Perverse sheaves and the decomposition theorem

Oxford Learning seminar

Week 1

Talk by Balázs Szendrői. In this talk, we give a general overview of classical results on the topology of algebraic varieties. We discuss smooth projective families of varieties and monodromy, and discuss how we could hope to generalise them to more general maps between varieties. For the most part, we will work over $\mathbf{C}$, which is by no means an essential assumption.

1.1 Topology of smooth proper varieties

In this section, we discuss *la Théorème de Lefschetz vache* and the *Hodge decomposition*. These two classical theorems form the starting point of the vast and exciting theory of intersection cohomology, perverse sheaves and the decomposition theorem.

Let us start with a non-singular projective variety $X^n \subset \mathbf{P}^N_{\mathbf{C}}$ of dimension n, and a "generic" hyperplane H. We consider the space $D^{n-1} := X \cap H$, which is non-singular by Bertini's theorem [Har77, Theorem 8.18]. This gives rise to a very ample line bundle $\mathscr{L} := \mathscr{O}_X(D)$ on X. Restriction of cochains now gives us a map from $\mathrm{H}^i(X) := \mathrm{H}^i(X;\mathbf{Q})$ to $\mathrm{H}^i(D)$.

Theorem 1 (Lefschetz hyperplane). *The map res* : $\mathrm{H}^i(X) \to \mathrm{H}^i(D)$ *is an isomorphism for* $i < n-1$. *When* $i = n-1$, *it is an injection.*

Two proofs is this theorem can be found in [GH78, Section 1.2]. One is based on the Kodaira vanishing theorem, the other uses Morse theory. We now give some examples.

Example 1. Let $C \subset \mathbf{P}^2$ be a non-singular projective curve. We wish to set $X = \mathbf{P}^2$ in the above theorem, and consider C as a hyperplane section. We can achieve this using the Veronese embedding, which is a homeomorphism onto its image and hence induces isomorphisms on cohomology groups. The Lefschetz hyperplane theorem now gives us that $\mathrm{H}^0(\mathbf{P}^2) = \mathrm{H}^0(C)$ and hence that C is connected. We will use the Veronese embedding in all the following examples without further explicit mention.

Example 2. Let $S \subset \mathbf{P}^3$ be a hypersurface, then we obtain an injection $\mathrm{H}^2(\mathbf{P}^3) \hookrightarrow \mathrm{H}^2(S)$. There is an isomorphism $\mathrm{H}^2(\mathbf{P}^3) = \mathbf{Q} \otimes \mathrm{Pic}(\mathbf{P}^3) = \mathbf{Q}$, and $\mathrm{H}^2(S) = \mathbf{Q} \otimes \mathrm{Pic}(S)$, but usually S will have many more line bundles than the ones coming from $\mathbf{P}^3$. For example, when S is a cubic hypersurface it is well-known that it is isomorphic to $\mathbf{P}^2$ blown up at 6 points, see [Vak14, Chapter 27]. From this, it follows immediately that its Picard group is isomorphic to $\mathbf{Z}^7$.

Example 3. Let $X \subset \mathbf{P}^4$ be a hypersurface, then we obtain an isomorphism $\mathrm{H}^2(\mathbf{P}^4) = \mathrm{H}^2(X)$, so $\mathrm{H}^2(X) = \mathbf{Q}$.

We now turn to the first of our two main theorems in this section, *la théorème de Lefschetz vache*. With the notation as above, we get a restriction map res $: \mathrm{H}^i(X) \to \mathrm{H}^i(D)$ for all i. As both topological spaces have the structure of compact C^∞-manifolds, we have Poincaré duality for their cohomology theories. We get a "wrong way" restriction map $\mathrm{res}^* : \mathrm{H}^{2n-(i+2)}(D)^* \to \mathrm{H}^{2n-(i+2)}(X)^*$ which gives us a Gysin map $\mathrm{H}^i(D) \to \mathrm{H}^{i+2}(X)$ by Poincaré duality as D is a smooth $2n-2$ dimensional manifold. It turns out that the composition of this Gysin map with the restriction map is the same as taking the cup product with the first Chern class of $\mathscr{L}$. The following theorem can be viewed as a concrete description of an isomorphism between two cohomology groups we know to be isomorphic by Poincaré duality.

Theorem 2 (Hard Lefschetz). *The i-fold cup product with the first Chern class of $\mathscr{L} = \mathscr{O}_X(D)$ gives an isomorphism*

$$c_1(\mathscr{L})^i \cup - : \mathrm{H}^{n-i}(X) \to \mathrm{H}^{n+i}(X),$$

whenever $0 \leq i \leq n$.

We finally recall a piece of extra structure on the cohomology of a smooth complex variety X.

Theorem 3 (Hodge decomposition). *We have a decomposition*

$$\mathrm{H}^k(X) \otimes \mathbf{C} \cong \bigoplus_{p+q=k} \mathrm{H}^{p,q}(X),$$

where $\mathrm{H}^{p,q}(X)$ consists of differential forms which locally look like the wedge product of p terms of the form dz_i, and q terms of the form $d\overline{z_i}$, and $\mathrm{H}^{p,q} \cong \mathrm{H}^q(X, \Omega_X^p)$. Here, the last cohomology group is that of holomorphic p-differential forms.

When we put the above two theorems together, we can cook up a bilinear form

$$\mathrm{H}^{n-i}(X) \times \mathrm{H}^{n-i}(X) \to \mathbf{Q} : (\alpha, \beta) \mapsto \int_X c_1(\mathscr{L})^i \cup \alpha \cup \beta.$$

It turns out that this bilinear form is non-degenerate. Moreover, the Hodge-Riemann relations say that its signature on (p,q)-pieces is definite.

1.2 Families of smooth projective varieties

In the previous section we exhibited lots of structure on the cohomology groups of a smooth projective variety. We now turn to the question of how this structure varies in families, and develop relative versions of the above results. We discuss classical work of Deligne and Griffiths, which can be viewed as the subject's state of the art in the early 70's.

Definitions. Consider a map $f : X \to Y$ with connected fibres, which comes with a fixed factorisation through $\mathbf{P}^N \times Y$ for some N. This makes all the fibres connected projective varieties. We assume henceforth that Y is connected. The Ehresmann fibration theorem [Voi03, Theorem I.9.3] states that analytically locally on the base, we can trivialise X as a C^∞ manifold. This implies that the C^∞-structure of $X_{y_0} := f^{-1}(y_0)$ is independent of the point $y_0 \in Y$. We call Z this C^∞-manifold. This gives us a family $\underline{\mathrm{H}}^i(Z)$ of $\mathbf{Q}$-vector spaces on the base Y. Hence, for any $y_0 \in Y$ this gives us a monodromy representation $\rho_{y_0,i} : \pi_1(Y; y_0) \to \mathrm{GL}(\mathrm{H}^i(Z))$.

Some results of Deligne. An important tool in topology is the Leray spectral sequence, which gives us a very powerful computational tool for analysing the cohomology of X in terms of the cohomology of Y and the cohomology of the fibres of f. It is a spectral sequence whose second page is given by

$$E_2^{i,j} := \mathrm{H}^i(Y, \underline{\mathrm{H}}^j(Z)),$$

which abuts to $\mathrm{H}^{i+j}(X)$. The Hodge decomposition is essentially the same as the degeneration of this spectral sequence at the second page, and Deligne proves this in the relative setting. More precisely, he proves

Theorem 4. *Assume X, Y are non-singular projective varieties, and that $f : X \to Y$ is smooth, then*

1. *The Leray spectral sequence is totally degenerate. This gives us a canonical filtration on the cohomology of X, and a non-canonical decomposition $\mathrm{H}^{i+j}(X) \cong \bigoplus_{i+j} \mathrm{H}^i(Y, \underline{\mathrm{H}}^j(Z))$.*
2. *Each ρ_i is semi-simple. This means that for every $y_0 \in Y$, the representation $\rho_{y_0,i}$ is a direct sum of irreducible representations.*
3. *(Invariant cycle theorem) We have $\mathrm{H}^i(X_{y_0})^{\pi_1(Y,y_0)} = \mathrm{Im}\left(res : \mathrm{H}^i(X) \to \mathrm{H}^i(X_{y_0})\right)$. This essentially says that the monodromy invariant subspace is as small as we would expect.*

Remark. This theorem gives us very strong restrictions on the topology of algebraic varieties. Note that it is not true for more general topological spaces. Indeed, consider the Hopf fibration $S^3 \to S^2$, which is a C^∞-map. If the above decomposition would hold, we would get that $\mathrm{H}^1(S^3) \cong \mathrm{H}^0(S^2, \underline{\mathrm{H}}^1(S^1)) \oplus \mathrm{H}^1(S^2, \underline{\mathrm{H}}^0(S^1))$, which is clearly false as S^3 and S^2 are simply connected.

Some results of Griffiths. We now investigate the question of *variation* of Hodge structures more closely. This brings us to the results of Griffiths, which describe some fundamental properties of the Gauss-Manin connection.

Theorem 5 (Griffiths)**.** *Let $\mathscr{E}$ be the vector bundle $\underline{\mathrm{H}}^n(Z) \otimes \mathscr{O}_Y$. It comes with a Hodge filtration $F^n\mathscr{E} \supseteq F^{n-1}\mathscr{E} \supseteq \ldots \supseteq F^0\mathscr{E} = 0$ by vector bundles, and a Gauss-Manin connection $\nabla_{GM} : \mathscr{E} \to \mathscr{E} \otimes \Omega^1_Y$, such that we have*

1. *an isomorphism $\left(F^i\mathscr{E}/F^{i-1}\mathscr{E}\right)_{y_0} \cong H^{i,n-i}(X_{y_0})$,*
2. *an inclusion $\nabla_{GM}(F^i\mathscr{E}) \subseteq F^{i+1}\mathscr{E} \otimes \Omega^1_Y$ (Griffiths transversality).*

1.3 Main questions

The results in the previous section are incredibly powerful, and we now turn to the question of whether and how we could relax some of the assumptions made above. At this point it becomes natural to wonder:

1. What happens when X or Y are singular?
2. What are the connections to the rest of mathematics? How could we apply these results to neighbouring disciplines?
3. What happens if we allow f to have singular fibres? Can we still get similar results for arbitrary maps?

Question 1. An approach to this question was proposed by Deligne, who introduced the notion of mixed Hodge structures on the cohomology of a variety X which might not be smooth or proper. This is done by approximating X by a variety which is proper and smooth, in some appropriate sense. We can embed X into a proper variety [+++]

A second approach to this problem is to instead modify the object of study, i.e. singular cohomology. Indeed, failure of duality theory for the cohomology groups of non-proper varieties motivates an interest in a construction of a cohomology theory that does satisfy such a duality, which recovers the duality of singular cohomology when the variety is proper and smooth. Goresky-MacPherson gave a purely topological construction of such a theory, which is called *intersection cohomology*. Its construction is very similar in flavour to that of singular cohomology, but it takes into account the stratification

$$X \supset \mathrm{Sing}(X) \supset \mathrm{Sing}(\mathrm{Sing}(X)) \supset \dots,$$

which is finite as X is Noetherian. The $\mathbf{Q}$-vector spaces $\mathrm{IH}^*(X)$ have a duality theory that recovers Poincaré duality when X is proper. Moreover, if X is non-singular, we have $\mathrm{IH}^*(X) \cong \mathrm{H}^*(X)$. The construction in general depends on a choice of *perversity*, but there is a canonical *middle* perversity for X.

Question 2. We mention a deep connection with Frobenius eigenvalues of étale cohomology, and the theory of weights. Say that X is defined over $\mathbf{Z}$, then we can reduce mod p and consider X_p as a scheme over $\overline{\mathbf{F}_p}$. We obtain the étale cohomology groups with compact support $\mathrm{H}^i_{\text{ét,c}}(X_p, \mathbf{Q}_l)$, which carry an action of Frob_p. There is a very strong analogy between the Hodge decomposition of $X_{/\mathbf{C}}$ and the weights of this Frobenius action on this cohomology. This is the starting point of the celebrated Weil conjectures, point counting over finite fields, and many deep theorems in number theory.

We also mentioned above that Hodge bundles come with a Gauss-Manin connection. This is a particular instance of the theory of D-modules in algbraic analysis, which is concerned with such objects in full generality. It studies non-algebraic functions, like the exponential function, in an algebraic way by instead using the systems of differential equations that they satisfy. These systems allow for an algebraic description, and there is an equivalence of categories between these flat connections on vector bundles with regular singularities, and local systems or finite-dimensional representations of the fundamental group (if the base is connected). This is called the *Riemann-Hilbert correspondence*.

Question 3. To answer this question, we take into account what we have learned so far. We are looking for $\mathbf{Q}$-vector spaces that form nice families over the base. We should take into account singular stratifications of the base, and in the context of singular spaces, it is natural to try and use intersection cohomology. The following theorem takes all these observations into account, and formulates a good answer to this question:

Theorem 6 (Intersection theorem, simplified). *Let $f : X \to Y$ be a map between complex algebraic varieties, then we have a decomposition*

$$\mathrm{IH}^k(X) \cong \bigoplus_a \mathrm{H}^{k-d_a}(\overline{Y_a}, L_a),$$

where the direct sum ranges over a finite number of triples (Y_a, L_a, d_a). Here, each Y_a is a locally closed, smooth, irreducible subvariety of Y, L_a is a $\mathbf{Q}$-local system on Y_a, and d_a is a natural number.

Remarks. The triples mentioned in the above theorem are essentially determined by f. Whereas decomposition above is not canonical, the corresponding filtration is. See [dCM09, Section 1.4.2] for more details. The above theorem hence gives us a good analogue of the Hodge decomposition for smooth projective varieties. We note that we can also prove satisfactory analogues of Lefschetz vache and the Hodge-Riemann bilinear relations for intersection cohomology.

Examples. If $f : X \to Y$ is a non-singular family of projective varieties, then each $Y_a = X$. We get a corresponding decomposition which recovers the degeneration of the Leray spectral sequence. On the other end of the spectrum, assume that $f : X \to Y$ is a resolution of singularities of a singular variety Y. Then the left hand side of the decomposition is simply $\mathrm{H}^*(X)$, whereas the right hand side contains $\mathrm{IH}^*(Y)$ as a direct summand; there is an a such that $Y_a = Y$ and $L_a = \mathbf{Q}$. See [dCM09, Example 1.8.1] for more details on what the extra piece of cohomology is.

Proofs. There are currently three known proofs of the decomposition theorem, which we will now briefly discuss. They have very different flavours, and use a large range of techniques from a variety of mathematical disciplines.

- **Beilinson-Bernstein-Deligne and Gabber [BBDG82].** This proof utilises techniques that could be classified as number theory. The main argument is via reduction mod p.
- **M. Saito [Sai88].** This is an approach rooted in algebraic analysis, that makes use of the idea of families of Hodge structures. It introduces the notion of Hodge modules. This is a very interesting proof, but it considerably more difficult to get a conceptual understanding of the ideas featuring in this work.
- **de Cataldo-Migliorini [dCM05].** This proof is the most recent one, and is topological in nature. It proceeds via an inductive argument, and makes use of techniques reminiscent of classical Hodge theory. This proof is well-understood by many mathematicians, and the ideas have been applied in a variety of settings.

Week 2

Talk by Jan Vonk. As mentioned last week, one of our main objects of interest is the study of families of varieties $f : X \to Y$. Today, we recall the story for smooth families and the Gauss-Manin connection. Starting from a hands-on investigation of monodromy for some hyperelliptic curves, we come to an algebraic description of the Gauss-Manin connection and its properties, following [KO68]. We then compute some more examples, and sketch the proof of the local monodromy theorem by Katz [Kat70], which could be viewed as an introduction to some of the arithmetic ideas in the proof of the decomposition theorem by Beilinson-Bernstein-Deligne and Gabber [BBDG82]. Finally, we mention how to cope with a lack of smoothness or properness via Deligne's canonical extensions, and discuss the Riemann-Hilbert correspondence, and D-modules.

2.1 Families and monodromy

The idea of deforming mathematical objects, and putting them in families is truly ubiquitous in mathematics. For instance, we could attempt to extract information about a variety by putting it in a family where one of the fibres is easier to understand, for instance because it is degenerate, has lots of extra automorphisms, or possesses some other desirable property that makes the task at hand considerably easier. Often, it is possible to deduce relevant information about our original variety if we understand how things vary in the family. Of course there are many other reasons that one would want to investigate families of mathematical objects, one might for instance want to classify a certain type of object by constructing its moduli space. Given the extremely diverse backgrounds of the people attending this seminar, I will now suppose that you each have your own motivation to study families of varieties, and are interested in how the different structures on those varieties, like cohomology, vary as a function of some parameters.

Example 1. Let us play around with a toy example. Consider the family of complex curves $E \to \mathbf{A}^1_t$ described by $y^2 = (x^2 - t)(x - 1)$, which degenerates to a nodal curve at $t = 0$. We are really taking the projective morphism corresponding to this affine equation!

[Insert picture here.]

By the Ehresmann fibration lemma, this is locally on the base a trivial C^∞ bundle, so for every $t_0 \neq 0$, we can pick a small neighbourhood $U \subset \mathbf{A}^1_t$ such that E_{t_0} is isomorphic to E_t as C^∞ manifolds for all $t \in U$. The choice of this morphism is not canonical, but if we shrink U to a simply connected open, this gives us a *canonical* morphism on the level of cohomology. Given a path in $\mathbf{A}^1_t$, this gives us a canonical way to identify the cohomology groups of the fibres of the endpoints. In particular, we get an action of $\pi_1(\mathbf{C}^\times; t) \cong \mathbf{Z}$ on $\mathrm{H}^1(E_t)$, which is called the *monodromy action*.

Let us compute this monodromy in our example. Pick a generator γ for $\pi_1(\mathbf{C}^\times; 1)$, which loops around the origin once, counterclockwise. The cycles that form a basis for H^1 can now easily be deformed by rotating the middle branch cut around once, swapping its end vertices. This shows that the clockwise monodromy around 0 is given by the matrix

$$\gamma \qquad \text{acts as} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

This matrix is of infinite order, reflecting the fact that this family never pulls back to a smooth family. The singular fibre at 0 is essential, but we could still have found the matrix using Picard-Lefschetz theory. We'll get back to this phenomenon later, but for now you should appreciate that it was very easy to deduce the matrix in this case.

Example 2. Now consider the family of complex curves $E \to \mathbf{A}^1_t$, described by the equation $y^2 = x^3 + t$. Above any point $t \in \mathbf{C}^\times$, the fibre is an elliptic curve, whereas we obtain a cuspidal curve above $t = 0$.

[Insert picture.]

We can picture $E_1 : y^2 = x^3 + 1$ by considering it as a double cover of $\mathbf{P}^1$, ramified above the points $\{\sqrt[3]{t}, \zeta_3\sqrt[3]{t}, \zeta_3^2\sqrt[3]{t}, \infty\}$. If t loops around the origin once, these points will rotate around the origin by $2\pi/3$, except for ∞, which is fixed. We can pick a basis for the cohomology by picking two loops as shown below, and rotate them along. This is a lot more complicated than before, and thanks go to Alex Betts for explaining to me how to do this from pictures of loops and branch cuts.

[Insert picture here.]

We can read off what the matrix of the monodromy transformation by γ is, if we know how to identify the loops we obtain with the right loops on the original branch cuts. This is clear up to sign, and to figure out the sign we can draw both branch cuts simultaneously. These now determine a region in the middle, which has to flip the cycles between the two sheets, as the point $(0, \sqrt{t})$ maps into this region, and is clearly swapped to the other sheet by γ. This reveals without ambiguity what the image of our basis is, and by taking intersections with the original cycles, we determine that

$$\gamma \qquad \text{acts as} \qquad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

with respect to our chosen basis. Note that this matrix has order 6, which we could have predicted from the fact that the family $E \to \mathbf{A}^1_t$ becomes smooth after base change along $\mathbf{A}^1 \to \mathbf{A}^1 : z \mapsto z^6$.

2.1.1 The Gauss-Manin connection

We will now turn to an algebraic description of the concept of variation in families, and discuss Gauss-Manin connections on the relative de Rham cohomology of a smooth morphism $X \to S$.

Connections. In the previous two examples we could easily find the matrix of the monodromy transformation using the techniques described. However, this should leave us hungry for more! We can treat families like $y^2 = x^5 + t$ or $y^2 = x(x^2 - t)(x - 1)$ similarly on more complicated pictures, but we ultimately rely on three crucial things here: Firstly, we are only dealing with hyperelliptic curves here, where we have a well-understood finite map to $\mathbf{P}^1$ of small degree. Secondly, we could describe the roots of the polynomials in our examples really explicitly, and picture how they moved around in the plane. And thirdly, we were dealing with varieties over the complex numbers. This naturally leads us to the question of whether we can describe a similar variation on cohomology for families of varieties over general base schemes, and whether we can compute it in an algebraic way.

The answer to all these questions is *yes*, and we will now give an algebraic description of this concept of variation. The notion we wish to introduce is that of a *connection*. Let S be a smooth scheme over some field k. If $\mathscr{F}$ is a quasi-coherent sheaf on S, then a connection on $\mathscr{F}$ is a morphism

$$\nabla : \mathscr{F} \to \Omega^1_S \otimes \mathscr{F},$$

such that for any function $f \in \mathscr{O}_S$ we have $\nabla(fs) = f\nabla(s) + df \otimes s$. Equivalently, a connection ∇ is given by the data of an $\mathscr{O}_S$-linear morphism $\phi : Der_k(\mathscr{O}_S) \to End_k(\mathscr{F})$ such that for any sections D of $Der_k(\mathscr{O}_S)$, s of $\mathscr{F}$ and f of $\mathscr{O}_S$ we have

$$\phi(D)(fs) = f\phi(D)(s) + D(f)s.$$

This tells us how tangent vector fields act as derivations on sections of $\mathscr{F}$, and hence how we should transport elements of stalks in a manner *parallel to the base*.

The Gauss-Manin connection. There is a wide class of examples of such connections, with very special properties. They arise from smooth morphisms $\pi : X \to S$ from a smooth k-scheme X. Let us fix such a morphism π. The quasi-coherent sheaf we will construct the connection on is the *relative de Rham cohomology*, which is a vector bundle defined by

$$\mathscr{H}^q_{dR}(X/S) := \mathbf{R}^q\pi_*(\Omega^\bullet_{X/S}).$$

We note that this is a very concrete object that we can easily compute in many examples, but more on this later. Now that we have our vector bundle, we have to pull the rabbit out of the hat and say how we construct our connection. The algebraic way to define it is by using the spectral sequence of a filtered complex. The map is then simply one of the maps on the second page. This is not as esotheric as it might seem!

First, we use the smoothness of π to define a filtration on the complex $\Omega^\bullet_{X/S}$. This smoothness guarantees exactness of the sequence

$$0 \to \pi^*\Omega^1_S \to \Omega^1_X \to \Omega^1_{X/S} \to 0,$$

so we can define a filtration $\Omega^\bullet_X = F^0\Omega^\bullet_X \supseteq F^1\Omega^\bullet_X \supseteq \dots$ by

$$F^i\Omega^\bullet_X := \mathrm{Im}(\pi^*\Omega^i_S \otimes \Omega^{\bullet-i}_{X/S} \to \Omega^\bullet_X).$$

We can also describe the graded pieces of this filtration by [Har77, Exercise III.5.16.d], and obtain

$$\mathrm{Gr}^i := F^i/F^{i+1} = \pi^*\Omega^i_S \otimes \Omega^{\bullet-i}_{X/S}.$$

Now that we have a filtration on our complex, we invoke the existence of a spectral sequence for filtered complexes. An excellent exposition can be found in [Ked07, Section 1.8]. The situation is somewhat atypical, as we do not really care what the spectral sequence degerenates to, or when it does! Instead, all we care about is the first page. So all we will use is the existence of a spectral sequence with first page

$$E^{p,q}_1 = \mathbf{R}^{p+q}\pi_*\mathrm{Gr}^p = \Omega^p_S \otimes \mathscr{H}^q_{dR}(X/S).$$

There is a non-trivial justification to be made for this second equality. Now we define the Gauss-Manin connection to be the differential $d^{0,q}$ on the first page, which is a morphism

$$\nabla_{GM} : \mathscr{H}^q_{dR}(X/S) \to \Omega^1_S \otimes \mathscr{H}^q_{dR}(X/S).$$

Some properties of ∇_{GM}. The connection we just described, arising from a smooth map $\pi : X \to S$ enjoys many remarkable properties. First of all, we know that the composite map

$$\nabla^1_{GM} \circ \nabla_{GM} : \mathscr{H}^q_{dR}(X/S) \to \Omega^2_S \otimes \mathscr{H}^q_{dR}(X/S),$$

where ∇^1_{GM} is defined by $f \otimes \omega \mapsto f \otimes d\omega - \omega \wedge \nabla(s)$, is just $d^{1,q} \circ d^{0,q}$, which is 0 as E is a spectral sequence. This property is called *integrability* of a connection. Secondly, we know that the vector bundle $\mathscr{H}^q(X/S)$ on S comes with a Hodge filtration from its calculation as hypercohomology of the Cech double complex. We denote this filtration as $\mathscr{H}^q(X/S) = F^0_{Hod}\mathscr{H}^q(X/S) \supseteq F^1_{Hod}\mathscr{H}^q(X/S) \supseteq \ldots$. The *Griffiths transversality theorem* now says that whereas ∇_{GM} does not respect this Hodge filtration, it almost does. More precisely, it states that

$$\nabla_{GM}(F^i_{Hod}) \subseteq F^{i+1}_{Hod}.$$

In the next section, we investigate monodromy again, this time by using the explicit and computable nature of de Rham cohomology. All the matrices we obtain will be of a very special form, i.e. they will be quasi-unipotent. We will discuss this further in the final section.

2.2 Some explicit examples.

We now turn to the explicit computation in some examples, to show that everything defined above is very tangible and concrete. We start with an example over the complex numbers that nonetheless has a very arithmetic flavour, following Katz [Kat73, Appendix A1.3].

The Tate curve. Recall that given τ in the complex upper half plane $\mathfrak{H}$, we obtain an elliptic curve $E_\tau = \mathbf{C}/(\mathbf{Z}+\tau\,\mathbf{Z})$. This elliptic curve has a Weierstrass equation $y^2 = 4x^3 - G_4 x - G_6$, where $(x, y) = (\wp(z), \wp'(z))$ where G_4, G_6 are certain numbers attached to the lattice $\mathbf{Z}+\tau\,\mathbf{Z}$, and $\wp$ its Weierstrass function. The de Rham cohomology of E has a basis given by $\omega := dx/y$ and $x\omega$.

First, we pick the basis γ_1, γ_2 for the singular homology of E_τ given by the line segments in $\mathbf{C}$ going from 0 to $\tau, 1$ respectively. By the definition of the Gauss-Manin connection, their duals form a set of ∇-invariant sections of $\mathrm{H}^1_{dR}(E)$. So all we need to do is calculate the periods of ω and $x\omega$ with respect to this basis, and derive them. This requires a little calculation relying on power series expansions of the Weierstrass zeta function ζ, which satisfies $-d\zeta = \wp(z)dz$, finally obtaining that

$$\frac{d}{d\tau} \quad \text{acts as} \quad \frac{1}{2\pi i}\begin{pmatrix} \frac{\pi^2 P}{3} & 1 \\ \frac{\pi^4 P^2}{9} - \frac{6P'}{\pi i} & -\frac{\pi^2 P}{3} \end{pmatrix},$$

where $P(\tau) = 1 - 24\sum_{n\geq 1}\sigma_1(n)e^{2\pi i n\tau}$ and $\sigma_1(n)$ is the sum of all the divisors of n. This form P is almost a modular form of weight 2. In fact, it is a p-adic modular form for every p, but it is never overconvergent or classical. Interpreted as a complex analytic function, it satisfies the transformation law

$$P\left(\frac{-1}{\tau}\right) = \tau^2 P(\tau) - \frac{6i\tau}{\pi},$$

as can be proved using Serre's cup product formula.

Remark. This example computed the Gauss-Manin connection based on the fact that we could easily describe the invariant cycles. We then simply expressed our favourite basis in terms of that by calculating the periods and deriving those. There is also an algebraic way to compute Gauss-Manin connections of smooth families. Kiran Kedlaya has a beautiful set of notes from the Arizona Winter School 2007 [Ked07] that tells you how to do it directly. It works especially well for families of hyperelliptic curves (over a general base ring) as we have an explicit reduction algorithm for de Rham cohomology that allows us to write an arbitrary differential form as a combination of a chosen basis, plus an exact form. The computation is by lifting the relevant differential forms, differentiating, and reducing in cohomology. I have written some code this week that allows you to compute these matrices for families of hyperelliptic curves, and will discuss some examples now.

Example 1. Let us revisit the second example we did, namely the family $E \to \mathbf{A}^1_t$ given by $y^2 = x^3 + t$. We compute the Gauss-Manin matrix using the methods in [Ked07] and obtain

$$\begin{pmatrix} \frac{1}{6t} & 0 \\ 0 & \frac{-1}{6t} \end{pmatrix}.$$

How do we obtain the monodromy matrix from this? All we need to do is solve the local system around 0 and let its solutions transfer our basis. This comes down to calculating $\exp(2\pi i A)$, where A is the matrix of residues of the Gauss-Manin matrix. In this example we obtain

$$\begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix},$$

which is the diagonalised version of the matrix we found earlier by complex analytic methods. We have not committed to a specific base field yet, though!

Example 2. Now let us revisit the very first example we did, where we considered the family $y^2 = (x^2 - t)(x - 1)$. We can compute the Gauss-Manin connection using the methods in [Ked07], and obtain

$$\begin{pmatrix} \frac{1}{4(t-1)} & \frac{1}{4(t-1)} \\ \frac{-1}{4t(t-1)} & \frac{-1}{4(t-1)} \end{pmatrix},$$

When we exponentiate $2\pi i \times$ the residue matrix, we obtain the monodromy matrix

$$\begin{pmatrix} 1 & 0 \\ \frac{-\pi i}{2} & 1 \end{pmatrix},$$

Example 3. Consider the family $y^2 = x^5 + 5x^2 + 3t$. Already it is not so obvious to me how to do this pictorially over the complex numbers! But I haven't really thought about this carefully. In any case, we can do things algebraically and compute the Gauss-Manin matrix

$$\begin{pmatrix} \frac{3t^2}{10t^3+40} & \frac{3t}{5t^3+20} & \frac{6}{5t^3+20} & \frac{-3t^2}{5t^3+20} \\ \frac{-1}{5t^3+20} & \frac{t^2}{10t^3+40} & \frac{t}{5t^3+20} & \frac{2}{5t^3+20} \\ \frac{t}{10t^3+40} & \frac{1}{5t^3+20} & \frac{-t^2}{10t^3+40} & \frac{-t}{5t^3+20} \\ \frac{-3}{5t^4+20t} & \frac{3t}{10t^3+40} & \frac{3}{5t^3+20} & \frac{-3t^2}{10t^3+40} \end{pmatrix},$$

so by taking the residue at 0 and exponentiating, we get the monodromy matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-3\pi i}{10} & 0 & 0 & 1 \end{pmatrix}.$$

There are a lot of interesting things to note about this matrix. First off, it has infinite order, implying that the family cannot be made smooth after a finite base change. Secondly, we see an honest *period* appear.

2.3 Nilpotent connections

In all the examples of monodromy matrices we computed, the end result was very special. We saw it was often of infinite order, unipotent, or a combination of the two. More precisely, the matrix M we obtained was always *quasi-unipotent*, which means there exist m, n such that $(M^m - 1)^n = 0$. This is not coincidence, and is in fact true for any Gauss-Manin connection arising from a 1-dimensional smooth family over a non-singular quasi-projective complex curve. This theorem is called the *local monodromy theorem*, often attributed to Griffiths-Landman-Grothendieck. We will sketch a sketch a proof due to Katz [Kat70], which is arithmetic in flavour.

Let S be a Zariski open subset of a projective non-singular connected curve,obtained by deleting the points $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$. Let $\pi : X \to S$ be proper and smooth, giving rise to the relative de Rham cohomology $\mathscr{H}^q_{dR}(X/S)$ and a Gauss-Manin connection ∇. Around any of the missing points $\mathfrak{m}_i$, we have a canonical generator for the fundamental group of a small punctured disk around it, whose matrix with respect to some basis of the fibre of the relative de Rham cohomology with respect to some point in the punctured disk is M. Then there is an m such that $(M^m - 1)^{q+1} = 0$. In other words, the monodromy acts quasi-unipotently, and we have control over the *exponent of nilpotence*. Katz [Kat70] adapts the following strategy to prove this. We first find a thickening of X, S and $\mathrm{Spec}(\mathbf{C})$ into $\mathbf{X}, \mathbf{S}$ and $\mathrm{Spec}(R)$ for some $R \subset \mathbf{C}$ which is finitely generated over $\mathbf{Z}$.

We can now reduce the problem mod $\mathfrak{p}$ for various primes of R. In characteristic p, we have an entirely new range of tools at our disposal, such as a Frobenius endomorphism and a Cartier operator. Using these tools, we prove an analogous nilpotency statement in characteristic p. The result can then be lifted with the aid of a theorem of Turrittin.

2.4 The Riemann-Hilbert correspondence

When we computed the Gauss-Manin matrix in examples, the poles of the entries were all at most simple. This was not coincidence, and in fact this is part of a very general theorem due to Deligne. He considered Gauss-Manin connections on varieties U that sit inside a smooth projective variety X, such that the complement of U in X is a strict normal crossings divisor on X. He proved that all such connections are *regular* in the sense that they extend to meromorphic connections on X with logarithmic poles along $X \backslash U$, or equivalently that we can choose a basis such that the matrix of the connection has at most simple poles.

Given a finite rank vector bundle with a flat connection on a smooth complex algebraic variety X, we get a system of differential equations corresponding to the connection on it. We can therefore look at its solutions, which gives rise to a *local system* on X. This is nothing more than a locally constant sheaf on X, or equivalently, a finite dimensional representation of $\pi_1(X)$. The *Riemann-Hilbert correspondence* says that this in fact defines an equivalence of categories!

Extending Riemann-Hilbert. Vector bundles with connections are a special case of a more general theory of D-modules. A D-module is a quasi-coherent sheaf with an action of the sheaf of derivations, satisfying certain conditions. If this quasi-coherent sheaf is in fact a vector bundle, the D-module structure is equivalent to the structure of a flat connection on it. We can now ask whether there is a similar convenient equivalence of categories for this more general class of objects? The notion that replaces that of local systems turns out to be a *perverse sheaf*. So we were talking about perverse sheaves all along!

Week 3

Talk by Brent Pym. Brent typed up some excellent notes, far superior to anything I could reproduce. They are linked to separately, on `http://people.maths.ox.ac.uk/~vonk/perversity/pympmyhomology.pdf`.

Week 4

Talk by Jan Vonk. This week, we will start a more systematic investigation of the sheaves $R^q f_* \mathbf{Q}_X$ that play such a crucial role in classical Hodge theory, but we will not insist on smoothness of f. This will bring us to the realms of derived categories and perverse sheaves. Finally, we come to a statement of the decomposition theorem, and illustrate this theorem with a few examples. These notes are sloppily written and should only be used as a reminder of what we talked about. It is by no means a substitute for the original source [dCM09], where the reader should learn the material.

4.1 Motivation

In classical Hodge theory, we investigate the cohomology groups of a smooth projective variety, or more general a smooth projective family $f : X \to Y$ over a smooth base scheme Y. A nice summary of the theory in that case was outlined in Balász's talk. The main role in the smooth story was played by the sheaf $R^q f_* \mathbf{Q}_X$, which was a local system on Y. We now turn to the question of what sort of sheaf this is for more general maps.

The first observation to make is that $R^q f_* \mathbf{Q}_X$ is not a local system for more general maps f. For instance, consider the resolution of a 2-dimensional cone on the one hand, and a degeneration of a smooth family of curves over a 1-dimensional base scheme:

[Insert pictures here.]

Note that in the first example, the dimension of $R^2 f_* \mathbf{Q}_X$ jumps up over the singular point. In the second example, the dimension of $R^1 f_* \mathbf{Q}_X$ jumps down at the degeneracy point. So these sheaves can exhibit different sorts of wild behaviour, and indeed fail to be local systems in very simple examples. So what are they? Recall the theorem of Whitney and Thom that was mentioned in Brent's talk last week: It says that Y can be stratified as $Y = \amalg Y_\alpha$, where the Y_α are regular, locally closed, such that $f^{-1}(Y_\alpha) \to Y_\alpha$ is a topological fibration. This at least ensures that $R^q f_* \mathbf{Q}_X |_{Y_\alpha}$ is a local system! Let us give this class of sheaves a name.

Definition. A *constructible sheaf* $\mathscr{F}$ on a scheme S is a sheaf such that $\mathscr{F} |_{S_\alpha}$ is a local system, for some decomposition $S = \amalg S_\alpha$ into locally closed subsets S_α.

At this point it is prudent to throw away as little information as possible. Therefore, we are motivated to consider the entire complex $Rf_* \mathbf{Q}_X$, and not just its cohomology sheaves $R^q f_* \mathbf{Q}_X$. This motivates our systematic investigation of the topology of f to take place in the derived category of sheaves. The complex $Rf_* \mathbf{Q}_X$ lives in a much smaller subcategory, which enjoys many nice properties and comes with a Verdier duality functor. This category is called the constructible derived category, which we will define in the next section.

4.2 Some category theory

In this section, we come to a definition of the constructible derived category and its properties. Recall that we just motivated an interest in a systematic investigation of the nature of the complex of sheaves $Rf_* \mathbf{Q}$. This object lives in the derived category of sheaves, so we first recall the theory of derived categories. We then define the constructible derived category and perverse sheaves. Finally, we state the decomposition theorem and give some examples.

4.2.1 Derived categories

We now recall the definition of the derived category of an abelian category $\mathscr{A}$, and Grothendieck's six operations.

Definition. Let $\mathscr{A}$ be an abelian category. The derived category $D(\mathscr{A})$ is the category whose objects are complexes of objects in $\mathscr{A}$, and in which a morphism from $A^\bullet$ to $B^\bullet$ is an equivalence class of diagrams

$$\begin{tikzcd} & X^\bullet \arrow[dl, "q"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where q is a quasi-isomorphism of complexes. Two such diagrams are considered equivalent if there exists a complex $Z^\bullet$ such that, with obvious notation, the following diagram commutes

$$\begin{tikzcd} & & Z^\bullet \arrow[dl, "q"'] \arrow[dr] & & \\ & X^\bullet \arrow[dl] \arrow[drrr] & & Y^\bullet \arrow[dlll] \arrow[dr] & \\ A^\bullet & & & & B^\bullet \end{tikzcd}$$

The derived category $D(\mathscr{A})$ naturally comes with the structure of a triangulated category. We will not define this precisely, but it has some rather familiar consequences, not in the least the familiar long exact sequence in cohomology. We will come back to this triangulated structure in more detail when we talk about t-structures. The bounded derived category $D^b(\mathscr{A})$ is the full subcategory of $D(\mathscr{A})$ obtained from complexes whose cohomology is zero in all but finitely many degrees.

One of the main uses of derived categories is that it puts the construction of derived functors and sheaf cohomology in a very illuminating perspective. It provides a very flexible framework for manipulating cohomology and obtaining duality theorems, as we will see shortly. We now recall the notion of *derived functors*. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor. To define its derived functor, we need the notion of an *adapted class* of objects for F. This is a class of objects $\mathscr{R}$ in $\mathscr{A}$ such that F is exact on $\mathscr{R}$, and such that it is *large enough* in the sense that for every object A of $\mathscr{A}$, there exists a monomorphism $A \to R$ for some R in $\mathscr{R}$.

Definition. The derived functor of F for the adapted class $\mathscr{R}$ is the functor $RF : D^b(\mathscr{A}) \to D^b(\mathscr{B})$ defined as follows. By largeness of $\mathscr{R}$, for any complex $A^\bullet$, there exists a quasi-isomorphism $A^\bullet \to R^\bullet$ to a complex consisting of objects in $\mathscr{R}$. Then $RF(A^\bullet) := F(R^\bullet)$, which is clearly well-defined up to quasi-isomorphism. This definition turns out to be independent of the choice of adapted class $\mathscr{R}$. We can define the derived functor of a right exact functor similarly.

Example 1. Let $\mathscr{A} = \mathfrak{Ab}$ be the category of abelian groups, and let $F : \mathfrak{Ab} \to \mathfrak{Ab}$ be the functor $(A \otimes -)$ for some abelian group A. This functor is right exact, and $\mathfrak{Ab}$ has enough projectives. Its derived functor, often denoted $(C \otimes^L -)$ has the property that, when evaluated at the complex B given by a single abelian group B in degree 0, has cohomology in degree $-q$ equal to $\mathrm{Tor}_q(A, B)$.

Example 2. Consider a continuous map of topological spaces $f : X \to Y$. Let $\mathscr{A}$ be the category of sheaves X, and $\mathscr{B}$ the category of sheaves on Y. Then we can consider the left exact functor $f_* : \mathscr{A} \to \mathscr{B}$. Consider the complex consisting of the sheaf $\mathbf{Q}_X$ in degree 0, then $Rf_* \mathbf{Q}_X$ is an element of $D^b(Y)$ with the property that its q-th cohomology is the sheaf $R^q f_* \mathbf{Q}_X$.

Example 3. Let X be a scheme, and consider the global sections functor Γ from the category of sheaves on X to the category of abelian groups. Then $R\Gamma\mathscr{F}$, where $\mathscr{F}$ is just a complex concentrated in degree 0, is a complex whose cohomology is given by the sheaf cohomology $\mathrm{H}^q(X, \mathscr{F})$.

4.2.2 Grothendieck's six operations

We now specialise our discussion the specific case of the category of sheaves of abelian groups on topological spaces. On sheaves, we have an interesting set of functors, often called *Grothendieck's six operations*. The philosophy is that any operation on sheaves of abelian groups we would reasonable be interested in, can be obtained from six fundamental operations, that come in three adjoint pairs. They are often denoted

$$(\otimes^L, R\mathscr{H}om), \qquad (f^{-1}, Rf_*), \qquad (Rf_!, f^!).$$

Let us first discuss the first adjoint pair. We define $R\mathscr{H}om(\mathscr{F}^\bullet, \mathscr{G}^\bullet)$ to be the *graded Hom* functor $\underline{\mathrm{Hom}}(\mathscr{F}^\bullet, \mathscr{I}^\bullet)$, where $\mathscr{I}^\bullet$ is an injective resolution of $\mathscr{G}^\bullet$. This graded Hom is nothing more that the complex whose degree q term equals $\bigoplus_{i-j=q} \mathrm{Hom}(\mathscr{F}^j, \mathscr{I}^i)$. The functor $\otimes^L$ has a similar construction: We define $\mathscr{F}^\bullet \otimes^L \mathscr{G}^\bullet$ to be the graded tensor product $\mathscr{F}^\bullet \otimes^L \mathscr{P}^\bullet$, where $\mathscr{P}^\bullet$ is a flat resolution of $\mathscr{G}^\bullet$. It can be shown that they form an adjoint pair of bifunctors, and preserve the bounded derived subcategory of sheaves $D^b(\mathfrak{Sh}_X)$ on a topological space X.

Given a continuous map $f : X \to Y$, we obtain 4 more important operations. The classical adjoint pair (f^{-1}, Rf_*) has showed up numerous times in these lectures before. We now discuss $f_!$ and $f^!$. Let $\mathscr{F}$ be a sheaf on X, then the *proper push-forward* of $\mathscr{F}$, denoted $f_!\mathscr{F}$ is the sheaf on Y defined by

$$f_!\mathscr{F}(U) := \{s \in f^{-1}(U) \ : \ f|_{Supp\ s} : \mathrm{Supp}\ s \to U \text{ is proper}\}.$$

So by definition, we have $f_!\mathscr{F} \subseteq f_*\mathscr{F}$. If the map f is proper, we see that the condition we impose on sections is automatically satisfied, as the support of a section is always a closed subset. In the special case where f is the inclusion of an open subset, the functor $f_!$ coincides with the functor of extension by zero. The functor $f_!$ is left exact, and flabby sheaves are an adapted class for it. We can therefore define the derived functor $Rf_!$.

The definition of the left adjoint to $Rf_!$ is by no means easy. We will not construct it here, but we will just say that it exists, and is usually called $f^!$. It has the property that when f is a closed immersion, it consists of sections with support on the closed subset, and when f is a smooth map of varieties, it coincides with f^* up to a shift by the relative dimension of f. We finally come to an important definition, which plays a central role in duality theorems.

Definition: The dualising complex. Let X be a topological space, and let $f : X \to \{pt\}$ be the constant map to a point. Then the *dualising complex* ω_X on X is the complex $f^! \mathbf{Q}$.

4.2.3 The constructible derived category

Recall that we discovered an interesting constructibility property of the complexes $Rf_* \mathbf{Q}_X$, for a proper map $f : X \to Y$. We now formalise this by defining the constructible derived category $D^b_c(Y)$ armed with our knowledge on derived categories, which provide the correct framework. By the comments in our introductory motivation, we can view $Rf_* \mathbf{Q}_X$ as an object in $D^b_c(Y)$.

Definition. The *constructible derived category* $D^b_c(X)$ of a complex algebraic variety X is the full subcategory of the bounded derived category of sheaves on X, whose objects are complexes of sheaves of finite dimensional $\mathbf{Q}$-vector spaces with constructible cohomology sheaves.

Recall Grothendieck's six operations, and in particular the functors $f^{-1}, Rf_*, Rf_!, f^!$ when $f : X \to Y$. It can be checked that these functors behave well with respect to constructibility. More precisely, this means that $f^! : D^b_c(X) \to D^b_c(Y)$ and analogously for the other functors.

Definition. For an object $\mathscr{F}^\bullet$ of $D^b_c(X)$, we define the *Verdier dual* $\mathbf{D}\mathscr{F}^\bullet := R\mathscr{H}om(\mathscr{F}^\bullet, \omega_X)$, where ω_X is the dualising complex of X as defined above.

The operation of Verdier duality is of tremendous importance. It has the property that $\mathscr{H}^q \mathbf{D}\mathscr{F}^\bullet = (\mathscr{H}^{-q}_c \mathscr{F}^\bullet)^*$. Many classical duality theorems can be deduced from its basic properties, with essentially entirely formal proofs. Of course we cheated, as we skipped the definition of $f^!$ as the right adjoint of $Rf_!$. If we are happy to accept this, many theorems can be proved entirely formally. We note that $\mathbf{D} f^{-1}\mathscr{F}^\bullet \simeq f^! \mathbf{D}\mathscr{F}^\bullet$, $\mathbf{D} Rf^! \mathscr{F}^\bullet \simeq Rf_* \mathbf{D}\mathscr{F}^\bullet$. Poincaré duality is now a formal consequence of the observation that on a smooth, oriented n-dimensional manifold X, we have $\omega_X \simeq \mathbf{Q}[n]$.

4.3 Perverse sheaves

We finally come to the definition of perverse sheaves. Recall that from an initial interest in the sheaves $R^i f_* \mathbf{Q}_X$, motivated by classical Hodge theory, we were naturally lead to the introduction of the constructible derived category, which comes with a set of important operators from general abstract theory by Grothendieck, and in particular comes with a Verdier duality functor.

Perverse sheaves form a subcategory of $D^b_c(X)$. To motivate their definition, we first take a small digression. Recall the *Lefschetz hyperplane theorem* from the first talk by Balász: Let X be a non-singular projective variety of dimension n, and D a generic hyperplane section. Then the natural map $\mathrm{H}^q(X) \to \mathrm{H}^q(X_s)$ is an isomorphism for $q < n-1$, and injective for $q = n-1$. Why is this true? If U is an affine non-singular variety of dimension n over $\mathbf{C}$, then U is homotopic to a CW complex of real dimension n by the theorem of Andreotti-Frankel. This implies that $H^q(U, \mathbf{Q}) = 0$ for $q > n$, and hence the excision exact sequence becomes

$$\ldots \to \mathrm{H}^q_c(X \backslash X_s) \to \mathrm{H}^q(X) \to \mathrm{H}^q(X_s) \to \ldots,$$

from which the result now follows by $\mathrm{H}^q_c(X \backslash X_s) \cong \mathrm{H}^{2n-q}(X \backslash X_s) = 0$.

Actually, we can prove a more general theorem. Let $\mathscr{F}$ be a constructible sheaf on an n-dimensional smooth affine variety U, then $\mathrm{H}^q(U, \mathscr{F}) = 0$ for $q > n$. In the spirit of the definitions above, we may wonder whether we get a similar result for objects in $D_c^b(X)$. That is, if $\mathscr{F}^\bullet$ is an object in the constructible derived category of sheaves on X, is it necessarily true that $R^q\Gamma(\mathscr{F}^\bullet) = 0$ whenever $q > n$? We could try and compute this cohomology explicitly with the Grothendieck spectral sequence, which computes $R^{p+q}\Gamma\mathscr{F}^\bullet$, and whose second page is given by

$$E_2^{p,q} = R^p\Gamma(\mathscr{H}^q\mathscr{F}^\bullet) = \mathrm{H}^p\left(\mathrm{Supp}\ \mathscr{H}^q, \mathscr{H}^q\mathscr{F}^\bullet\right),$$

which is zero for whenever $p > \dim \mathrm{Supp}\mathscr{H}^q$, because the sheaves $\mathscr{H}^q$ are constructible by definition. This spectral sequence abuts to $R^{p+q}\Gamma\mathscr{F}^\bullet$, so if $\mathscr{F}^\bullet$ is chosen such that $\dim \mathrm{Supp}\ \mathscr{H}^q + q \leq n$, then we obtain that $\mathrm{H}^q(U, \mathscr{F}^\bullet) = 0$ for $q > n$.

This analogue of the classical statement of cohomological dimension of an affine scheme seems like a desirable property to have, but does not hold in general. This motivates an interest in the subcategory of $D_c^b(X)$ for which this theorem holds, which by the above happens whenever we impose an additional condition on the dimension of the support of the cohomlogy sheaves of the complex. The following definition is exactly that, with the complexes shifted by n. This is a convention we will adopt, which roughly "centers cohomology around degree 0".

Definition. The category $\mathscr{P}_X$ of *perverse sheaves* on X is the full subcategory of $D_c^b(X)$ consisting of the complexes $\mathscr{F}^\bullet$ such that $\dim \mathrm{Supp}\ \mathscr{H}^q\mathscr{F}^\bullet \leq -q$ and $\dim \mathrm{Supp}\ \mathscr{H}^q\mathbf{D}\mathscr{F}^\bullet \leq -q$, for all $q \in \mathbf{Z}$.

We remark that the conditions in the definition of perverse sheaves, often called the *support and co-support conditions*, give rise to a t-structure on $D_c^b(X)$ which we call the *perverse t-structure*. This gives rise to truncation functors ${}^p\tau_{\leq 0}$ and ${}^p\tau_{\geq 0}$ and perverse cohomology sheaves ${}^p\mathscr{H}^k :={}^p \tau_{\geq 0} \circ^p \tau_{\leq 0} \circ [k]$.

4.3.1 Intersection cohomology

We have just defined the category $\mathscr{P}_X$ of perverse sheaves on X. By definition, it is self-dual under Verdier duality $\mathbf{D}$. By the deep work of Beilinson-bernstein-Deligne [BBDG82], this category is Artinian and Noetherian, so every object has a finite Jordan-Hölder filtration. It is the replacement of the category of local systems we were considering in classical Hodge theory. We now turn to a more careful study of the simple objects in this category, which will turn out to give rise to familiar objects that we met in Brent's talk last week.

Let X be a complex manifold with locally closed smooth subvariety S, and simple local system L on S. We define the *intersection cohomology complex* to be

$$\mathscr{IC}_X(\overline{S}, L) := i_* j_{!*} L[\dim_{\mathbf{C}} S],$$

where $i : \overline{S} \hookrightarrow X$ is the closed embedding of the closure $\overline{S}$ of S in X, and $j : S \hookrightarrow \overline{S}$ is the natural inclusion. Here, $j_{!*}$ is the *Goresky-MacPherson extension* or *middle extension*, which can be defined as $j_{!*}P := \mathrm{Im}\left({}^p\mathscr{H}^0 f_! P \to^p \mathscr{H}^0 f_* P\right)$, for any perverse sheaf on S. It can be checked that $\mathscr{IC}_X(\overline{S}, L)$ is indeed a perverse sheaf. Moreover, it is a simple object in the category $\mathscr{P}_X$.

Note that by definition, we know that $\mathscr{IC}_X(S, L)|_S = L[\dim S]$ and $\mathbf{D}\mathscr{IC}_X(\overline{S}, L) = \mathscr{IC}_X(\overline{S}, L^*)$. To familiarise ourselves further with this definition, let us look at some examples. Note that in general, $\mathscr{IC}$ is **very hard** to compute. There are only a few well-understood cases.

Warning: In what follows, I will make use of the notation $\tau_{\leq q}$, which is an operation on the derived constructible category $D_c^b(X)$ that kills the cohomology in degrees larger than q. They are often called truncation functors.

Example 1. If X is smooth of dimension n, we obtain $\mathscr{IC}_X(X,\mathbf{Q}) = \mathbf{Q}_X[n]$, as the identity map is proper.

Example 2. Let X be a complex variety whose singular locus is a finite set of points. Let $j : X_{reg} \hookrightarrow X$ be the inclusion of the regular locus into X, then

$$\mathscr{IC}_X(\overline{X_{reg}}, \mathbf{Q}_{X_{reg}}) = \tau_{\leq -1} Rj_* \mathbf{Q}_{X_{reg}}[n].$$

If we let x be a singularity, we have that $\mathscr{H}^q(Rj_* \mathbf{Q}_{X_{reg}}[n])_x = \lim_{x \in U} \mathrm{H}^{q+n}(U \backslash \{x\}, \mathbf{Q})$. This property shows that intersection cohomology encodes the singularity in a natural way.

Example 3. Let L be a non-trivial local system on an open subset U of a non-singular projective curve X, and let i be the inclusion. Then

$$\mathscr{IC}_X(\overline{U}, L) = (j_* L)[1] \qquad \text{and so} \qquad (\mathscr{H}^{-1}\mathscr{IC}_X(L))_x = (\mathscr{H}^0 j_* L)_x = \mathrm{Ker}(N - I),$$

where N is the monodromy operator around the point $x \in X \backslash U$. This shows that intersection cohomology encodes the monodromy in a natural way.

Example 4. Let L be a local system on the complement of a normal crossing divisor with irreducible components Z_i on a variety X. Assume that the monodromy operator N_i of Z_i is unipotent for all i, which we know is not a big assumption from my previous talk. There is a wonderful paper by Cattani-Kaplan-Schmid that analyses the intersection cohomology complex in this case.

Finally we explain the link with the intersection cohomology as defined in Brent's talk. Recall that he defined them as groups, and in his notes there is a more general construction of intersection cohomology sheaves $IC^\bullet_X(L)$ with coefficients in a local system L. The links between these definitions is that $IC^\bullet_X(L) = \mathscr{H}^\bullet \mathscr{IC}_X(L)$.

4.3.2 The decomposition theorem

In this final paragraph, we come to the statement of the decomposition theorem.

Theorem 7. *Let $f : X \to Y$ be a proper map of complex algebraic varieties. There is an isomorphism in $D^b_c(Y)$*

$$Rf_* \mathscr{IC}_X \simeq \bigoplus_{i \in \mathbf{Z}}{}^p \mathscr{H}^i(Rf_* \mathscr{IC}_X)[-i],$$

where the perverse sheaves on the right hand side are semi-simple. The category of perverse sheaves is Noetherian and Artinian, and the simple objects are the intersection complexes $\mathscr{IC}_X(\overline{X_\alpha}, L)$, where L is a simple local system on a locally closed smooth subvariety X_α.

Remarks. To compute these things, we will often rely on the notion of semi-small maps. The map f is semi-small if $\dim X \times_Y X = \dim X$. If this is the case, the only term in the decomposition is ${}^p\mathscr{H}^0(Rf_* \mathscr{IC}_X)$.

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