A superficial introduction to Langlands functoriality

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Preliminaries remarks on representations: the algebraic case

G/K reductive algebraic group over an algebraically closed field K of characteristic zero.

The *Borel-Weil-Bott* theorem classifies algebraic representations of G, that is, algebraic homomorphisms

$$\rho : G \longrightarrow \mathrm{Aut}(V),$$

where V is a finite-dimensional K-vector space.

In fact, suffices to classify irreducible ones, since G is reductive (e.g. GL_n).

Preliminary remarks on representations: the algebraic case

Let $T \subset B \subset G$ be a maximal torus and a Borel subgroup (e.g. $\mathbb{G}_m^n \subset \{\text{upper triangular matrices}\} \subset GL_n$).

Because $B \simeq U \rtimes T$, any character

$$\chi : T \longrightarrow \mathbb{G}_m$$

or *weight* can be extended to B.

Define

$$V_\chi := \{f : G \longrightarrow V \mid f(bg) = \chi(b)f(g)\}.$$

Then V_χ has a G-action defined by

$$f^g(h) = f(hg).$$

Here, f is required to be algebraic. Thus, this construction could be called *algebraic induction*.

Preliminary remarks on representations: the algebraic case

BWB theorem:

Suppose χ is a dominant weight. Then V_χ is a non-zero irreducible representation. All irreducible representations are obtained this way. If $\chi \neq \chi'$ are dominant, then V_χ is not isomorphic to $V_{\chi'}$.

Thus, the set of irreducible reps of G are parametrized by $X^*(T)_+$, the set of dominant weights of T.

Example: $G = GL_n$.

$$\chi = \prod_i t_i^{w_i} : \mathbb{G}_m^n \longrightarrow \mathbb{G}_m$$

is dominant if $w_1 \geq w_2 \geq \cdots \geq w_n$,

Preliminary remarks on representations: some notation

T: torus.

The weight lattice $X^*(T)$ denotes the group of characters

$$T \longrightarrow \mathbb{G}_m.$$

The coweight lattice $X_*(T)$ denotes the group of cocharacters

$$\mathbb{G}_m \longrightarrow T.$$

Note that there is a pairing

$$\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \longrightarrow \mathbb{Z}.$$

Remark: $T = \mathrm{Spec}(K[X^*(T)])$.

Preliminary remarks on representations: some notation

The *roots* $\Phi \subset X^*(T)$ of G are the characters appearing in the representation of T on $\mathfrak{g} = LieG$.

One can define a set coroots $\Phi^* \subset X_*(T)$ together with a bijection

$$\Phi \simeq \Phi^*,$$

$$\alpha \mapsto \alpha^*$$

such that $s_\alpha(x) = x - \langle x, \alpha^* \rangle \alpha$ is a reflection of the weight lattice.

The quadruple

$$(X^*(T), \Phi, X_*(T), \Phi^*)$$

is the *root datum* of G.

Preliminary remarks on representations: some notation

Example:
$G = GL_n$.
$X^*(T) \simeq \mathbb{Z}^n$, $\Phi = \{e_i - e_j\}$.
$X_*(T) \simeq \mathbb{Z}^n$, $\Phi^* = \{e_i - e_j\}$.

In general, B determines a notion of positivity for roots via

$$\mathfrak{b} = \mathfrak{t} \oplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha,$$

as well as a notion of dominance:
A weight χ is dominant if

$$\langle \chi, \alpha^* \rangle \geq 0$$

for all positive α.

Preliminary remarks on representations: alternative description of the parametrisation

The Weyl group

$$W = N(T)/T$$

acts on $X^*(T)$ by conjugation, and for every weight χ, there is a unique dominant weight in its W orbit.

That is, we can identify the set of dominant weights with the W orbits in $X^*(T)$, which then gives another description of the parameter space for representations.

The dual torus T^* is defined as

$$T^* = \mathrm{Spec}(K[X_*(T)]).$$

Thus,

$$X^*(T^*) = X_*(T); \quad X_*(T^*) = X^*(T).$$

Preliminary remarks on representations: alternative description of the parametrisation

So representations of G are parametrized by W orbits of homomorphisms

$$\mathbb{G}_m \longrightarrow T^*.$$

In fact, there is a *Langlands dual* group $G^* \supset T^*$ such that the root datum for G^* is

$$(X_*(T), \Phi^*, X^*(T), \Phi)$$

and $W = N(T^*)/T^*$.

The union of the conjugates of T^* are exactly the semi-simple elements $[G^*]^{ss}$ of G^*. Thus, we can view the representations as being parametrized by G^*-orbits of homomorphisms

$$\mathbb{G}_m \longrightarrow [G^*]^{ss}.$$

Denote a homomorphism corresponding to the representation ρ by $\ell(\rho)$.

Preliminary remarks on representations: 'functoriality'

Algebraic functoriality:

For reductive groups G_1 and G_2, a homomorphism

$$f : G_1^* \longrightarrow G_2^*$$

induces a transfer

$$\rho \mapsto f_*(\rho)$$

from irreducible representations of G_1 to irreducible representations of G_2

$$\mathbb{G}_m \xrightarrow{\ell(\rho)} [G_1^*]^{ss} \xrightarrow{f} [G_2^*]^{ss}.$$

Preliminary remarks on representations: 'functoriality'

A subtle point:

Suppose G is define over a number field F and we are interested in F-rational representations

$$\rho : G \longrightarrow \mathrm{Aut}(V).$$

Clearly, we need to start with a χ defined over F to get V_χ defined over F. Thus, we need to consider the action of $\Gamma_F := \mathrm{Gal}(\bar{F}/F)$ on

$$(X^*(T), \Phi, X_*(T), \Phi^*).$$

This induces an action on $G^*/\bar{F}$, and it becomes useful to consider the *L-group*

$${}^L G = G^* \rtimes \Gamma_F.$$

Langlands functoriality: big picture

G/F reductive algebraic group over a number field F. We are interested in complex automorphic representations of $G(\mathbb{A}_F)$.

We will also denote $G(\mathbb{A}_F)$ by just G and the set of isomorphism classes of irreducible automorphic representations of G by

$$\mathcal{A}(G).$$

Goal (fantasy): Parametrize automorphic representations of G via conjugacy classes of admissible homomorphisms

$$\mathcal{L} \longrightarrow {}^L G(\mathbb{C}) = G^*(\mathbb{C}) \rtimes \Gamma_F,$$

where $\mathcal{L}$ is the *Langlands group*.
If G is quasi-split, then every continuous algebraic homomorphism should be admissible.

Langlands functoriality: big picture

The Langlands group is supposed the have quotient groups as follows:

$$\mathcal{L} \twoheadrightarrow G_M \twoheadrightarrow \Gamma_F,$$

where G_M is the *motivic Galois group* over F.

Thus, Galois representations

$$\Gamma_F \longrightarrow GL_n(\mathbb{C})$$

and more general motives

$$G_M \longrightarrow GL_n(\mathbb{C})$$

over F are supposed to have automorphic representations

$$\mathcal{L} \longrightarrow GL_n(\mathbb{C})$$

of $GL_n(\mathbb{A}_F)$ associated to them.

Langlands functoriality: big picture

If the goal were realized, then given a homomorphism

$$f : {}^L G_1 \longrightarrow {}^L G_2,$$

the parameter

$$\mathcal{L} \longrightarrow {}^L G_1,$$

for any automorphic representation could be composed

$$\mathcal{L} \longrightarrow {}^L G_1 \xrightarrow{f} {}^L G_2.$$

Langlands Functoriality

A homomorphism $f : {}^L G_1 \longrightarrow {}^L G_2$ with G_2 quasi-split, induces a map

$$f_* : \mathcal{A}(G_1) \longrightarrow \mathcal{A}(G_2).$$

Langlands functoriality: big picture

Examples include

– Jacquet Langlands correspondence: $G_1 = D^*$ for a quaternion algebra D and $G_2 = GL_2$.

– Base-change.

–Symmetric powers: $f : GL(V) \longrightarrow GL(\mathrm{Sym}^k(V))$, e.g. $GL_2 \longrightarrow GL_{k+1}$.

Big picture: small improvement

Break up $G(\mathbb{A}_F)$ as

$$\prod_v{}' G(F_v).$$

Representation $\pi \in \mathcal{A}(G)$ can be written as a restricted tensor product

$$\pi \simeq \otimes_v' \pi(v),$$

where $\pi(v)$ is an *admissible representation* of $G_v = G(F_v)$ and most of them are *unramified*.

Local Langlands correspondence

Proposes to parametrize admissible representations of $G(F_v)$ in terms of admissible homomorphisms

$$WD_v \longrightarrow {}^L G(\mathbb{C}),$$

where WD_v is the *Weil-Deligne group* of F_v.

Local Langlands correspondence: a few definitions

A representation

$$\pi : G_v \longrightarrow Aut(V)$$

on a complex vector space V is admissible if

(1) For any compact open subgroup $J \subset G_v$, V^J is finite-dimensional.

(2) For any $v \in V$, the stabilizer of v is open in G_v.

Local Langlands correspondence: a few definitions

The Weil group of F_v is

$$W_v = I_v \sigma^{\mathbb{Z}} \subset \mathrm{Gal}(\bar{F}_v/F_v),$$

where I_v is the inertia subgroup and σ is a Frobenius element.

Topologize as $W_v \simeq I_{F_v} \rtimes \mathbb{Z}$. Note that

$$W^{ab} \simeq I_{F_v}^{ab} \times \mathbb{Z} \simeq \mathcal{O}_v^* \times \mathbb{Z} \simeq F_v^*.$$

The Weil-Deligne group is

$$WD_v = \mathbb{G}_a \rtimes W_{F_v},$$

where $w \in W_{F_v}$ acts on $\mathbb{G}_a$ by

$$wx = |w|x.$$

Here, $|\cdot|$, the norm on W_{F_v}, is defined by

$$W_{F_v} \longrightarrow W_{F_v}^{ab} \simeq F_v^* \longrightarrow q^{\mathbb{Z}},$$

where $q = |\mathcal{O}_v/m_v|$.

Local Langlands correspondence: a few definitions

A homomorphism $\rho : WD_v \longrightarrow {}^L G$ is admissible if
(1)

$$WD_v \longrightarrow {}^L G \longrightarrow \Gamma_F$$

is the composition

$$WD_v \longrightarrow W_v \hookrightarrow \mathrm{Gal}(\bar{F}_v/F_v) \hookrightarrow \Gamma_F.$$

(2) ρ is continuous;
(3) $\rho(\mathbb{G}_a)$ is unipotent;
(4) $\rho(\sigma)$ is semi-simple;
(5) A certain relevance condition having to do with the field of definition of parabolic subgroups. (Ignore for quasi-split groups.)

Local Langlands correspondence: a few definitions

An admissible ρ is in bijection with pairs

$$(\phi, N)$$

in G^* such that ϕ is semi-simple, N is nilpotent, and

$$\phi N \phi^{-1} = qN.$$

Local Langlands correspondence for GL_n

There is a bijection:

$$\boxed{\text{Irreducible admissible representations } \pi \text{ of } GL_n(F_v)}$$

$$\updownarrow$$

$$\boxed{\text{Admissible homomorphisms } \rho : WD_v \longrightarrow GL_n(\mathbb{C})}.$$

$$\updownarrow$$

$$\boxed{(\phi, N) \in GL_n(\mathbb{C}),\ \phi \text{ semi-simple, } N \text{ nilpotent, } \phi N \phi^{-1} = qN}.$$

Denote by $(\phi(\pi), N(\pi))$ the pair, the *Langlands parameter* corresponding to an admissible representation π.

For a general group, one Langlands parameter is supposed to correspond to several admissible representations, an *L-packet*.

Local Langlands correspondence: a few definitions

Remark:

A continuous l-adic Galois representation

$$\mathrm{Gal}(\bar{F}_v/F_v) \longrightarrow GL_n(\bar{\mathbb{Q}}_l)$$

gives rise to a complex WD representation. When it arises from H^1 of a variety, it is admissible. Hence, there is a corresponding admissible representation of $GL_n(F_v)$.

Local Langlands correspondence: Examples

$n = 1$.

The objects are supposed to be irreducible admissible reps of $GL_1(F_v) = F_v^*$ and continuous homomorphisms $W_v \longrightarrow GL_1(\mathbb{C})$, which all factor to $W_v^{ab} \longrightarrow \mathbb{C}^*$.

But irreducible admissible reps of F_v^* are necessarily 1-dim, so the correspondence in this case reduces to local class field theory $W_v^{ab} \simeq F_v^*$.

Note that for F_v^*, the admissible 1-dim reps are those characters $\chi : F^* \longrightarrow \mathbb{C}^*$ such that $\chi(1 + m_v^n) = 1$ for some n.

Also, for any $GL_n(F_v)$, we have the admissible rep

$$\chi \circ \det .$$

Local Langlands correspondence: Examples

For $n = 2$, need to construct a substantial family of admissible representations of G_v.

$K = GL_2(\mathcal{O}_V)$
$J_n = I + t_v^n M_2(\mathcal{O}_v)$, where $t_v \in m_v \subset \mathcal{O}_v$ denotes a generator of the maximal ideal.
T: diagonal matrices
B: upper-triangular matrices.
U: (identity)+(strictly upper-triangular).
Thus, $B = U \rtimes T$. Also $G_v = BK$.
For

$$b = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

$\delta(b) = |a/c|^{1/2}$.

Local Langlands correspondence: Examples

From χ_1, χ_2, two admissible characters of F_v^*, we can form the character $\chi = \chi_1\chi_2$ of T and hence B.

Then $P(\chi_1, \chi_2)$ consists of the locally constant functions

$$f : G_v \longrightarrow \mathbb{C}$$

such that

$$f(bg) = \chi(b)\delta(b)f(g).$$

The action of G_v is defined by $(gf)(h) = f(hg)$.

Local Langlands correspondence: Examples

Theorem
$P(\chi_1, \chi_2)$ *is an admissible representation.*

Proof.
$P(\chi_1, \chi_2)$ injects by restriction into the locally constant functions on K. Since K is compact, for each f, there is an open J such that f is constant on the left coset of J. Hence, f is fixed by J.

On the other hand, for any open J, let $f \in P(\chi_1, \chi_2)^J$. Then $f|K$ factors through K/J, which is finite. Thus, $P(\chi_1, \chi_2)^J$ is finite-dimensional. □

Local Langlands correspondence: Examples

In fact, $P(\chi_1, \chi_2)$ is irreducible if $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$. We then denote the representation by $\pi(\chi_1, \chi_2)$. Call these the *principal series*.

If $\chi_1/\chi_2 = |\cdot|$, then

$$P(\chi_1, \chi_2) = P(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2}) \simeq P(|\cdot|^{1/2}, |\cdot|^{-1/2}) \otimes (\chi \circ \det).$$

Similarly, if $\chi_1/\chi_2 = |\cdot|^{-1}$, then

$$P(\chi_1, \chi_2) = P(\chi|\cdot|^{-1/2}, \chi|\cdot|^{1/2}) \simeq P(|\cdot|^{-1/2}, |\cdot|^{1/2}) \otimes (\chi \circ \det).$$

The representation $P(|\cdot|^{1/2}, |\cdot|^{-1/2})$ has an irreducible quotient by a one-dim subspace, called the Steinberg representation, denoted

$$St.$$

Similarly, $P(|\cdot|^{-1/2}, |\cdot|^{1/2})$ has a one-dim quotient and an irreducible subspace also isomorphic to St. Thus, we get a collection of *special irreducible representations*

$$\pi(\chi) = St \otimes (\chi \circ \det).$$

Local Langlands correspondence: Examples

There is another family of *supercuspical representations* for G_v that do not occur in the principal series in any way. They correspond to admissible characters χ of L^*, where L/F_v is a quadratic extension, where χ is required not to come from F_v^*.

Local Langlands correspondence: Examples

The Langlands correspondence in this case works as follows:

1. $\chi \circ \det$ corresponds to $\chi|\cdot|^{1/2} \oplus \chi|\cdot|^{-1/2}$.
2. $\pi(\chi_1, \chi_2)$ for $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$ correspond to the rep. $\chi_1 \oplus \chi_2$ of W_v, $(N = 0)$.
3. $St \otimes (\chi \circ \det)$ corresponds to $\chi \oplus \chi|\cdot|$ with $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,
4. The supercuspidal representation associated to a character χ of L^* corresponds to $Ind_{W_v(L)}^{W_v}$.

Local Langlands correspondence: Examples

$V = T_l E \otimes \mathbb{Q}_p$: Galois representation corresponding to an elliptic curve E over $\mathbb{Q}$.

l-adic representation of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$;

$\longrightarrow$ admissible represention $WD_p \longrightarrow GL_2(\mathbb{C})$;

$\longrightarrow$ admissible representation π of $GL_2(\mathbb{Q}_p)$.

Local Langlands correspondence: Examples

Facts:

(1) π is an unramified principal series iff E has good reduction at p.

(2) π is special iff E has potentially semi-stable reduction at p;

(3) π is unramified special iff E has semi-stable reduction at p;

(4) π is ramified principal or supercuspidal iff E has bad but potentially good reduction at p.

(4') π is a ramified principal series iff E has good reduction over an abelian extension of $\mathbb{Q}_p$.

Local Langlands correspondence: Examples

Even for a fairly general group G, there is one family of representations relatively easy to parametrize.

These are the *unramified* representations. That is, we assume $K \subset G_v$ is a hyperspecial subgroup, i.e., like $GL_n(\mathcal{O}_v) \subset GL_n(F_v)$.

An irreducible representation V is unramified if $V^K \neq 0$. In fact, V^K must be an irreducible representation of the spherical Hecke algebra $\mathcal{H}(G_v, K)$ consisting of locally constant functions on G that are bi-invariant under K.

Thus, V determines

$$c : \mathcal{H}(G, K) \longrightarrow \mathbb{C}^*$$

and is determined by it.

Local Langlands correspondence: Examples

But in fact,

$$\mathcal{H}(G,K) \simeq \mathbb{C}[X_*(T)]^W.$$

To get a sense of this when $G = GL_n$, note the Cartan decomposition

$$GL_n(F_v) = \cup_w KwK,$$

where w consists of matrices of the form

$$w = \mathrm{diag}(t_v^{w_1}, t_v^{w_2}, \ldots, t_v^{w_n})$$

with

$$w_1 \geq w_2 \cdots \geq w_n.$$

Local Langlands correspondence: Examples

Thus,

unramified representations V of G_v are in bijection with

algebra homomorphisms $\mathcal{H}(G_v, K) \longrightarrow \mathbb{C}$, which are in bijection with

algebra homomorphisms $\mathbb{C}[X_*(T)]^W \longrightarrow \mathbb{C}$, which are in bijection with

points of T^*/W, which are in bijection with

conjugacy classes of semi-simple elements in G^*.

Back to global conjectures

Recall that the functoriality conjecture proposes that

$$f : {}^L G_1 \longrightarrow {}^L G_2$$

induces a map

$$f_* : \mathcal{A}(G_1) \longrightarrow \mathcal{A}(G_2)$$

as least for G_2 quasi-split.

How to think of this in absence of Langlands group?

Back to global conjectures

Given an automorphic rep π_1 of G_1, can associate a family of local admissible reps

$$(\pi_1(v)) \in \prod_v{}' A([G_1]_v)$$

Thus, get a collection of Langlands parameters

$$(\phi(\pi_1(v)), N(\pi_1(v))),$$

in G_1^*.

Using f, we then get a collection

$$(f(\phi(\pi_1(v))), f(N(\pi_1(v)))$$

of Langlands parameters in G_2^*.

Back to global conjectures

We would like to know that $(f(\phi(\pi_1(v))), f(N(\pi_1(v)))$ corresponds to a global $\pi_2 \in \mathcal{A}(G_2)$.

This might follow from *converse theorems*.

-Hecke

-Weil

-Cogdell, Piatetskii-Shapiro, H. Kim

-Applications to functoriality due to Cogdell, Piatetskii-Shapiro, Shahidi, H. Kim

Back to global conjectures

That is, if $r : G^L \longrightarrow GL_n(\mathbb{C})$ is a representation and $\pi(v)$ is a local admissible rep, then have an L-function

$$L(\pi(v), r, s) = \det(I - q_v^{-s} r(\phi(\pi(v)))|E^{r(N(\pi(v))})^{-1}.$$

Conjecture: If

$$\prod_v L(\pi(v), r, s)$$

is nice for every r, then the collection $\pi(v)$ comes from a global representation. (See, for example, the conjecture of Piatetskii-Shapiro.)

Back to global conjectures

However, the transfer f_* preserves local L-functions, that is, for a representation

$$r :^L G_2 \longrightarrow GL_n(\mathbb{C})$$

of ${}^L G_2$,

$$r \circ f :^L G_1 \longrightarrow GL_n(\mathbb{C})$$

is a representation of ${}^L G_1$.

Back to global conjectures

Furthermore,

$$L(\pi_1(v), r \circ f, s) = L(f_*(\pi_1(v)), r, s).$$

So, nice properties for

$$\prod_v L(f_*(\pi_1(v))), r, s)$$

should follow from those for

$$\prod_v L(\pi_1(v)), r \circ f, s) = L(\pi_1, r, s).$$

Back to global conjectures

That is, we are supposed to have something like a fiber product diagram

$$
\begin{array}{ccc}
\mathcal{A}(G) & \xrightarrow{\ L\ } & \text{Nice entire functions} \\
\downarrow & & \downarrow \\
\prod_v{}' A(G_v) & \xrightarrow{\ L\ } & \text{Euler products}
\end{array}
$$

Back to global conjectures

$$
\begin{array}{ccccc}
\mathcal{A}(G_1) & & \mathcal{A}(G_2) & \xrightarrow{L} & \text{Nice entire functions} \\
\downarrow & & \downarrow & & \downarrow \\
\prod_v{}' A(G_v) & \xrightarrow{f_*} & \prod_v A(G_v) & \xrightarrow{L} & \text{Euler products}
\end{array}
$$

Back to global conjectures

$$
\begin{array}{ccccc}
\mathcal{A}(G_1) & \xrightarrow{f_*} & \mathcal{A}(G_2) & \xrightarrow{L} & \text{Nice entire functions} \\
\downarrow & & \downarrow & & \downarrow \\
\prod_v{}' A(G_v) & \xrightarrow{f_*} & \prod_v{}' A(G_v) & \xrightarrow{L} & \text{Euler products}
\end{array}
$$

Back to global conjectures

In practice, Langlands expects the implication to go the other way: Use functoriality to show that general automorphic L-functions are nice.

He also seems to place much more hope in the trace formula approach to functoriality than converse theorems.

Motivation: Diophantine geometry

X/F variety.
We would like to understand

$$X(F) \subset X(\mathbb{A}_F).$$

Construct a family of motives parametrized by X:

$$Z \longrightarrow X$$

A point $(x_v) \in X(\mathbb{A}_F)$ gives a family of motives (Z_{x_v}) over $\mathbb{A}_F$.

If $(x_v) = x \in X(F)$, then there is a global motive Z_x such that $Z_{x_v} = Z_x \otimes F_v$.

So the local-to-global principle becomes encoded into the problem of whether or not the adelic collection (Z_{x_v}) is global.

Motivation: Diophantine geometry

If all of Langlands work out, there is a reductive group G (for example GL_n) and for each Z_{x_v} an admissible representation $\pi(v)$ of G_v.

But then, if $(x_v) = x \in X(F)$, then there should be a global automorphic π corresponding to it.
That is, we get the following kind of obstruction theory

$$\begin{array}{ccccc} X(F) & \longrightarrow & A(G) & \longrightarrow & \boxed{\text{Nice entire functions}} \\ \downarrow & & \downarrow & & \downarrow \\ X(\mathbb{A}_F) & \longrightarrow & \prod_v' A(G_v) & \longrightarrow & \boxed{\text{Euler Products}} \end{array}$$

Currently, desirable to generalize Z to a family of *mixed motives*. But then, the automorphic theory doesn't work so well, so needs to be generalized to non-reductive groups.

Motivation from Diophantine geometry

More precisely, the Langlands-Hasse-Weil diagram is supposed to be like

$$\begin{array}{ccc} \{\text{global pure motive}\} & \longrightarrow & \text{Nice entire functions} \\ \downarrow & & \downarrow \\ \{\text{family of local pure motives}\} & \longrightarrow & \text{Euler products} \end{array}$$

What we more or less understand is the situation where M is a global motive and we would just like to understand the extensions $\mathrm{Ext}(1, M)$. Then we know an obstruction theory for

$$\mathrm{Ext}(1, M) \longrightarrow {\prod}' \mathrm{Ext}_v(1, M) \cdots\cdots\to$$

Motivation: Diophantine geometry

Would like an amalgamation like

$$\begin{array}{ccc} \{\text{global mixed motives}\} & \longrightarrow & \boxed{?} \\ \downarrow & & \downarrow \\ \{\text{family of local mixed motives}\} & \longrightarrow & \boxed{?} \end{array}$$