

Intersection homology

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Abstract

We define the most basic ingredients of the decomposition theorem: pseudomanifolds and their intersection homology.

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Warning: These notes are mostly unedited; please notify the author of typos/corrections.

1 Introduction

In the seminar, we are following [1] and heading towards the proof of the

Theorem 1. (*Decomposition and semisimplicity theorems*) *Let $f : X \rightarrow Y$ be a proper map of complex algebraic varieties. Then there is an isomorphism in the constructible bounded derived category of abelian sheaves on Y :*

$$Rf_*IC_X \cong \bigoplus_{i \in \mathbb{Z}} {}^p\mathcal{H}^i(Rf_*IC_X)[-i]$$

Moreover, the perverse sheaves ${}^p\mathcal{H}^i(Rf_*IC_X)$ are semisimple in the sense that there exists a decomposition $Y = \bigsqcup S_\beta$ of Y into disjoint locally-closed subvarieties equipped with local systems L_β such that

$${}^p\mathcal{H}^i(Rf_*IC_X) \cong \bigoplus_{\beta} IC_{S_\beta}^-(L_\beta)$$

Our goal in this talk is to define the most basic ingredients in this formula: the *intersection (co)homology complexes* IC_X . These notes roughly follow the book [2] of Kirwan and Woolf.

Intersection homology was invented by Goresky and MacPherson in the 1970s; the goal was to produce a homology theory that behaves as well for singular spaces as it does for manifolds, in the sense that basic properties such as Poincaré duality and the Lefschetz theorems hold even for singular projective varieties. Intersection homology is defined for a class of spaces known as topological pseudo-manifolds, which we now define.

2 Stratified spaces and pseudomanifolds

An m -dimensional topologically stratified space is a para-compact Hausdorff space X equipped with a filtration

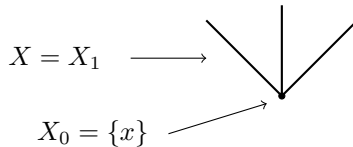
$$X = X_m \supset X_{m-1} \supset \cdots \supset X_1 \supset X_0$$

by closed subsets, where X_0 is the “most singular part”. The open strata $X_k^\circ = X_k \setminus X_{k-1}$ are allowed to be empty, but if they are nonempty then they are required to be manifolds of dimension k , and, moreover, the singularities of X in a neighbourhood of X_k° are controlled in a precise sense that we will make clear shortly. A topological pseudo-manifold is a topological space X that admits a stratification as above such that $X_{m-1} = X_{m-2}$ and X_m° is dense. Note that a given pseudo-manifold may admit many different stratifications.

Before giving the precise constraints on the singularities, let us look at some low-dimensional examples:

Example 1. A 0-dimensional stratified space $X = X_0$ is simply a countable set with the discrete topology. \square

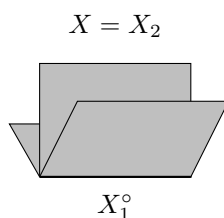
Example 2. A 1-dimensional stratified space has the form $X = X_1 \supset X_0$, where $X_1^\circ = X_1 \setminus X_0$ is a 1-dimensional manifold, and X_0 is a collection of isolated points where X may fail to be a manifold. We require that every point $x \in X_0$ has a neighbourhood that is homeomorphic to the (open) cone over a finite collection of points:



Such an X will be a pseudo-manifold if and only if $X_0 = \emptyset$. In other words, every 1-dimensional pseudo-manifold is a 1-dimensional manifold. \square

Example 3. A 2-dimensional stratified space has a stratification of the form $X = X_2 \supset X_1 \supset X_0$ such that

- X_2° is a 2-dimensional manifold
- Every point in the 1-dimensional manifold X_1° has a neighbourhood homeomorphic to $\mathbb{R} \times Cone(L)$, where L is a finite collection of points, and X_1° corresponds to $\mathbb{R} \times \{\text{the vertex}\}$.



- Every point in X_0 has a neighbourhood homeomorphic to $Cone(L)$ where $L = L_1 \supset L_0$ is a compact 1-dimensional stratified space. The points in X_0 correspond to the vertex of the cone over L and the points in X_1 correspond to the cone over L_0 .

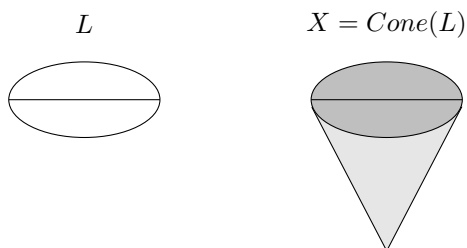


Figure 1: A compact 1-dimensional stratified space and the two-dimensional stratified space that is its cone

Such a space will be a pseudomanifold exactly when we can take L to be non-singular (otherwise X_1° will be nonempty as above). Hence a 2-dimensional pseudomanifold locally looks like \mathbb{R}^2 or $Cone(L)$ where L is a disjoint union of circles.

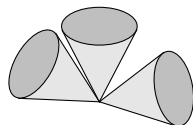


Figure 2: A 2-dimensional pseudomanifold

□

Example 4 (Plane curves (Milnor)). Every plane algebraic curve $X \subset \mathbb{C}^2$, singular or not, is a 2-dimensional pseudomanifold: given a point $x \in X$, consider a very small three-sphere $S \subset \mathbb{C}^2$ centred at X . Then as long as S is small enough, the intersection $X \cap S$ will be transverse, and hence it will be a disjoint union of circles (a “link”). A neighbourhood of x in X is then homeomorphic to the cone over $X \cap S$. For example the nodal curve $\{z_1 z_2 = 0\} \subset \mathbb{C}^2$ is locally the cone over two disjoint circles. \square

In general, for a filtration

$$X = X_m \supset X_{m-1} \supset \cdots \supset X_1 \supset X_0$$

by closed subsets to define a stratification, we require that for every $x \in X_j^\circ = X_j \setminus X_{j-1}$, there is a neighbourhood N_x of x in X and a compact $(m-j-1)$ -dimensional topologically stratified space L with a filtration

$$L = L_{m-j-1} \supset \cdots \supset L_0$$

and a homeomorphism

$$\phi : N_x \rightarrow \mathbb{R}^j \times \text{Cone}(L)$$

that restricts to homeomorphisms

$$X_j \cap N_x \cong \mathbb{R}^j \times \{\text{vertex of } \text{Cone}(L)\}$$

and

$$X_{j+i+1} \cap N_x \cong \mathbb{R}^j \times \text{Cone}(L_i)$$

for all $i \geq 0$.

We are interested in pseudo-manifolds because of the following

Theorem 2 (Borel, Whitney). *Every quasi-projective variety X of pure dimension n over the complex numbers admits a topological stratification*

$$X = X_{2n} \supset X_{2n-1} \supset \cdots \supset X_1 \supset X_0$$

where $X_{2i} = X_{2i+1}$ are closed subvarieties for all i .

3 Intersection homology

Let us begin by recalling the definition of the singular chain complex of a space X . For $i \geq 0$ a **singular i -simplex in X** is a continuous map $\sigma : \Delta^i \rightarrow X$ from the standard i -simplex $\Delta^i \subset \mathbb{R}^i$ into X . Restricting σ to a face of Δ^i gives a singular $(i-1)$ -simplex in X , and summing over these restrictions with appropriate signs we obtain the boundary operator

$$\partial : C_i(X) \rightarrow C_{i-1}(X),$$

giving a chain complex

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \longrightarrow 0.$$

If X is a pseudomanifold equipped with a stratification, the intersection homology of X is obtained as the homology of a subcomplex of $C_\bullet(X)$ that consists of chains whose intersection with the strata is “not too bad”. Here by “bad”, we essentially mean “not transverse”. Intuitively, what we want is that only the low-dimensional faces of Δ^i can map into low-dimensional strata of X .

We make this precise by introducing a function $p : \mathbb{Z}_{\geq 2} \rightarrow \mathbb{Z}_{\geq 0}$ called a **perversity** which we will use to measure the failure of transversality. We require that $p(2) = 0$ and $p(k+1) \in \{p(k), p(k+1)\}$ for all $k \in \mathbb{Z}_{\geq 2}$. In other words, we require p to be a non-decreasing function that jumps by at most 1 at every step. We will mainly make use of the **lower-middle perversity**

$$p(k) = \left\lfloor \frac{k}{2} \right\rfloor - 1$$

although there are many others.

Given a perversity p , we say that a singular i -simplex $\sigma : \Delta^i \rightarrow X$ is **p -allowable** if $\sigma^{-1}(X_{m-k}^\circ)$ is contained in the $(i - k + p(k))$ -skeleton of Δ^i for all $k \geq 2$. A singular chain $\xi = \sum_j a_j \sigma_j$ is **p -allowable** if every singular simplex appearing with a nonzero coefficient in ξ or $\partial\xi$ is allowable.

Example 5. Let us consider the algebraic curve $X \subset \mathbb{P}^2$ that is the union of two distinct lines, which has a nodal singularity at the intersection point $x_0 \in X$. Topologically, it is a pair of spheres that meet in a point and the local model near the singular point is the cone over a pair of circles. The stratification has the form $X = X_2 \supset X_0$ so we only need to consider $k = 2$. Since $p(2) = 0$ for any perversity, all perversities will give the same notion of allowability. Allowability puts the following constraints on simplices:

Dimension of simplex	Pre-image of X_0 contained in
0	(-2) -skeleton (empty)
1	(-1) -skeleton (empty)
2	0-skeleton
3	1-skeleton
\vdots	\vdots

In other words, 0- and 1-simplices are allowable if and only if they do not intersect X_0 , while 2-simplices are allowable if and only if the only points where they intersect X_0 are vertices.

A similar picture is valid locally for any algebraic curve. □

It follows from the definition that the set $IC_\bullet^p(X) \subset C_\bullet(X)$ of p -allowable chains is actually a subcomplex, called the **intersection chain complex**. The **intersection homology of X with perversity p** is the homology $I^p H_\bullet(X)$ of this complex. If p is the middle perversity, we denote $I^p C_\bullet(X)$ and $I^p H_\bullet(X)$ simply by $IC_\bullet(X)$ and $IH_\bullet(X)$.

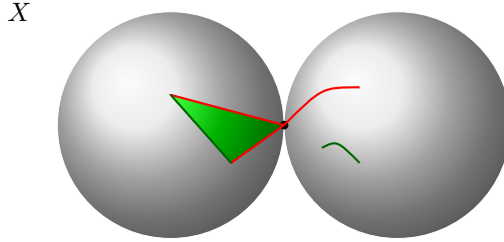


Figure 3: Some allowable simplices are shown in green while unallowable ones are shown in red. Notice that the two-simplex is allowable as a simplex but not as a chain because its boundary contains non-allowable 1-simplices.

Remark 1. The notion of allowability of chains clearly depends on the choice of stratification. However, the groups $I^p H_\bullet(X)$ are actually topological invariants of X for any perversity: different stratification of X produce canonically isomorphic intersection homology groups and hence homeomorphic pseudomanifolds have isomorphic intersection homologies. \square

Remark 2. This was the definition of intersection homology via singular chains. One can do a simplicial version using triangulations, although some care is required to get a result that is independent of the triangulation. \square

Example 6. We return to the curve $X \subset \mathbb{P}^2$ that is the union of two lines. It is homeomorphic to a pair of tetrahedra joined at a point and we will compute IH using this triangulation.

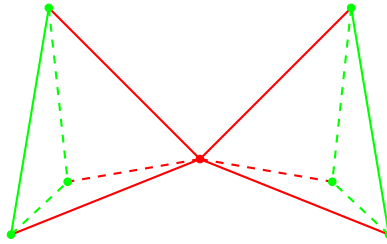


Figure 4: A triangulation of X by 2-simplices. Allowable 0- and 1-simplices are shown in green and unallowable ones are red. All 2-simplices are allowable.

The group of two-cycles on X is generated by the fundamental cycles of the two tetrahedra. These two-cycles are allowable chains because all of the two-simplices are allowable, and the boundary of each chain is trivial. (Notice, however, that the individual simplices in the chain are not all allowable; only their linear combination is!) Since there are no three-simplices, there are no 2-boundaries, and so we find $IH_2(X) = \mathbb{Z} \oplus \mathbb{Z}$ with one copy of \mathbb{Z} for each irreducible component of X .

The only allowable one-cycles in X are linear combinations of the ones shown in green in the diagram. (All other 1-cycles pass through the singular point of X and are therefore not allowable.) The green cycles are both boundaries of faces and hence they are trivial in homology, giving $IH_1(X) = 0$.

Finally, the allowable zero-cycles are the green dots—the vertices that are not the singular point. Any two green vertices in the leftmost tetrahedron give the same homology class because their difference is the boundary of an allowable 1-simplex (a green line). Similarly, the green vertices on the right all define the same homology class. However, a green vertex on the left cannot be connected to a green vertex on the right by a collection of allowable 1-simplices. Hence they define distinct classes in homology and we find $IH_0(X) = \mathbb{Z} \oplus \mathbb{Z}$. We conclude that

$$IH_i(X) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 2 \end{cases}$$

the homology of a disjoint unions of two spheres. Notice that $IH_0 = IH_2$ so we have a form of Poincaré duality.

Basically the same argument gives the more general

Proposition 1 (Prop. 4.4.1 in [2]). *Suppose that X is a pseudomanifold of dimension $2n$ with an isolated singular point, i.e., $X_j = X_0 = \{x\}$ for $j < 2n$. Using the proposition, we see that the lower middle perversity intersection homology is given by*

$$IH_i(X) = \begin{cases} H_i(X) & i > n \\ \text{img}(H_i(X \setminus \{x\}) \rightarrow H_i(X)) & i = n \\ H_i(X \setminus \{x\}) & i < n. \end{cases}$$

One readily recovers the calculation in the example above from this result.

Example 7 (The nodal cubic). The curve $\{y^2z = x^2(x+z)\} \subset \mathbb{P}^2$, a nodal cubic, is a pinched torus, i.e. a sphere with two points identified. Applying the proposition we find

$$IH_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z} & i = 2 \end{cases}$$

the homology of a sphere. Again, Poincaré duality holds. \square

More generally we have the following result of Goresky–MacPherson:

Proposition 2 (see Prop. 4.5.2 in [2]). *Let X be a quasi-projective complex algebraic variety, and let \tilde{X} be its normalization. Then we have a natural isomorphism $I^p H_\bullet(X) \cong I^p H_\bullet(\tilde{X})$ for all perversities p .*

This explains the calculations above since the normalization of the union of two distinct lines in \mathbb{P}^2 is simply $\mathbb{P}^1 \sqcup \mathbb{P}^1$, while the normalization of a nodal cubic is \mathbb{P}^1 .

The appearance of Poincaré duality in these example is not coincidental. Indeed, we have the following remarkable

Theorem 3. *The “Kähler package” (Poincaré duality, Lefschetz theorems, Hodge decomposition, etc.) holds for intersection homology of projective varieties.*

We hope to better understand this result by the end of the seminar.

4 Generalizations and the IC sheaves

In order to get to the statement of the decomposition theorem, we need to beef up the definition of intersection homology in two ways: first, we need to allow for coefficients in a local system, and second, we need to sheafify everything.

4.1 Locally constant coefficients

Recall that if X is a space and A is an abelian group, then the constant sheaf A_X is the sheaf that assigns to every open set $U \subset X$ the set of locally constant maps $U \rightarrow A$ with the obvious restriction morphisms. A sheaf \mathcal{L} of abelian groups on X is **locally constant** if for every $x \in X$ there exists an open neighbourhood N_x of x in X such that $\mathcal{L}|_{N_x}$ is isomorphic to a constant sheaf on N_x . (In this case the underlying group is canonically isomorphic to the stalk \mathcal{L}_x of \mathcal{L} at x .)

Suppose that \mathcal{L} is a locally constant sheaf and that $\sigma : \Delta^i \rightarrow X$ is a singular i -simplex in X . Then the pullback $\sigma^*\mathcal{L}$ is locally constant, and since Δ^i is simply connected, it is actually constant. We denote by $\Gamma(\sigma, \mathcal{L})$ the space of global sections of $\sigma^*\mathcal{L}$. Notice that if $\tau : \Delta^{i-1} \rightarrow X$ is a face of σ , then we have a canonical restriction isomorphism $\Gamma(\sigma, \mathcal{L}) \cong \Gamma(\tau, \mathcal{L})$,

An *i -chain on X with values in \mathcal{L}* is an element of the group

$$C_i(X, \mathcal{L}) = \bigoplus_{\sigma: \Delta^i \rightarrow X} \Gamma(\sigma, \mathcal{L}).$$

In other words, it is a linear combination of simplices σ where the coefficient of σ , rather than being a number, is a section of the local system over σ . The boundary map ∂ is obtained by combining the usual sum-with-signs and the restriction of section to faces.

Now suppose that $X = X_m \supset X_{m-1} = X_{m-2} \supset \cdots \supset X_0$ is a stratified pseudomanifold and that \mathcal{L} is a local system on the open dense stratum X_m° . No matter which perversity we use, the image of any allowable simplex must intersect X_m° and hence we can make the same definition as above without extending the local system to all of X . In this way, we obtain the **intersection homology $I^p H_\bullet(X, \mathcal{L})$ with coefficients in \mathcal{L}** .

4.2 Chains with locally finite support and \mathcal{IC} sheaves

Recall that there is another version of homology called homology with locally finite supports (aka Borel–Moore homology), in which we replace $C_i(X)$ with the larger group $C_i((X))$ which consists of possibly infinite linear combinations of simplices that are “locally finite”. Here locally finite means that for any point $x \in X$, there is a neighbourhood N_x that intersects only finitely many of the simplices in the chain. When X is compact the usual homology and the homology with locally finite support are isomorphic. Similarly, we can talk about locally finite allowable chains, and therefore obtain the intersection homology with locally finite supports.

Given a pseudomanifold X with a stratification, we define a complex of sheaves \mathcal{IC}_X on X by the formula

$$\mathcal{IC}_X^{-i}(U) = I^p C_i((U))$$

the intersection chains with locally finite support on U , but with the degree reversed to get a cochain complex instead of a chain complex. This grading convention is chosen simply because it is typical to use cohomological gradings in sheaf theory.

Notice that if $V \subset U$ is an inclusion of open sets, it is not at all clear why there should be a restriction map $r : \mathcal{IC}_X(U) \rightarrow \mathcal{IC}_X(V)$. For this it is enough to define the restriction of a single singular simplex σ , which we do by performing a bunch of barycentric subdivisions to break σ into pieces that lie only in V . (We will be more precise below.) Since barycentric subdivision is a chain map, the restriction map is a morphism of chain complexes and hence we obtain a sheaf of cochain complexes on X . We can also make the same definition with coefficients in a local system \mathcal{L} on X_m° .

To define the restriction map, we begin with a singular simplex σ on U . We will use σ to define a set $J(\sigma)$ of simplices on V and set $r(\sigma) = \sum_{\sigma' \in J(\sigma)} \sigma'$. The procedure is inductive:

1. If the image of σ lies entirely in $U \setminus V$ then $J(\sigma) = \emptyset$.
2. If the image of σ lies entirely in V , then $J(\sigma) = \{\sigma\}$.
3. If the image of σ intersects both V and $U \setminus V$, then we inductively define $J(\sigma) = J(\eta_0) \cup J(\eta_1) \cup \dots \cup J(\eta_i)$, where the simplices η_0, \dots, η_i are the i -simplices appearing in the barycentric subdivision of σ .

This process will, in general, continue forever, which is why we need infinite chains. It is a good exercise to convince oneself that the resulting chain, while possibly infinite, is nevertheless locally finite as required.

Theorem 4. *The sheaves $\mathcal{I}^p \mathcal{C}_X^i$ are soft for all $i \leq 0$ and all perversities p , and hence their hypercohomology is simply the cohomology of global sections*

$$\mathbb{H}^{-i}(X, \mathcal{I}^p \mathcal{C}_X^\bullet) = I^p H_i^{cl}(X),$$

the intersection homology with locally finite supports.

References

- [1] M. A. A. de Cataldo and L. Migliorini, *The decomposition theorem, perverse sheaves and the topology of algebraic maps*, *Bull. Amer. Math. Soc. (N.S.)* **46** (2009), no. 4, 535–633.
- [2] F. Kirwan and J. Woolf, *An introduction to intersection homology theory*, second ed., Chapman & Hall/CRC, Boca Raton, FL, 2006.