Notes on classification of irreducible admissible $\mathrm{GL}_2(\mathbb{R})$-modules

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1   Introduction

In order to begin the archimedean part of automorphic theory, we want to understand certain infinite-dimensional irreducible representations of Lie groups G, in particular for us the group $\mathrm{GL}_2 = \mathrm{GL}_2(\mathbb{R})$. Two natural strategies present themselves.

Firstly, we could mirror the proof of the classification of supercuspidal representations in the nonarchimedean case by examining the restriction of the representation to a maximal compact Lie subgroup K, for instance the subgroup $\mathrm{O}_2 \leq \mathrm{GL}_2$. The reason this seems promising is that the theory of continuous representations of compact groups is well-understood:

Lemma 1.1 (Representations of compact groups)**.** *Let (π, V) be a continuous representation of a compact group K on a Hilbert space. Then*

1. *the inner product on V may be chosen, without changing the topology, so that the K-action is unitary;*

2. *should V be irreducible, it is necessarily finite-dimensional;*

3. *in general, V is completely reducible: it is the closure of the (orthogonal) direct sum of some irreducible K-subrepresentations.*

Most often we will write the final condition as follows: we let (γ) be a complete list of (finite-dimensional) irreducible representations of K, and $V(\gamma)$ the γ-isotypic component of V. Then

$$V = \overline{\bigoplus_\gamma V(\gamma)}$$

The second possibility is to mimic the relationship between the representations of the Lie group and the Lie algebra found in the theory of finite-dimensional representation theory.

Lemma 1.2 (Representations of Lie groups). *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then*

1. *any finite-dimensional continuous representation (π_G, V) of G can be made a representation of $\mathfrak{g}$ by*

$$\pi_{\mathfrak{g}}(x) \cdot v := \lim_{t \to 0} \frac{1}{t}\left(\pi_G(\exp(tx)) - 1\right) \cdot v$$

2. *from this representation of $\mathfrak{g}$, the original representation can be recovered on the identity component of G by*

$$\pi_G(\exp(x)) \cdot v = \sum_{r=0}^{\infty} \frac{\pi_{\mathfrak{g}}(x)^r}{r!} \cdot v$$

3. *for G connected, (π_G, V) is irreducible iff $(\pi_{\mathfrak{g}}, V)$ is.*

Example 1.3 (Finite-dimensional representations of GL_2^+ and GL_2). Recall (modified from the representation theory of $\mathfrak{sl}_2$) that the finite-dimensional irreducible representations of $\mathfrak{gl}_2 \cong \mathfrak{sl}_2 \times \mathbb{R}$ are given by the degree k homogenous polynomials in x, y, with $\mathfrak{gl}_2$ action given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = x\frac{\mathrm{d}}{\mathrm{d}y}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = y\frac{\mathrm{d}}{\mathrm{d}x}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = x\frac{\mathrm{d}}{\mathrm{d}x} - y\frac{\mathrm{d}}{\mathrm{d}y}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mu$$

for some scalar μ. A more enlightened way of phrasing this is that the irreducible representations are given by $\mathrm{Symm}^k(V_1) \otimes \mu \mathrm{tr}$ where V_1 is the standard 2-dimensional representation.

Exponentiating up this tells us a classification of the finite-dimensional irreducible representations of *the identity component of* GL_2, namely GL_2^+, the group of positive-determinant matrices. Specifically, it tells us that its irreducible representations are given by $\mathrm{Symm}^k(V_1) \otimes \chi \circ \det$, where now V_1 is the standard 2-dimensional representation of GL_2^+ and $\chi : \mathbb{R}^{>0} \to \mathbb{C}^{\times}$ is a quasicharacter.

Finally we need to extend this to all of GL_2, i.e. we need to worry about the action of $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. However, we can always extend the action on $\mathrm{Symm}^k(V_1) \otimes \chi \circ \det$ by specifying that

$$\eta \cdot f(x, y) = \pm f(x, -y)$$

to produce an irreducible GL_2-representation. The choice of sign can be subsumed into the character χ, so we have again produced irreducible representations $\text{Symm}^k(V_1) \otimes \chi \circ \det$ where now V_1 is the standard representation of GL_2 and $\chi : \mathbb{R}^\times \to \mathbb{C}^\times$ is any character. To see that this is all irreducible representations, one can use a straightforward Frobenius argument.

The realisation (due to Harish-Chandra) that allows us to proceed with a classification of infinite-dimensional representations is that we need to consider both these ideas simultaneously in order to be able to get a handle on the behaviours involved.

2 Reduction to $(\mathfrak{g}, K)$-modules

The general setup we will be considering is that of a continuous action π by a Lie group G on a complex Hilbert space V. We will let K be a maximal compact subgroup (so that we may assume K, but not necessarily G, acts unitarily on V). For our purposes though, we will only need the case when $G = GL_2$ and $K = O_2$ (or $G = GL_2^+$ and $K = SO_2$), so not all these proofs may work in complete generality.

From hereon, a representation of a Lie group G will always mean a continuous representation on a Hilbert space. We may occasionally assume that the action of a maximal compact subgroup K is unitary, since this can always be ensured.

In order to make the classification problems tractable (and because these are many of the examples we see), we introduce the following

Definition 2.1. Let (π, V) be a representation of G. We say (π, V) is *admissible* just when each K-isotypic component $V(\gamma)$ is finite-dimensional. We say that (π, V) is *irreducible* just when it has no non-trivial *closed* invariant subspaces.

Remark 2.2. All finite-dimensional representations are admissible, as are all irreducible unitary representations. In some sense, admissibility is the smallest sensible property which subsumes both of these.

One problem that immediately presents us is that we can't manufacture an action of $\mathfrak{g}$ on all of V. For example, $L^2(S^1)$ with the right regular action of S^1 is a representation of the circle group, and $v = \sum_{r>0} r^{-\frac{4}{3}} z^r$ is a perfectly good element of it, but if we try to define an action of $i \in i\mathbb{R} = T_1(S^1)$ on v, then we should calculate this to be

$$\lim_{t \to 0} \sum_{r>0} r^{-\frac{4}{3}} \frac{e^{irt} - 1}{t} z^r = \sum_{r>0} i r^{-\frac{1}{3}} z^r$$

which is not square-integrable.

However, we can make some headway by looking at a restricted (non-closed!) subspace of V.

Definition 2.3. Let (π, V) be a representation of G. We define the subspace

$$V^{\mathrm{fin}} := \bigoplus_{\gamma} V(\gamma) \leq V = \overline{\bigoplus_{\gamma} V(\gamma)}$$

(the K-isotypic decomposition) so that $V^{\mathrm{fin}} \leq V$ is dense and K-stable. Equivalently, V^{fin} is the set of all vectors such that $\pi(K) \cdot v$ only spans a finite-dimensional subspace (such vectors are called K-finite).

Proposition 2.4 (Smoothness of K-finite vectors)**.** *Let (π, V) be an admissible representation of G, and let $v \in V$ be a K-finite vector. Then the map $G \to V$ given by $X \mapsto \pi(X) \cdot v$ is smooth, i.e. it is infinitely differentiable. Such a vector is referred to as a* smooth *vector.*

Sketch proof, see also Bump Proposition 2.4.5: Recall that $C_c^{\infty}(G)$ acts on V by

$$\pi(f) \cdot v = \int_G f(X^{-1})\pi(X) \cdot v \ \mathrm{d}X$$

where the integral is taken in the sense of Riemann with respect to left Haar measure. The vectors $\pi(f) \cdot v$ are always smooth (their derivatives can be written down explicitly in terms of those of f).

On the other hand, if we let $\chi : K \to \mathbb{C}$ be the character associated to the irreducible K-representation γ, i.e. $\chi(Y) = \dim(\gamma)\mathrm{tr}(\gamma(Y))$, then the vector

$$\int_K \chi(Y^{-1})\pi(Y) \cdot v \ \mathrm{d}Y$$

always lies in $V(\gamma)$. This is the compact group version of the idempotent decomposition.

These two identities can be combined usefully. Let $\phi_0 \in C_c^{\infty}(G)$ and let $\phi = \chi *_K \phi_0$ be the convolution, i.e.

$$\phi(X) = \int_K \chi(Y^{-1})\phi_0(XY) \ \mathrm{d}Y$$

so that ϕ is also C^{∞} and compactly supported (it's supported in $\mathrm{supp}(\phi_0)K$). Now we have the identity

$$\begin{aligned}
\pi(\phi) \cdot v &= \int_G \left(\int_K \chi(Y^{-1})\phi_0(X^{-1}Y) \ \mathrm{d}Y \right) \pi(X) \cdot v \ \mathrm{d}X \\
&= \int_K \int_G \chi(Y^{-1})\phi_0(X^{-1}Y)\pi(X) \cdot v \ \mathrm{d}X \ \mathrm{d}Y \\
&= \int_K \int_G \chi(Y^{-1})\phi_0(Z^{-1})\pi(YZ) \cdot v \ \mathrm{d}Z \ \mathrm{d}Y \\
&= \int_K \chi(Y^{-1})\pi(Y) \cdot \left(\int_G \phi_0(Z^{-1})\pi(Z) \cdot v \right) \ \mathrm{d}Y
\end{aligned}$$

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From the top line we see that $\pi(\phi) \cdot v$ is always smooth, and from the bottom line we see that it is always in $V(\gamma)$. If now $v \in V(\gamma)$ itself, then we shall choose some delta-sequence of ϕ_0 (i.e. positive smooth functions of integral 1 whose support shrinks to $\{1\}$). Then we see that $\pi(\phi_1) \cdot v \to v$, so

$$\pi(\phi) \cdot v \to \int_K \chi(Y^{-1})\pi(Y) \cdot v \ \mathrm{d}Y = v$$

so we see that v is a limit of smooth vectors in $V(\gamma)$. In other words, the smooth vectors in $V(\gamma)$ are dense, so that every vector in $V(\gamma)$ is smooth, since it is finite-dimensional. This concludes the proof. $\qquad\square$

Corollary 2.5 ($\mathfrak{g}$-action on K-finite vectors)**.** *Let* (π, V) *be an admissible representation of G. Then $\mathfrak{g}$ acts on V^{fin} by the formula*

$$\pi(x) \cdot v = \lim_{t \to 0} \frac{1}{t}\left(\exp(tx) - 1\right) \cdot v$$

V^{fin} *is $\mathfrak{g}$-stable with this action (though it needn't be G-stable!) and satisfies*

1. *for all $v \in V^{\mathrm{fin}}$, the K-span of v is finite-dimensional, and the K-action thereon is continuous;*

2. *the infinitesimal K-action agrees with that of $\mathfrak{g}$, i.e. if $y \in \mathfrak{k}$ is in the Lie algebra of K then*

$$\pi(y) \cdot v = \lim_{t \to 0} \frac{1}{t}\left(\exp(ty) - 1\right) \cdot v$$

3. *the $\mathfrak{g}$-action is compatible with the adjoint action of K on $\mathfrak{g}$, i.e. for $x \in \mathfrak{g}$ and $Y \in K$ we have*

$$\pi(YxY^{-1}) = \pi(Y)\pi(x)\pi(Y)^{-1}$$

Remark 2.6. A structure obeying the above conditions is referred to as a $(\mathfrak{g}, K)$-module, and is termed *admissible* just when the K-isotypic components of the $(\mathfrak{g}, K)$-module are all finite-dimensional. Notice that the notion of a $(\mathfrak{g}, K)$-module features essentially no analysis or topology.

In the absence of admissibility, one can still recover a $(\mathfrak{g}, K)$-module by instead taking $V^{\mathrm{fin}} \cap V^{\infty}$, the space of all *smooth* K-finite vectors. However, in general we will only obtain admissible $(\mathfrak{g}, K)$-modules from admissible G-representations.

The main reason that this structure is useful to us is that it is sensitive to the submodule structure of our representation. Specifically

Theorem 2.7 (G-submodules and $(\mathfrak{g}, K)$-submodules)**.** *Let* (π, V) *be an admissible G-representation. Then there is a bijection between closed G-subrepresentations of V and $(\mathfrak{g}, K)$-submodules of V^{fin}, given on the one hand by $U \mapsto U^{\mathrm{fin}} = U \cap V^{\mathrm{fin}}$ and on the other $W \mapsto \overline{W}$.*

Proof. Proving that the operations are mutually inverse is not difficult. The equality $\overline{U^{\text{fin}}} = U$ we have already seen, when we remarked that V^{fin} was dense in V. To prove the other equality, we may suppose that π is unitary as a K-representation, so that the K-isotypic decomposition

$$V = \overline{\bigoplus_\gamma V(\gamma)}$$

is orthogonal. We can then write $\overline{W} = W^{\perp\perp}$, so that $\overline{W}(\gamma) = W(\gamma)^{\perp\perp}$ (where we restrict the inner product to $V(\gamma)$). Yet $V(\gamma)$ is finite-dimensional, so $W(\gamma)^{\perp\perp} = W(\gamma)$, and so we've seen $\overline{W}(\gamma) = W(\gamma)$, i.e. $\overline{W}^{\text{fin}} = W$.

The subtlety in this theorem is in proving that $\overline{W}$ is always a G-representation. This is true for general Lie groups, but for GL_2 (or any GL_n) we can remove some of the technical details, by using the identity (Exercise 2.4.2. of Bump)

$$\pi(\exp(x)) \cdot v = \left(\sum_0^\infty \frac{1}{r!} \pi(x)^r \right) \cdot v$$

valid whenever $x \in \mathfrak{g}$ and v a smooth vector (in particular for $v \in W$). In particular, this directly tells us that $\pi(\exp(x)) \cdot W \subseteq \overline{W}$, so that $\overline{W}$ is stable under the action of all $\exp(x)$, i.e. the action of the identity component GL_2^+ of GL_2. To complete the proof, just note that $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{O}_2$, so that W, and hence $\overline{W}$, are already stable under the action of η, and hence of all of GL_2. $\square$

Corollary 2.8. *An admissible G-representation is irreducible iff its associated $(\mathfrak{g}, K)$-representation is.*

Remark 2.9. Because of the utility of working with $(\mathfrak{g}, K)$-modules, we often only try to classify G-representations up to *infinitesimal equivalence*, i.e. up to isomorphism of their associated $(\mathfrak{g}, K)$-modules. In fact, for GL_2 this doesn't lose us anything: two GL_2-representations are isomorphic iff they are infinitesimally equivalent (although this is highly GL_2-specific).

3 Classification of GL_2^+-representations

3.1 Understanding $(\mathfrak{gl}_2, \text{SO}_2)$-modules

With the theoretical machinery developed, we will be able to classify irreducible admissible $(\mathfrak{gl}_2, \text{SO}_2)$- and $(\mathfrak{gl}_2, \text{O}_2)$-modules, and later on see that these come from bona fide GL_2^+- and GL_2-representations. Since $(\mathfrak{g}, K)$-modules are essentially algebraic objects, we are happy to consider them as modules over $\mathcal{U}$, the universal enveloping algebra of the complexification of $\mathfrak{g}$. The classification will involve a degree of (in?) computation, for which we adopt the following

Notation. We use the following basis of the complexification of $\mathfrak{gl}_2$:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$E = \frac{1}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$$

$$F = \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$$

(note that (H, E, F) is the usual basis of $\mathfrak{sl}_2$ conjugated by $\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$, the same base-change that simultaneously diagonalises SO_2). We let

$$\Delta = \frac{-1}{4} \left(H^2 + 2EF + 2FE \right) = \frac{-1}{4} \left(H^2 + 2H + 4FE \right) = \frac{-1}{4} \left(H^2 - 2H + 4EF \right)$$

denote the Casimir operator, so that the centre of $\mathcal{U}$ is a 2-variable polynomial ring generated by I and Δ.

Before we launch into the calculations we'll need a preliminary lemma:

Lemma 3.1 (Schur's lemma). *Let V be an irreducible admissible $(\mathfrak{g}, K)$-module. Then every endomorphism of V is given by multiplication by a scalar. In particular, the centre of $\mathcal{U}$ acts on V by scalars.*

Proof. Exercise. $\qquad\square$

Proposition 3.2 (Preparatory calculations). *Let V be an admissible $(\mathfrak{gl}_2, SO_2)$-module. Since the irreducible representations of SO_2 are just one-dimensional, given by characters $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto e^{ik\theta}$ for $k \in \mathbb{Z}$, we know that*

$$V = \bigoplus_k V(k)$$

For all k, H acts on each $V(k)$ by multiplication by k, and $E \cdot V(k) \subseteq V(k+2)$, $F \cdot V(k) \subseteq V(k-2)$.

If additionally both I and Δ act by scalars μ, λ respectively (such a representation is called quasi-simple*), then EF and FE on each $V(k)$ by multiplication by scalars, namely $\frac{k(2-k)}{4} - \lambda$ and $-\frac{k(k+2)}{4} - \lambda$ respectively.*

Proof. The crucial calculation is that of the action of H on $V(k)$: we know that $\exp(i\theta H) = \exp \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ so that for $v \in V(k)$ we have

$$iH \cdot v = \lim_{\theta \to 0} \frac{1}{\theta} \left(\exp(i\theta H) - 1 \right) \cdot v = \lim_{\theta \to 0} \frac{1}{\theta} (e^{ik\theta} - 1) \cdot v = ikv$$

so that $H \cdot v = kv$. The fact that $E \cdot V(k) \subseteq V(k+2)$ and $F \cdot V(k-2)$ is immediate from their commutation relations with H.

In the quasi-simple case, the calculations of the actions of EF and FE on $V(k)$ are immediate from the equations

$$-4\Delta = H^2 + 2H + 4FE = H^2 - 2H + 4EF$$

and the fact that Δ acts like the scalar λ. $\qquad\square$

Theorem 3.3 (Classification of $(\mathfrak{gl}_2, \mathrm{SO}_2)$-modules – uniqueness). *Let V be an irreducible admissible $(\mathfrak{gl}_2, \mathrm{SO}_2)$-module, so that (by Schur) I and Δ act by scalars μ, λ respectively. Then in the decomposition $V = \bigoplus_k V(k)$, each $V(k)$ is at most one-dimensional, and the k for which $V(k) \neq 0$ all have the same parity ϵ (called the* parity *of V).*

If λ is not of the form $k(k-2)$ for $k \in \mathbb{Z}$ of the same parity as V, then all $V(k)$ with k the same parity as V appear in V, and there is at most one $(\mathfrak{gl}_2, \mathrm{SO}_2)$-module with these parameters. We call this module $P_\mu(\lambda, \epsilon)$, and if $\mu = 0$ refer to it as principal series.

If $\lambda = k(k-2)$ for some $k \equiv \epsilon \bmod 2$ (we may suppose wlog $k \geq 1$), then there are three possibilities for the set $\Sigma(V)$ of l with $V(l) \neq 0$, namely

$$\Sigma^0(k) = \{l \equiv k \bmod 2 : -k < l < k\}$$
$$\Sigma^+(k) = \{l \equiv k \bmod 2 : l \geq k\}$$
$$\Sigma^-(k) = \{l \equiv k \bmod 2 : l \leq -l\}$$

(note that the first is zero for $k = 1$). In this case, the parameters μ, k and a choice of $* \in \{0, +, -\}$ uniquely determines V, and we call this module $D_\mu^*(k)$. If $* = \pm$ we refer to this as discrete series *for $k > 1$ and* limit of discrete series *for $k = 1$.*

Proof. Firstly, pick some $v \in V(l_0)$ non-zero. Then, since we know that SO_2, H, EF and FE act by scalars on all $V(l)$, it is clear that the $\mathbb{C}$-span of $\{v; Ev, E^2v, \ldots; Fv, F^2v, \ldots\}$ is a submodule of V, hence all of V, so we have proven the first part (note that $E^r \cdot v \in V(l_0 + 2r)$ and similarly for $F^r \cdot v$, so that all the $V(l)$ appearing have the same parity).

For the second part, it follows from the calculations of the action of EF on $V(l)$ that in this case both E and F are invertible, and so V contains a non-zero element in each $V(l_0 \pm 2r)$ as desired. For uniqueness, we just note that specifying that F acts invertibly and the (scalar) action of H and EF on each $V(l)$ in enough to reconstruct V.

For the final part, we know that $EF = 0$ on $V(k)$, so that either $F = 0$ on $V(k)$ or $E = 0$ on $V(k-2)$. If the former held, then we can see that $\bigoplus_{l \in \Sigma^+(k)} V(l)$ is a submodule of V, so that it is either 0 or all of V. In other words we see that either $\Sigma(V) \subseteq \Sigma^+(k)$ or $\Sigma(V) \subseteq \Sigma^0(k) \cup \Sigma^-(k)$. In the latter case, the same argument pertaining to $\bigoplus_{l \in \Sigma^0(k) \cup \Sigma^-(k)} V(l)$ establishes the same conclusion.

Similarly, $EF = 0$ on $V(2-k)$, so by the same argument $\Sigma(V) \subseteq \Sigma^-(k)$ or $\Sigma(V) \subseteq \Sigma^0(k) \cup \Sigma^+(k)$.

We've seen that $\Sigma(V)$ is certainly contained in one of the $\Sigma^*(k)$, but we also know that EF acts invertibly on all other $V(l)$ ($l \neq k, 2-k$), so that $\Sigma(V) = \Sigma^*(k)$.

Finally, to prove uniqueness, note that our choices specify the actions of H and EF on each $V(l)$, and specify exactly when $E \cdot v = 0$ and when $F \cdot v = 0$, so that V is determined by these data. $\qquad \square$

3.2 GL_2^+-representations

With the preceding classification result, two questions now naturally present themselves. Firstly, do all of these supposed $(\mathfrak{gl}_2, \mathrm{SO}_2)$-modules actually occur? Secondly, can all of these be produced from genuine GL_2^+-representations? It transpires that the answer to both of these questions is "yes", and moreover we can produce the desired GL_2^+-representations from representations induced from the Borel subgroup of GL_2^+ (what would be called "principal series" in the nonarchimedean case).

To construct these representations, we fix complex numbers s_1 and s_2, and a parity $\epsilon \in \{0,1\}$, which together uniquely specify a character of the Borel subgroup by

$$\chi : \begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} = \mathrm{sgn}(a_1)^\epsilon |a_1|^{s_1} |a_2|^{s_2}$$

We want to induce this character up to GL_2^+, so as to obtain a Hilbert space representation of GL_2^+. The right way to do this is to look at the representation[1]

$$\mathfrak{H}(\chi) = \left\{ f \in L^2(G) : f\left(\begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} g \right) = \mathrm{sgn}(a_1)^\epsilon |a_1|^{s_1+\frac{1}{2}} |a_2|^{s_2-\frac{1}{2}} f(g) \right\}$$

endowed with the right regular action of G and inner product

$$\langle f_1, f_2 \rangle = \int_K f_1(Y) \overline{f_2(Y)} \; \mathrm{d}Y$$

The extra factor of $|a_1|^{\frac{1}{2}} |a_2|^{-\frac{1}{2}}$ that has appeared comes from the module of the Borel subgroup, and its usage makes our induction functor better behaved – for example it will preserve unitaricity of the representation.

We can now try to analyse these representations $\mathfrak{H}(\chi)$: the important point is that we have a strong description of a sensible basis in the following

Proposition 3.4 (Structure of $\mathfrak{H}(\chi)$)**.** *Let* $V = \mathfrak{H}(\chi)$ *be as above and write* $\mu = s_1 + s_2$, $s = \frac{1}{2}(s_1 - s_2 + 1)$ *and* $\lambda = s(1-s)$.

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[1]or, more precisely, the set of all square-integrable functions satisfying the desired identity, identifying those that agree almost everywhere

Then the spaces $V(k)$ are zero if $k \not\equiv \epsilon$ mod 2, and if $k \equiv \epsilon$ mod 2 then they are one-dimensional, spanned by

$$\phi_k\left(\begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = \mathrm{sgn}(a_1)^\epsilon |a_1|^{s_1+\frac{1}{2}}|a_2|^{s_2-\frac{1}{2}}e^{ik\theta}$$

$\mathfrak{H}^{\mathrm{fin}}(\chi)$ *is quasi-simple, with I acting like μ and Δ acting like λ respectively.*

Proof. Calculation. $\qquad\square$

Corollary 3.5 (Irreducible admissible GL_2^+-representations)**.** *There is a symmetry (up to isomorphism) in interchanging s_1 and s_2 in our definitions, so we shall assume for simplicity that $\Re s_1 \geq \Re s_2$, so that $\Re s \geq \frac{1}{2}$.*

In light of the preceding proposition and the earlier classification theorem, we see that if s is not of the form $\frac{k}{2}$ where $k \equiv \epsilon$ mod 2, then $\mathfrak{H}^{\mathrm{fin}}(\chi)$ is irreducible, isomorphic to $P_\mu(\lambda,\epsilon)$. In particular, $\mathfrak{H}(\chi)$ is an irreducible admissible GL_2^+-representation.

If however $s = \frac{k}{2}$ where $k \equiv \epsilon$ mod 2, then $\mathfrak{H}^{\mathrm{fin}}(\chi)$ has length three, with irreducible factors $\mathfrak{H}_*^{\mathrm{fin}}(\chi)$ for $* \in \{0, +, -\}$ isomorphic to $D_\mu^*(k)$ (except that when $k = 1$ the factor $\mathfrak{H}_0^{\mathrm{fin}}(\chi) = 0$ does not appear). In particular, $\mathfrak{H}(\chi)$ has length three (or two) as a GL_2^+-representation, with factors $\mathfrak{H}_*(\chi)$ for $* \in \{0, +, -\}$.

Moreover, each pair λ, μ arises from a unique pair s_1, s_2 with $\Re s_1 \geq \Re s_2$, so we see that in our classification theorem, all of the identified irreducible $(\mathfrak{gl}_2, \mathrm{SO}_2)$-modules do actually exist, and arise from genuine GL_2^+-representations, which we have an explicit description of. We have thus classified all irreducible admissible GL_2^+-representations, up to infinitesimal equivalence.

Exercise. Determine which of the factors $\mathfrak{H}_*^{\mathrm{fin}}(\chi)$ in the second case appear as submodules or as quotients of $\mathfrak{H}^{\mathrm{fin}}(\chi)$. How does this change when $\Re s_1 \leq \Re s_2$?

4 Representations of GL_2

There are now a variety of ways of extending our analysis to GL_2-representations. It is possible to do a similar study of $(\mathfrak{gl}_2, \mathrm{O}_2)$-modules, but for our purposes it is perhaps easier to just directly induce up representations from GL_2^+.

The key point here is the representations $\mathfrak{H}(\chi)$ can naturally have their action extended to GL_2 in two distinct ways. There are two distinct ways of lifting χ to a character of the Borel subgroup of GL_2, namely

$$\chi_1 \otimes \chi_2 \begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} = \mathrm{sgn}(a_1)^{\epsilon_1}|a_1|^{s_1}\mathrm{sgn}(a_2)^{\epsilon_2}|a_2|^{s_2}$$

for some choice of $\epsilon 1, \epsilon_2 \in \{0, 1\}$ with sum ϵ mod 2. We write $\chi_i = \mathrm{sgn}^{\epsilon_i}|\cdot|^{s_i}$.

Inducing this up to GL_2, we obtain the representations

$$\mathfrak{H}(\chi_1, \chi_2) = \left\{f \in L^2(\mathrm{GL}_2) : f\left(\begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix}g\right) = \chi_1(a_1)\chi_2(a_2)\left(\frac{a_1}{a_2}\right)^{\frac{1}{2}}f(g)\right\}$$

Such an f is uniquely determined by its restriction to GL_2^+, so that $\mathfrak{H}(\chi_1, \chi_2) \to \mathfrak{H}(\chi)$ is an isomorphism of GL_2^+-representations.

To understand these representations, we consider the action of $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on the representation. A key observation is that (since O_2 is infinite dihedral), the action of η on the SO_2-isotypic components of any $(\mathfrak{gl}_2, O_2)$-module must interchange $V(k)$ and $V(-k)$.

Now in the general case (s not of the form $\frac{k}{2}$ where $k \equiv \epsilon \bmod 2$) $\mathfrak{H}(\chi)$ is an irreducible GL_2^+-representation, so $\mathfrak{H}(\chi_1, \chi_2)$ is an irreducible GL_2-representation. In the remaining case, our calculation of the action of η tells us that it interchanges $\mathfrak{H}_{\pm}(\chi)$ (which are submodules of $\mathfrak{H}(\chi)$ and preserves $\mathfrak{H}_0(\chi)$ (which is a quotient). Thus we have in this case two GL_2-representations, namely $\mathfrak{D}_\mu(k) = \mathfrak{H}_-(\chi) \oplus \mathfrak{H}_+(\chi)$, and $\mathfrak{H}_0(\chi_1, \chi_2) = \mathfrak{H}_0(\chi)$. Again by considering the action of η, these two are clearly irreducible (since their underlying $(\mathfrak{gl}_2, O_2)$-modules are irreducible).

Thus we have found a large collection of irreducible admissible representations of GL_2, and we want to check firstly when these are isomorphic (this is straightforward, since most of them are already nonisomorphic as GL_2^+-representations), and secondly that we have found all such GL_2-representations (at least up to infinitesimal equivalence). This can be done in a variety of ways, for example either using Frobenius reciprocity, or by using a similar argument in the world of $(\mathfrak{gl}_2, O_2)$-representations to deduce that the underlying $(\mathfrak{gl}_2, O_2)$-modules of the listed representations is a complete list of the irreducible admissible $(\mathfrak{gl}_2, O_2)$-modules. After an argument of this form, we find

Theorem 4.1 (Classification of irreducible admissible GL_2-representations). *Pick s_1, s_2 complex numbers with $\Re s_1 \geq \Re s_2$ and pick $\epsilon_1, \epsilon_2 \in \{0, 1\}$. Denote by χ_i the character $\mathrm{sgn}^{\epsilon_i} |\cdot|^{s_i}$ and write $s = \frac{1}{2}(s_1 - s_2 + 1)$, $\mu = s_1 + s_2$, $\lambda = s(1-s)$ and $\epsilon = \epsilon_1 + \epsilon_2$. Then*

- *if s is not of the form $\frac{k}{2}$ where $k \equiv \epsilon \bmod 2$, then $\mathfrak{H}(\chi_1, \chi_2)$ is an irreducible representation;*

- *if $s = \frac{k}{2}$ for some such k then $\mathfrak{H}(\chi_1, \chi_2)$ has two irreducible factors: $\mathfrak{H}_0(\chi_1, \chi_2)$ is finite-dimensional and appears as a quotient; and $\mathfrak{D}_\mu(k)$ is infinite-dimensional and appears as a submodule (and is referred to as a* discrete series *representation;*

- *if $k = 1$ in the above case, then note that $\mathfrak{H}(\chi_1, \chi_2) = 0$ and the representation $\mathfrak{D}_\mu(k)$ is referred to instead as* limit of discrete series.

The above are nonisomorphic except that in the second case interchanging ϵ_1 and ϵ_2 does not change the representation $\mathfrak{D}_\mu(k)$, and together these constitute a complete list of irreducible admissible representations of GL_2, up to infinitesimal equivalence (indeed isomorphism).