

MIXED HODGE MODULES

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1. MIXED HODGE THEORY

1.1. **Pure Hodge structures.** Let X be a smooth projective complex variety and Ω_X^\bullet the complex of sheaves of holomorphic differential forms with the de Rham differential.

We have two structures on the de Rham cohomology $H_{dR}^n(X) = H^n(X, \Omega_X^\bullet)$:

- We have a period isomorphism

$$\begin{aligned} H_{dR}^n(X) &\rightarrow H^n(X, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C} \\ \omega &\mapsto \int_{\gamma} \omega \end{aligned}$$

This gives a \mathbf{Q} -sublattice $H^n(X, \mathbf{Q}) \subset H^n(X, \Omega_X^\bullet)$

- We have a decreasing filtration

$$\Omega_X^\bullet = F^0(\Omega_X^\bullet) \supset F^1(\Omega_X^\bullet) \supset \dots$$

where

$$F^p \Omega_X^\bullet = (\Omega_X^p \rightarrow \Omega_X^{p+1} \rightarrow \dots)[-p].$$

This defines a *Hodge filtration* F on $H_{dR}^n(X)$.

We can also define the conjugate filtration \bar{F} using the real structure coming from the \mathbf{Q} -sublattice. These two pieces of data define a *pure Hodge structure* of weight n : we have

$$\mathrm{gr}_F^p \mathrm{gr}_{\bar{F}}^q H_{dR}^n(X) = 0$$

unless $p + q = n$.

1.2. **Example.** Consider the twice-punctured torus:

picture

It seems different cycles land in $H_1(X, \mathbf{Q})$ for different reasons: some come from encircling the boundary (α, β) and some come from the actual homology (γ, γ') . Note that this does not give a decomposition of the homology since $\gamma' = \gamma + 2\alpha + 2\beta$. However, this gives a cofiltration:

$$H_1(X) \twoheadrightarrow H_1^{\mathrm{pure}}(X),$$

where $H_1^{\mathrm{pure}}(X)$ is the one-dimensional vector space spanned by γ (or γ').

More precisely, let $\bar{X} \supset X$ be the torus with the punctures filled in. Then we have a long exact sequence

$$H_2(\bar{X}, X) \rightarrow H_1(X) \rightarrow H_1(\bar{X}) \rightarrow H_1(\bar{X}, X) \rightarrow \dots$$

The pure part $H_1^{\mathrm{pure}}(X)$ is the kernel

$$H_1^{\mathrm{pure}}(X) = \mathrm{Ker}(H_1(\bar{X}) \rightarrow H_1(\bar{X}, X)).$$

1.3. Mixed Hodge structures. Let us switch to cohomology to get a filtration instead of a cofiltration. Let X be a smooth quasi-projective complex variety with $\bar{X} \supset X$ a smooth compactification with the boundary $D = \bar{X} - X$ being a smooth divisor (more generally, a smooth normal crossing divisor).

We still have the period isomorphism $H_{dR}^n(X) \cong H^n(X, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$, but we want to put extra structure on the left-hand side.

Let $\Omega_{\bar{X}}^{\bullet}(\log D)$ be the log de Rham complex of meromorphic differential forms ω on \bar{X} which are holomorphic on X and such that for every point $x \in D$ and a local equation $z = 0$ of the divisor, the differential forms $z\omega$ and $zd\omega$ are holomorphic at x .

Proposition (Deligne). *The restriction map*

$$H^n(\bar{X}, \Omega_{\bar{X}}^{\bullet}(\log D)) \rightarrow H^n(X, \Omega_X^{\bullet})$$

is an isomorphism.

We have an increasing filtration on $\Omega_{\bar{X}}^{\bullet}(\log D)$ called the *weight* filtration. It is defined as follows:

$$W_p \Omega_{\bar{X}}^n(\log D) = \begin{cases} \Omega_{\bar{X}}^n, & p \leq 0 \\ \Omega_{\bar{X}}^n(\log D), & p \geq n \\ \Omega_{\bar{X}}^{n-p} \wedge \Omega_{\bar{X}}^p(\log D), & 0 < p < n \end{cases}$$

This defines a mixed Hodge structure on $H_{dR}^n(X)$.

Definition. A *mixed Hodge structure* on a complex vector space V is a collection of the following data:

- a decreasing Hodge filtration $F^p V$,
- an increasing weight filtration $W_p V$,
- a \mathbf{Q} -lattice $V_{\mathbf{Q}} \subset V$

satisfying the condition that the graded pieces $\text{gr}_W^p(V)$ are pure Hodge structures of weight p .

One can easily see that the mixed Hodge structure we have defined on $H_{dR}^n(X)$ has weights starting at n . More generally, we have the following heuristics:

- If X is projective, $H_{dR}^n(X)$ contains weights $0, \dots, n$.
- If X is not smooth, $H_{dR}^n(X)$ contains weights $n, \dots, 2n$.

Example: $X = \mathbf{C}^{\times}$. The first de Rham cohomology is represented by the differential form $\frac{dx}{x}$ which implies that $H_{dR}^1(X)$ is pure of weight 2. We call this the Tate Hodge structure $\mathbf{Q}(-1)$. Note that it coincides with the Hodge structure on $H_{dR}^2(\mathbf{CP}^1)$.

2. HODGE MODULES

2.1. Variations of Hodge structures. Previously we have looked at the case of a single complex variety X . Now consider a proper smooth morphism $f: X \rightarrow S$ of smooth complex varieties.

Then $V_{\mathbf{Q}} := \mathbb{R}^n f_* \mathbf{Q}$ is a local system of \mathbf{Q} -vector spaces and $V = \mathbb{R}^n f_*(\Omega_{X/S}^{\bullet})$ is a vector bundle with a flat Gauss–Manin connection ∇ satisfying the Griffiths transversality condition.

Definition. A *variation of a Hodge structure* of weight n on a smooth complex variety S is a collection of the following data:

- a flat vector bundle (V, ∇)
- a local system $V_{\mathbf{Q}}$ of \mathbf{Q} -vector spaces together with an identification

$$V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{O}_S$$

of flat vector bundles

- a decreasing Hodge filtration F^\bullet on V

satisfying

- (1) Griffiths transversality: $\nabla(F^p V) \subset F^{p-1} V \otimes \Omega_S^1$
- (2) Opposedness: $\mathrm{gr}_F^p \mathrm{gr}_{\overline{F}}^q V = 0$ unless $p + q = n$.

Note that Griffiths transversality shows that the filtration is not flat with respect to ∇ and so does not descend to a filtration on the local system $V_{\mathbf{Q}}$.

One can also define a *variation of a mixed Hodge structure* by combining the notions of a variation of Hodge structure and mixed Hodge structure. Given a morphism $f: X \rightarrow S$, then generically (i.e. over a smooth open $U \subset S$) $\mathbb{R}^n f_* \mathbf{Q}$ underlines a variation of a mixed Hodge structure.

2.2. Polarization. When one defines a projective variety, one simply says that there is an embedding $X \subset \mathbf{P}^N$ without actually specifying it as a data. The data of the embedding is equivalent to the data of an ample line bundle $\mathcal{O}(1)$. It has a first Chern class $\omega \in F^1 H_{dR}^2(X)$ which is, moreover, integral (i.e. comes from a class in $H^2(X, \mathbf{Z})$).

Define a pairing

$$H_{dR}^n(X) \otimes H_{dR}^n(X) \rightarrow \mathbf{C}$$

by

$$\alpha, \beta \mapsto \int_X \alpha \wedge \beta \wedge \omega^{d-n},$$

where d is the dimension of X .

It defines a morphism

$$H_{dR}^n(X) \otimes H_{dR}^n(X) \rightarrow \mathbf{Q}(-n)$$

of pure Hodge structures.

The morphism is skew-symmetric (symmetric for n even and antisymmetric for n odd) and non-degenerate.

Definition. A *polarization* of a pure Hodge structure V of weight n is a (graded)-symmetric morphism

$$V \otimes V \rightarrow \mathbf{Q}(-n)$$

inducing an isomorphism

$$V \rightarrow V^\vee(-n).$$

We say that a pure Hodge structure is *polarizable* if it admits a polarization. A mixed Hodge structure V is *graded-polarizable* if $\mathrm{gr}_W^n(V)$ are polarizable pure Hodge structures.

2.3. Hodge modules. A variation of a Hodge structure has an underlying flat vector bundle and a local system of flat sections.

Let us try to extend this definition to include regular holonomic \mathcal{D} -modules and perverse sheaves. Before we do that, let us make an observation regarding the Griffiths transversality condition. It says that

$$\nabla(F^p V) \subset F^{p-1} V \otimes \Omega^1.$$

In particular, for a vector field v we have

$$\nabla_v(F^p V) \subset F^{p-1} V.$$

Thus, the increasing filtration $F_{-p} V = F^p V$ is compatible with the natural filtration on the sheaf of differential operators \mathcal{D} by degree. In fact, the filtration F_\bullet on the \mathcal{D} -module V is an example of a *good* filtration which we will not define.

Let X be a complex variety.

Definition. A *rational filtered \mathcal{D} -module* on X is a collection of the following data:

- A regular holonomic \mathcal{D} -module M
- A perverse sheaf of \mathbf{Q} -vector spaces $M_{\mathbf{Q}}$ together with an isomorphism $\mathrm{DR}(M) \cong M_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$
- A good filtration F_\bullet on M .

The category of rational filtered \mathcal{D} -modules is closed under Verdier duality and nearby and vanishing cycles.

The full subcategory $\mathrm{HM}(X, n)$ of Hodge modules of weight n on X is defined inductively.

Definition. A rational filtered \mathcal{D} -module M has *strict support* on a subvariety $Z \subset X$ if it is supported on Z and no quotient or sub object has a smaller support.

We let

$$\mathrm{HM}(X, n) = \bigoplus_{Z \subset X} \mathrm{HM}_Z(X, n),$$

where $\mathrm{HM}_Z(X, n)$ is the category of Hodge modules with a strict support on Z .

The induction procedure is as follows:

- Hodge modules of weight n supported on a point are the same as pure Hodge structures of weight n
- Suppose $U \subset X$ is an open subvariety and $f: U \rightarrow \mathbf{C}$ a holomorphic function. For a rational filtered \mathcal{D} -module M , the module of nearby cycles $\psi_f(M)$ has a weight filtration and we say $M \in \mathrm{HM}(X, n)$ if $\mathrm{gr}_k^W(\psi_f(M))$ are Hodge modules of weight k with support on $f^{-1}(0)$.

We will not define the category of mixed Hodge modules $\mathrm{MHM}(X)$, but we will just state its properties:

- There are forgetful functors $\mathrm{MHM}(X) \rightarrow \mathrm{Perv}(X, \mathbf{Q})$ and $\mathrm{MHM}(X) \rightarrow \mathcal{D}_{rh}(X)$ compatible with the Riemann–Hilbert correspondence.
- Objects in $\mathrm{MHM}(X)$ admit a weight filtration.
- We have the six-functor formalism on the bounded derived category compatible with the forgetful functors to perverse sheaves and \mathcal{D} -modules.
- The functors $f_!$, f^* do not increase weights, the functors f_* , $f^!$ do not decrease weights.

Let $f: X \rightarrow \text{pt}$ be the natural projection. Then we have the Tate Hodge structure $\mathbf{Q}(0) \in \text{MHM}(\text{pt})$. Therefore, $f_*f^*\mathbf{Q}(0)$ is a complex of mixed Hodge modules and so we have a mixed Hodge structure on $H^*(X, \mathbf{Q})$.

Suppose X is smooth. Then $f^*\mathbf{Q}(0) = \mathbf{Q}_X(0)$ has weights ≥ 0 . Similarly, $f^!\mathbf{Q}(0) = \mathbf{Q}_X(\dim X)[2\dim X]$ has weights ≤ 0 , or, equivalently, $\mathbf{Q}_X(0)$ has weights ≤ 0 . Therefore, $\mathbf{Q}_X(0)$ is pure of weight 0. If f is proper, $f_* = f_!$ and so it sends pure Hodge modules to pure Hodge modules. Therefore, the cohomology $H^*(X, \mathbf{Q})$ has a pure Hodge structure in this case.

For an open immersion $j: U \rightarrow X$ of the smooth locus we can define the intermediate extension functor $j_{!*}$. We define the IC sheaves $\text{IC}_U \in \text{MHM}(X)$ to be

$$\text{IC}_U := j_{!*}(\mathbf{Q}_U(0)[\dim X]).$$

Proposition. *The Hodge module IC_U is of pure weight 0. More precisely, $\text{gr}_{\dim X} H^{\dim X} \mathbf{Q}_X(0) = \text{IC}_U$.*

In particular, the n -th intersection cohomology of X has a pure Hodge structure of weight n .

2.4. Towards the decomposition theorem. Let us go back to the setting of (pure) Hodge modules. We can define polarizable Hodge modules similarly to polarizable variations of Hodge structure by replacing linear duality by Verdier duality.

The Chern class of a line bundle on X can be represented as a morphism $\mathbf{Z}_X \rightarrow \mathbf{Z}_X(1)[2]$ in the derived category of mixed Hodge modules on X .

Consider a projective morphism $f: X \rightarrow Y$ with a relatively ample line bundle \mathcal{L} and the associated Lefschetz operator on Hodge modules.

Proposition (Saito). *Suppose M is a polarizable Hodge module of weight n on X . Then:*

- (1) *The pushforward $\mathbb{R}^k f_* M$ is a Hodge module of weight $n + k$ on Y .*
- (2) *(Hard Lefschetz) The Lefschetz operator induces an isomorphism*

$$\mathbb{R}^{-i} f_* M \rightarrow \mathbb{R}^i f_* M(i).$$

- (3) *The primitive part of $\mathbb{R}^{-i} f_* M$ is a polarized Hodge module.*

Proposition (Deligne). *The hard Lefschetz property implies that*

$$\mathbb{R} f_* M \cong \bigoplus_i \mathbb{R}^i f_* M[-i].$$

To derive the decomposition theorem, the only thing left is the structure theorem for polarizable Hodge modules.

Proposition (Saito). *Every polarizable Hodge module of weight n on X comes as an intermediate extension of a variation of Hodge structure of weight $n - \dim Z$ on an smooth open subset of Z for some closed subvariety $Z \subset X$.*

Given a variation of Hodge structure V on an open subset of Z , we denote by $\text{IC}_Z(V)$ the intermediate extension as a Hodge module on X .

Combining the previous theorems, we get:

Theorem (Decomposition theorem). *Let $f: X \rightarrow Y$ be a projective morphism. Then*

$$(1) \quad \mathbb{R}f_* \mathrm{IC}_X(V) = \bigoplus_i \mathbb{R}^i f_* \mathrm{IC}_X(V)[-i].$$

$$(2) \quad \mathbb{R}^i f_* \mathrm{IC}_X(V) \cong \bigoplus_{Z \subset X} \mathrm{IC}_Z(V_Z^i)$$

for some variations of Hodge structure V_Z^i .