Jacquet-Langlands and cyclic base change

Oxford Learning seminar

Week 1

Talk by Andrew. This talk gave a general overview of the results we will be discussing in this seminar.

Motivation. Why are Jacquet-Langlands and cyclic base change such central results? We start with some history: In the 30's, Hasse and Davenport remarked that for an algebraic variety, its zeta function should have an analytic continuation and functional equation. Gauss' calculations already prove this for CM elliptic curves, Hasse and Davenport prove it for all elliptic curves. In the 50's, Eichler realised that $X_0(11)$ gives rise to a zeta function which is easy to analytically continue due to properties of modular forms.

Later, Artin and Grothendieck introduced étale cohomology. Given $X_{/F}$, we can associate the cohomology groups $\mathrm{H}^i_{et}(X_{/\overline{F}}, \mathbf{Q}_l)$, which come with a Galois action. So we get

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbf{Q}_l),$$

which provides us with a more systematic way to understand the ζ-function of X. We even end up with a better question than the one we started with: What are these systems of Galois representations that we obtain? Shimura later realised that many automorphic forms can be understood as sitting inside the cohomology of Shimura varieties. This gives us Galois representations again, so it is natural to ask for a direct way of obtaining Galois representations from automorphic forms.

There are three prototype problems in the Langlands programme that we will discuss in this seminar:

- Let f be an eigenform on GL_2. Given an extension of number fields F/F_0, this should give rise to $f_{/F}$.
- The form f should give rise to $\mathrm{Sym}^n f$, for instance by taking the symmetric power $\mathrm{GL}_2 \to \mathrm{GL}_n$.
- Artin conjecture: Let $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbf{C})$ be an irreducible representation, then

  $$L(s, \rho) = \prod_p \det\left(I - \mathrm{Frob}_p \mid_{V^{I_p}} p^{-s}\right)^{-1}$$

  should be analytic. (We know it is meromorphic from Brauer induction)

1.1 Hilbert modular forms

Let us make these prototype problems more precise for Hilbert modular forms. Let F be a totally real number field of degree d over $\mathbf{Q}$, and let $\mathrm{GL}_2^+(F)$ be the subgroup of $\mathrm{GL}_2(F)$ consisting of matrices which have totally positive determinant.

Let $f : \mathfrak{H}^d \to \mathbf{C}$ be a holomorphic function, then define an action of $\mathrm{GL}_2^+(F)$ on such function by

$$(f\|_k\alpha)(z) = \left(\prod_{i=1}^{d} \det{}^{k/2}(\alpha_i)(c_i z + d_i)^{-k}\right) f(\alpha(z)), \qquad \text{where } \alpha_i = \sigma_i(\alpha) = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right).$$

Pick $\mathscr{I} \lhd \mathscr{O}_F$ integral ideal, and let $\Gamma_1(\mathscr{I}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+\mathscr{O}_F \text{where } c \equiv 0 \pmod{\mathscr{I}},\ a \equiv 1 \pmod{\mathscr{I}}\}$, then we define

$$M_k(\mathscr{I}) = \left\{f : \mathfrak{H}^d \cup \{\text{cusps}\} \to \mathbf{C} \text{ hol. } : f\|_k\alpha = f\ \ \forall \alpha \in \Gamma_1(\mathscr{I})\right\}$$

We define the operator $T(\pi)$ for $\pi \lhd \mathscr{O}_F$ principal and totally positive, $\pi \nmid \mathscr{I}$, by the usual rule

$$f \mapsto \sum_{a \in \mathscr{O}_F/\pi} f\left(\frac{z+a}{\pi}\right) + p^{k-1} f(\pi z)$$

There is a much better definition using adèles. Consider $f : \mathrm{GL}_2(\mathbf{A}_F) \to \mathbf{C}$ such that

- $f(\alpha x) = f(x)$ for $\alpha \in \mathrm{GL}_2(F)$,
- $f(x\mathfrak{u}) = f(x)$ for $\mathfrak{u} \in \prod_v \mathfrak{U}_v(\mathscr{I})$, where $\mathfrak{U}_v(\mathscr{I})$ is defined analogously to Γ_1 above,
- $\forall x \in \mathrm{GL}_2(\mathbf{A}_F)$, $f(xy^{-1})$ is equal to $(f_x\|_k y)(i)$ for some Hilbert modular form f_x and for all $y \in \mathrm{GL}_2^+(\mathbf{R})^d$.

Hecke operators are defined by double quotients

$$T(\mathfrak{p})f = \sum_{a \pmod{\pi}} f\left(x\left(\begin{smallmatrix} \pi & a \\ 0 & 1 \end{smallmatrix}\right)_{\mathfrak{p}}\right) + f\left(x\left(\begin{smallmatrix} \pi & a \\ 0 & 1 \end{smallmatrix}\right)_{\mathfrak{p}}\right)$$

Galois representations. Eigenforms of Hecke operators give rise to semi-simple Galois representations

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(K_{f,\lambda}),$$

for each λ of $\mathscr{O}_{K_f}$, where $\mathrm{Frob}_{\mathfrak{p}}$ for $\mathfrak{p} \nmid \mathscr{I}$ or $N\lambda$ satisfies

$$x^2 - a(\mathfrak{p}, \mathfrak{f})x + \psi_f(\mathfrak{p})\mathrm{N}\mathfrak{p}^{k-1}.$$

1.2 Base change

The base change conjecture now predicts that for a totally real field extension F/F_0, the form $f_{/F_0}$ should give rise to a form $\tilde{f}_{/F}$, whose associated Galois representation is just restriction to $\mathrm{Gal}(\overline{F}/F)$ of the Galois representation attached to $f_{/F_0}$. Such a form arising from base change therefore satisfies $\tilde{f}_{/F}^{\sigma} = \tilde{f}_{/F}$ for all $\sigma \in \mathrm{Gal}(F/F_0)$, and we may wonder whether the converse holds. The answer is **no** in general. Invariant Galois representations in general do not descend. They do if they are project, however.

Let $\tilde{\rho}_{f,\lambda} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{PGL}_2(K_{f,\lambda})$ be the projective representation attached to $\tilde{f}_{/F}$. If this is Galois-invariant, we know it descends to a representation $\tilde{\rho}_0 : \mathrm{Gal}(\overline{F}_0/F_0) \to \mathrm{PGL}_2(K_{f,\lambda})$.

Theorem 1 (Tate)**.** There exists a $\rho_0 : \mathrm{Gal}(\overline{F}_0/F_0) \to \mathrm{GL}_2(K_{f,\lambda})$ lifting $\tilde{\rho}_0$.

This theorem of Tate tells us that the Galois representation does indeed descend, but this representation ρ_0 might only be a twist of $\rho_{f,\lambda}$. So we obtain $\rho_0 |_K = \rho_{f,\lambda} \otimes \chi$.

There is a descent conjecture that says that in the Galois case, for every invariant f on $\mathrm{GL}_{2,F}$ there is an invariant χ such that $f \otimes \chi$ is the base change of some form over F_0. For cyclic extensions, we do not need a χ.

Cyclic base change. The case of base change for Hilbert modular forms over a cyclic Galois extension was basically done by Saito. Later, Shintani put this into representation theory language, and Langlands proved the full result for GL_2. This establishes cyclic base change in lots of generality, as it allows for arbitrary number fields and Maaßforms as well. Any argument we usually give relies on geometry, and therefore we are often clueless when the field involved are not CM. The power of Langlands' method lies in its systematic use of the trace formula. Langlands also realised that there is an application of these methods to the Artin conjecture.

We note that the case of solvable base change is much harder, and follows by no means from the cyclic case. The work of Langlands-Tunnell however gives us the solvable GL_2 Artin conjecture.

1.3 Jacquet-Langlands

As mentioned before, we can attach 2-dimensional Galois representations to Hilbert modular eigenforms. Our first reflex might be to look for them in the cohomology of Hilbert modular varieties, but unfortunately this only gives us $2d$-dimensional representations, where d is the degree of the totally real field over $\mathbf{Q}$. This $2d$-dimensional representation we retrieve is the tensor product of all the Galois conjugates of the 2-dimensional representation we are looking for, and unfortunately it is impossible to recover the 2-dimensional one from this. So how do we get them?

Let $D = \mathcal{O}_F[\sqrt{-\alpha}, \sqrt{-\beta}, \sqrt{\alpha\beta}]$. Depending on the number of embeddings where α, β are positive, we get an embedding

$$\mathbf{R} \otimes D \hookrightarrow \mathrm{GL}_2(\mathbf{R})^{d_1} \times D^{d_2}.$$

We will ignore the d_2-part on the right hand side in this discussion.

When $d_1 = 1$, we get a discrete subgroup Γ of $\mathrm{GL}_2(\mathbf{R})$, and $\Gamma\backslash\mathfrak{H}$ is a Shimura curve. This is where we can find the 2-dimensional representations we were looking for! Now a theorem of Jacquet-Langlands-Eichler-Shimizu says that any eigenform f on Γ gives rise to an eigenform $\tilde{f}$ on $\mathrm{GL}_2(F)$, whose eigenvalues are the same away from the ramification of D. Jacquet and Langlands characterise exactly what the image of this association is. Some forms on $\mathrm{GL}_2(F)$ do not arise this way, but we can still patch them.

The case $d_1 = 0$ is also important. Now we obtain the class set, a 0-dimensional Shimura variety. This becomes even more important for the case of GL_n, as they help us understand Hecke rings.

Week 2

Talk by Minhyong. This talk gave an overview of Langlands functoriality. Slides are available at `http://people.maths.ox.ac.uk/vonk/automorphic`.

Week 3

Talk by Netan. This talk gave an overview of some important definitions in the representation theory of locally profinite groups. We discussed the representation theory of GL_2 over finite fields, and introduced the Jacquet functor.

Notation. Throughout F will denote a mixed characteristic local field with ring of integers $\mathcal{O}$, prime $\mathfrak{p}$ and residue field k of characteristic p. G will denote $GL_2(F)$. G, B, T and N will denote the groups $GL_2(F)$, $\begin{pmatrix} * & * \\ & * \end{pmatrix}$ $\begin{pmatrix} * & \\ & * \end{pmatrix}$ and $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ unless otherwise stated.

3.1 Smooth representations of $GL_2(F)$ and the Hecke algebra

We recall some definitions and constructions from last week.

Definition. A complex representation (π, V) of G is *smooth* if the map $\cup_K V^K \to V$ is an isomorphism. For a smooth representation (π, V), its *smooth dual*, or *contragredient*, is defined to be $\check{V} := \cup_K (V*)^K$, where V^* is the whole space of linear functionals and as before the union is over all open compacts. $\check{(.)}$ defines a contravariant functor from the category of smooth representations to itself.

Lemma 1. The natural map $V \to \check{\check{V}}$ is an isomorphism if and only if V is admissible.

3.1.1 Induction and compact induction

Recall that we saw last time that a good way to find interesting representations of GL_2 was by inducing from the Borel subgroup

Proposition 1 (Cartan decomposition)**.** Let $K = GL_2(\mathcal{O})$. Then

$$G = \bigcup K \begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix} K$$

where the union is over all $a \leq b$ in $\mathbb{Z}$.

Proof. Convince yourself that any g in $GL_2(F)$ can be written as $g_1 \begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix} g_2$ for some g_1 and g_2

in K. Then note that the determinant of g and the index $[K : gKg^{-1} \cap K]$ uniquely determine the a and the b. ☐

3.1.2 Hecke algebras

Define the *Hecke algebra* $\mathscr{H}(G)$ to be the space of locally constants functions $G \to \mathbb{C}$ of compact support. Hence an element of $\mathscr{H}(G)$ is just a linear combination of indicator functions $\sum \lambda_i \mathbf{1}_{g_i K_i}$ for some g_i in G, K_i open compact and λ_i in $\mathbb{C}$.

For f_1, f_2 in $\mathscr{H}(G,K)$, define the *convolution* of f_1 and f_2 to be

$$f_1 * f_2 : g \mapsto \int f_1(x) f_2(x^{-1}g) d\mu(x)$$

This gives the Hecke algebra the structure of an associative algebra.

Example. If f_1 and f_2 have support on an open compact K, and factor through $K \to K/K'$, then $*$ is just the usual notion of convolution from the complex representation theory of finite groups.

Definition. For a compact open subgroup K, define the subalgebra$\mathscr{H}(G,K)$ of $\mathscr{H}(G)$ to be the $\mathbb{C}$-vector space of compactly supported functions $K \backslash G / K \to \mathbb{C}$.

3.1.3 Spherical representations

In the case when $K = GL_2(\mathscr{O})$, this is referred to as the *spherical Hecke algebra*

Lemma 2. The spherical Hecke algebra is commutative.

Proof. Define the (vector space) involution $f \mapsto \widetilde{f}$ by

$$\widetilde{f}(x) := f(x^t)$$

Then

$$\widetilde{f_1 * f_2} = \widetilde{f}_2 * \widetilde{f}_1$$

On the other hand, the Cartan decomposition gives a basis of $\mathscr{H}(G,K)$ which is fixed by this involution, so we must have

$$f_1 * f_2 = f_2 * f_1$$

☐

3.2 Principal series representations of $GL_2(k)$

In this section we go over the complex representation theory of GL_2 of the finite field k, which will serve as a guide for classifying admissible representations of $GL_2(F)$. We'll denote by G, B, N and T the corresponding groups of k-points. Recall that since G is now finite the number of isomorphism classes of irreducible representations is just the number of conjugacy classes.

Lemma 3. G has $q^2 - 1$ conjugacy classes.

Recall from last week that given characters χ_1 and ξ_2, the representation $\chi = \chi_1 \otimes \chi_2$ of B was the composition

$$B \to B/N \to \mathbb{C}^\times$$

sending $\begin{pmatrix} a & \\ & b \end{pmatrix}$ to $\chi_1(a) \otimes \chi_2(b)$.

Lemma 4. An irreducible representation of G is contained in $Ind_B^G \chi$ (for some χ) if and only if its restriction to N contains the trivial character.

The representations $Ind_B^G \chi$ and their irreducible summands are referred to as *principal series* representations of G.

Irreducible representations of G which do not contain the trivial character of G are referred to as *cuspidal* representations. The construction of these is harder

3.3 The Jacquet Functor and Principal series

Recall that a fruitful method for constructing representations of GL_2 (in various contexts is to construct representations of T, lift them to representations of B via the projection $B \to T$ and then induce up to G. The Jacquet functor allows one to go the other way.

For a smooth representation (π, V) define

$$V(N) := \{v - \pi(x)v : v \in V, x \in N\}$$

Define $V_N := V/V(N)$. This is the maximal quotient of V (in the category of B-representations) on which N acts trivially. The functor

$$(\pi, V) \mapsto (\pi_N, V_N)$$

is an exact functor from smooth representations of G to smooth representations of T. Now suppose (σ, W) is a smooth representation of T. We lift this in the usual way to a representation of B trivial on N and then take $Ind_B^G \pi$. Frobenius reciprocity gives

$$Hom_G(\pi, Ind\sigma) = Hom_B(\pi, \sigma)$$

but since N acts trivially on σ, we get

$$Hom_G(\pi, Ind\sigma) = Hom_T(\pi_N, \sigma)$$

Definition. Let (π, V) be an irreducible smooth representation of G. We say π is *cuspidal* if $V_N = 0$. If V_N is nonzero say π is *principal series*.

3.4 Models of admissible representations

Fix a character ϑ of N. We say that an irreducible smooth representation (π, V) admits a *Whittaker model* if it there is a non-zero homomorphism

$$V \hookrightarrow Ind_N^G \vartheta$$

The corresponding subspace of $Ind_N^G \vartheta$ is denoted $W(\pi, \vartheta)$. By Frobenius reciprocity the existence of a Whittaker model for π is equivalent to the existence of a nonzero N homomorphism $V \to \vartheta$. This is equivalent to a nonzero homomorphism $V_\vartheta \to \vartheta$.

Theorem 2. For any infinite dimensional irreducible smooth representation of G, $dim V_\vartheta = 1$.

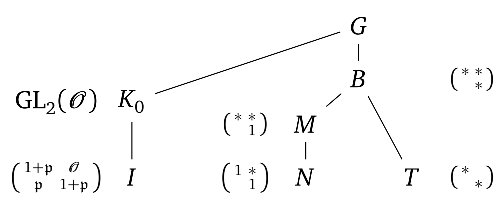
This implies the existence of a Whittaker model for any irreducible smooth representation, and the 'local multiplicity one' result that the Whittaker model is unique. A smooth representation admits a *Whittaker model* if there is an inclusion of G-representation

Week 4

Talk by Jan. Let F be a non-Archimedean local field. In this talk, we will classify all the irreducible smooth representations of $GL_2(F)$ that appear in representations induced from T. These representations are called *principal series*. In the second half of the talk, we will start constructing *supercuspidal* representations. We will not finish their classification until the end of the next talk. Almost all the material here is shamelessly taken from the excellent book [BH06].

4.1 Recap of Netan's talk

Recall that we are trying to classify all the irreducible smooth representations of $G = GL_2(F)$. This group is quite difficult to understand directly, so our first reflex is to study representations of a subgroup of G whose representations are easier to classify, and then induce those representations to G. A diagram of some of the subgroups we will use is



We will use representations of these subgroups, which are often easier to understand, to construct representations of G. By inflating characters on T and inducing them to G, we obtain the *principal series* representations. They come from $\mathrm{GL}_1(F)$, and are hence the easiest ones to get our hands on. We will classify them in the next section. The representations that are not obtained this way, and hence could be seen as native to $\mathrm{GL}_2(F)$, are called the *supercuspidal* representations. They are harder to construct and classify.

The case of $\mathrm{GL}_2(\mathbf{F}_q)$

To see some tangible examples of these representations, let us compute some examples of representations of $GL_2(\mathbf{F}_q)$ first.

The group $GL_2(\mathbf{F}_2)$**.** This group is not abelian and of order 6, so it is isomorphic to S_3. Its character table is

	e	(**)	(***)
$\mathbf{1}$	1	1	1
ε	1	-1	1
ρ	2	0	-1

,

and by Frobenius reciprocity we have $\mathrm{Ind}_B^G \mathbf{1}_B = \mathbf{1}_G \oplus \rho$. So ρ is the Steinberg representation St_G. Furthermore, T is trivial, so there is a unique supercuspidal representation ε.

The group $\mathrm{GL}_2(\mathbf{F}_3)$**.** This group is a lot bigger, size 48 and 8 conjugacy classes by Netan's talk. He also gave us all the conjugacy classes explicitly. The determinant composed with $\pm 1 \hookrightarrow \mathbf{C}$ gives us a character. More generally, $\mathrm{PGL}_2(\mathbf{F}_3)$ acts triply transitively on $\mathbf{P}^1_{\mathbf{F}_3}$, which consists of 4 points, it must be isomorphic to S_4. We can therefore inflate all the representations of S_4 to G. The remaining three representations can be found by standard cleverness, and we obtain the character table

Order:	1	2	2	3	4	6	8	8
Size:	1	1	12	8	6	8	6	6
	1	1	1	1	1	1	1	1
	1	1	-1	1	1	1	-1	-1
	2	2	0	-1	2	-1	0	0
	2	-2	0	-1	0	1	$\sqrt{2}$	$-\sqrt{2}$
	2	-2	0	-1	0	1	$-\sqrt{2}$	$\sqrt{2}$
	3	3	-1	0	-1	0	1	1
	3	3	1	0	-1	0	-1	-1
	4	-4	0	1	0	-1	0	0

Now which of those two 3-dimensional ones is the Steinberg representation of G (Exercise)? As T is isomorphic to D_{12}, we can easily construct its character table. Restricting representations to T reveals by Frobenius reciprocity that the representations of dimensions $1, 3$ and 4 are principal series. Note that there are indeed $5 = \frac{1}{2}(3^2 + 3) - 1$ of them, as predicted by Netan. This means that the 3 representations of dimension 2 are the supercuspidal ones. Note that one of these was obtained by inflating from $\mathrm{PGL}_2(\mathbf{F}_3) \simeq S_4$, whereas the others were constructed by ad hoc combinatorics.

Back to $\mathrm{GL}_2(F)$

Let $G = \mathrm{GL}_2(F)$, where F is a non-Archimedean local field. We will classify the irreducible smooth representations of G that occur in $\mathrm{Ind}_B^G \chi$ for some character χ of T, in the next section. Let us naively approach this problem first, using intuition from the finite field case.

First attempt. Mimicking the representation theory for finite groups, we now try and characterise irreducible pieces of $\mathrm{Ind}_B^G \chi$. What properties did they have in the finite field case? One thing that made them special is that they contained a subspace that was fixed by N. However, the topology on G makes the same condition really very restrictive!

Exercise. *Let (π, V) be an irreducible smooth representation of G with a non-trivial $\pi(N)$-fixed vector. Show that $\pi = \phi \circ \det$, for some character ϕ of $F^\times$.*

In fact, any *finite dimensional* irreducible smooth representation (π, V) of G must be $\phi \circ \det$ for some character ϕ of $F^\times$. Indeed, the finite dimensionality forces the kernel of π to contain an open subgroup by continuity, and hence an element $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ in N acts trivially on every vector if x is small enough. By conjugating with $\begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix}$, we show that in fact all of N is in the kernel of π. It follows that $\pi = \phi \circ \det$ for some character ϕ of $F^\times$.

Second attempt. Instead, the property we want to use to detect what representations of G can be found by induction from characters of T is that of admitting a non-zero N-trivial quotient. Indeed, this was exactly the definition of the Jacquet functor that Netan introduced. Recall that when (π, V) is a smooth representation of G, its Jacquet module (π_N, V_N) is the largest N-trivial quotient of π, i.e. $V_N := V/\{v - \pi(n)v\}_{v,n}$. Recall that this defines an exact, additive functor

$$\mathrm{Rep}^{sm}(G) \to \mathrm{Rep}^{sm}(T),$$

which is called the *Jacquet functor*. This functor does exactly what we wanted.

Proposition 2. Let (π, V) be an irreducible smooth representation of G. Then π is isomorphic to a G-subspace of $\mathrm{Ind}_B^G \chi$ for some character χ of T if and only if $V_N \neq 0$.

Proof. Frobenius reciprocity gives us that

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \chi) = \mathrm{Hom}_B(\pi, \chi) = \mathrm{Hom}_T(\pi_N, \chi),$$

where the last equality comes from the fact that χ is trivial on N, so any morphism of B-representations into χ factors through the Jacquet module.

If π appears in $\mathrm{Ind}_B^G \chi$, then clearly this implies that $\pi_N \neq 0$. Conversely, assume that $V_N \neq 0$. The irreducibility of V implies that for any non-zero vector $v \in V$, the set $\pi(g)v$ spans V as g runs through G. As π is smooth, v is fixed by a finite index subgroup of K_0, and since $G = BK_0$ we find a finite set $\{v_1, \ldots, v_r\}$ with the properties

- it generates V over B,
- its reduction in V_N generates V_N over T.

Now choose a minimal generating set $\{u_1, \ldots, u_t\}$, and pick the maximal T-subspace U of V_N for the conditions $\{u_1, \ldots, u_{t-1}\} \subseteq U$ and $u_t \notin U$, which exists by Zorn's lemma. There can be no strict T-subspace of V_N strictly constaining U, so V_N/U is an irreducible T-representation, which must be a character. ☐

The irreducible representations (π, V) of G with $V_N \neq 0$ are called *principal series* representations. We say an irreducible representation is *supercuspidal* if $V_N = 0$. These are quite different in nature, and we will turn to them later. First, let us classify the principal series representations of G.

4.2 Classification of principal series

We outline the classification of irreducible smooth principal series representations of G in this section. One reason we never explicitly mention the admissibility condition defined by Netan is that principal series representations automatically satisfy it. This is also true for supercuspidal representations, but the proof is quite different and will be given in the next section.

Proposition 3. Let (π, V) be an irreducible smooth representation of G which is not supercuspidal, then π is an admissible representation.

Proof. As V is a subrepresentation of $\mathrm{Ind}_B^G\chi$ for some character χ, we will prove that this latter representation is admissible. Fix a compact open $K \subseteq \mathrm{GL}_2(\mathscr{O})$, then $B\backslash G/K$ is finite. By definition, we have

$$\left(\mathrm{Ind}_B^G\chi\right)^K = \{f : G \to \mathbf{C} \mid f(bgk) = \chi(b)f(g), \forall b \in B, g \in G, k \in K\},$$

so every double coset in $B\backslash G/K$ has a one-dimensional space of functions supported on it. This shows that $\mathrm{Ind}_B^G\chi$ is admissible. □

The main ingredient we need to classify principal series representations is a good understanding of when $\mathrm{Ind}_B^G\chi$ is irreducible. This is exactly the content of the irreducibility criterion.

Remarks. Recall that $\|x\|$ is defined by $q^{-v_F(x)}$. Also recall that for any left Haar measure μ_G on G we can consider the functional $C_c^\infty(G) \to \mathbf{C}$ defined by

$$f \mapsto \int_G f(xg)d\mu_G(x),$$

for some $g \in G$, which defines another left Haar measure. Hence there is a unique $\delta_G(g) \in \mathbf{R}_+^\times$ such that

$$\delta_G(g)\int_G f(xg)d\mu_G(x) = \int_G f(x)d\mu_G(x).$$

The character δ_G is called the *module* of G. The module δ_B of B is equal to

$$\delta_B : B \to \mathbf{R}_+^\times : tn \mapsto \|t_2/t_1\|, \qquad n \in N,\ t = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}.$$

Indeed, the integral computing $\delta_B(tn)$ can be split up into an integral over T, composed with an integral over N. The integral over N can then easily be computed using $N \simeq F$, and using a Haar measure on F. See [BH06, Proposition 7.6]. We will introduce one more piece of notation. Let $\iota_B^G\sigma = \mathrm{Ind}_B^G(\delta_B^{-1/2} \otimes \sigma)$, for any smooth representation σ of T. This is an exact functor $\mathrm{Rep}^{sm}(T) \to \mathrm{Rep}^{sm}(G)$, which we call the *normalised* or *unitary* smooth induction. This normalisation is very convenient as it simplifies notation greatly and $(\iota_B^G\chi)^\vee \simeq \iota_B^G\check{\chi}$.

Theorem 3 (Irreducibility)**.** Let $\chi = \chi_1 \otimes \chi_2$ be a character of T, then $\iota_B^G\chi$ is reducible if and only if $\chi_1\chi_2^{-1}$ is equal to $x \mapsto \|x\|^{\pm 1}$. Moreover, if it is reducible we have:

- Its G-composition length is 2.
- One composition factor is 1-dimensional, the other infinite dimensional.
- The 1-dimensional factor is a subspace if and only if $\chi_1\chi_2^{-1}(x) = \|x\|^{-1}$, and it is a quotient if and only if $\chi_1\chi_2^{-1}(x) = \|x\|$.

The classification. We can now determine the dimension of $\mathrm{Hom}_G(\iota_B^G\chi, \iota_B^G\xi)$, which is equal to $\mathrm{Hom}_T((\iota_B^G\chi)_N, \xi)$. This boils down to a good understanding of the composite of induction and the Jacquet functor. We get a canonical map $\iota_B^G\chi \to \chi$, given by $f \mapsto f(1)$, which induces a map $\alpha_\chi : (\iota_B^G\chi)_N \to \chi$. The *Restriction-Induction Lemma* [BH06, Section 9.3] proves that this is a surjection, and its kernel is exactly equal to $\chi^w \otimes \delta_B^{-1/2}$. Therefore we get a short exact sequence of B-representations

$$0 \to \chi^w \otimes \delta_B^{-1/2} \to (\iota_B^G\chi)_N \to \chi \otimes \delta_B^{-1/2} \to 0,$$

which splits if $\chi^w \neq \chi$. If the sequence above does not split, we must have $\chi^w = \chi$, and hence $\iota_B^G \chi$ is irreducible by Theorem 3. In both cases, we conclude that $\mathrm{Hom}_G(\iota_B^G \chi, \iota_B^G \xi)$ is zero, unless $\xi = \chi^w$ or χ, in which case it is 1-dimensional.

Whenever $\iota_B^G \chi$ is *reducible*, we can determine its irreducible factors using Theorem 3. In the special case where $\chi = \mathbf{1}_T$, we get an irreducible quotient of $\iota_B^G \chi$, which we call the *Steinberg representation* St_G of G. More generally, for any character ϕ of $F^\times$ we get short a exact sequence

$$0 \to \phi \circ \det \to \iota_B^G(\phi \otimes \phi) \to \phi \cdot St_G \to, 0$$

which we interpret as the definition of $\phi \cdot St_G$.

We summarise the discussion above into a formal classification theorem.

Theorem 4 (Classification theorem for principal series)**.** Any irreducible smooth representation of G which is not cuspidal is isomorphic to one of the following.

- $\iota_B^G \chi$ for $\chi \neq \phi \cdot \delta_B^{\pm 1/2}$ for any character ϕ of $F^\times$,
- $\phi \circ \det$ for some character ϕ of $F^\times$,
- $\phi \cdot \mathrm{St}_G$ for some character ϕ of $F^\times$.

Moreover, all the representation listed above are irreducible and distinct, apart from $\iota_B^G \chi \simeq \iota_B^G \chi^w$.

4.3 Supercuspidal representations

We now turn to the classification of irreducible smooth representations (π, V) whose Jacquet module V_N is zero. First, we establish the same helpful finiteness condition we had before, so we can essentially drop admissibility from our assumptions.

Proposition 4. Every irreducible supercuspidal representation of G is admissible.

We briefly sketch the proof of this theorem, as it relies on an analysis of the important concept of matrix coefficients. For any smooth representation (π, V) of G and vectors $v \in V, \check{v} \in \check{V}$ we get a smooth function

$$\gamma_{\check{v} \otimes v} : g \mapsto \langle \check{v}, \pi(g) v \rangle,$$

which span the space $\mathscr{C}(\pi)$ of *matrix coefficients*. It has a natural action of $G \times G$. When π is irreducible, Schur's Lemma implies that the centre Z acts via a character ω_π, and because $\gamma(zg) = \omega_\pi(z)\gamma(g)$ the support of a matrix coefficient is invariant under translation by Z.

The point of this definition is that we can characterise supercuspidal representations in terms of the support of their matrix coefficients. Let (π, V) be an irreducible supercuspidal representation of G, γ a matrix coefficient for π, and set $t = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. First note that since $V_N = 0$, every $v \in V$ can be written as a finite sum of elements of the form $v_i - \pi(n_i)v_i$. Taking N_0 to be a compact open subgroup of N containing all the n_i, then

$$\int_{N_0} \pi(x) v \, dx = 0.$$

For all a large enough, $t^a N_0 t^{-a}$ fixes $\check{v}$, and hence

$$\langle \check{v}, \pi(t^a)v\rangle = C\int_{t^a N_0 t^{-a}} \langle \check{\pi}(x^{-1})\check{v}, \pi(t^a)v\rangle dx = C'\int_{N_0} \langle \check{\pi}(t^{-a})\check{v}, \pi(x)v\rangle dx = 0$$

Setting K be an open normal subgroup of K_0 fixing v and $\check{v}$, we may use the Cartan decomposition from Netan's talk to show that $k_i^{-1}t^n k_j$ forms a complete set of double coset representatives of $ZK\backslash G/K$, where k_i is a finite set of coset representatives of K_0/K. However, the function $x \mapsto \gamma(k_i x k_j^{-1})$ vanishes when $x = t^n$ for n big enough by the above calculation. Therefore γ is compact modulo center.

From the result that irreducible supercuspidal representations have matrix coefficients with compact support, it follows that they are admissible. If we would be able to find a compact open K such that V^K was infinite-dimensional, then its dimension must be countable and hence $\check{V}^K$ would be uncountably infinite-dimensional. On the other hand, we have a map $\check{V}^K \to \mathscr{C}(\pi)$ given by $\check{v} \mapsto \gamma_{\check{v}\otimes v}$, which is injective because the translates gv span V. This is a contradiction, as $\mathscr{C}(\pi)$ is countably infinite because it consists of functions supported on a finite union of cosets ZK_0gK_0 which satisfy an equivariance condition with respect to Z and K_0.

We have established that irreducible supercuspidal representations are admissible and have matrix coefficients which are compactly supported modulo center. It turns out that the condition on matrix coefficients characterises supercuspidals amongst irreducible admissible smooth representations of G.

Theorem 5. Let (π, V) be an irreducible admissible representation of G. If some non-zero coefficient in $\mathscr{C}(\pi)$ has compact support modulo centre, then all of them do, and π is supercuspidal.

The reason these matrix coefficients are so important, is that they show us the true nature of supercuspidal representations. The above theorem may be interpreted as saying that supercuspidal representations behave as if they were representations of a compact group. This is a key observation that shows it is a good idea to investigate representations of K instead, which in turn are governed by the case of representations of $\mathrm{GL}_2(\mathbf{F}_q)$! Indeed, the following theorem shows that this is a good way to produce irreducible supercuspidal representations.

Theorem 6. Let $K \supset Z$ be a compact open subgroup of G, compact modulo Z. Let (ρ, W) be an irreducible smooth representation of K such that $\mathrm{Hom}_{K^g\cap K}(\rho^g, \rho) \neq 0$ for some $g \in G$ if and only if $g \in K$. Then $\mathrm{c}-\mathrm{Ind}_K^G\rho$ is an irreducible supercuspidal representation of G.

Remark. When $\mathrm{Hom}_{K^g\cap K}(\rho^g, \rho) \neq 0$, we say that g *intertwines* ρ. This condition might seem very strange and unmotivated, so let us explain how to think about this. Generalising Netan's Hecke algebra construction, we define the *ρ-spherical Hecke algebra* or *intertwining algebra* $\mathscr{H}(G, \rho)$ to be the space of functions $f : G \to \mathrm{End}_{\mathbf{C}}(W)$ which are compactly supported modulo Z and satisfy $f(k_1 g k_2) = \rho(k_1)f(g)\rho(k_2)$ for all $k_i \in K$ and $g \in G$. This has the structure of a unital associative $\mathbf{C}$-algebra by

$$\phi_1 * \phi_2(g) = \int_{G/Z} \phi_1(x)\phi_2(x^{-1}g)d\dot{\mu}(x),$$

where $\dot{\mu}$ is a Haar measure on G/Z. This definition is clearly a natural analogue of Netan's definition of the spherical Hecke algebra, now that we have motivated an interest in functions with compact support module center based on the properties of matrix coefficients for supercuspidals. There is a canonical isomorphism between $\mathrm{Hom}_{K^g\cap K}(\rho, \rho^g)$ and the space of functions in $\mathscr{H}(G, \rho)$ supported on KgK, resulting in the following lemma, [BH06, Lemma 11.2].

Lemma 5. Let $g \in G$, then there exists $\phi \in \mathscr{H}(G, \rho)$ with support KgK if and only is g intertwines ρ.

Week 5

Talk by Jan. We discuss how to construct supercuspidal representations of $G = \mathrm{GL}_2(F)$ using the Weil representation.

Status. Recall some of the highlights we've proved so far. Firstly, we've proved that any irreducible smooth representation of G is admissible. This implies many good finiteness properties, for instance the existence of a trace. Secondly, we've shown that there are two main classes of representations, which we called the principal series and supercuspidal representations. We've completely classified the principal series, and understand very well when they are irreducible (most of the time) and when they are not. We don't know very much about the supercuspidals, but we have shown that they are characterised by the compactness modulo centre of the support of their matrix coefficients. This prompted us to look for representations that can be (compactly) induced from subgroups that are compact modulo centre, and we know that these are all irreducible supercuspidal under some mysterious *intertwining* condition.

5.1 Back to finite fields (once more!)

Last week we calculated some explicit examples, to see which representations were principal series, and which were supercuspidal. The methods we used to construct the character tables were ad-hoc, and only the construction of principal series provided us with a systematic way to obtain representations of $\mathrm{GL}_2(\mathbf{F}_q)$. The construction of all irreducible representations was first achieved by Green, and there is an algebro-geometric construction in the case of finite groups of Lie type by Deligne-Lusztig. We will follow the treatment of Weil, which we will be able to generalise for a non-Archimedean local field.

First we investigate $S = \mathrm{SL}_2(\mathbf{F}_q)$. It has a very explicit presentation, generated by elements $t(x)$ for $x \in \mathbf{F}_q^\times$, $n(x)$ for $x \in F$, and w, subject to the relations

- $t(x_1)t(x_2) = t(x_1x_2)$
- $n(x_1)n(x_2) = n(x_1 + x_2)$
- $t(x_1)n(x_2)t(x_1^{-1}) = n(x_1^2x_2)$
- $wt(x)w = t(-x^{-1})$
- $wn(x)w = t(-x^{-1})n(-x)wn(-x^{-1})$.

The isomorphism of the group defined by this presentation with S is given by $t(x) \mapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}$, $n(x) \mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ and $w \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We will use this presentation to describe the Weil representation of S. This is a representation that depends on a choice of a 2-dimensional commutative semi-simple $\mathbf{F}_q$-algebra E (hence *split* $E = \mathbf{F}_q \oplus \mathbf{F}_q$ or *anisotropic* $E = \mathbf{F}_{q^2}$) and a choice of a non-trivial additive character $\psi : \mathbf{F}_q \to \mathbf{C}^\times$.

Fourier transforms. We fix once and for all such an E and ψ. Let W be the q^2-dimensional $\mathbf{C}$-vector space of functions $E \to \mathbf{C}$. This will be the vector space underlying the Weil representation. We have a notion of a Fourier transform. First of all, we have a natural involution ι on E, which is conjugation when E is a field, and swaps the factors when $E = \mathbf{F}_q \oplus \mathbf{F}_q$. For $f \in W$, we define

$$\widehat{f}(x) = \epsilon q^{-1} \sum_{y \in E} f(y) \psi(\mathrm{Tr}(x^\iota y)),$$

where ϵ is 1 for E split, and -1 for E anisotropic. We can check that $\widehat{\widehat{f}}(x) = f(-x)$.

The Weil representation. Using the above presentation, we can now show the existence of a representation of $\mathrm{SL}_2(\mathbf{F}_q)$ satisfying certain conditions on the above set of generators.

Theorem 7. There is a unique representation $\mathrm{SL}_2(\mathbf{F}_q) \to \mathrm{Aut}_{\mathbf{C}}(W)$ satisfying:

- $\big(t(x) \cdot f\big)(e) = f(xe)$ for all $x \in F^\times$
- $\big(n(x) \cdot f\big)(e) = \psi(x \, \mathrm{Nm}(e)) f(e)$ for all $x \in F$
- $(w \cdot f)(e) = \widehat{f}(e)$.

Decomposition. Now that we've defined the Weil representation of $\mathrm{SL}_2(\mathbf{F}_q)$ attached to (E, ψ), we wonder how it decomposes. Are all its simple factors principal series, or do we get supercuspidals as well? First off, we note that this representation has dimension q^2. We will see in a second that in fact this can be promoted to a G-representation, and as such it must split from our experience with toy examples. Let E_1 denote the kernel of the norm map $\mathrm{Nm} : E \to \mathbf{F}_q^\times$, and χ a character of $E^\times$ that does not factor through E_1. We define

$$W(\chi) := \{f \in W \mid f(e_1 e) = \chi(e_1)^{-1} f(e), \; e_1 \in E_1\}$$

These are $\mathrm{SL}_2(\mathbf{F}_q)$-subspaces, which can be checked on generators. For instance, w preserves $W(\chi)$ because

$$(w \cdot f)(e_1 e) = \epsilon q^{-1} \sum_{x \in E} f(x) \psi\left(\mathrm{Tr}((e_1 e)^\iota x)\right) = \epsilon q^{-1} \sum_{x/e_1 \in E} f\left(e_1 \frac{x}{e_1}\right) \psi\left(\mathrm{Tr}(e^\iota \frac{x}{e_1})\right) = \chi(e_1)^{-1} (w \cdot f)(e).$$

Extension to a G-representation. We have constructed Weil representation of $\mathrm{SL}_2(\mathbf{F}_q)$, but we can easily make this into a G-representation. We pick a χ as above, and define the action of G on $W(\chi)$ by decomposing $g \in G$ as $\gamma \cdot \begin{pmatrix} d & 0 \\ & 1 \end{pmatrix}$ with $\gamma \in \mathrm{SL}_2(\mathbf{F}_q)$ and setting

$$(g \cdot f)(e) = \chi(b) f(bx), \qquad \text{where } \; b \in E \; \text{ such that } \; \mathrm{Nm}(b) = d.$$

Clearly this defines an action of G on $W(\chi)$.

Case (a). Assume we are in the split case, and hence $E = \mathbf{F}_q \oplus \mathbf{F}_q$. In this case, we check that $W(\chi)$ has dimension $q + 1$. Now χ decomposes as the sum of two characters, and it is not hard to show that $W(\chi)$ is just $\mathrm{Ind}_B^G(\chi)$, see [Bum98, Proposition 4.1.4]. These hence correspond to principal series, and we know all we need to know about when they are irreducible.

Case (b). When we are in the anisotropic case, $E = \mathbf{F}_{q^2}$, we find that the dimension of $W(\chi)$ is $q-1$. Indeed, let $f \in W(\chi)$, then $f(0) = 0$ and f is determined by its value on a set of representatives for $E^\times/E_1$. It is not so hard to show that $W(\chi)$ is irreducible, see [Bum98, Proposition 4.1.6]. These representations must therefore be supercuspidals!

If we count how many representations this procedure gives us, we see that we have in fact constructed all the irreps of G! By identifying $\mathbf{F}_q \oplus \mathbf{F}_q$ and $\mathbf{F}_{q^2}$ with the split and non-split Cartan subgroups of G, we now see that all representations of G are induced from characters on tori in G. This is going to generalise to the case where F is a non-Archimedean field. As a sanity check, let us have another look at the character table of $\mathrm{GL}_2(\mathbf{F}_q)$ and see so many things fall into place. How much of it can you reconstruct without looking?

Order:	1	2	2	3	4	6	8	8
Size:	1	1	12	8	6	8	6	6
	1	1	1	1	1	1	1	1
	1	1	−1	1	1	1	−1	−1
	2	2	0	−1	2	−1	0	0
	2	−2	0	−1	0	1	$\sqrt{2}$	$-\sqrt{2}$
	2	−2	0	−1	0	1	$-\sqrt{2}$	$\sqrt{2}$
	3	3	−1	0	−1	0	1	1
	3	3	1	0	−1	0	−1	−1
	4	−4	0	1	0	−1	0	0

5.2 Supercuspidal representations of $\mathrm{GL}_2(F)$

We now try to construct all the supercuspidal representations of $G = \mathrm{GL}_2(F)$, mimicking the situation for finite fields. This can be found in great detail in Jacquet-Langlands [JL70, Section 1]. Everything will be modelled on the case of finite fields, so far any definition below you should ask yourself what it is trying to mirror!

The Weil representation for $\mathrm{SL}_2(F)$**.** Let K be one of the following four F-algebras: (a) $F \oplus F$; (b) a separable quadratic extension of F; (c) the unique non-split quaternion algebra D_F over F; or (d) the matrix algebra $M_2(F)$. Each of these come with an obvious involution ι, as well as a trace map Tr and a norm map Nm defined by

$$\mathrm{Tr}(k) = k + \iota(k), \qquad \mathrm{Nm}(k) = k \cdot \iota(k).$$

We define $\omega : F^\times \to \pm 1$, which is the quadratic character associated to K in case (b), and trivial otherwise. We also pick a non-trivial locally constant $\tau : F \to \mathbf{C}^\times$, and set $\tau_K = \tau \circ \mathrm{Nm}$.

Analogously to the Weil representation we constructed over finite fields, we can attach to the pair (K, τ) a representation (ρ, W) of $\mathrm{SL}_2(F)$ satisfying the analogous conditions on generators.

Theorem 8. There is a unique representation of $\mathrm{SL}_2(F)$ on $C_c^\infty(K)$ satisfying

- $(t(x) \cdot f)(k) = \omega(x)|k|_K^{1/2} f(xk)$
- $(n(x) \cdot f)(k) = \tau_F(x\,\mathrm{Nm}(k)) f(k)$
- $(w \cdot f)(k) = \gamma \int_K f(x) \tau_K(kx) d\mu_K(x)$.

Here γ is an explicit constant, depending on K.

Proof. The presentation for $\mathrm{SL}_2(\mathbf{F}_q)$ given in the previous section remains valid for any base field, so all we need to do is check that the above constraints on the representation respect the relations of that presentation. This can be done case by case. □

Extension to G. We now extend the Weil representation of $\mathrm{SL}_2(F)$ to G. Let G_+ be the subgroup of G consisting of all elements with determinant in $\mathrm{Nm}(K^\times)$. This group is equal to G, except when we are in case (b) and K is a separable quadratic extension of F.

Let (π, U) be a finite-dimensional smooth irreducible representation of $K^\times$, and let $C_c^\infty(K,\pi)$ be the subspace of $C_c^\infty(K) \otimes U$ consisting of functions f such that $f(xk_1) = \pi(k_1)^{-1} f(x)$ for all $x \in K$ and $k_1 \in K_1$, where K_1 is the subset of $K^\times$ consisting of all elements of norm 1. We make the $\mathrm{SL}_2(F)$-representation $C_c^\infty(K,\pi)$ into a G_+-representation by setting

$$\left(\begin{pmatrix} \mathrm{Nm}(k) & 0 \\ 0 & 1 \end{pmatrix} \cdot f \right)(x) = |k|_K^{1/2} \pi(k) f(xk), \qquad x \in K,\ k \in K^\times.$$

Finally, to obtain a representation of G, just induce up from G_+.

Case (a). We assume that $K = F \oplus F$. In this case we might expect the Weil representation to give rise to principal series. Indeed, they do, but because we no longer have semi-simplicity we can only obtain them as quotients of the Weil representation. See the discussion in [Bum98], starting on page 543. We might think that we wouldn't really care about this case, as we have a much simpler way to construct these representations. However, there is a notion of *Whittaker models* of representations, and we obtain Whittaker models for the principal series by constructing them through the Weil representation.

Case (b). We assume that K is a separable quadratic extension of F, then we wonder what representation of G we have constructed. This is described in [JL70, Theorem 4.6].

Theorem 9. The representation $C_c^\infty(K,\pi)$ is admissible and irreducible. If π factors through the norm map Nm, so that $\pi = \chi \circ \mathrm{Nm}$ for some character χ of $F^\times$, then it is isomorphic to $\iota_B^G(\chi \otimes \chi\omega)$. Otherwise, $C_c^\infty(K,\pi)$ is supercuspidal.

The supercuspidal representations constructed this way are called *dihedral*. Tunnell proved in his doctoral thesis that in odd residue characteristic, every supercuspidal representation is dihedral. So in this case, we again obtain that irreducible admissible representations are induced from characters of maximal tori of G!

Case (c). Now assume that K is the non-split quaternion algebra over F. The following theorem is [JL70, Theorem 4.2].

Theorem 10 (Jacquet-Langlands)**.** The representation $C_c^\infty(K,\pi)$ is admissible, and is isomorphic to $\dim(U)$ copies of an irreducible G-representation $\mathrm{JL}(\pi)$. If $\dim(U) \geq 2$, then $\mathrm{JL}(\pi)$ is supercuspidal, and if $\dim(U) = 1$ then $\pi = \chi \circ \mathrm{Nm}$ for χ a character of $F^\times$ and $\mathrm{JL}(\pi)$ is isomorphic to $\chi \cdot St_G$.

So the two cases that corresponded to the maximal tori of G gave rise to several irreducible smooth representations through this Weil representation construction, but suddenly we've now produced many of the same representations, starting with finite dimensional representations of a quaternion algebra. There is a precise way to match up both ways to obtain the same representations.

Week 6

Talk by Alex. This talk marks the start of the Archimedean part of the representation theory of GL_2. It discusses the classification of irreducible admissible $\mathrm{GL}_2(\mathbf{R})$-modules.

6.1 Introduction

In order to begin the archimedean part of automorphic theory, we want to understand certain infinite-dimensional irreducible representations of Lie groups G, in particular for us the group $\mathrm{GL}_2 = \mathrm{GL}_2(\mathbf{R})$. Two natural strategies present themselves.

Firstly, we could mirror the proof of the classification of supercuspidal representations in the nonarchimedean case by examining the restriction of the representation to a maximal compact Lie subgroup K, for instance the subgroup $\mathrm{O}_2 \leq \mathrm{GL}_2$. The reason this seems promising is that the theory of continuous representations of compact groups is well-understood:

Lemma 6.1.1 (Representations of compact groups)**.** *Let (π, V) be a continuous representation of a compact group K on a Hilbert space. Then*

1. *the inner product on V may be chosen, without changing the topology, so that the K-action is unitary;*
2. *should V be irreducible, it is necessarily finite-dimensional;*
3. *in general, V is completely reducible: it is the closure of the (orthogonal) direct sum of some irreducible K-subrepresentations.*

Most often we will write the final condition as follows: we let (γ) be a complete list of (finite-dimensional) irreducible representations of K, and $V(\gamma)$ the γ-isotypic component of V. Then

$$V = \overline{\bigoplus_{\gamma} V(\gamma)}$$

The second possibility is to mimic the relationship between the representations of the Lie group and the Lie algebra found in the theory of finite-dimensional representation theory.

Lemma 6.1.2 (Representations of Lie groups)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then*

1. *any finite-dimensional continuous representation* (π_G, V) *of* G *can be made a representation of* $\mathfrak{g}$ *by*

$$\pi_{\mathfrak{g}}(x)\cdot v := \lim_{t\to 0}\frac{1}{t}\left(\pi_G(\exp(tx)) - 1\right)\cdot v$$

2. *from this representation of* $\mathfrak{g}$*, the original representation can be recovered on the identity component of* G *by*

$$\pi_G(\exp(x))\cdot v = \sum_{r=0}^{\infty}\frac{\pi_{\mathfrak{g}}(x)^r}{r!}\cdot v$$

3. *for* G *connected,* (π_G, V) *is irreducible iff* $(\pi_{\mathfrak{g}}, V)$ *is.*

Example 6.1.3 (Finite-dimensional representations of GL_2^+ and GL_2)**.** Recall (modified from the representation theory of $\mathfrak{sl}_2$) that the finite-dimensional irreducible representations of $\mathfrak{gl}_2 \cong \mathfrak{sl}_2 \times \mathbf{R}$ are given by the degree k homogenous polynomials in x, y, with $\mathfrak{gl}_2$ action given by

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = x\frac{\mathrm{d}}{\mathrm{d}y}$$
$$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = y\frac{\mathrm{d}}{\mathrm{d}x}$$
$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = x\frac{\mathrm{d}}{\mathrm{d}x} - y\frac{\mathrm{d}}{\mathrm{d}y}$$
$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mu$$

for some scalar μ. A more enlightened way of phrasing this is that the irreducible representations are given by $\mathrm{Symm}^k(V_1)\otimes \mu\mathrm{tr}$ where V_1 is the standard 2-dimensional representation.

Exponentiating up this tells us a classification of the finite-dimensional irreducible representations of *the identity component of* GL_2, namely GL_2^+, the group of positive-determinant matrices. Specifically, it tells us that its irreducible representations are given by $\mathrm{Symm}^k(V_1)\otimes \chi\circ\det$, where now V_1 is the standard 2-dimensional representation of GL_2^+ and $\chi : \mathbf{R}^{>0} \to \mathbb{C}^\times$ is a quasicharacter.

Finally we need to extend this to all of GL_2, i.e. we need to worry about the action of $\eta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. However, we can always extend the action on $\mathrm{Symm}^k(V_1)\otimes \chi\circ\det$ by specifying that

$$\eta\cdot f(x,y) = \pm f(x,-y)$$

to produce an irreducible GL_2-representation. The choice of sign can be subsumed into the character χ, so we have again produced irreducible representations $\mathrm{Symm}^k(V_1)\otimes\chi\circ\det$ where now V_1 is the standard representation of GL_2 and $\chi : \mathbf{R}^\times \to \mathbf{C}^\times$ is any character. To see that this is all irreducible representations, one can use a straightforward Frobenius argument.

The realisation (due to Harish-Chandra) that allows us to proceed with a classification of infinite-dimensional representations is that we need to consider both these ideas simultaneously in order to be able to get a handle on the behaviours involved.

6.2 Reduction to $(\mathfrak{g}, K)$-modules

The general setup we will be considering is that of a continuous action π by a Lie group G on a complex Hilbert space V. We will let K be a maximal compact subgroup (so that we may assume K, but not necessarily G, acts unitarily on V). For our purposes though, we will only need the case when $G = \mathrm{GL}_2$ and $K = \mathrm{O}_2$ (or $G = \mathrm{GL}_2^+$ and $K = \mathrm{SO}_2$), so not all these proofs may work in complete generality.

From hereon, a representation of a Lie group G will always mean a continuous representation on a Hilbert space. We may occasionally assume that the action of a maximal compact subgroup K is unitary, since this can always be ensured.

In order to make the classification problems tractable (and because these are many of the examples we see), we introduce the following

Definition 6.2.1. Let (π, V) be a representation of G. We say (π, V) is *admissible* just when each K-isotypic component $V(\gamma)$ is finite-dimensional. We say that (π, V) is *irreducible* just when it has no non-trivial *closed* invariant subspaces.

Remark 6.2.2. All finite-dimensional representations are admissible, as are all irreducible unitary representations. In some sense, admissibility is the smallest sensible property which subsumes both of these.

One problem that immediately presents us is that we can't manufacture an action of $\mathfrak{g}$ on all of V. For example, $L^2(S^1)$ with the right regular action of S^1 is a representation of the circle group, and $v = \sum_{r>0} r^{-\frac{4}{3}} z^r$ is a perfectly good element of it, but if we try to define an action of $i \in i\,\mathbf{R} = \mathrm{T}_1(S^1)$ on v, then we should calculate this to be

$$\lim_{t \to 0} \sum_{r>0} r^{-\frac{4}{3}} \frac{e^{irt} - 1}{t} z^r = \sum_{r>0} i r^{-\frac{1}{3}} z^r$$

which is not square-integrable.

However, we can make some headway by looking at a restricted (non-closed!) subspace of V.

Definition 6.2.3. Let (π, V) be a representation of G. We define the subspace

$$V^{\mathrm{fin}} := \bigoplus_{\gamma} V(\gamma) \leq V = \overline{\bigoplus_{\gamma} V(\gamma)}$$

(the K-isotypic decomposition) so that $V^{\mathrm{fin}} \leq V$ is dense and K-stable. Equivalently, V^{fin} is the set of all vectors such that $\pi(K) \cdot v$ only spans a finite-dimensional subspace (such vectors are called K-finite).

Proposition 5 (Smoothness of K-finite vectors)**.** Let (π, V) be an *admissible* representation of G, and let $v \in V$ be a K-finite vector. Then the map $G \to V$ given by $X \mapsto \pi(X) \cdot v$ is smooth, i.e. it is infinitely differentiable. Such a vector is referred to as a *smooth* vector.

Sketch proof, see also Bump Proposition 2.4.5: Recall that $C_c^\infty(G)$ acts on V by

$$\pi(f) \cdot v = \int_G f(X^{-1}) \pi(X) \cdot v \, \mathrm{d}X$$

where the integral is taken in the sense of Riemann with respect to left Haar measure. The vectors $\pi(f) \cdot v$ are always smooth (their derivatives can be written down explicitly in terms of those of f).

On the other hand, if we let $\chi : K \to \mathbf{C}$ be the character associated to the irreducible K-representation γ, i.e. $\chi(Y) = \dim(\gamma)\mathrm{tr}(\gamma(Y))$, then the vector

$$\int_K \chi(Y^{-1})\pi(Y)\cdot v \ \mathrm{d}Y$$

always lies in $V(\gamma)$. This is the compact group version of the idempotent decomposition.

These two identities can be combined usefully. Let $\phi_0 \in C_c^\infty(G)$ and let $\phi = \chi *_K \phi_0$ be the convolution, i.e.

$$\phi(X) = \int_K \chi(Y^{-1})\phi_0(XY) \ \mathrm{d}Y$$

so that ϕ is also C^∞ and compactly supported (it's supported in $\mathrm{supp}(\phi_0)K$). Now we have the identity

$$\begin{aligned}
\pi(\phi)\cdot v &= \int_G \left(\int_K \chi(Y^{-1})\phi_0(X^{-1}Y) \ \mathrm{d}Y\right)\pi(X)\cdot v \ \mathrm{d}X \\
&= \int_K\int_G \chi(Y^{-1})\phi_0(X^{-1}Y)\pi(X)\cdot v \ \mathrm{d}X \ \mathrm{d}Y \\
&= \int_K\int_G \chi(Y^{-1})\phi_0(Z^{-1})\pi(YZ)\cdot v \ \mathrm{d}Z \ \mathrm{d}Y \\
&= \int_K \chi(Y^{-1})\pi(Y)\cdot\left(\int_G \phi_0(Z^{-1})\pi(Z)\cdot v\right) \ \mathrm{d}Y
\end{aligned}$$

From the top line we see that $\pi(\phi)\cdot v$ is always smooth, and from the bottom line we see that it is always in $V(\gamma)$. If now $v \in V(\gamma)$ itself, then we shall choose some delta-sequence of ϕ_0 (i.e. positive smooth functions of integral 1 whose support shrinks to $\{1\}$). Then we see that $\pi(\phi_1)\cdot v \to v$, so

$$\pi(\phi)\cdot v \to \int_K \chi(Y^{-1})\pi(Y)\cdot v \ \mathrm{d}Y = v$$

so we see that v is a limit of smooth vectors in $V(\gamma)$. In other words, the smooth vectors in $V(\gamma)$ are dense, so that every vector in $V(\gamma)$ is smooth, since it is finite-dimensional. This concludes the proof. □

Corollary 1 ($\mathfrak{g}$-action on K-finite vectors)**.** Let (π, V) be an admissible representation of G. Then $\mathfrak{g}$ acts on V^{fin} by the formula

$$\pi(x)\cdot v = \lim_{t\to 0}\frac{1}{t}\left(\exp(tx) - 1\right)\cdot v$$

V^{fin} is $\mathfrak{g}$-stable with this action (though it needn't be G-stable!) and satisfies

1. for all $v \in V^{\mathrm{fin}}$, the K-span of v is finite-dimensional, and the K-action thereon is continuous;
2. the infinitesimal K-action agrees with that of $\mathfrak{g}$, i.e. if $y \in \mathfrak{k}$ is in the Lie algebra of K then

$$\pi(y)\cdot v = \lim_{t\to 0}\frac{1}{t}\left(\exp(ty) - 1\right)\cdot v$$

3. the $\mathfrak{g}$-action is compatible with the adjoint action of K on $\mathfrak{g}$, i.e. for $x \in \mathfrak{g}$ and $Y \in K$ we have

$$\pi(YxY^{-1}) = \pi(Y)\pi(x)\pi(Y)^{-1}$$

Remark 6.2.4. A structure obeying the above conditions is referred to as a $(\mathfrak{g}, K)$-module, and is termed *admissible* just when the K-isotypic components of the $(\mathfrak{g}, K)$-module are all finite-dimensional. Notice that the notion of a $(\mathfrak{g}, K)$-module features essentially no analysis or topology.

In the absence of admissibility, one can still recover a $(\mathfrak{g}, K)$-module by instead taking $V^{\text{fin}} \cap V^{\infty}$, the space of all *smooth* K-finite vectors. However, in general we will only obtain admissible $(\mathfrak{g}, K)$-modules from admissible G-representations.

The main reason that this structure is useful to us is that it is sensitive to the submodule structure of our representation. Specifically

Theorem 6.2.5 (G-submodules and $(\mathfrak{g}, K)$-submodules)**.** *Let (π, V) be an* admissible *G-representation. Then there is a bijection between closed G-subrepresentations of V and $(\mathfrak{g}, K)$-submodules of V^{fin}, given on the one hand by $U \mapsto U^{\text{fin}} = U \cap V^{\text{fin}}$ and on the other $W \mapsto \overline{W}$.*

Proof. Proving that the operations are mutually inverse is not difficult. The equality $\overline{U^{\text{fin}}} = U$ we have already seen, when we remarked that V^{fin} was dense in V. To prove the other equality, we may suppose that π is unitary as a K-representation, so that the K-isotypic decomposition

$$V = \overline{\bigoplus_{\gamma} V(\gamma)}$$

is orthogonal. We can then write $\overline{W} = W^{\perp\perp}$, so that $\overline{W}(\gamma) = W(\gamma)^{\perp\perp}$ (where we restrict the inner product to $V(\gamma)$). Yet $V(\gamma)$ is finite-dimensional, so $W(\gamma)^{\perp\perp} = W(\gamma)$, and so we've seen $\overline{W}(\gamma) = W(\gamma)$, i.e. $\overline{W}^{\text{fin}} = W$.

The subtlety in this theorem is in proving that $\overline{W}$ is always a G-representation. This is true for general Lie groups, but for GL_2 (or any GL_n) we can remove some of the technical details, by using the identity (Exercise 2.4.2. of Bump)

$$\pi(\exp(x)) \cdot v = \left(\sum_{0}^{\infty} \frac{1}{r!} \pi(x)^r \right) \cdot v$$

valid whenever $x \in \mathfrak{g}$ and v a smooth vector (in particular for $v \in W$). In particular, this directly tells us that $\pi(\exp(x)) \cdot W \subseteq \overline{W}$, so that $\overline{W}$ is stable under the action of all $\exp(x)$, i.e. the action of the identity component GL_2^+ of GL_2. To complete the proof, just note that $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{O}_2$, so that W, and hence $\overline{W}$, are already stable under the action of η, and hence of all of GL_2. □

Corollary 2. An admissible G-representation is irreducible iff its associated $(\mathfrak{g}, K)$-representation is.

Remark 6.2.6. Because of the utility of working with $(\mathfrak{g}, K)$-modules, we often only try to classify G-representations up to *infinitesimal equivalence*, i.e. up to isomorphism of their associated $(\mathfrak{g}, K)$-modules. In fact, for GL_2 this doesn't lose us anything: two GL_2-representations are isomorphic iff they are infinitesimally equivalent (although this is highly GL_2-specific).

6.3 Classification of GL_2^+-representations

6.3.1 Understanding $(\mathfrak{gl}_2, \mathrm{SO}_2)$-modules

With the theoretical machinery developed, we will be able to classify irreducible admissible $(\mathfrak{gl}_2, \mathrm{SO}_2)$- and $(\mathfrak{gl}_2, \mathrm{O}_2)$-modules, and later on see that these come from bona fide GL_2^+- and GL_2-representations. Since $(\mathfrak{g}, K)$-modules are essentially algebraic objects, we are happy to consider them as modules over $\mathscr{U}$, the universal enveloping algebra of the complexification of $\mathfrak{g}$. The classification will involve a degree of (in?) computation, for which we adopt the following

Notation. We use the following basis of the complexification of $\mathfrak{gl}_2$:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$E = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$$
$$F = \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$$

(note that (H, E, F) is the usual basis of $\mathfrak{sl}_2$ conjugated by $\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$, the same base-change that simultaneously diagonalises SO_2). We let

$$\Delta = \frac{-1}{4}\left(H^2 + 2EF + 2FE\right) = \frac{-1}{4}\left(H^2 + 2H + 4FE\right) = \frac{-1}{4}\left(H^2 - 2H + 4EF\right)$$

denote the Casimir operator, so that the centre of $\mathscr{U}$ is a 2-variable polynomial ring generated by I and Δ.

Before we launch into the calculations we'll need a preliminary lemma:

Lemma 6.3.1 (Schur's lemma)**.** *Let V be an irreducible admissible $(\mathfrak{g}, K)$-module. Then every endomorphism of V is given by multiplication by a scalar. In particular, the centre of $\mathscr{U}$ acts on V by scalars.*

Proof. Exercise. □

Proposition 6 (Preparatory calculations)**.** Let V be an admissible $(\mathfrak{gl}_2, \mathrm{SO}_2)$-module. Since the irreducible representations of SO_2 are just one-dimensional, given by characters $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto e^{ik\theta}$ for $k \in \mathbb{Z}$, we know that

$$V = \bigoplus_k V(k)$$

For all k, H acts on each $V(k)$ by multiplication by k, and $E \cdot V(k) \subseteq V(k+2)$, $F \cdot V(k) \subseteq V(k-2)$.

If additionally both I and Δ act by scalars μ, λ respectively (such a representation is called *quasi-simple*), then EF and FE on each $V(k)$ by multiplication by scalars, namely $\frac{k(2-k)}{4} - \lambda$ and $-\frac{k(k+2)}{4} - \lambda$ respectively.

Proof. The crucial calculation is that of the action of H on $V(k)$: we know that $\exp(i\theta H) = \exp\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ so that for $v \in V(k)$ we have

$$iH \cdot v = \lim_{\theta\to 0} \frac{1}{\theta}\left(\exp(i\theta H) - 1\right) \cdot v = \lim_{\theta \to 0} \frac{1}{\theta}(e^{ik\theta} - 1) \cdot v = ikv$$

so that $H \cdot v = kv$. The fact that $E \cdot V(k) \subseteq V(k+2)$ and $F \cdot V(k-2)$ is immediate from their commutation relations with H.

In the quasi-simple case, the calculations of the actions of EF and FE on $V(k)$ are immediate from the equations

$$-4\Delta = H^2 + 2H + 4FE = H^2 - 2H + 4EF$$

and the fact that Δ acts like the scalar λ. □

Theorem 6.3.2 (Classification of $(\mathfrak{gl}_2, \mathrm{SO}_2)$-modules – uniqueness)**.** *Let V be an irreducible admissible $(\mathfrak{gl}_2, \mathrm{SO}_2)$-module, so that (by Schur) I and Δ act by scalars μ, λ respectively. Then in the decomposition $V = \bigoplus_k V(k)$, each $V(k)$ is at most one-dimensional, and the k for which $V(k) \neq 0$ all have the same parity ϵ (called the* parity *of V).*

If λ is not of the form $k(k-2)$ for $k \in \mathbb{Z}$ of the same parity as V, then all *$V(k)$ with k the same parity as V appear in V, and there is at most one $(\mathfrak{gl}_2, \mathrm{SO}_2)$-module with these parameters. We call this module $P_\mu(\lambda, \epsilon)$, and if $\mu = 0$ refer to it as* principal series*.*

If $\lambda = k(k-2)$ for some $k \equiv \epsilon$ mod 2 (we may suppose wlog $k \geq 1$), then there are three possibilities for the set $\Sigma(V)$ of l with $V(l) \neq 0$, namely

$$\begin{aligned} \Sigma^0(k) &= \{l \equiv k \textit{ mod } 2 : -k < l < k\} \\ \Sigma^+(k) &= \{l \equiv k \textit{ mod } 2 : l \geq k\} \\ \Sigma^-(k) &= \{l \equiv k \textit{ mod } 2 : l \leq -l\} \end{aligned}$$

(note that the first is zero for $k = 1$). In this case, the parameters μ, k and a choice of $* \in \{0, +, -\}$ uniquely determines V, and we call this module $D^*_\mu(k)$. If $* = \pm$ we refer to this as discrete series *for $k > 1$ and* limit of discrete series *for $k = 1$.*

Proof. Firstly, pick some $v \in V(l_0)$ non-zero. Then, since we know that SO_2, H, EF and FE act by scalars on all $V(l)$, it is clear that the **C**-span of $\{v; Ev, E^2v, \ldots; Fv, F^2v, \ldots\}$ is a submodule of V, hence all of V, so we have proven the first part (note that $E^r \cdot v \in V(l_0 + 2r)$ and similarly for $F^r \cdot v$, so that all the $V(l)$ appearing have the same parity).

For the second part, it follows from the calculations of the action of EF on $V(l)$ that in this case both E and F are invertible, and so V contains a non-zero element in each $V(l_0 \pm 2r)$ as desired. For uniqueness, we just note that specifying that F acts invertibly and the (scalar) action of H and EF on each $V(l)$ in enough to reconstruct V.

For the final part, we know that $EF = 0$ on $V(k)$, so that either $F = 0$ on $V(k)$ or $E = 0$ on $V(k-2)$. If the former held, then we can see that $\bigoplus_{l \in \Sigma^+(k)} V(l)$ is a submodule of V, so that it is either 0 or all of V. In other words we see that either $\Sigma(V) \subseteq \Sigma^+(k)$ or $\Sigma(V) \subseteq \Sigma^0(k) \cup \Sigma^-(k)$. In the latter case, the same argument pertaining to $\bigoplus_{l \in \Sigma^0(k) \cup \Sigma^-(k)} V(l)$ establishes the same conclusion.

Similarly, $EF = 0$ on $V(2-k)$, so by the same argument $\Sigma(V) \subseteq \Sigma^-(k)$ or $\Sigma(V) \subseteq \Sigma^0(k) \cup \Sigma^+(k)$.

We've seen that $\Sigma(V)$ is certainly contained in one of the $\Sigma^*(k)$, but we also know that EF acts invertibly on all other $V(l)$ ($l \neq k, 2-k$), so that $\Sigma(V) = \Sigma^*(k)$.

Finally, to prove uniqueness, note that our choices specify the actions of H and EF on each $V(l)$, and specify exactly when $E \cdot v = 0$ and when $F \cdot v = 0$, so that V is determined by these data. □

6.3.2 GL_2^+-representations

With the preceding classification result, two questions now naturally present themselves. Firstly, do all of these supposed $(\mathfrak{gl}_2, \mathrm{SO}_2)$-modules actually occur? Secondly, can all of these be produced from genuine GL_2^+-representations? It transpires that the answer to both of these questions is "yes", and moreover we can produce the desired GL_2^+-representations from representations induced from the Borel subgroup of GL_2^+ (what would be called "principal series" in the nonarchimedean case).

To construct these representations, we fix complex numbers s_1 and s_2, and a parity $\epsilon \in \{0, 1\}$, which together uniquely specify a character of the Borel subgroup by

$$\chi : \begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} = \mathrm{sgn}(a_1)^\epsilon |a_1|^{s_1} |a_2|^{s_2}$$

We want to induce this character up to GL_2^+, so as to obtain a Hilbert space representation of GL_2^+. The right way to do this is to look at the representation[1]

$$\mathfrak{H}(\chi) = \left\{ f \in L^2(G) : f\left(\begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} g \right) = \mathrm{sgn}(a_1)^\epsilon |a_1|^{s_1+\frac{1}{2}} |a_2|^{s_2-\frac{1}{2}} f(g) \right\}$$

endowed with the right regular action of G and inner product

$$\langle f_1, f_2 \rangle = \int_K f_1(Y) \overline{f_2(Y)} \, \mathrm{d}Y$$

The extra factor of $|a_1|^{\frac{1}{2}} |a_2|^{-\frac{1}{2}}$ that has appeared comes from the module of the Borel subgroup, and its usage makes our induction functor better behaved – for example it will preserve unitaricity of the representation.

We can now try to analyse these representations $\mathfrak{H}(\chi)$: the important point is that we have a strong description of a sensible basis in the following

Proposition 7 (Structure of $\mathfrak{H}(\chi)$)**.** Let $V = \mathfrak{H}(\chi)$ be as above and write $\mu = s_1 + s_2$, $s = \frac{1}{2}(s_1 - s_2 + 1)$ and $\lambda = s(1-s)$.

Then the spaces $V(k)$ are zero if $k \not\equiv \epsilon \bmod 2$, and if $k \equiv \epsilon \bmod 2$ then they are one-dimensional, spanned by

$$\phi_k \left(\begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right) = \mathrm{sgn}(a_1)^\epsilon |a_1|^{s_1+\frac{1}{2}} |a_2|^{s_2-\frac{1}{2}} e^{ik\theta}$$

$\mathfrak{H}^{\mathrm{fin}}(\chi)$ is quasi-simple, with I acting like μ and Δ acting like λ respectively.

Proof. Calculation. □

[1] or, more precisely, the set of all square-integrable functions satisfying the desired identity, identifying those that agree almost everywhere

Corollary 3 (Irreducible admissible GL_2^+-representations)**.** There is a symmetry (up to isomorphism) in interchanging s_1 and s_2 in our definitions, so we shall assume for simplicity that $\Re s_1 \geq \Re s_2$, so that $\Re s \geq \frac{1}{2}$.

In light of the preceding proposition and the earlier classification theorem, we see that if s is not of the form $\frac{k}{2}$ where $k \equiv \epsilon \bmod 2$, then $\mathfrak{H}^{\text{fin}}(\chi)$ is irreducible, isomorphic to $P_\mu(\lambda, \epsilon)$. In particular, $\mathfrak{H}(\chi)$ is an irreducible admissible GL_2^+-representation.

If however $s = \frac{k}{2}$ where $k \equiv \epsilon \bmod 2$, then $\mathfrak{H}^{\text{fin}}(\chi)$ has length three, with irreducible factors $\mathfrak{H}_*^{\text{fin}}(\chi)$ for $* \in \{0, +, -\}$ isomorphic to $D_\mu^*(k)$ (except that when $k = 1$ the factor $\mathfrak{H}_0^{\text{fin}}(\chi) = 0$ does not appear). In particular, $\mathfrak{H}(\chi)$ has length three (or two) as a GL_2^+-representation, with factors $\mathfrak{H}_*(\chi)$ for $* \in \{0, +, -\}$.

Moreover, each pair λ, μ arises from a unique pair s_1, s_2 with $\Re s_1 \geq \Re s_2$, so we see that in our classification theorem, all of the identified irreducible $(\mathfrak{gl}_2, SO_2)$-modules do actually exist, and arise from genuine GL_2^+-representations, which we have an explicit description of. We have thus classified all irreducible admissible GL_2^+-representations, up to infinitesimal equivalence.

Exercise. Determine which of the factors $\mathfrak{H}_*^{\text{fin}}(\chi)$ in the second case appear as submodules or as quotients of $\mathfrak{H}^{\text{fin}}(\chi)$. How does this change when $\Re s_1 \leq \Re s_2$?

6.4 Representations of GL_2

There are now a variety of ways of extending our analysis to GL_2-representations. It is possible to do a similar study of $(\mathfrak{gl}_2, O_2)$-modules, but for our purposes it is perhaps easier to just directly induce up representations from GL_2^+.

The key point here is the representations $\mathfrak{H}(\chi)$ can naturally have their action extended to GL_2 in two distinct ways. There are two distinct ways of lifting χ to a character of the Borel subgroup of GL_2, namely

$$\chi_1 \otimes \chi_2 \begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} = \text{sgn}(a_1)^{\epsilon_1} |a_1|^{s_1} \text{sgn}(a_2)^{\epsilon_2} |a_2|^{s_2}$$

for some choice of $\epsilon 1, \epsilon_2 \in \{0, 1\}$ with sum ϵ mod 2. We write $\chi_i = \text{sgn}^{\epsilon_i} |\cdot|^{s_i}$.

Inducing this up to GL_2, we obtain the representations

$$\mathfrak{H}(\chi_1, \chi_2) = \left\{ f \in L^2(GL_2) : f\left(\begin{pmatrix} a_1 & b \\ & a_2 \end{pmatrix} g \right) = \chi_1(a_1)\chi_2(a_2) \left(\frac{a_1}{a_2} \right)^{\frac{1}{2}} f(g) \right\}$$

Such an f is uniquely determined by its restriction to GL_2^+, so that $\mathfrak{H}(\chi_1, \chi_2) \to \mathfrak{H}(\chi)$ is an isomorphism of GL_2^+-representations.

To understand these representations, we consider the action of $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on the representation. A key observation is that (since O_2 is infinite dihedral), the action of η on the SO_2-isotypic components of any $(\mathfrak{gl}_2, O_2)$-module must interchange $V(k)$ and $V(-k)$.

Now in the general case (s not of the form $\frac{k}{2}$ where $k \equiv \epsilon \bmod 2$) $\mathfrak{H}(\chi)$ is an irreducible GL_2^+-representation, so $\mathfrak{H}(\chi_1, \chi_2)$ is an irreducible GL_2-representation. In the remaining case, our calculation of the action of η tells us that it interchanges $\mathfrak{H}_\pm(\chi)$ (which are submodules of $\mathfrak{H}(\chi)$ and preserves $\mathfrak{H}_0(\chi)$ (which is a quotient).

Thus we have in this case two GL_2-representations, namely $\mathfrak{D}_\mu(k) = \mathfrak{H}_-(\chi) \oplus \mathfrak{H}_+(\chi)$, and $\mathfrak{H}_0(\chi_1, \chi_2) = \mathfrak{H}_0(\chi)$. Again by considering the action of η, these two are clearly irreducible (since their underlying $(\mathfrak{gl}_2, \mathrm{O}_2)$-modules are irreducible).

Thus we have found a large collection of irreducible admissible representations of GL_2, and we want to check firstly when these are isomorphic (this is straightforward, since most of them are already nonisomorphic as GL_2^+-representations), and secondly that we have found all such GL_2-representations (at least up to infinitesimal equivalence). This can be done in a variety of ways, for example either using Frobenius reciprocity, or by using a similar argument in the world of $(\mathfrak{gl}_2, \mathrm{O}_2)$-representations to deduce that the underlying $(\mathfrak{gl}_2, \mathrm{O}_2)$-modules of the listed representations is a complete list of the irreducible admissible $(\mathfrak{gl}_2, \mathrm{O}_2)$-modules. After an argument of this form, we find

Theorem 6.4.1 (Classification of irreducible admissible GL_2-representations)**.** *Pick s_1, s_2 complex numbers with $\Re s_1 \geq \Re s_2$ and pick $\epsilon_1, \epsilon_2 \in \{0, 1\}$. Denote by χ_i the character $\mathrm{sgn}^{\epsilon_i} |\cdot|^{s_i}$ and write $s = \frac{1}{2}(s_1 - s_2 + 1)$, $\mu = s_1 + s_2$, $\lambda = s(1-s)$ and $\epsilon = \epsilon_1 + \epsilon_2$. Then*

- *if s is not of the form $\frac{k}{2}$ where $k \equiv \epsilon \bmod 2$, then $\mathfrak{H}(\chi_1, \chi_2)$ is an irreducible representation;*
- *if $s = \frac{k}{2}$ for some such k then $\mathfrak{H}(\chi_1, \chi_2)$ has two irreducible factors: $\mathfrak{H}_0(\chi_1, \chi_2)$ is finite-dimensional and appears as a quotient; and $\mathfrak{D}_\mu(k)$ is infinite-dimensional and appears as a submodule (and is referred to as a* discrete series *representation;*
- *if $k = 1$ in the above case, then note that $\mathfrak{H}(\chi_1, \chi_2) = 0$ and the representation $\mathfrak{D}_\mu(k)$ is referred to instead as* limit of discrete series.

The above are nonisomorphic except that in the second case interchanging ϵ_1 and ϵ_2 does not change the representation $\mathfrak{D}_\mu(k)$, and together these constitute a complete list of irreducible admissible representations of GL_2*, up to infinitesimal equivalence (indeed isomorphism).*

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