disc}}(\pi')$ is the multiplicity of π' in the decomposition of R_0^+.

3.2. **Characters.**

The main purpose of this section is to sketch the proof of the following statement; let (π_i, V_i) be inequivalent irreducible admissible representations of G a locally profinite toplogical group, then their characters are linearly independent. The statement is also true for GL_n over $\mathbb{R}$.

Let (π, V) be an irreducible admissible representation of G and let K be an open compact subgroup of G. We have the space V^K and

$$V(K) = \{v - \pi(k)v : k \in K\}$$

One can define an action of a double coset KgK on V^K by

$$\pi(KgK) \cdot v = \sum_{i=1}^{n} \pi(g_i)v \text{ where } KgK = \bigsqcup_{i=1}^{n} g_i K$$

One can then define its action on V by first projecting to V^K, the projection operator is $\pi(K1K) = \pi(K)$. In fact we get the direct sum decomposition

$$V = V^K \oplus V(K)$$

To see this note that any $v \in V$ is fixed by some K', and set $K'' = K \cap K'$. Thus we have that

$$\frac{1}{[K : K'']} \sum_{K/K''} \pi(k)v = \pi(K) \cdot v$$

This is clear as the map on the left clearly fixes V^K, and has image contained in it so has the correct image. Trivially we also have

$$\frac{1}{[K : K'']} \sum_{K/K''} v = v$$

Thus we see that

$$v - \pi(K) \cdot v = \frac{1}{[K : K'']} \sum_{K/K''} (v - \pi(k)v) \in V(K)$$

We use this fact to show that V^K is irreducible. If it were not then suppose $U \leq V^K$ was $K\backslash G/K$-stable and non-zero. Then as V is irreducible U must G-generate V, thus we can write any $v \in V$ in the form

$$v = \sum_j c_j \pi(g_j) u_j$$

If $v \in V^K$ then we have that

$$v = \pi(K) \cdot v = \sum_j c_j \pi(K) \cdot (\pi(g_j) u_j)$$

Now for $g \in G$, $u \in U$, we have that
(3.2.1)

$$\pi(K) \cdot (\pi(g)u) = \frac{1}{n} \sum_{K/(K \cap gKg^{-1})} \pi(k)\pi(g)u = \frac{1}{n} \sum_{i=1}^{n} \pi(k_i g)u = \pi(KgK) \cdot u \in U$$

Thus $V^K = U$ so it is irreducible. Now suppose we have a pair of inequivalent irreducible representations (π_i, V_i) for $i = 1, 2$, with K open compact s.t. $V_i^K \neq 0$. Suppose V_1^K and V_2^K were equivalent under the $K\backslash G/K$ action. Then we'd have an invertible linear map $T : V_1^K \to V_2^K$ s.t.

$$T(\pi_1(KgK) \cdot v) = \pi_2(KgK) \cdot Tv$$

As V_i^K generates V_i we can define a G-isomorphism[1]

$$\tilde{T}(v) = \tilde{T}(\sum_j c_j \pi_1(g_j) v_j) := \sum_j c_j \pi_2(g_j) T(v_j) \text{ where } v_j \in V_1^K$$

This means V_1 and V_2 would be equivalent as G-representations which is a contradiction, so the V_i^K must also be inequivalent.

We have shown that distinct irreducible (π_i, V_i) give rise to distinct irreducible V_i^K. Note that for any $v \in V^K$

$$\pi(\mathbf{1}_{KgK})v = \int_{KgK} \pi(x) v dx = \mathrm{vol}(K)\pi(KgK)v$$

Therefore the distinct irreducible (π_i, V_i) give rise to distinct irreducible representations of the Hecke algebra $\mathcal{H}(G, K)$, where K is chosen s.t. each $V_i^K \neq 0$. As distinct irreducible finite dimensional representations of an alegbra have linearly independent characters, the result is proved (the key fact for algebras is that if ρ is irreducible and finite dimensioanl the map $\rho : A \to \mathrm{End}(V)$ is surjective).

3.3. **The Proof.**

We are now ready to prove the Jacquet-Langlands correspodence. Let π be a cuspidal automorphic representation of G. We know that outside our set S we have an isomorphism to GL_2. We therefore break the representation down into two parts; $\pi = \pi_S \otimes \tau^S$ where

$$\pi_S = \bigotimes_{v \in S} \pi_v$$

We are looking for an automorphic representation π' of $G' = $ GL2 where $\pi' \cong \pi'_S \otimes \tau^S$ for some π'_S. Using the decomposition $\mathrm{tr}\pi(f) = \mathrm{tr}\pi_S(f_S)\mathrm{tr}\tau^S(f^S)$ and the definition of f'_v for $v \notin S$ we have re-writing (3.1.4) that

$$\sum_{\tau^S} \sum_{\pi_S} a^G(\pi_S \otimes \tau^S)\mathrm{tr}\pi_S(f_S)\mathrm{tr}\tau^S(f^S) = \sum_{\tau^S} \sum_{\pi'_S} a^{G'}_{\mathrm{disc}}(\pi'_S \otimes \tau^S)\mathrm{tr}\pi'_S(f'_S)\mathrm{tr}\tau^S(f^S)$$

[1] There is something to check here, the important thing is it sends any representative of the zero vector to zero.

which re-arranges to

$$\sum_{\tau^S}\left\{\sum_{\pi_S} a^G(\pi_S\otimes\tau^S)\mathrm{tr}\pi_S(f_S)-\sum_{\pi'_S} a^{G'}_{\mathrm{disc}}(\pi'_S\otimes\tau^S)\mathrm{tr}\pi'_S(f'_S)\right\}\mathrm{tr}\tau^S(f^S)=0 \tag{3.3.1}$$

The previous subsection means we can use linear independence of characters to obtain

$$\sum_{\pi_S} a^G(\pi_S\otimes\tau^S)\mathrm{tr}\pi_S(f_S)=\sum_{\pi'_S} a^{G'}_{\mathrm{disc}}(\pi'_S\otimes\tau^S)\mathrm{tr}\pi'_S(f'_S) \tag{3.3.2}$$

The correspondence follows immediately from this; we know there is an automorphic $\pi=\pi_S\otimes\tau^S$ so the left hand side of (3.3.2) is not identically zero. We can therefore pick f_S s.t. it doesn't vanish, and again using (3.3.2) we know this transfers to a function f'_S so the right hand side doesn't vanish. In particular there must be at least one non-zero term on the right hand side,

$$a^{G'}_{\mathrm{disc}}(\pi'_S\otimes\tau^S)\mathrm{tr}\pi'_S(f'_S)\neq 0$$

We have therefore found an automorphic $\pi'=\pi'_S\otimes\tau^S$, this is our transfer. It is unique by strong multiplicity one.

A more careful analysis of the transfer can refine it to tell us exactly which automorphic representations of GL_2 come from inner forms. One can also use the result to give a local transfer, by letting $G=B^\times$ be non-split at $\{r,\infty\}$.

References

[1] J. Arthur, *An Introduction to the Trace Formula.*
[2] B. Casselman, *Essays on representations of p-adic groups: Smooth representations.*
[3] S. Gelbart, *Automorphic forms on Adele Groups.*
[4] H. Jacquet & R. Langlands, *Automorphic forms on* GL(2).