SINGULAR MODULI FOR REAL QUADRATIC FIELDS

LEARNING SEMINAR IAS/PRINCETON UNIVERSITY

Singular moduli are special values of the modular j-invariant at arguments in imaginary quadratic fields. They enjoy many beautiful properties, putting them at the centre of the classical theory of complex multiplication. For instance, Weber computed that

$$j(\sqrt{-14}) = 2^3 \left(323 + 228\sqrt{2} + (231 + 161\sqrt{2})\sqrt{2\sqrt{2} - 1}\right)^3,$$

and this number has several interesting features, of which we will highlight three particular ones:

(1) It generates the Hilbert class field over $K = \mathbf{Q}(\sqrt{-14})$.
(2) Its norm has the prime factorisation

$$\mathrm{Nm}\, j(\sqrt{-14}) = 2^{24} \cdot 11^6 \cdot 17^3 \cdot 41^3$$

(3) Let $E_s(z_1, z_2)$ be the real analytic family of Eisenstein series over the quadratic field $K = \mathbf{Q}(\sqrt{42})$, attached to the genus character that cuts out the extension generated by $\sqrt{-14}$ over K. Then the quantity $\mathrm{Nm}\, j(\sqrt{-14})$ is related to the first Fourier coefficient of the holomorphic projection of

$$\left(\frac{\partial}{\partial s} E_s(z, z)\right)_{s=0}$$

The first of these properties illustrates the central role of the j-function in the theory of complex multiplication, and the *Kronecker Jugendtraum* in the context of Hilbert's 12^{th} problem. The second and third properties correspond to the algebraic and analytic parts of the paper of Gross–Zagier [GZ85], which was the gateway to their landmark results in Gross–Zagier [GZ86] and Gross–Kohnen–Zagier [GKZ87]. An explicit formula for the factorisation of the norm was given after reinterpreting the exponent of a prime q as a certain arithmetic intersection number of two points on the (0-dimensional) Shimura variety associated to a definite quaternion algebra $B_{q\infty}$ ramified at q and ∞. The third property is exploited to give an analytic proof of the same explicit formula, and is of particular importance for our seminar.

In this seminar, we will study real quadratic analogues of singular moduli. A basic obstacle to using the j-invariant is the fact that its domain is the Poincaré upper half plane $\mathcal{H}_\infty$, which contains no real quadratic points. An interesting approach initiated by Kaneko [Kan09] and Duke–Imamoḡlu–Tóth [DIT11a, DIT11b] defines values of the j-function at real quadratic arguments through the procedure of regularised cycle integrals. The invariants that are obtained this way do not appear to have the algebraic properties (1) and (2) in the above list, though they do exhibit behaviour similar to that in property (3).

The alternative approach in [DV] based on the theory of *rigid meromorphic cocycles* seems to yield invariants that satisfy all three of the above properties. This approach is fundamentally p-adic, and starts by replacing $\mathcal{H}_\infty$ by the Drinfeld upper half plane $\mathcal{H}_p$, which contains many real quadratic points. The theory revolves around the *Ihara group* $\Gamma = \mathrm{SL}_2\, \mathbf{Z}[1/p]$, acting on meromorphic functions $\mathcal{M}^\times$ on $\mathcal{H}_p$ via linear fractional transformations. A rigid meromorphic cocycle J defines an element

$$J \in \mathrm{H}^1(\Gamma, \mathcal{M}^\times)$$

satisfying a finiteness condition close to parabolicity. A key point in the theory is that such a cocycle can be meaningfully evaluated at a real quadratic argument τ, yielding invariants

$$J[\tau] \ \in \ \mathbf{P}^1(\mathbf{C}_p)$$

through a simple procedure that is independent of the choice of argument τ in its Γ-orbit, therefore behaving as if they were the value of some Γ-invariant meromorphic function on $\mathcal{H}_p$. However, no such functions exist (other than constants)! The quantities $J[\tau]$ behave in many ways like singular moduli, and in this seminar we will learn more about their properties. We will discuss a wide variety of related topics along the way that are of independent interest for a wide number-theoretic audience, as outlined below.

We will focus on three separate themes, which involve rather different flavours of ingredients:

- The construction and classification of the subspace of rigid *analytic* cocycles $\mathrm{H}^1(\Gamma, \mathcal{A}^\times)$, which is more classical and reinterprets via the Schneider–Teitelbaum lift earlier constructions of Gross–Stark units and and Stark–Heegner points on elliptic curves in [Dar01, DD06].
- The construction and classification of rigid meromorphic cocycles, and their conjectural algebraicity properties analogous to (1) and (2) above. In analogy with the results of Gross–Zagier [GZ85], the exponent of a prime q in the factorisation in (2) is described in terms of certain arithmetic intersection numbers of real quadratic geodesics on the Shimura curve attached to the quaternion algebra B_{pq} ramified only at the primes p and q. The main source here is [DV].
- The modularity of the generating series of RM values of the winding cocycle, analogous to (3). Here, the real analytic family of Eisenstein series is replaced by a p-adic one, and the deformation theory of Galois representations available in the p-adic setting leads to proofs of special cases of some of the above conjectures. The main source here is [DPV].

Contents

1. Differences of singular moduli

Speaker: John Halliday

Before their monumental and celebrated papers [GZ86, GKZ87] on height pairings of Heegner points, Gross and Zagier wrote a paper on differences of singular moduli [GZ85]. The results in this paper correspond to those contained in the follow-up papers, in the special case of the modular curve $X(1)$, and the arguments take a particularly simple and appealing form, while still retaining many of the crucial ideas that will be recurrent themes throughout this seminar. Since this curve is of genus 0, the height pairing is necessarily trivial, and as a pleasant consequence we get an equality between the local contributions to the height pairing of two CM points τ_1, τ_2 at the archimedean and non-archimedean places, leading to two different proofs of an explicit formula for the factorisation of

$$\mathrm{Nm}\big(j(\tau_1) - j(\tau_2)\big). \tag{1}$$

The first proof is algebraic, and uses CM theory of elliptic curves to compute the arithmetic intersection numbers of τ_1 and τ_2. The second proof is analytic, and uses the diagonal restriction of a real analytic family of Hilbert Eisenstein series attached to the genus character defined by the pair (τ_1, τ_2).

This talk will start by recalling the main statements of CM theory for elliptic curves, guided by explicit examples whenever possible. Some good sources for this material are [Cox89, Chapter 3], [Sil94, Chapter 2] and [Shi71, Chapter 5]. The main topic of this talk is the paper of Gross–Zagier [GZ85] on differences of singular moduli, with a special focus on those arguments that can be interpreted conceptually. Of crucial importance for this seminar is a thorough discussion of the analytic arguments involving families of diagonal restrictions of Eisenstein series around weight 2.

Part 1. Rigid analytic cocycles

2. The Dedekind–Rademacher symbol

Speaker: Jonathan Love

This talk will start by discussing a classical result of Milnor [Mil71] identifying the space $\mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R})$ with a three-dimensional sphere minus a trefoil knot (depicted below). A beautiful argument of Ghys [Ghy07] attaches to any hyperbolic matrix a knot in this threefold, whose linking number with the trefoil is shown to be described by the Dedekind–Rademacher symbol

$$\phi_{\mathrm{DR}} : \mathrm{SL}_2(\mathbf{Z}) \longrightarrow \mathbf{Z},$$

which is defined using the transformation law of the Dedekind η-function. It would be nice to discuss this also in the light of the forthcoming work of Bergeron–Charollois–Garcia–Venkatesh [BCG19].

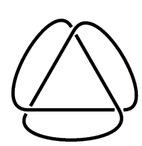

Figure 1. The trefoil knot

After a brief recap of the theory of period polynomials and the Eichler–Shimura theorem, following for instance [KZ84, §1], we turn to the case of weight 2 forms on $\Gamma_0(p)$, realising the duality with homology relative to the cusps $\{0, \infty\}$ through the modular symbols attached to eigenforms, and the Dedekind–Rademacher homomorphism $\Phi_p : \Gamma_0(p) \to \mathbf{Z}$. A good source for this material is [Maz79].

3. The p-adic upper half plane

Speaker: Boya Wen

In this talk, we will start developing the necessary background on the p-adic upper half plane $\mathcal{H}_p$. This is a rigid analytic variety which plays the principal role in the theory of rigid meromorphic cocycles. For our purposes, it is most fruitful to think of it as a tubular neighbourhood of a combinatorial structure known as the *Bruhat–Tits tree*, depicted below.

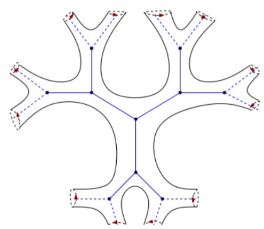

Figure 2. The Bruhat–Tits tree

In this talk, the basic properties of $\mathcal{H}_p$ will be developed, for instance following [DT08]. In preparation of the definition of rigid meromorphic cocycles, we will discuss in some detail the main statements in the theory of Mumford curves and p-adic uniformisation, using the tool of theta functions as discussed for instance in [GvdP80]. This theory can be very nicely explained and illustrated on the two examples of the Tate curve, and quaternionic Shimura curves, see for instance [DT08, Tei88] and [Tei90, §3].

4. Gross–Stark units and Stark–Heegner points

Speaker: Gyujin Oh

The goal of this talk is to cover the description of $\mathrm{H}^1(\Gamma, \mathcal{A}^\times / \mathbf{C}_p^\times)$ in [DV, Chapter 2]. Explicit elements can be constructed from the modular symbols attached to eigenforms, and the Dedekind–Rademacher cocycle discussed two lectures ago, via the *Schneider–Teitelbaum lift*. We then discuss the RM values of these cocycles, which are given by Gross–Stark units and Stark–Heegner points. The latter provide us with a set of conjecturally global points on elliptic curves. We will return to the former later.

Finally, we end by constructing an explicit cocycle J_w, defining a class in $\mathrm{H}^1(\Gamma, \mathcal{A}^\times / \mathbf{C}_p^\times)$ which we refer to as the *winding cocycle*. This cocycle has an extremely concrete and simple description, and will play a central role later in the seminar. We discuss its spectral decomposition following [DPV, §3].

Part 2. Rigid meromorphic cocycles

5. Rational cocycles

Speaker: Amina Abdurrahman

In preparation of our definitions of rigid meromorphic cocycles, we start by discussing the work of Knopp [Kno78] on *rational cocycles*, which define classes in

$$\mathrm{H}^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(x))$$

with values in the group of rational functions $\mathbf{C}(x)$ on $\mathbf{P}^1_{\mathbf{C}}$. These cocycles are attached to narrow ideal classes in real quadratic fields, and their definition involves intersection numbers of geodesics on $\mathcal{H}_\infty$. They were classified completely in the work of Choie–Zagier [CZ93].

Going back to the setting of the threefold $\mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R})$ and its modular knots, the following natural question arises: Now that we know the linking number of these modular knots and the trefoil knot, we would like to describe the linking number of two, suitably symmetrised, modular knots with each other. We will discuss the work of Duke–Imamoḡlu–Tóth [DIT17] which shows that the answer is similar to that of Ghys [Ghy07], where the Dedekind–Rademacher cocycle is replaced by Knopp's rational cocycles.

6. Rigid meromorphic cocycles

Speaker: Michele Fornea

We now come to the main definitions given in [DV], where by a process of taking suitable infinite products, one obtains cocycles which are multiplicative lifts of a p-adically enriched version of the rational cocycles of Knopp [Kno78]. This yields a very explicit range of cocycles

$$J_\tau \in \mathrm{H}^1(\Gamma, \mathcal{M}^\times / \mathbf{C}_p^\times)$$

attached to a choice of RM point τ, whose definition is very similar to that of the p-adic theta functions that arose in the theory of p-adic uniformisation of Mumford curves several lectures ago.

In this talk, we will discuss the basic theory of rigid meromorphic cocycles, focussing on the important aspect of the lifting obstructions of the classes J_τ defined above, which can be interpreted as a Stark–Heegner point on a modular Jacobian $J_0(p)$. In cases where p is a prime that divides the order of the Monster group, the lifting obstructions of certain cocycles J_τ^+ take on a particularly compelling form.

7. Singular moduli for real quadratic fields

Speaker: Evan O'Dorney

The real interest in the definition of rigid meromorphic cocycles lies in the apparent arithmetic significance of their RM values, defined by

$$J[\tau] = J(\gamma_\tau)(\tau) \quad \in \mathbf{C}_p \cup \{\infty\}$$

where γ_τ is the automorph attached to the RM point τ. This yields a quantity that is independent of the choice of representative J of its cohomology class, and independent of the choice of τ in its Γ-orbit. To give a random example, for $p = 19$ we compute numerically that

$$J^{+}_{11\sqrt{2}}\left[\frac{1+\sqrt{5}}{2}\right] = \frac{209711 - 130467\sqrt{-11}}{2 \cdot 5^2 \cdot 59 \cdot 163}$$

We then turn to the conjectures stated in [DV] about the arithmetic information that is encoded in these RM values of rigid meromorphic cocycles, stating suitable analogues of properties (1) and (2) above. The factorisation conjecture is phrased in terms of arithmetic intersection numbers of geodesics on quaternionic Shimura curves attached to the quaternion algebra B_{pq} ramified at p and some variable prime q.

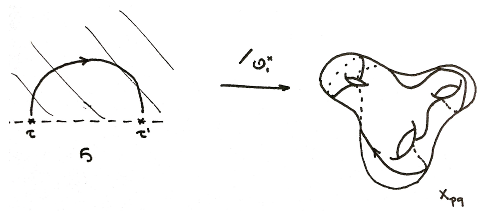

FIGURE 3. Geodesics on X_{pq}

Part 3. Generating series of the winding cocycle

8. OVERCONVERGENT MODULAR FORMS AND GALOIS DEFORMATIONS

Speaker: Kim Tuan Do

In this talk, we introduce the theory of overconvergent modular forms, the ordinary projection, and Hida families. To avoid getting bogged down in unnecessary technicalities, it would be nice to aim for a discussion where the general theory is illustrated with the aid of explicit examples and numerical computations. We also explain the salient points of the theory of p-adic families of overconvergent Hilbert modular forms over real quadratic fields.

Then, we discuss how deformations of modular forms in (parallel) weight 1 can be related to class field theory, making explicit such families to first order. In the elliptic case, many beautiful results are obtained through this principle, e.g [DLR15b, DLR15a], but for us the most important case is that of weight $(1, 1)$ Hilbert Eisenstein series as discussed in Darmon–Dasgupta–Pollack [DDP11].

9. COHERENT AND INCOHERENT EISENSTEIN SERIES

Speaker: Linus Hamann

We now explore the analogue of property (3) in the setting of rigid analytic cocycles, introducing the tool of p-adic families of overconvergent forms to study RM values of rigid meromorphic cocycles. We will focus on the case of the *winding cocycle* defined in the third lecture on analytic cocycles.

We follow [DPV] and study the diagonal restrictions $G_{2k}(\psi)$ of the p-adic families of Eisenstein series attached to an unramified character ψ, restricted to the diagonal. We see that the splitting behaviour of p creates a dichotomy in the behaviour of the specialisation in weight $2 = 1 + 1$. The case that is of most interest to us is the *incoherent case* where p is inert, in which case the diagonal restriction vanishes. This is entirely analogous to the phenomenon that occurs also in the work of Gross–Zagier [GZ85] as we discussed in the first talk of this seminar.

10. Concluding remarks

Speaker: Jan Vonk

This talk will wrap up the seminar by giving an overview of all the talks, and finishing the proof of the algebraic nature of $J_{\mathrm{DR}}[\tau]$ started last week. We make some concluding remarks.

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