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Rigid cocycles and singular moduli for real quadratic fields

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Abstract. These lectures aim to give an introduction to the theory of rigid cocycles, and to discuss their role in the analytic construction of singular moduli for real quadratic fields. Special emphasis will lie on the computational techniques and experiments that informed the development of the subject.

Contents

1. Singular moduli for imaginary quadratic fields

The theme of this PCMI is "number theory informed by computation". One mathematical story that may be described by this phrase is the theory of complex multiplication (CM). Several of its most celebrated results have been discovered or directly informed by explicit experimentation and computation. Notably, this includes the work of Gross–Zagier on heights of Heegner points.

The goal of these notes is to illustrate the role of computations in the development of CM theory, and to discuss how this tradition continues in recent investigations of a nascent RM theory. The focus lies on the theory of rigid cocycles.

Outline. After a brief introduction on CM theory, we recall some reduction theory of binary quadratic forms in §2. We define *rational cocycles* for $\mathrm{SL}_2(\mathbf{Z})$ in §3, and in §4 we consider their p-adic limits, defining *rigid cocycles* for $\mathrm{SL}_2(\mathbf{Z}[1/p])$. Special values of rigid cocycles at RM points define invariants which behave like RM counterparts of the differences of singular moduli of Gross and Zagier.

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1.1. Singular moduli. The story begins with classical results from the late 19th century German school, on the theory of complex multiplication. Several comprehensive treatments exist: inspiring historical sources the reader may wish to consult are Klein–Fricke [47,48] and Fricke–Klein [32,33] as well as the insightful account of the works of Eisenstein and Kronecker by Weil [67].

Our primary interest is Klein's modular j-invariant

$$\begin{aligned} j(q) &= \left(1+240\sum_{n\geqslant 1}\frac{n^3q^n}{1-q^n}\right)^3 \div \left(q\prod_{n\geqslant 1}(1-q^n)^{24}\right) \\ &= q^{-1}+744+196884q+21493760q^2+\dots \end{aligned}$$

with $q=\exp(2\pi i\tau)$. It is a holomorphic function (though with a simple pole at the cusp ∞), invariant under the action of $SL_2(\mathbf{Z})$ by linear fractional transformations on the argument τ, defined on the Poincaré or Lobachevsky hyperbolic plane

$$\begin{aligned} \mathfrak{H}_\infty &:= \{\tau\in\mathbf{C} \,:\, \operatorname{Im}(\tau)>0\} \\ &= \{q\in\mathbf{C} \,:\, |q|<1\}. \end{aligned}$$

We will study *singular moduli*, which are the values of the j-function at CM points τ, i.e. elements $\tau\in\mathfrak{H}_\infty$ that satisfy a quadratic equation over $\mathbf{Q}$, necessarily of negative discriminant. Singular moduli are always algebraic integers. Let us naively tabulate some singular moduli at purely imaginary CM points:

τ	$j(\tau)$	Trace	Norm
$\sqrt{-1}$	12^3	$2^6\cdot 3^3$	$2^6\cdot 3^3$
$\sqrt{-2}$	20^3	$2^6\cdot 5^3$	$2^6\cdot 5^3$
$\sqrt{-3}$	54000	$2^4\cdot 3^3\cdot 5^3$	$2^4\cdot 3^3\cdot 5^3$
$\sqrt{-4}$	66^3	$2^3\cdot 3^3\cdot 11^3$	$2^3\cdot 3^3\cdot 11^3$
$\sqrt{-5}$	$(26\sqrt{5}+20)^3$	$2^7\cdot 5^3\cdot 79$	$2^{12}\cdot 5^3\cdot 11^3$
$\sqrt{-6}$	$1707264\sqrt{2}+2417472$	$2^7\cdot 3^3\cdot 1399$	$2^{12}\cdot 3^6\cdot 17^3$
$\sqrt{-7}$	255^3	$3^3\cdot 5^3\cdot 17^3$	$3^3\cdot 5^3\cdot 17^3$
$\sqrt{-8}$	$(130\sqrt{2}+190)^3$	$2^4\cdot 5^6\cdot 11\cdot 19$	$2^6\cdot 5^6\cdot 23^3$
$\sqrt{-9}$	$44330496\sqrt{3}+76771008$	$2^7\cdot 3^2\cdot 133283$	$2^{12}\cdot 3^3\cdot 11^3\cdot 23^3$
$\sqrt{-10}$	$(162\sqrt{5}+390)^3$	$2^7\cdot 3^3\cdot 5^2\cdot 13\cdot 379$	$2^{12}\cdot 3^6\cdot 5^3\cdot 29^3$

The next entry of the table would be $j(\sqrt{-11})$, whose minimal polynomial

$$f(x)=x^3-1122662608x^2+270413882112x-653249011576832$$

has a non-abelian splitting field over $\mathbf{Q}$ with Galois group S_3. It is the Hilbert class field of $\mathbf{Q}(\sqrt{-11})$. We note for future reference that the norm of $j(\sqrt{-11})$, which is the absolute value of the constant coefficient, has prime factorisation

$$653249011576832 = 2^{12} \cdot 11^3 \cdot 17^3 \cdot 29^3.$$

These examples invite several observations and results, as noticed by early practitioners of the theory of complex multiplication; we single out two:

(1) **Field of definition.** A singular modulus generates the *ring class field* of the quadratic order of its argument τ, of discriminant $\Delta < 0$. These fields are important, for instance, in the classical question of the representability of primes by quadratic forms, e.g. characterising primes of the form

$$p = x^2 + ny^2.$$

Class field theory reduces this question to the explicit knowledge of a set of generators of the ring class field. Specifically: excluding divisors of $4n$, it is precisely (see Cox [10]) the set of primes p that *split completely* in the ring class field of discriminant $-4n$, namely

$$\mathbf{Q}\left(\sqrt{-n}, j(\sqrt{-n})\right).$$

Historically, a set of generators was first described in situations where the ring class field is abelian over $\mathbf{Q}$. This is the subject of *genus theory*. The most classical version is due to Gauß [34], and it covers all ten entries in the above table. To give an example, the fact that the ring class field of conductor -20 is $\mathbf{Q}(\sqrt{-5}, \sqrt{5})$, where the set of split primes p is characterised by the simultaneous solubility of $x^2 + 5$ and $x^2 - 5$ over $\mathbf{F}_p$, allows one to deduce a famous conjecture of Euler, stating that

$$p = x^2 + 5y^2 \iff p = 5 \quad \text{or} \quad p \equiv 1, 9 \pmod{20}.$$

In general, a characterisation by a simple congruence condition on the prime p does not exist, owing to the fact that the ring class field is not generally abelian, i.e. not contained in a cyclotomic field. We observed for instance that the ring class field of discriminant -44 is an S_3-extension. Two other singular moduli of historical notability are

$$\begin{aligned} j(\sqrt{-14}) &= 2^3\left(323 + 228\sqrt{2} + (231 + 161\sqrt{2})\sqrt{2\sqrt{2} - 1}\right)^3, \\ j(\sqrt{-27}) &= 2^4 \cdot 3 \cdot 5^3\left(5285131824\sqrt[3]{2} + 6658848836\sqrt[3]{2} + 8389623817\right). \end{aligned}$$

The former was computed by Weber [66, Section 144]. It is an integer in a cyclic extension of degree 4 of $\mathbf{Q}(\sqrt{-14})$; the ring class field of discriminant -56 is a degree 8 dihedral extension of $\mathbf{Q}$. The latter is a generator of the number field $\mathbf{Q}(\sqrt[3]{2})$, proving a criterion conjectured by Euler:

$$p = x^2 + 27y^2 \iff \begin{cases} p \equiv 1 \pmod 3 \\ x^3 - 2 \equiv 0 \pmod p \text{ has a solution.} \end{cases}$$

(2) **Arithmetic factorisations.** Arguably the most powerful applications of singular moduli came from a detailed study of their prime factorisations, which was first undertaken entirely experimentally. An early collection of tables may be found in the survey by Greenhill [36], published in 1889. The rich arithmetic properties that reside in these prime factorisations are clearly acknowledged in the experimental work of Berwick [1], who tabulated the prime factorisations of the two quantities

$$j(\tau) \quad \text{and} \quad j(\tau) - 1728$$

whenever the singular modulus $j(\tau)$ is of degree at most three. As Berwick notes, several striking patterns arise from these tables. For instance:

> 10. An inspection of the results of §§ 4, 5, 8, 9 suggests the possibility of several general theorems, unattempted here, concerning the factors of j and $j-1728$.
>
> (i) $\Delta \equiv 3 \pmod 8, \quad j \equiv 0 \pmod{2^{15}}$;
>
> $\frac{1}{4}\Delta \equiv 3 \pmod 8, \quad j \equiv 0 \pmod{2^4}$;
>
> Δ or $\frac{1}{4}\Delta$ or $\frac{1}{16}\Delta \equiv 7 \pmod 8, \quad j \not\equiv 0 \pmod 2$;

Figure 1.1. An observation of Berwick [1].

Berwick makes several observations, and we will pick out a few that fit into the themes that reappear later in our discussion of RM theory.

First, the question of *which primes* can arise in the factorisation. Inspecting the ten entries of our own table above, we might for instance conjecture that for primes q we have

$$q \mid \operatorname{Nm} j(\sqrt{-n}) \quad \Rightarrow \quad \begin{cases} q \equiv 0, 2 \pmod 3 \\ q < 3n. \end{cases}$$

This was also noticed by Berwick. Furthermore, he notes that such primes satisfy a congruence condition modulo the discriminant $\Delta = -4n$ summarised by a single condition on the Kronecker symbol:

$$\left(\frac{\Delta}{q}\right) \neq 1.$$

Second, for a given prime q that arises in the factorisation, the q-adic valuation of that norm can be studied. Berwick makes several conjectures for small values of q, predicting a lower bound on this valuation. The lower bound for the case $q = 2$ appears in the excerpt Figure 1.1.

1.2. The work of Gross–Zagier. After the observations of Berwick, the study of singular moduli took a spectacular turn several decades later, in the modern era. This time around, in the context of the Birch–Swinnerton-Dyer conjecture, with the landmark results of Gross and Zagier [39, 40].

Documented in the 1983 letter of Zagier to Gross [68], we find an elaboration of the computational experiments of Berwick. We can infer from the letter that Gross and Zagier had already proved several observations of Berwick about

$$\mathrm{Nm}\, \mathrm{j}(\tau) \qquad \text{and} \qquad \mathrm{Nm}(\mathrm{j}(\tau) - 1728). \tag{1.2}$$

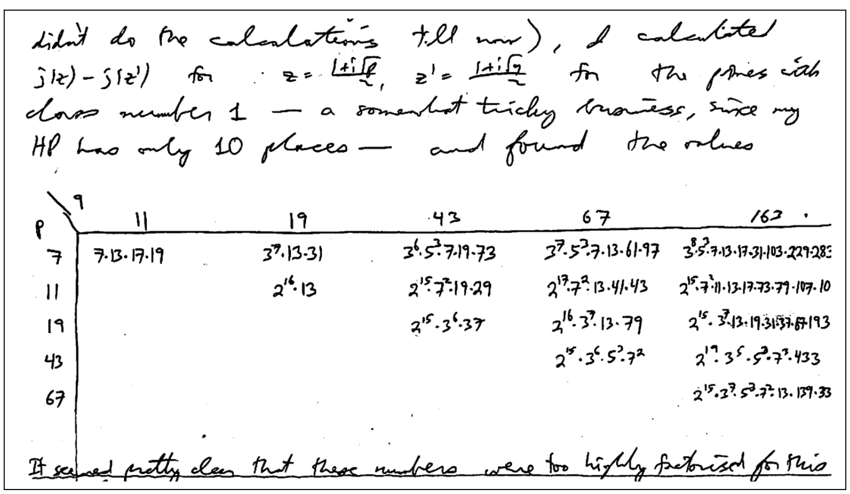
In his letter, Zagier discovers a more general phenomenon. Observe that both quantities in (1.2) are the norms of *differences* of singular moduli, since

$$\mathrm{j}\left(\frac{1+\sqrt{-3}}{2}\right) = 0, \qquad \mathrm{j}(\sqrt{-1}) = 1728.$$

One may wonder whether the factorisations of general differences of singular moduli behave similarly. Consider a pair of CM points τ_1 and τ_2 of coprime fundamental discriminants $\Delta_1, \Delta_2 < 0$, and define the integer

$$\mathrm{J}_\infty(\tau_1, \tau_2) := \mathrm{Nm}_{\mathbf{Q}}\big(\mathrm{j}(\tau_1) - \mathrm{j}(\tau_2)\big) \qquad \in \mathbf{Z}.$$

One the first page of Zagier's letter, we find the following table of some of these integers, associated to CM points whose discriminant has class number one.


didn't do the calculations till now), I calculated $j(z) - j(z')$ for $z = \frac{1+i\sqrt{p}}{2}$, $z' = \frac{1+i\sqrt{q}}{2}$ for the primes with class number 1 — a somewhat tricky business, since my HP has only 10 places — and found the values

p \ q	11	19	43	67	163
7	$7\cdot13\cdot17\cdot19$	$3^7\cdot13\cdot31$	$3^6\cdot5^3\cdot7\cdot19\cdot73$	$3^7\cdot5^3\cdot7\cdot13\cdot61\cdot97$	$3^8\cdot5^3\cdot7\cdot13\cdot17\cdot31\cdot103\cdot229\cdot283$
11		$2^{16}\cdot13$	$2^{15}\cdot7^2\cdot19\cdot29$	$2^{17}\cdot7^2\cdot13\cdot41\cdot43$	$2^{15}\cdot7^2\cdot11\cdot13\cdot17\cdot73\cdot79\cdot107\cdot10$
19			$2^{15}\cdot3^6\cdot37$	$2^{16}\cdot3^7\cdot13\cdot79$	$2^{15}\cdot3^7\cdot13\cdot19\cdot31\cdot37\cdot67\cdot193$
43				$2^{15}\cdot3^6\cdot5^3\cdot7^2$	$2^{17}\cdot3^5\cdot5^3\cdot7^3\cdot433$
67					$2^{15}\cdot3^7\cdot5^3\cdot7^2\cdot13\cdot139\cdot33$

It seemed pretty clear that these numbers were too highly factorised for this

Figure 1.3. Excerpt from the letter of Zagier [68].

A crucial observation, scribbled at the bottom, is that the prime divisors q of these integers are small. Zagier also observes congruences modulo $\Delta_1\Delta_2$ satisfied by those primes q, and proceeds to completely determine the q-adic valuation

$$\mathrm{ord}_q\, \mathrm{J}_\infty(\tau_1, \tau_2).$$

In his reply, dated 18 February 1983, Gross gives a different determination of this q-adic valuation, using properties of CM elliptic curves. This leads to two independent proofs of the same result, which have the following features:

- The proof in Zagier's letter is analytic, and studies the Fourier coefficients of a family of Hilbert modular forms introduced by Hecke [43]. Denote $L = \mathbf{Q}(\sqrt{\Delta_1}, \sqrt{\Delta_2})$ and F for its real quadratic subfield. Write χ for the genus character of F corresponding to L/F. Hecke defines a non-holomorphic Eisenstein family in the variables z_1, z_2 on $\mathfrak{H}_\infty$ by

$$E_s(z_1, z_2) := \sum_{[\mathfrak{a}] \in \mathrm{Cl}_F^+} \chi(\mathfrak{a}) \mathrm{Nm}(\mathfrak{a})^{1+2s} E_s^{\mathfrak{a}}(z_1, z_2),$$

where the non-holomorphic series $E_s^{\mathfrak{a}}(z_1, z_2)$ is defined as follows. Set

$$y_1 := \mathrm{Im}(z_1), \qquad y_2 := \mathrm{Im}(z_2)$$

and denote the algebraic conjugation of $F/\mathbf{Q}$ by $a \mapsto a'$. Define

$$E_s^{\mathfrak{a}}(z_1, z_2) \ := \sum_{(m,n) \in \mathfrak{a}^2/\mathcal{O}_F^\times} \frac{y_1^s y_2^s}{(mz_1 + n)(m'z_2 + n')|mz_1 + n|^{2s}|m'z_2 + n'|^{2s}}$$

where the sum extends over non-zero pairs (m, n) of elements in $\mathfrak{a}$, modulo the diagonal action of totally positive units in F. This sum converges absolutely for real values of $s > 0$, and transforms under $\mathrm{SL}_2(\mathcal{O}_F)$ like a modular form of parallel weight one. The argument relies on the computation of the Fourier expansion of

(1) its diagonal restriction $E_s(z, z)$ (vanishes at $s = 0$)
(2) its analytic first order derivative with respect to s
(3) its holomorphic projection, contained in the space of holomorphic forms of weight two for the full modular group, which is trivial:

$$M_2(\mathrm{SL}_2(\mathbf{Z})) = \{0\}.$$

Through an ingenious direct computation, the argument shows that its first Fourier coefficient can be expressed in the form

$$\log J_\infty(\tau_1, \tau_2) \ - \ \sum_q \mathrm{Int}_q \cdot \log(q)$$

for an explicit character sum Int_q. Since this Fourier coefficient is zero, this gives an explicit description of the integer $J_\infty(\tau_1, \tau_2)$. It is striking that this proof does not use CM theory, and *neither uses nor implies* that the quantities $j(\tau_1) - j(\tau_2)$ (without the norm) are algebraic.

- The proof in the letter of Gross offers a different set of insights, and relies on a study of the pair of CM elliptic curves (E_1, E_2) associated to (τ_1, τ_2). Note that both E_1 and E_2 have potentially good reduction, and since the reduction map induces an injection on endomorphism rings, they can only reduce to the same curve $\overline{E}$ in finite characteristic q if that curve is supersingular, i.e. we get injections

$$\mathcal{O}_{K_1}, \mathcal{O}_{K_2} \hookrightarrow R := \mathrm{End}(\overline{E}) \subset B_{\infty q}$$

where R is a maximal order in the definite quaternion algebra $B_{\infty q}$ ramified at $\{q, \infty\}$. Gross inverts this procedure and reduces the problem of computing the q-adic valuation

$$\mathrm{ord}_q \; J_\infty(\tau_1, \tau_2)$$

to a counting problem of conjugacy classes of such pairs of embeddings. When unfolded and carefully counted, this yields precisely the same expressions as the ideal sums Int_q appearing in the analytic proof.

These two proofs involve very different ideas. When combined, they amount to the computation of the Néron height pairing $\langle P_1, P_2 \rangle$ of the Heegner divisors

$$P_1 = \sum_{\sigma \in \mathrm{Pic}(\mathcal{O}_{K_1})} (\sigma\tau_1) - (\infty), \qquad P_2 = \sum_{\sigma \in \mathrm{Pic}(\mathcal{O}_{K_2})} (\sigma\tau_2) - (\infty)$$

on the modular curve $X(1)$. The contribution at finite primes q is given by the intersection multiplicity Int_q whereas the archimedean contribution is $\log J_\infty(\tau_1, \tau_2)$. The series in $M_2(\mathrm{SL}_2(\mathbf{Z}))$ appearing in Zagier's letter is therefore

$$\mathrm{Proj}_{\mathrm{hol}} \left[\frac{\partial}{\partial s} E_s(z, z) \right]_{s=0} = \sum_{n \geqslant 1} \langle P_1, T_n P_2 \rangle q^n.$$

Since the modular curve $X_0(1)$ has genus zero the global height is trivial, and this series vanishes. In later work, Gross–Zagier [40] and Gross–Kohnen–Zagier [41] consider instead the height pairing of Heegner divisors on $X_0(N)$, which does not necessarily vanish, and relate the series to derivatives of L-functions.

1.3. Real quadratic fields. It is natural to wonder what can be said on the results discussed so far for pairs $\Delta_1, \Delta_2 > 0$ of *positive* discriminants. The goal of these notes is to discuss a recent conjectural approach to singular moduli for *real quadratic fields*, based on the notion of rigid meromorphic cocycles.

The question of generalisation to real quadratic, or arbitrary number fields, has been around since Kronecker, and is the objective of Hilbert's 12^{th} problem. To discuss related progress, it is important to clarify exactly what properties of singular moduli one is trying to generalise to other number fields.

If our goal is to find explicit generators for abelian extensions of number fields, an important approach came from the conjectures of Stark [63] which predict that leading terms of L-series of Artin representations produce generators via a refinement of the Dirichlet class number formula. Little is known about Stark's conjecture; the case of real quadratic fields remains open. A refinement of the p-adic version of Gross [37] was recently proved by Dasgupta–Kakde [25,26], giving an analytic formula for p-units in abelian extensions of totally real fields, and thus a satisfactory answer to the problem of finding *generators* for those fields.

If, on the other hand, we want the rich arithmetic factorisations which gave rise to the results of Gross–Zagier, we need a different approach. The p-units provided by the refinements of Gross–Stark based on L-functions have trivial factorisations at primes $q \neq p$. Purely archimedean approaches have been attempted, and cycle integrals of the j-function were shown to mirror in several key aspects the analytic properties of generating series of singular moduli as the appear in Kudla–Rapoport–Yang [51] by Kaneko [46] and Duke–Imamoḡlu–Tóth [28, 29]. So far, cycle integrals of the j-function have not yielded algebraic numbers that reflect the arithmetic of singular moduli. We mention also a conjectural programme based on non-commutative geometry and C^*-algebras due to Manin [55].

1.4. Computational tools. It is remarkable how the study of singular moduli has been directly informed by computation, using tools of ever increasing technological sophistication. The 1889 article of Greenhill [36] gives an overview of the fruits of the manual labour of the likes of Abel, Jacobi, Kronecker, Weber, and others. Powerful methods to obtain closed expressions in radicals for singular moduli are discussed at length in Weber's 1908 *Lehrbuch der Algebra* [66].

Two decades later, in the 1928 work of Berwick [1] we read:

> The author has carried out the heavier numerical work involved in the preparation of these results on a Trinks-Brunsviga calculating machine in the mathematical laboratory of Leeds University.

Figure 1.4. Berwick's acknowledgement [1].

Over half a century later, Zagier acknowledges in his letter an HP with 10 decimal places, see Figure 1.3. Not only does this take advantage of a computing power much superior to that of the Trinks–Brunsviga calculating machine used by Berwick, but also of its greater portability, since Zagier was writing his letter while he was travelling in Japan in early 1983.

Since those days, the degree of mechanisation has transformed beyond recognition. This greatly enhances our ability to continue, as well as expand, the tradition of explicit experimentation established by many generations before us. Perhaps this ability should even be viewed as a duty. We possess, after all, an immense privilege of which early practitioners in CM theory could only dream.

2. Binary quadratic forms

In this section, we recall some classical aspects of the theory of quadratic forms, including reduction theory, with an emphasis on indefinite forms. Most results mentioned here are due to Gauß [34].

2.1. Binary quadratic forms A (binary integral) quadratic form is an element

$$\langle a, b, c\rangle := aX^2 + bXY + cY^2 \quad \in \mathbf{Z}[X, Y].$$

It is called *primitive* if $\gcd(a, b, c) = 1$. There is a right $SL_2(\mathbf{Z})$-action on $\mathbf{Z}[X, Y]$ by ring automorphisms, defined on generators by

$$\gamma \,:\, \begin{cases} X & \longmapsto & pX + qY \\ Y & \longmapsto & rX + sY \end{cases} \qquad \forall \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbf{Z}).$$

This action preserves the set of quadratic forms, respects primitivity, and preserves the *discriminant* $\Delta = b^2 - 4ac$ of a quadratic form $\langle a, b, c\rangle$. A form of discriminant Δ is called *definite* if $\Delta < 0$ and *indefinite* if $\Delta > 0$. The set of all primitive quadratic forms with fixed discriminant Δ, satisfying $a > 0$ when $\Delta < 0$, is denoted by $\mathcal{F}_\Delta$. To a quadratic form $F = \langle a, b, c\rangle$ in $\mathcal{F}_\Delta$ we associate

$$A_F := \begin{pmatrix} -b & -2c \\ 2a & b \end{pmatrix}, \qquad A_F^2 = \Delta.$$

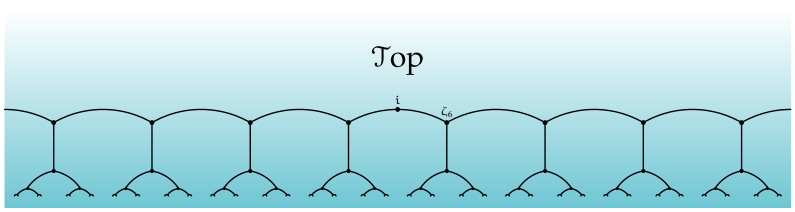
When $\Delta \neq 0$ the matrix A_F has two eigenlines, with eigenvalues[1] $\sqrt{\Delta}$ and $-\sqrt{\Delta}$ respectively. They define canonical elements r_F and r'_F in $\mathbf{P}^1(\mathbf{C})$, respectively, which we call the *first* and *second* roots of F. When Δ is not a square, we have

$$r_F = \frac{-b + \sqrt{\Delta}}{2a}, \qquad r'_F = \frac{-b - \sqrt{\Delta}}{2a} \quad \in \mathbf{C}.$$

The group $SL_2(\mathbf{Z})$ acts on $\mathcal{F}_\Delta$, where the orbits are typically infinite. Any orbit has a convenient visualisation via the *Conway topograph* [9, 42]. It is a tree in $\mathcal{H}_\infty$ together with a labelling of the connected components of its complement. The tree is the inverse image of the closed interval $[0, 1728]$ for the j-function, i.e.

$$\mathcal{T}op \;:=\; j^{-1}\left([0, 1728]\right),$$

It is the planar embedding of a 3-regular tree, and has a simply transitive action of $SL_2(\mathbf{Z})$ on the set of *oriented* edges, see Serre [61]. It is depicted here.



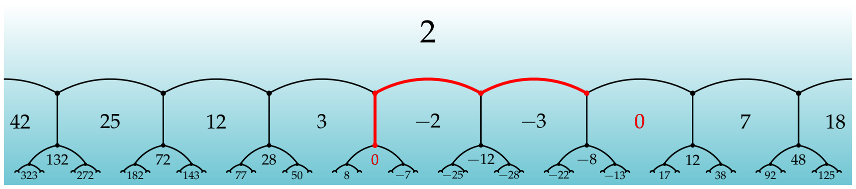
[1] Throughout these notes, we fix embeddings of $\overline{\mathbf{Q}}$ into both $\mathbf{C}$ and $\mathbf{C}_p$.

Each connected component C of $\mathcal{H}_\infty \setminus \mathcal{T}op$ determines a unique adjacent cusp $(u : v) \in \mathbf{P}^1(\mathbf{Q})$. Normalise u, v to be coprime integers, and label component C by the integer $F(u, v) = au^2 + buv + cv^2$. In particular, the regions adjacent to the cusps $\infty = (1 : 0)$ and $(0 : 1)$ are labelled by the integers a and c respectively.

Example 2.1. Consider the indefinite form $F = \langle 2, -3, -2 \rangle$ of discriminant $\Delta = 25 = 5^2$. Since this discriminant is square (such forms are sometimes called *isotropic*), its first and second roots are rational, in this case given by the cusps

$$r_F = (-1 : 2), \qquad r'_F = (2 : 1) \in \mathbf{P}^1(\mathbf{Q})$$

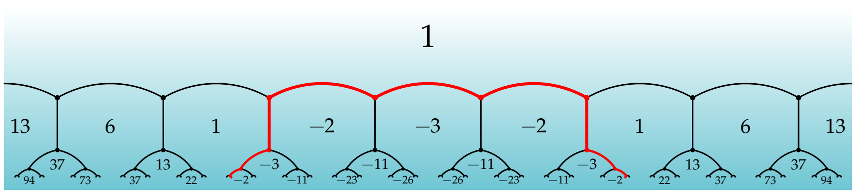
The form takes both negative and positive values at the cusps, and the topograph has a finite set of edges (labelled in red in the picture below) separating the corresponding regions according to sign, and connecting the two components labelled with zero, corresponding to the two roots of F. We depict the topograph here.



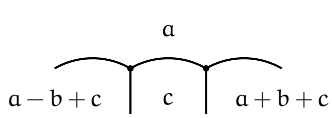
Example 2.2. Consider the indefinite form $F = \langle 1, 0, -3 \rangle$ of discriminant $\Delta = 12$. Its first and second roots are irrational, in this case given by

$$r_F = \sqrt{3}, \qquad r'_F = -\sqrt{3}$$

The form takes both negative and positive values at the cusps, and the topograph has an infinite set of edges (labelled in red in the picture below) separating the regions according to sign. This infinite path of edges in the topograph of an indefinite form with non-square discriminant is called the *river*.



Remark 2.3. From three pairwise adjacent numbered regions, one easily reconstructs the entire topograph, and the associated quadratic forms, using a very simple rule [9, p.9]. Around the oriented edge corresponding to $\langle a, b, c \rangle$ (which one can always transform to the standard edge) the topograph looks like



2.2. Class groups For a fixed discriminant $\Delta \neq 0$, the set of orbits $\mathcal{F}_\Delta / \mathrm{SL}_2(\mathbf{Z})$ is finite and endowed with the structure of an abelian group. More specifically:

- When $\Delta = n^2$ with $n \geqslant 1$, every $\mathrm{SL}_2(\mathbf{Z})$-orbit contains a unique element of the form $\langle a, n, 0\rangle$ with $0 \leqslant a < n$ (see exercises) and the map that sends an orbit to the class of a modulo n gives a bijection

$$\begin{array}{ccc} \mathcal{F}_\Delta / \mathrm{SL}_2(\mathbf{Z}) & \xrightarrow{\sim} & (\mathbf{Z}/n\mathbf{Z})^\times \\ \langle a, n, 0\rangle & \longmapsto & [a] \end{array}$$

- When Δ is not a square, there is a bijection between $\mathcal{F}_\Delta / \mathrm{SL}_2(\mathbf{Z})$ and the narrow class group of the quadratic order of discriminant Δ, given by

$$\begin{array}{ccc} \mathcal{F}_\Delta / \mathrm{SL}_2(\mathbf{Z}) & \xrightarrow{\sim} & \mathrm{Pic}^+\left(\mathbf{Z}\left[\frac{\Delta+\sqrt{\Delta}}{2}\right]\right) \\ \langle a, b, c\rangle & \longmapsto & \left[\left(a, \frac{-b+\sqrt{\Delta}}{2}\right)\right] \end{array}$$

In both cases, the set of $\mathrm{SL}_2(\mathbf{Z})$-orbits of primitive forms inherits the structure of a finite abelian group. It should be noted that the composition law on quadratic forms historically predates the introduction of ideal class groups, and when endowed with this operation, the above bijections become isomorphisms.

2.3. Reduction theory The aim of reduction theory is to identify distinguished elements inside $\mathrm{SL}_2(\mathbf{Z})$-orbits of primitive quadratic forms. The notion is qualitatively different for definite and indefinite forms.

Definition 2.4. Let $F = \langle a, b, c\rangle$ be a quadratic form of discriminant $\Delta \neq 0$.

- When $\Delta < 0$, we say F is *reduced* if

$$|b| \leqslant a \leqslant c,$$
with $b \geqslant 0$ if either equality holds.

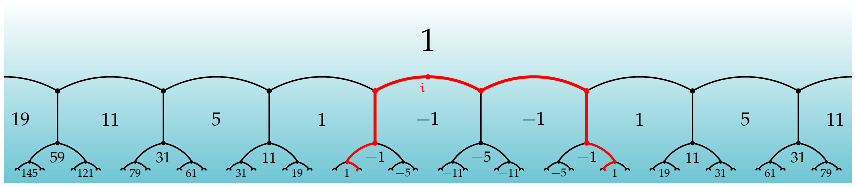
- When $\Delta > 0$, we say that F is

nearly reduced if $ac < 0$,
reduced if $ac < 0$ and $b > |a + c|$.

When $\Delta < 0$, a form F is reduced if and only if its first root r_F is contained in the standard fundamental domain for $\mathrm{SL}_2(\mathbf{Z})$. When $\Delta > 0$ non-square, F is nearly reduced if and only if $r_F r_F' < 0$, and reduced if and only if

$$r_F r_F' < 0 \quad \text{and} \quad |r_F| < 1 < |r_F'|.$$

Reducedness for $\Delta > 0$ non-square may therefore be rephrased in terms of the topograph as the property that the *river* passes through i, and flows along one of the vertical edges with $\mathrm{Re}(z) = 1/2$ or $-1/2$, but not both. As an example, we depict here the case of the reduced form $F = \langle 1, -1, -1\rangle$ of discriminant $\Delta = 5$.



The following classical result is due to to Gauß, and its proof is well-known.

Proposition 2.5. *Suppose Δ is a non-square discriminant. There are finitely many reduced forms of discriminant Δ, and any $\mathrm{SL}_2(\mathbf{Z})$-orbit of $\mathcal{F}_\Delta$ contains at least one.*

The finiteness is easily seen. When $\Delta > 0$ there are clearly finitely many forms with $ac < 0$, and when $\Delta < 0$ a reduced form satisfies $|b| \leqslant a$ and $3a^2 \leqslant |\Delta|$ and therefore there are finitely many options for each of the coefficients.

The proof that every $\mathrm{SL}_2(\mathbf{Z})$-orbit contains at least one reduced form is constructive, and uses an explicit *reduction algorithm*. It proceeds as follows.

(1) Apply the unique power of the translation matrix

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \;:\; \langle a, b, c\rangle \longmapsto \langle a, b+2a, a+b+c\rangle$$

that yields a quadratic form $\langle a, b, c\rangle$ satisfying

$$\begin{cases} -|a| < b \leqslant |a| & \text{if } \Delta > 0 \text{ and } |a| \geqslant \sqrt{\Delta}, \text{ or } \Delta < 0 \\ \sqrt{\Delta} - 2|a| < b \leqslant \sqrt{\Delta} & \text{if } \Delta > 0 \text{ and } |a| < \sqrt{\Delta}. \end{cases}$$

(2) If the form is reduced, stop. Otherwise, repeat the previous step after applying the matrix

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;:\; \langle a, b, c\rangle \longmapsto \langle c, -b, a\rangle.$$

Assume now that F is an element of $\mathcal{F}_\Delta$ where $\Delta > 0$ is a non-square discriminant. One can show that this algorithm produces a reduced form after at most

$$\frac{1}{2}\log_2\left(\frac{|a|}{\sqrt{\Delta}}\right) + 2$$

steps. When continued, it produces all the reduced forms in the $\mathrm{SL}_2(\mathbf{Z})$-orbit. From the powers of the translation T that were applied at each step, one also enumerates all *nearly* reduced forms in the orbit (see exercises). Likewise, from these transformations one computes a generator

$$\gamma_F = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

of the stabiliser of F in $\mathrm{SL}_2(\mathbf{Z})$. The quantity $\varepsilon := t \cdot r_F + u$ is a fundamental unit of norm 1 in the quadratic order of discriminant Δ. The generator γ_F for which $\varepsilon > 1$, is called the *automorph* of F. Indefinite reduction theory is discussed in detail in Gauß [34], Buell [6, Chapter 3], and Buchmann–Vollmer [5, Chapter 6].

3. Rational cocycles

To prepare us for the rigid cocycles and RM singular moduli discussed in § 4, we first explore all the structural steps for the simpler *rational cocycles*, which are 1-cocycles for the modular group $\mathrm{SL}_2(\mathbf{Z})$ valued in rational functions on $\mathbf{P}^1(\mathbf{C})$.

There are three fundamental steps in the process of producing arithmetic invariants. First, we consider *additive* cocycles, valued in the additive group of rational functions $\mathbf{C}(z)$. The main actor is a cocycle defined by Knopp [49, 50]. We then discuss *multiplicative lifts* of these additive cocycles with respect to the logarithmic derivative

$$\mathrm{dlog} : \mathbf{C}(z)^\times \longrightarrow \mathbf{C}(z).$$

Finally, we explain how to *evaluate* the resulting multiplicative cocycle, yielding a systematic supply of well-defined arithmetic invariants.

General notation. Let G be a group and M a left G-module. Denote the action by $\star$. The group of 1-cocycles $Z^1(G, M)$ is the set of maps $\varphi : G \to M$ that satisfy

$$\varphi(\gamma_1\gamma_2) = \varphi(\gamma_1) + \gamma_1 \star \varphi(\gamma_2) \qquad \forall\, \gamma_1, \gamma_2 \in G. \tag{3.1}$$

The subgroup of 1-coboundaries $B^1(G, M)$ consists of all maps $\varphi : G \to M$ for which there exists $g_\varphi \in M$ satisfying

$$\varphi(\gamma) = (1-\gamma) \star g_\varphi \qquad \forall\, \gamma \in G.$$

The first cohomology group is defined by $H^1(G, M) := Z^1(G, M)/B^1(G, M)$.

In this chapter we will consider $G = \mathrm{SL}_2(\mathbf{Z})$ acting on $M = \mathbf{C}(z)$ or $\mathbf{C}(z)^\times$. This eliminates some technicalities that we encounter for $G = \mathrm{SL}_2\,\mathbf{Z}[1/p]$ and its action on meromorphic functions $M = \mathrm{Mer}_p$ or $\mathrm{Mer}_p^\times$ on $\mathcal{H}_p$ considered in § 4.

3.1. Additive cocycles.

Consider $\mathbf{C}(z)$, the additive group of rational functions on the Riemann sphere $\mathbf{P}^1(\mathbf{C})$ with coordinate z. It is a left $\mathrm{GL}_2(\mathbf{C})$-module endowed with the weight two action, defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star f(z) \;:=\; \frac{ad-bc}{(-cz+a)^2} \cdot f\left(\frac{dz-b}{-cz+a}\right) \tag{3.2}$$

Definition 3.3. *Rational cocycles* are elements of $Z^1\big(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)\big)$. The subgroup of *parabolic* cocycles $Z^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z))$ consists of those 1-cocycles φ that are trivial on the subgroup of translations, i.e. that satisfy

$$\varphi(T) = 0, \qquad \text{where } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Finally, we define *rational coboundaries* to be the elements of $B^1\big(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)\big)$.

We first investigate a toy example of a rational cocycle, which is in fact a cocycle for the larger group $\mathrm{GL}_2(\mathbf{C})$. Its arithmetic interest is very limited, but it gives us opportunity to introduce some useful tools for the richer cocycles to follow.

Toy cocycle

Choose a base point $b = (u : v) \in \mathbf{P}^1(\mathbf{C})$, and define

$$p_b : \mathrm{GL}_2(\mathbf{C}) \longrightarrow \mathbf{C}(z), \quad \gamma \longmapsto L(b) - L(\gamma b), \qquad \text{where } L((r:s)) := \frac{s}{sz - r}.$$

We will prove the following claims.

- The map p_b defines a cocycle.
- This rational cocycle only depends on $b \in \mathbf{P}^1(\mathbf{C})$ up to a coboundary.

Whereas both may be checked by a direct calculation, this is neither particularly pleasant, nor enlightening. We use instead the following elementary lemma.

Lemma 3.4. *Let G be a group endowed with*

- *a left action on a non-empty set X,*
- *a left action on a module M.*

Let $m : X \times X \longrightarrow M$ be a G-equivariant map such that for all $x, y, z \in X$

$$\begin{cases} m(x,x) = 0 & \text{“antisymmetry”} \\ m(x,z) = m(x,y) + m(y,z) & \text{“additivity”} \end{cases} \tag{3.5}$$

Then for any $x \in X$, we obtain a 1-cocycle $\varphi_x \in Z^1(G, M)$ defined by

$$\varphi_x : \gamma \longmapsto m(x, \gamma x),$$

whose cohomology class $[\varphi_x] \in H^1(G, M)$ is independent of the choice of $x \in X$.

Proof. Clearly φ_x is a 1-cocycle, since for every $\gamma_1, \gamma_2 \in G$ we have

$$\begin{aligned} \varphi_x(\gamma_1\gamma_2) &= m(x, \gamma_1\gamma_2 x) \\ &= m(x, \gamma_1 x) + m(\gamma_1 x, \gamma_1\gamma_2 x) \\ &= m(x, \gamma_1 x) + \gamma_1 \star m(x, \gamma_2 x) \\ &= \varphi_x(\gamma_1) + \gamma_1 \star \varphi_x(\gamma_2). \end{aligned}$$

The independence of the cohomology class follows from

$$\varphi_x(\gamma) - \varphi_y(\gamma) = (1 - \gamma) \star m(x, y). \qquad \square$$

Remark 3.6. Note that the proof is essentially the standard passage between homogeneous and non-homogeneous cocycles from homological algebra. The lemma extends formally to higher cocycles $Z^n(G, M)$ from G-equivariant maps $X^n \to M$ satisfying an appropriate homogeneous cocycle condition.

To establish the required claims about the toy example p_b, it suffices to apply the lemma to the function

$$\begin{array}{rrcl} m: & \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C}) & \longrightarrow & \mathbf{C}(z) \\ & (r,s) & \longmapsto & L(r) - L(s). \end{array}$$

where $G = \mathrm{GL}_2(\mathbf{C})$ acts on $\mathbf{P}^1(\mathbf{C})$ by Möbius transformations. The antisymmetry and additivity (3.5) are clear, so it remains to verify the G-equivariance of m.

Lemma 3.7. *For any pair of base points* $r, s \in \mathbf{P}^1(\mathbf{C})$ *we have the relation*

$$\gamma \star (L(r) - L(s)) = L(\gamma r) - L(\gamma s), \qquad \forall\, \gamma \in \mathrm{GL}_2(\mathbf{C}).$$

Proof. Whereas this lemma may be proved by a direct calculation, it is more pleasantly proved using the space of meromorphic differentials Ω^1 on $\mathbf{P}^1_{\mathbf{C}}$, viewed as a $\mathrm{GL}_2(\mathbf{C})$-module for the action defined by

$$\gamma \star f(z)dz := f(\gamma^\dagger z)d(\gamma^\dagger z), \qquad \text{where } \gamma^\dagger := \det(\gamma)\gamma^{-1}.$$

It is easily seen from the definition that this makes the following map into an isomorphism of $\mathrm{GL}_2(\mathbf{C})$-modules:

$$\mathrm{diff} : \mathbf{C}(z) \longrightarrow \Omega^1 \; ; \; f(z) \longmapsto f(z)dz.$$

Note that for any $r, s \in \mathbf{P}^1(\mathbf{C})$ the differential

$$\omega_{r,s} := \mathrm{diff}(L(r) - L(s))$$

is of the third kind (meaning it has only simple poles). Taking residue divisors $\mathrm{div} : \Omega^1 \to \mathrm{Div}^0\ \mathbf{P}^1(\mathbf{C})$, we obtain for any $\gamma \in \mathrm{GL}_2(\mathbf{C})$ that

$$\begin{aligned} \mathrm{div}(\gamma \star \omega_{r,s}) &= (\gamma r) - (\gamma s) \\ \mathrm{div}(\omega_{\gamma r, \gamma s}) &= (\gamma r) - (\gamma s) \end{aligned}$$

Since both $\gamma \star \omega_{r,s}$ and $\omega_{\gamma r,\gamma s}$ are differentials of the third kind, and there are no non-zero holomorphic differentials on $\mathbf{P}^1(\mathbf{C})$, they must be equal, as required. □

More interesting examples of rational cocycles were constructed by Knopp [49, 50]. They are associated to quadratic forms $F \in \mathcal{F}_\Delta$ with $\Delta > 0$. We first make some definitions on geodesics. For any indefinite form Q, we write $\mathrm{geo}(Q)$ to denote the oriented geodesic in the extended Poincaré upper half plane

$$\mathcal{H}^*_\infty := \mathcal{H}_\infty \cup \mathbf{P}^1(\mathbf{Q}),$$

which is an oriented semicircle running from the second root r'_Q to the first root r_Q. Fix a form $F \in \mathcal{F}_\Delta$ with $\Delta > 0$, and define the set

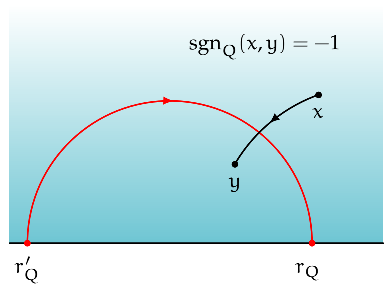
$$X_F := \mathcal{H}^*_\infty \setminus \left\{\mathrm{geo}(Q) : Q \sim F\right\},$$

i.e. the complement in the extended Poincaré upper half plane of the (measure zero) $\mathrm{SL}_2(\mathbf{Z})$-orbit of $\mathrm{geo}(F)$. The set X_F is uncountably infinite, and contains the rational cusps $\mathbf{P}^1(\mathbf{Q})$ if and only if the discriminant Δ of F is non-square.

To define the Knopp cocycle, we need a notion of intersection numbers of geodesics. Choose $x, y \in X_F$, and define

$$\mathrm{sgn}_Q(x, y) \in \{0, 1, -1\}$$

to be the intersection number of $\mathrm{geo}(Q)$ and the hyperbolic geodesic $\mathrm{geo}(x, y)$ in $\mathcal{H}_\infty^*$ from x to y, taken with respect to a fixed orientation of the plane. The following figure illustrates this for the right handed orientation.



The $\mathrm{SL}_2(\mathbf{Z})$-orbit of F is infinite, but for a fixed $x, y \in X_F$ it contains only finitely many forms Q whose geodesic $\mathrm{geo}(Q)$ intersects $\mathrm{geo}(x, y)$ non-trivially.

Lemma 3.8. *Let $F \in \mathcal{F}_\Delta$ with $\Delta > 0$, and $x, y \in X_F$. Then*

$$\left| \left\{ Q \sim F \ : \ \mathrm{sgn}_Q(x, y) \neq 0 \right\} \right| < \infty.$$

Proof. The images of $\mathrm{geo}(F)$ and $\mathrm{geo}(x, y)$ on the orbifold modular curve

$$X(1) := \mathrm{SL}_2(\mathbf{Z}) \backslash \mathcal{H}_\infty^*$$

are compact geodesics, and thus have a finite set S of intersection points. The quotient map from the set of intersection points

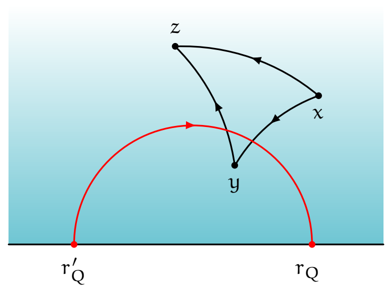
$$\{\mathrm{geo}(Q) \cap \mathrm{geo}(x, y) : Q \sim F\}$$

to S is finite to one, since the stabiliser in $\mathrm{SL}_2(\mathbf{Z})$ of any point in $\mathcal{H}_\infty$ is finite. □

The crucial property, on which the entire theory hinges, is the *additivity* of the intersection numbers. Let $x, y, z \in X_F$, then we have (for any $Q \sim F$) that

$$\mathrm{sgn}_Q(x, y) + \mathrm{sgn}_Q(y, z) = \mathrm{sgn}_Q(x, z) \tag{3.9}$$

since the composition of $\mathrm{geo}(x, y)$ and $\mathrm{geo}(y, z)$ is homotopic to $\mathrm{geo}(x, z)$.



The additivity is crucial, and allows us by Lemma 3.4 to define two cocycles by averaging over the infinite orbit $\{Q \sim F\} = F \cdot SL_2(\mathbf{Z})$.

- The function

$$\begin{array}{ccc} X_F \times X_F & \longrightarrow & \mathbf{Z} \\ (x,y) & \longmapsto & \sum_{Q\sim F} \mathrm{sgn}_Q(x,y) \end{array}$$

  is clearly antisymmetric and $SL_2(\mathbf{Z})$-equivariant (where $\mathbf{Z}$ has the trivial action). By (3.9) it is also additive. As a consequence, we obtain for any $b \in X_F$ a homomorphism $SL_2(\mathbf{Z}) \longrightarrow \mathbf{Z}$ defined by

$$\gamma \longmapsto \sum_{Q\sim F} \mathrm{sgn}_Q(b, \gamma b) = 0 \tag{3.10}$$

  which must necessarily be identically zero, since the abelianisation of $SL_2(\mathbf{Z})$ is finite. This vanishing property for sums of intersection numbers underlies the later convergence properties of rigid cocycles.

- The function

$$\begin{array}{ccc} X_F \times X_F & \longrightarrow & \mathbf{C}(z) \\ (x,y) & \longmapsto & \sum_{Q\sim F} \mathrm{sgn}_Q(x,y) \cdot L(r_Q). \end{array}$$

  is clearly antisymmetric. It is $SL_2(\mathbf{Z})$-invariant by Lemma 3.7, and it is additive by (3.9). We now formally obtain the *Knopp cocycle* from this function by virtue of Lemma 3.4.

Knopp cocycle

Let $F \in \mathcal{F}_\Delta$ with $\Delta > 0$. Choose a base point $b \in X_F$ and define

$$\begin{array}{cccc} \mathrm{kn}_{b,F} : & SL_2(\mathbf{Z}) & \longrightarrow & \mathbf{C}(z) \\ & \gamma & \longmapsto & \sum_{Q\sim F} \mathrm{sgn}_Q(b,\gamma b) \cdot L(r_Q) \end{array}$$

The Knopp cocycle $\mathrm{kn}_{b,F}$ is a rational cocycle, whose cohomology class

$$[\mathrm{kn}_{b,F}] \in H^1(SL_2(\mathbf{Z}), \mathbf{C}(z))$$

is independent of the base point b. It is frequently convenient to choose the base point $b = \infty := (1:0) \in \mathbf{P}^1(\mathbf{C})$, and we know precisely when this is possible by

$$\infty \in X_F \iff \Delta \text{ non-square.}$$

Whenever Δ is not a square, we may therefore choose $b = \infty$. In this case, we will drop the base point from our notation and simply write kn_F for the Knopp cocycle $\mathrm{kn}_{\infty,F}$. Note that this cocycle is parabolic, i.e.

$$\mathrm{kn}_F \in Z^1_{\mathrm{par}}\big(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)\big).$$

The cocycle kn_F can be efficiently computed, in terms of the reduction theory of the quadratic form F discussed in § 2. More precisely, the group $\mathrm{SL}_2(\mathbf{Z})$ is generated by the matrices S and T, where the cocycle has the values

$$\begin{aligned} \mathrm{kn}_F(T) &= 0 && \text{where } T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \\ \mathrm{kn}_F(S) &= \sum_{Q=\langle a,b,c\rangle \in \Sigma_F} \frac{\mathrm{sgn}\,(a)}{z - r_Q}, && \text{where } S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \end{aligned}$$

where Σ_F is the set of *nearly reduced* forms in the orbit of F:

$$\Sigma_F := \big\{\langle a, b, c\rangle \sim F \;:\; ac < 0\big\}.$$

The reduction algorithm of § 2.3 computes the set Σ_F efficiently. It is instructive to compute a few non-trivial examples of Knopp cocycles, and to verify the resulting functional equations, as in the following (simplest) example.

Example 3.11. Consider the form $F = \langle 1, 1, -1\rangle$ of discriminant $\Delta = 5$. We have

$$\Sigma_F = \big\{\langle -1, 1, 1\rangle, \langle -1, -1, 1\rangle, \langle 1, -1, -1\rangle, \langle 1, 1, -1\rangle\big\}$$

so that the value of the Knopp cocycle at S is given by

$$\mathrm{kn}_F(S) = \frac{\sqrt{5}}{z^2 - z - 1} + \frac{\sqrt{5}}{z^2 + z - 1}.$$

The fact that this rational function is the special value at S of a parabolic rational cocycle translates into the following identities

$$\begin{aligned} (1 + S) \star \mathrm{kn}_F(S) &= 0, \\ (1 + (ST) + (ST)^2) \star \mathrm{kn}_F(S) &= 0. \end{aligned}$$

These functional equations impose very restrictive conditions on $\mathrm{kn}_F(S)$. In fact, any rational function satisfying them arises as the value at S of a parabolic rational cocycle, providing a means to classify all parabolic cocycles [7] (see exercises).

Remark 3.12. Let φ be a rational cocycle. A *modular integral* for φ is a holomorphic function $\mathcal{G}$ on $\mathcal{H}^*_\infty$ that satisfies

$$\varphi(\gamma) = \mathcal{G}(z) \;\big\|\; (1 - \gamma^{-1}), \qquad {}^\forall\gamma \in \mathrm{SL}_2(\mathbf{Z})$$

where we use the (right) weight two slash action. Note that the space of invariant forms $M_2(\mathrm{SL}_2(\mathbf{Z})) = 0$ is trivial, so modular integrals are unique if they exist.

The modular integral of the toy cocycle p is the Eisenstein series of weight two. Recall that this is the holomorphic function $G_2(z)$ on $\mathcal{H}^*_\infty$ defined by

$$\begin{aligned} G_2(z) &:= \sum_{a\in\mathbf{Z}} \sum_{b\in\mathbf{Z}}{}' \frac{1}{(az+b)^2} \\ &= \frac{\pi^2}{3}\Big(1-24\sum_n \sigma_1(n)q^n\Big), \qquad \sigma_1(n)=\sum_{d|n} d, \end{aligned}$$

where $q=\exp(2\pi i z)$. Its transformation law is given by

$$G_2(z) \parallel (1-\gamma^{-1}) = 2\pi i \cdot p(\gamma)$$

where $p=p_\infty$ is the toy cocycle corresponding to the base point $b=\infty\in\mathbf{P}^1(\mathbf{C})$. This transformation law of $G_2(z)$ is equivalent to Legendre's period relation, and occupies a central place in the arithmetic theory of elliptic curves [67].

The modular integral of the Knopp cocycle lies much deeper, and its arithmetic properties were investigated by Duke–Imamoḡlu–Tóth [28, 30]. They construct a modular integral $\mathcal{G}_F$ with q-expansion

$$\mathcal{G}_F = \sum_{n\geqslant 1}\left(\int_{z_0}^{\gamma_F z_0} j_n(z)\frac{dz}{F(z)}\right)q^n, \qquad j_n(q)=q^{-n}+O(q) \ \in \mathbf{C}(j)$$

where γ_F is the automorph of F. In other words, it is a generating series for the cycle integrals of the j-function, mentioned in the introduction. We have

$$\mathcal{G}_F \parallel (1-\gamma^{-1}) = kn_F(\gamma)-kn_{-F}(\gamma).$$

3.2. Multiplicative cocycles We now consider the *multiplicative* group $\mathbf{C}(z)^\times$ of non-zero rational functions on $\mathbf{P}^1(\mathbf{C})$. It is a left $GL_2(\mathbf{C})$-module for the weight zero action, defined by

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\cdot f(z) \ := \ f\left(\frac{dz-b}{-cz+a}\right).$$

The multiplicative module $\mathbf{C}(z)^\times$ is related to the additive module $\mathbf{C}(z)$ by the logarithmic derivative, which is a morphism of $GL_2(\mathbf{C})$-modules

$$\mathrm{dlog} \ : \ \mathbf{C}(z)^\times \longrightarrow \mathbf{C}(z)\ ;\ f(z)\longmapsto \left(\frac{d}{dz}f(z)\right)\cdot f(z)^{-1}$$

whose kernel is the subgroup of constant functions $\mathbf{C}^\times \subset \mathbf{C}(z)^\times$.

We will now investigate whether the additive cocycles we constructed in § 3.1 can be lifted to multiplicative cocycles under the dlog map. Note that the rational function $L(c)$, for any $c=(u:v)\in\mathbf{P}^1(\mathbf{C})$ is the image of

$$[z-c] := vz-u \in \mathbf{C}(z)^\times/\,\mathbf{C}^\times$$

under dlog. Therefore both the toy cocycle p_b and the Knopp cocycle $kn_{b,F}$ are valued in the image of the logarithmic derivative

$$\mathrm{dlog}: \mathbf{C}(z)^\times/\,\mathbf{C}^\times \hookrightarrow \mathbf{C}(z),$$

and as a consequence they lift formally to multiplicative cocycles modulo scalars. Such a multiplicative lift is easily found explicitly:

$$\left.\begin{array}{llll}\text{Toy:} & \gamma & \longmapsto & [z-\gamma b] \\ \text{Knopp:} & \gamma & \longmapsto & \displaystyle\prod_{Q\sim F}[z-r_Q]^{\mathrm{sgn}_Q(b,\gamma b)}\end{array}\right\} \in Z^1_{\mathrm{par}}\left(\mathrm{SL}_2(\mathbf{Z}), \frac{\mathbf{C}(z)^\times}{\mathbf{C}^\times}\right)$$

An important question is whether we can resolve the scalar ambiguity, and lift to multiplicative cocycles valued in $\mathbf{C}(z)^\times$ rather than merely $\mathbf{C}(z)^\times/\mathbf{C}^\times$.

The toy cocycle. The multiplicative lift of the toy cocycle p_b can be written down explicitly. We will consider the quotient map

$$\begin{array}{rccc}\pi : & \mathbf{A}^2(\mathbf{C})\setminus\{0\} & \longrightarrow & \mathbf{P}^1(\mathbf{C}) \\ & (u,v) & \longmapsto & (u:v)\end{array}$$

which is a morphism of left $\mathrm{GL}_2(\mathbf{C})$-modules for the natural left action on $\mathbf{A}^2(\mathbf{C})$. By lifting our earlier definitions for the morphism π, we will be able to lift p_b to a multiplicative cocycle for the logarithmic derivative dlog.

Lemma 3.13. *Consider* $X = \mathbf{A}^2(\mathbf{C})\setminus\{0\}$ *and define the map*

$$\begin{array}{rccc}m: & X\times X & \longrightarrow & \mathbf{C}(z)^\times \\ & (r,s),(u,v) & \longmapsto & (r-sz)/(u-vz)\end{array}$$

Then m *is antisymmetric, additive, and* $\mathrm{GL}_2(\mathbf{C})$*-equivariant.*

Proof. It is clear that m is antisymmetric and additive. Standard properties of the determinant imply that m is G-equivariant, since

$$\gamma\star m(x,y) = \frac{\det\left[\begin{pmatrix} r\\ s\end{pmatrix}, \gamma^{-1}\begin{pmatrix} z\\ 1\end{pmatrix}\right]}{\det\left[\begin{pmatrix} u\\ v\end{pmatrix}, \gamma^{-1}\begin{pmatrix} z\\ 1\end{pmatrix}\right]} = \frac{\det\left[\gamma\begin{pmatrix} r\\ s\end{pmatrix}, \begin{pmatrix} z\\ 1\end{pmatrix}\right]}{\det\left[\gamma\begin{pmatrix} u\\ v\end{pmatrix}, \begin{pmatrix} z\\ 1\end{pmatrix}\right]} = m(\gamma x, \gamma y)$$

□

For any lift $\widetilde{b}\in\mathbf{A}^2\setminus\{0\}$ of the base point $b\in\mathbf{P}^1(\mathbf{C})$, we may apply Lemma 3.4 to the function m in order to obtain a completely explicit associated cocycle

$$P_b^\times : \mathrm{GL}_2(\mathbf{C}) \longrightarrow \mathbf{C}(z)^\times \tag{3.14}$$

The cocycle $P_b^\times$ is independent of the chosen lift $\widetilde{b}$ of b, and we easily see that

$$\mathrm{dlog}(P_b^\times) = p_b.$$

Remark 3.15. Note that Lemma 3.13 implies Lemma 3.7 by applying the logarithmic derivative. Arguably, the proof we give here of the (stronger) Lemma 3.13 is also simpler and more elementary than the proof of Lemma 3.7 above.

The Knopp cocycle. Constructing an explicit multiplicative lift of the Knopp cocycle is possible but less straightforward, see [20]. We instead take the abstract but more general route, merely proving its existence. Define

$$Z^1_{\mathsf{f}}\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^{\times}\right) \subset Z^1\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^{\times}\right)$$

to be the subgroup of cocycles which are parabolic *modulo scalars*, i.e. become parabolic after they are composed with the natural projection map

$$\mathbf{C}(z)^{\times} \longrightarrow \mathbf{C}(z)^{\times}/\,\mathbf{C}^{\times}\,. \tag{3.16}$$

Lemma 3.17. *The natural projection map* (3.16) *induces an isomorphism:*

$$12Z^1_{\mathsf{f}}\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^{\times}\right) \stackrel{\sim}{\longrightarrow} 12Z^1_{\mathrm{par}}\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^{\times}/\,\mathbf{C}^{\times}\right).$$

Proof. We first prove a corresponding cohomological result. The Mayer–Vietoris sequence [61] applied to the amalgamated product

$$\mathrm{SL}_2(\mathbf{Z}) = (\mathbf{Z}/4\mathbf{Z}) *_{(\mathbf{Z}/2\mathbf{Z})} (\mathbf{Z}/6\mathbf{Z})$$

gives us, by reduction to the cohomology of finite cyclic groups, that

$$\begin{aligned} \mathrm{H}^1(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}^{\times}) &= \mathbf{Z}/12\mathbf{Z} \\ \mathrm{H}^2(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}^{\times}) &= 0. \end{aligned}$$

Now consider the short exact sequence of $\mathrm{SL}_2(\mathbf{Z})$-modules

$$1 \longrightarrow \mathbf{C}^{\times} \longrightarrow \mathbf{C}(z)^{\times} \longrightarrow \mathbf{C}(z)^{\times}/\,\mathbf{C}^{\times} \longrightarrow 1.$$

From the associated long exact sequence in cohomology, we extract

$$\mathbf{Z}/12\mathbf{Z} \longrightarrow \mathrm{H}^1\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^{\times}\right) \longrightarrow \mathrm{H}^1\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^{\times}/\mathbf{C}^{\times}\right) \longrightarrow 0.$$

Note that the following two groups of parabolic coboundaries are trivial:

$$B^1_{\mathsf{f}}\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^{\times}\right) = B^1_{\mathrm{par}}\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^{\times}/\mathbf{C}^{\times}\right) = 0$$

To see this, note that a rational function (modulo scalars) whose divisor is invariant under translation must be constant, so that the associated coboundary is trivial. This means that any parabolic cocycle valued in $\mathbf{C}(z)^{\times}/\mathbf{C}^{\times}$ may be lifted to a cocycle in $\mathbf{C}(z)^{\times}$ whose 12-th power is unique. The statement follows. □

By virtue of Lemma 3.17 we may uniquely lift any parabolic cocycle modulo scalars, after we take the 12-th power (and even without the 12th power, at the cost of uniqueness). In the case of our two running examples, we denote

$$\mathsf{p}^{\times}, \mathsf{kn}^{\times}_{\mathsf{F}} \in Z^1_{\mathsf{f}}\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^{\times}\right)$$

for the unique lifts of 12th power of p and kn_{F}, respectively. Note that we already constructed $\mathsf{p}^{\times}$ explicitly, it is the 12th power of $\mathsf{P}^{\times}_{(1:0)}$ in (3.14).

3.3. Values of cocycles Now that we have constructed an infinite supply of multiplicative rational cocycles, it remains to explain how they may be used to produce well-defined invariants. This will be done by assigning *values* to a cocycle, which we associate to indefinite quadratic forms of non-square discriminant.

Suppose we are given a pair (φ, G) where

- $\varphi \in Z^1(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^\times)$ a multiplicative rational cocycle,
- $G \in \mathcal{F}_\Delta$ a quadratic form with $\Delta > 0$ non-square,

then we define the *value* of φ at G to be the number

$$\varphi[G] := \varphi(\gamma_G)(r_G) \quad \in \mathbf{P}^1(\mathbf{C})$$

where γ_G is the automorph of the quadratic form G, which is the distinguished free generator for the stabiliser of G in $\mathrm{SL}_2(\mathbf{Z})$ we defined in § 2.3.

4. Singular moduli for real quadratic fields

We have defined an infinite collection of rational cocycles, and discussed a three-step procedure to produce meaningful invariants. To wit;

(1) we started by defining *additive* cocycles in § 3.1,
(2) then, we lifted them to *multiplicative* cocycles in § 3.2,
(3) finally, we *evaluated* the resulting cocycle at a quadratic form in § 3.3.

The invariants we are interested in are p-adic limits of these values. The p-adic limit allows the values to converge to global elements outside the biquadratic field. We perform numerical experiments, and make observations that resemble closely the properties of CM singular moduli discussed in the introduction.

4.1. Rigid cocycles We begin by explaining how to construct p-adic limits of the Knopp cocycles [20, 21]. The definitions are largely the same as those in § 3, and proceed by repeating the three-step procedure while replacing

$$\begin{array}{rrcl} & \mathrm{SL}_2(\mathbf{Z}) & \text{acting on} & \mathbf{C}(z)^\times \\ \text{by} & \Gamma := \mathrm{SL}_2(\mathbf{Z}[1/p]) & \text{acting on} & \mathrm{Mer}^\times \end{array}$$

where $\mathrm{Mer}^\times$:= non-zero meromorphic functions on the p-adic half plane $\mathcal{H}_p$. At this point assume some more background from our readers. An introduction to the geometry of the p-adic half plane can be found in [27].

Consider the projective line $\mathbf{P}^1$ over $\mathbf{Q}_p$ with homogeneous variables $(z_1 : z_2)$, and for any integer $n \geqslant 0$ define the affinoid open subset

$$\mathcal{H}_p^{\leqslant n} := \left\{ (z_1, z_2) \text{ primitive: } \begin{array}{c} |sz_1 + rz_2|_p \geqslant p^{-n} \\ \forall (r,s) \in \mathbf{Z}^2 \text{ primitive} \end{array} \right\} \subset \mathbf{P}^1(\mathbf{C}_p)$$

where a pair of numbers in $\mathbf{C}_p$ is called *primitive* if they are integral and at least one of them is a p-adic unit. The affinoid $\mathcal{H}_p^{\leqslant n}$ is obtained from $\mathbf{P}^1(\mathbf{C}_p)$ by removing open disks of radius p^{-n} around rational points. The increasing union of these affinoids forms an admissible open covering of the p-adic half plane

$$\mathcal{H}_p := \lim_{n\to\infty} \mathcal{H}_p^{\leqslant n},$$

a rigid analytic space over $\mathbf{Q}_p$. The set of its $\mathbf{C}_p$-points is

$$\mathcal{H}_p(\mathbf{C}_p) = \mathbf{P}^1(\mathbf{C}_p) \setminus \mathbf{P}^1(\mathbf{Q}_p).$$

It is often referred to as "upper" half plane, though $\mathcal{H}_p$ is connected, unlike its archimedean counterpart $\mathbf{P}^1(\mathbf{C})\setminus\mathbf{P}^1(\mathbf{R})$ which is the disjoint union of "upper" and "lower" half planes. The p-adic half plane has a natural reduction map to the *Bruhat–Tits* tree. It is a $(p+1)$-regular tree. The reduction map sends the affinoid $\mathcal{H}_p^{\leqslant 0}$ to a distinguished vertex v_0, and the affinoid $\mathcal{H}_p^{\leqslant n}$ to the finite subtree spanned by the vertices of distance at most n to v_0, see [27, § 1.3].

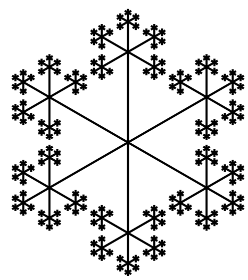


Figure 4.1. The Bruhat–Tits tree for $p = 5$.

Definition 4.2. A meromorphic function on $\mathcal{H}_p$ is the uniform limit, with respect to the supremum norm, of rational functions on each affinoid $\mathcal{H}_p^{\leqslant n} \subset \mathbf{P}^1(\mathbf{C}_p)$ for $n \geqslant 0$. The space of meromorphic functions defined over a field extension $L \supset \mathbf{Q}_p$ is denoted by Mer_L. When $L = \mathbf{C}_p$ we simply write $\mathrm{Mer} := \mathrm{Mer}_{\mathbf{C}_p}$.

We endow the additive group $(\mathrm{Mer}, +)$ of meromorphic functions with the left weight two action of $\mathrm{GL}_2(\mathbf{C}_p)$, defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star f(z) \quad := \quad \frac{ad - bc}{(-cz + a)^2} \cdot f\left(\frac{dz - b}{-cz + a}\right)$$

whereas the multiplicative group $\mathrm{Mer}^\times$ of non-zero meromorphic functions is a left $\mathrm{GL}_2(\mathbf{C}_p)$-module, with respect to the weight zero action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(z) \quad := \quad f\left(\frac{dz - b}{-cz + a}\right)$$

so that the logarithmic derivative provides a morphism of left $\mathrm{GL}_2(\mathbf{C}_p)$-modules

$$\begin{array}{rccc} \mathrm{dlog}: & \mathrm{Mer}^\times & \longrightarrow & \mathrm{Mer} \\ & f & \longmapsto & f'/f \end{array}$$

Remark 4.3. Additive cocycles are only considered for the weight two action here. We refer to Negrini [56] for a systematic treatment of higher weight additive cocycles, which admit an analogue of the Shimura–Shintani correspondence.

Arithmetic phenomena transpire after restricting to the *Ihara group*

$$\Gamma := \mathrm{SL}_2(\mathbf{Z}[1/p]) \leqslant \mathrm{GL}_2(\mathbf{C}_p).$$

It is a discrete subgroup of $\mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{Q}_p)$ which is dense in each factor. Analogies of Γ with a Hilbert modular group were studied by Ihara [44, 45]. As before, we consider cocycles modulo scalars, and attempt to lift the scalar ambiguity. This is more involved than it was for the group $\mathrm{SL}_2(\mathbf{Z})$. Define the groups of

$$\begin{array}{rcl} \text{rigid cocycles} & := & \mathrm{Z}^1(\Gamma, \mathrm{Mer}^\times) \\ \cap & & \cap \\ \text{theta cocycles} & := & \mathrm{Z}^1(\Gamma, \mathrm{Mer}^\times / \mathbf{C}_p^\times). \end{array}$$

For $\mathrm{SL}_2(\mathbf{Z})$ the space of lifting obstructions $\mathrm{H}^2(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}^\times) = 0$ is trivial. The Ihara group Γ is cohomologically richer; in general a theta cocycle does not lift to a rigid cocycle. Its action on the Bruhat–Tits tree gives [61, II.1.4]

$$\Gamma \simeq \mathrm{SL}_2(\mathbf{Z}) *_{\Gamma_0(p)} \mathrm{SL}_2(\mathbf{Z})$$

so that we obtain from Mayer–Vietoris an exact sequence

$$(\mathbf{Z}/12\,\mathbf{Z})^2 \longrightarrow \mathrm{H}^1(\Gamma_0(p), \mathbf{C}_p^\times) \longrightarrow \mathrm{H}^2(\Gamma, \mathbf{C}_p^\times) \longrightarrow 0$$

Since $\mathrm{H}^1(\Gamma_0(p), \mathbf{Z})$ has rank $2g+1$, where g is the genus of $X_0(p)$, we see that the lifting obstruction lies in a $\mathbf{C}_p^\times$-torus of the same rank. This presents issues in defining the RM values of a theta cocycle, which we circumvent as follows.

Let G be a quadratic form of positive discriminant for which p is inert in the associated quadratic order. Its stabiliser in the Ihara group Γ is of rank one

$$\mathrm{Stab}_\Gamma(G) = \{\pm 1\} \times \gamma_G^{\mathbf{Z}} \;\leqslant\; \mathrm{SL}_2(\mathbf{Z}) \;\leqslant\; \Gamma$$

Let φ be any theta cocycle, then we may lift its *restriction to* $\mathrm{SL}_2(\mathbf{Z})$ to a cocycle $\widetilde{\varphi} \in \mathrm{Z}^1\left(\mathrm{SL}_2(\mathbf{Z}), \mathrm{Mer}^\times\right)$. The RM value of φ at G is the well-defined quantity

$$\varphi[G] := \widetilde{\varphi}^{12}(\gamma_G)(r_G) \in \mathbf{P}^1(\mathbf{C}_p).$$

Remark 4.4. We note that RM values may be defined more generally at any indefinite form G discriminant $\Delta > 0$ such that p is non-split, see [22]. When we numerically compute RM values, as a rule we omit the 12th power appearing in the definition to assure well-definedness. The reason is that algebraic recognition routines based on LLL [54] are more likely to succeed when the height of the algebraic number is small, making the 12th power undesirable.

4.2. The p-adic Knopp cocycle We already know one example of a rigid cocycle: the construction of the toy cocycle goes through without any changes. Its RM values are algebraic, but of limited interest (see exercises). To construct more interesting theta cocycles, we adapt the Knopp cocycle in § 3.

Fix a quadratic form $F \in \mathcal{F}_\Delta$ with $\Delta > 0$, and a base point $b \in X_F$.

Case 1. Assume first that p is **nonsplit** in the quadratic algebra of F, i.e. that the Kronecker symbol (Δ/p) is not one. Define the *additive* cocycle Θ_F by

$$\Theta_F : \gamma \longmapsto \sum_{Q \in F \cdot \Gamma} \mathrm{sgn}_Q(b, \gamma b) L(r_Q), \qquad \Theta_F \in Z^1(\Gamma, \mathrm{Mer}).$$

Note that the only difference with the definition of the Knopp cocycle is in the index set; this time Q runs over all quadratic forms in the Γ-orbit of F. As before, we then lift the additive cocycle to a *multiplicative* theta cocycle

$$\Theta_F^\times : \gamma \longmapsto \prod_{Q \in F \cdot \Gamma} [z - r_Q]^{\mathrm{sgn}_Q(b, \gamma b)}, \qquad \Theta_F^\times \in Z^1(\Gamma, \mathrm{Mer}^\times / \mathbf{C}_p^\times). \tag{4.5}$$

The cohomology class is independent of the choice of base point, and since p is nonsplit, the cusp ∞ is contained in X_F and the cohomology class is parabolic. We may represent it uniquely by a parabolic cocycle, valued in $\mathrm{Mer}^\times / \mathbf{C}_p^\times$.

Remark 4.6. When $\mathrm{SL}_2(\mathbf{Z})$-equivalence is replaced by Γ-equivalence, the statement of Lemma 3.8 is false, and the sums and products occurring in the above definitions are *infinite*. The concomitant convergence issues, which we have kept from the reader, are slightly subtle, see [20]. We content ourselves here by pointing out the two main ingredients for this convergence:

- The divisor of Θ_F and $\Theta_F^\times$ is discrete, Lemma 3.8 can be used to show that the index set is naturally filtered by *finite* subsets

$$\mathcal{S}_n := \left\{ Q \in F \cdot \Gamma \ : \ \mathrm{sgn}_Q(b, \gamma b) \neq 0, \ \ r_Q \in \mathcal{H}_p^{\leqslant n} \right\}. \tag{4.7}$$

- When restricted to the affinoids $\mathcal{H}_p^{\leqslant n}$, the divisors of Θ_F and $\Theta_F^\times$ are of degree zero, since the vanishing of (3.10) can be used to show that

$$\sum_{Q \in \mathcal{S}_n} \mathrm{sgn}_Q(b, \gamma b) = 0. \tag{4.8}$$

Convergence is easier when one makes a symmetrisation $\Theta_{F,\mathrm{sym}}$ of the cocycle, as we do when p splits. It is similar to the symmetrised (rational) cocycles of Duke–Imamoḡlu–Tóth [30], where the corresponding symmetrisation of the knot assures that it is null-homologous. We mention the work of Simon [62] and Rickards [60], who determine the linking numbers without symmetrisation.

Case 2. When p is **split** in the quadratic algebra of F, i.e. when $(\Delta/p) = 1$, we define only the *symmetrised* additive cocycle $\Theta_{F,\mathrm{sym}} \in Z^1(\Gamma, \mathrm{Mer})$ by

$$\Theta_{F,\mathrm{sym}} : \gamma \longmapsto \sum_{Q \in F \cdot \Gamma} \mathrm{sgn}_Q(b, \gamma b) \left(L(r_Q) - L(r_Q') \right),$$

The convergence of this sum is manifest, since the terms uniformly converge to zero on the affinoids $\mathcal{H}_p^{\leqslant n}$. We lift the symmetrised additive cocycle to a symmetrised multiplicative theta cocycle $\Theta^{\times}_{F,\mathrm{sym}} \in Z^1(\Gamma, \mathrm{Mer}^{\times}/\mathbf{C}_p^{\times})$ by

$$\Theta^{\times}_{F,\mathrm{sym}} : \gamma \longmapsto \sum_{Q \in F\cdot\Gamma} \left(\frac{[z - r_Q]}{[z - r'_Q]}\right)^{\mathrm{sgn}_Q(b,\gamma b)}.$$

The cohomology class is independent of the choice of base point, but it is not necessarily parabolic. When Δ is not a square, the cusp ∞ is contained in X_F and there is a unique parabolic representative. When Δ is a square this is false, e.g. for the *winding cocycle* appearing below, corresponding to the case $\Delta = 1$.

The RM values of the p-adic Knopp cocycles associated to a p-adically non-split form F (Case 1) are of primary interest. Henceforth we write

$$\Theta^{\times}_p[F, G] := \Theta^{\times}_F[G] \qquad \in \ \mathbf{P}^1(\mathbf{Q}_{p^2}),$$

where $\mathbf{Q}_{p^2}$ is the unramified quadratic extension of $\mathbf{Q}_p$, which contains all the roots of the forms in the Γ-orbits of F and G. These RM values satisfy:

- The invariant is (multiplicatively) anti-symmetric, in the sense that
$$\Theta^{\times}_p[F, G] = \Theta^{\times}_p[G, F]^{-1}.$$
- The invariant only depends on the $\mathrm{SL}_2(\mathbf{Z})$-orbits of the forms F and G, giving us for a fixed pair of discriminants $\Delta_1, \Delta_2 > 0$ with
$$\left(\frac{\Delta_1}{p}\right) = \left(\frac{\Delta_2}{p}\right) = -1$$
a finite collection
$$\left\{ \Theta^{\times}_p[F, G] \ : \ \begin{array}{ccc} \mathrm{disc}(F) & = & \Delta_1 \\ \mathrm{disc}(G) & = & \Delta_2 \end{array} \right\} \subset \mathbf{P}^1(\mathbf{C}_p)$$
canonically indexed by $\mathrm{Cl}^+_1 \times \mathrm{Cl}^+_2$, the product of (narrow) class groups of the quadratic orders of discriminants Δ_1 and Δ_2 respectively.
- We may define an involution
$$w_\infty : \langle a, b, c\rangle \longmapsto \langle -a, b, -c\rangle$$
on the set of quadratic forms of any fixed discriminants. This involution changes the first root r_Q to minus the second root $-r'_Q$. It reflects the geodesic $geo(Q)$ along the y-axis and negates intersection numbers. Therefore, the effect of this involution on the RM values is
$$\Theta^{\times}_p[w_\infty F, w_\infty G] = \Theta^{\times}_p[F, G]^{-1}$$

4.3. Computational observations The explicit nature of the theta cocycles constructed in § 4.2 makes it possible to experiment with examples. In early 2017, we computed a first example of an RM value of a p-adic Knopp cocycle. We chose the prime $p = 3$ and the indefinite quadratic forms

$$\begin{aligned} F &= \langle 1, 1, -1\rangle \quad \text{discriminant } \Delta_1 = 5, \\ G &= \langle 1, 8, -4\rangle \quad \text{discriminant } \Delta_2 = 80. \end{aligned}$$

The RM value of the 3-adic rigid cocycle associated to F at G, was found to satisfy

$$\Theta_3^{\times}[F, G] \equiv \frac{24\sqrt{-1} - 7}{25} \pmod{3^{200}}.$$

The appearance of an algebraic number was encouraging. What seemed striking is that it generates the narrow ring class field of conductor 4 of $\mathbf{Q}(\sqrt{5})$, and its factorisation involves primes above $5 \neq 3$. This motivated a systematic computational exploration, to understand the arithmetic of these invariants.

The enormous benefit of historical hindsight on experimentation with CM singular moduli (discussed in the introduction) facilitates informed guesses for phenomena that we might expect. The reader is encouraged to use the algorithms [19] to make their own observations from experiments.

Remark 4.9. The infinite product $\Theta_F^{\times}$ may be computed directly up to any desired precision $O(p^n)$. However, the size of the set $\mathcal{S}_n$ in (4.7) grows exponentially in n. An algorithm running polynomially in n is described in [20], which computes the product (4.5) restricted to $\mathcal{S}_n$ iteratively in n, through a recursion formula. The price we pay for this polynomial running time, is that the output is only the restriction of $\Theta_F^{\times}$ to the standard affinoid $\mathcal{H}_p^{\leqslant 0}$, and that its RM values are only correct modulo powers of the fundamental unit ε_F in the quadratic order of F.

Example 4.10. Let $p = 2$ and choose the quadratic forms

$$\begin{aligned} F &= \langle 1, 1, -1\rangle & &\text{discriminant } \Delta_1 = 5, \\ G &= \langle 1, 3, -3\rangle \text{ and } \langle -1, 3, 3\rangle & &\text{discriminant } \Delta_2 = 21. \end{aligned}$$

The class group of quadratic forms of discriminant 21 is isomorphic to $\mathbf{Z}/2\mathbf{Z}$, and the two choices of G are representatives of the two classes. Using the polynomial time algorithm for computing RM invariants [20] we compute the quantity

$$\Theta_2^{\times}[F, G] \pmod{2^{300}}.$$

We suspect this is an algebraic number, and use the LLL agorithm [54] to attempt to find an algebraic relation between small powers of this number, see [8, § 2.7] for more on this method. The algorithm returns the quartic polynomial

$$91x^4 + 112x^3 + 123x^2 + 112x + 91. \tag{4.11}$$

The two choices for G yield different 2-adic numbers $\Theta_2^{\times}[F, G]$ computed to precision $O(2^{300})$, but the algebraic recognition nonetheless yields the same polynomial (4.11). The polynomial (4.11) has 4 roots, and further experimentation reveals that the two "missing" roots are accounted for by the values

$$\Theta_2^{\times}[w_2 F, G] = \Theta_2^{\times}[F, w_2 G], \qquad \text{where } w_p : \langle a, b, c\rangle \mapsto \langle a, pb, p^2 c\rangle.$$

These invariants can also be computed using the algorithms of [20], and appear to give a full set of Galois conjugates over $\mathbf{Q}$. The splitting field of (4.11) is

$$\mathbf{Q}(\sqrt{-3}, \sqrt{-35}).$$

Its roots are generators for H_1H_2 over K_1K_2; the compositum of the narrow Hilbert class fields of $K_1 = \mathbf{Q}(\sqrt{5})$ and $K_2 = \mathbf{Q}(\sqrt{21})$, which are given by, respectively,

$$H_1 = \mathbf{Q}(\sqrt{5}), \qquad H_2 = \mathbf{Q}(\sqrt{-3}, \sqrt{-7}).$$

Example 4.12. We go a little bit deeper into the same theme as the previous example, and explore the arithmetic of the algebraic numbers we find on a slightly larger (and richer) example. This time, we let $p = 3$ and choose quadratic forms

$$F = \langle 1, -1, -1 \rangle \quad \text{discriminant } \Delta_1 = 5,$$
$$G = \langle 1, 6, -2 \rangle \quad \text{discriminant } \Delta_2 = 44.$$

Using LLL, we find that $\Theta_3^{\times}[F, G] \pmod{3^{200}}$ is the root of the polynomial

$$48841x^8 + 115280x^6 + 164562x^4 + 115280x^2 + 48841.$$

Once again, $\Theta_3^{\times}[F, G]$ is found to be a generator of H_1H_2 over K_1K_2, the splitting field of this polynomial is the triquadratic field

$$K = \mathbf{Q}(\sqrt{5}, \sqrt{11}, \sqrt{-1}).$$

Now let us look at the arithmetic properties of this generator. Its prime factorisation is concentrated at primes dividing the constant term

$$48841 = 13^2 \cdot 17^2.$$

We note that both these primes are inert in both K_1 and K_2. This observation is strikingly similar to what was noticed by Berwick, Gross, and Zagier, and encourages us to tabulate the positive integers of the form

$$N_x := \frac{\Delta_1\Delta_2 - x^2}{4}$$

for which we find

x	N_x	x	N_x	x	N_x
0	$5 \cdot 11$	6	$2 \cdot 23$	12	19
2	$2 \cdot 3^3$	8	$3 \cdot \mathbf{13}$	14	$2 \cdot 3$
4	$3 \cdot \mathbf{17}$	10	$2 \cdot 3 \cdot 5$		

This reflects what Gross and Zagier observed in their experiments. We may further sharpen it by noticing the special role of the prime $p = 3$. Indeed, it seems that in fact $pq \mid N_x$ for all the primes q that divide our invariant, i.e. the factorisation is concentrated above primes dividing a positive integer of the form

$$\frac{\Delta_1\Delta_2 - x^2}{4p}. \tag{4.13}$$

Remark 4.14. The reader may have wondered why the RM invariants we find experimentally are never algebraic *integers*. After all, this was true for the quantities $j(\tau_1) - j(\tau_2)$ studied by Gross–Zagier. One explanation is that complex conjugation plays a different role in RM theory, acting on these invariants via

$$\overline{\Theta_p^\times[F,G]} = \Theta_p^\times[w_\infty F, w_\infty G] = \Theta_p^\times[F,G]^{-1}.$$

This explains why all the polynomials we find appear to be *palindromic*.

Example 4.15. The previous example shows that the prime p is reflected in the arithmetic of our RM invariants. The setup of Gross–Zagier corresponds to the choice $p = \infty$, and the fact that we are able to vary the prime p begs the question what the corresponding variation in the RM invariants looks like. To investigate this, we compute $\Theta_p^\times[F,G]$ in a few instances, where we fix the quadratic form

$$F = \langle 1, 3, -1 \rangle \qquad \text{discriminant} \;\; \Delta_1 = 13.$$

We now take forms G of discriminant $\Delta_2 > 0$ and compute each time the p-adic invariant for two different choices of p, where p is chosen to be inert with respect to both discriminants Δ_1 and Δ_2.

For G of discriminant $\Delta_2 = 12$ we find the following values of $\Theta_p^\times[F,G]$:

$p = 5$	$p = 7$
$\frac{1 \pm 4\sqrt{-3}}{7}$	$\frac{3 \pm 4\sqrt{-1}}{5}$

For G of discriminant $\Delta_2 = 45$ we find the following values of $\Theta_p^\times[F,G]$:

$p = 2$	$p = 7$
$\frac{150824917 \pm 100674475\sqrt{-3}}{2 \cdot 7^2 \cdot 13 \cdot 37 \cdot 67 \cdot 73}$	$\frac{1 \pm \sqrt{-15}}{2^2}$

For G of discriminant $\Delta_2 = 108$ we find the following values of $\Theta_p^\times[F,G]$:

$p = 5$	$p = 7$
$\frac{1237487 \pm 857860\sqrt{-3}}{7^2 \cdot 19 \cdot 31 \cdot 67}$	$\frac{128 \pm 2046\sqrt{-1}}{2 \cdot 5^2 \cdot 41}$

A few observations are in order here:

- It appears that the factorisations are more rich for smaller values of p. This is in line with our previous observations; for a fixed Δ_1 and Δ_2, there are fewer positive integers of the form (4.13) when p is larger.
- When a prime q divides an integer of the form $(\Delta_1\Delta_2 - x^2)/4p$, then the same is obviously true with the roles of p and q reversed. Inspecting the above examples, we see that there even appears to be a relation between

$$\mathrm{ord}_p\, \Theta_q^\times[F,G] \;\leftrightarrow\; \mathrm{ord}_q\, \Theta_p^\times[F,G].$$

To better understand the prime factorisations of the RM invariants $\Theta_p^\times[F,G]$, we compute a larger example that involves rich factorisations with large exponents. This example will also provide algebraic invariants that do not live in genus fields, i.e. whose splitting fields are non-abelian over $\mathbf{Q}$.

Example 4.16. We now investigate the set of RM invariants for

$$(\Delta_1, \Delta_2) = (13, 621).$$

We expect rich factorisations by inspecting the positive integers of the form

$$N_x = \frac{\Delta_1\Delta_2 - x^2}{4}$$

which we tabulate here for future reference:

x	N_x	x	N_x	x	N_x	x	N_x	x	N_x
1	$2\cdot 1009$	19	$2^3\cdot 241$	37	$2^2\cdot 419$	55	$2\cdot 631$	73	$2\cdot 7^3$
3	$2^5\cdot 3^2\cdot 7$	21	$2^2\cdot 3^2\cdot 53$	39	$2\cdot 3^2\cdot 7\cdot 13$	57	$2\cdot 3^2\cdot 67$	75	$2^2\cdot 3^2\cdot 17$
5	$2^2\cdot 503$	23	$2\cdot 23\cdot 41$	41	$2\cdot 17\cdot 47$	59	$2^2\cdot 7\cdot 41$	77	$2^3\cdot 67$
7	$2\cdot 17\cdot 59$	25	$2\cdot 7^2\cdot 19$	43	$2^2\cdot 389$	61	$2^6\cdot 17$	79	$2\cdot 229$
9	$2\cdot 3^3\cdot 37$	27	$2^2\cdot 3^3\cdot 17$	45	$2^3\cdot 3^3\cdot 7$	63	$2\cdot 3^3\cdot 19$	81	$2\cdot 3^3\cdot 7$
11	$2^2\cdot 7\cdot 71$	29	$2^4\cdot 113$	47	$2\cdot 733$	65	$2\cdot 13\cdot 37$	83	$2^3\cdot 37$
13	$2^3\cdot 13\cdot 19$	31	$2\cdot 7\cdot 127$	49	$2\cdot 709$	67	$2^7\cdot 7$	85	$2^2\cdot 53$
15	$2\cdot 3^2\cdot 109$	33	$2\cdot 3^2\cdot 97$	51	$2^3\cdot 3^2\cdot 19$	69	$2^2\cdot 3^2\cdot 23$	87	$2\cdot 3^2\cdot 7$
17	$2\cdot 7\cdot 139$	35	$2^4\cdot 107$	53	$2^2\cdot 7\cdot 47$	71	$2\cdot 379$	89	$2\cdot 19$

Note that the narrow class group $\mathrm{Cl}_1^+ = 1$ of discriminant $\Delta_1 = 13$ is trivial, whereas the narrow class group $\mathrm{Cl}_2^+ \simeq \mathbf{Z}/6\mathbf{Z}$ of discriminant $\Delta_2 = 621$ is cyclic of order 6. We therefore produce, for every choice of prime p that is non-split for both discriminants, a total of twelve p-adic RM invariants

$$\left\{\Theta_p^\times[F,G]\right\} \cup \left\{\Theta_p^\times[w_pF,G]\right\}.$$

We wish to recognise them as algebraic numbers. This is a challenging task; our previous observations and the entries in the above table cause us to expect algebraic numbers of very large height, particularly when $p = 2$. We noticed that the heights are more manageable for the symmetrised invariants

$$\frac{\Theta_p^\times[w_pF,G]}{\Theta_p^\times[F,G]}.$$

We will now compute these six symmetrised invariants for a variety of small primes p that are inert for both discriminants $(\Delta_1, \Delta_2) = (13, 621)$. The prime $p = 2$ presents special difficulties, since it produces algebraic numbers of immense height. Computing to precision $O(2^{1000})$, we recognised the 2-adic symmetrised invariants to be roots of the following degree six polynomial:

$$\begin{aligned}
&532662811974216268987046368230622959690070361192975999934916\,x^6\\
&-278367526244451072555505377961835322613068104302177742390746\,x^5\\
&-292977016274297008338188853638915462702409980987593341 48135\,x^4\\
&+879582695503881002603098558912072457112885628056565606298 05\,x^3\\
&-292977016274297008338188853638915462702409980987593341 48135\,x^2\\
&-278367526244451072555505377961835322613068104302177742390746\,x\\
&+532662811974216268987046368230622959690070361192975999934916
\end{aligned}$$

The discriminant of the number field defined by this polynomial is $3^7 \cdot 23^2$, and its Galois closure is the narrow ring class field H_2 of discriminant 621; it is a dihedral extension of $\mathbf{Q}$ of degree 12. The factorisation of the constant term is

$$2^2 \cdot 7^7 \cdot 19^4 \cdot 37^5 \cdot 47^2 \cdot 59^2 \cdot 67^3 \cdot 97^2 \cdot 109^2 \cdot 229 \cdot 241 \cdot 379 \cdot 631 \cdot 709 \cdot 733 \cdot 1009.$$

For different choices of primes p, the reader may prefer to try to predict some of its factorisations based on the above table of integers N_x and then compare with the LLL recognition, which yield the following algebraic polynomials:

p	Minimal polynomial	Factorisation
7	$4378144x^6 + 5762700x^5 + 9490680x^4 + 11616641x^3$ $+9490680x^2 + 5762700x + 4378144$	$2^5 \cdot 41 \cdot 47 \cdot 71$
19	$64x^6 + 72x^5 + 207x^4 + 142x^3 + 207x^2 + 72x + 64$	2^6
41	$7x^6 + 6x^5 + 6x^4 + 10x^3 + 6x^2 + 6x + 7$	7
47	$28x^6 + 54x^5 + 39x^4 + 14x^3 + 39x^2 + 54x + 28$	$2^2 \cdot 7$
59	$4x^6 + 3x^4 + 2x^3 + 3x^2 + 4$	2^2
71	$7x^6 + 6x^5 + 6x^4 + 10x^3 + 6x^2 + 6x + 7$	7

We invite the reader to reflect on these numerical examples, and to compare to previously made observations. Several things may be noticed that confirm our earlier observations, and several more can be made. The next section contains an overview of the main conjectures in [20] that were informed by computations of this sort. The amount of rich arithmetic appearing in this data does not exclude the possibility that an industrious reader could expand these observations to include some that eluded the authors of [20]. The theme of this PCMI Summer School is “number theory informed by computation”, and is perhaps best enjoyed by actively taking part in the act of being informed by computation; be it by the data included here, or by new experiments using the algorithms [19].

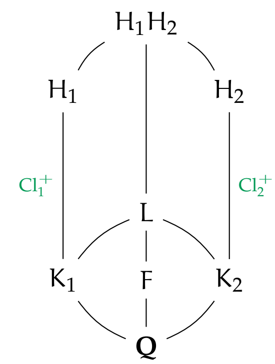
4.4. Main conjectures. We now state the main conjectures of [20], which were informed by multitudes of examples of the sort we discussed above. We do not attempt to be maximally general in the statements, in favour of simplicity. Let $\Delta_1, \Delta_2 > 0$ be coprime discriminants and p a prime such that

$$\left(\frac{\Delta_1}{p}\right) = \left(\frac{\Delta_2}{p}\right) = -1.$$

Consider the real quadratic fields

$$K_1 = \mathbf{Q}(\sqrt{\Delta_1}), \qquad K_2 = \mathbf{Q}(\sqrt{\Delta_2}),$$

whose compositum L is a real biquadratic field. Let H_1 and H_2 be the narrow ring class fields of discriminants Δ_1 and Δ_2 respectively. The Galois groups $G_i := \mathrm{Gal}(H_i/\mathbf{Q})$ are generalised dihedral, canonically split by Frobenius at p, and



$$G_1 \simeq \mathrm{Cl}_1^+ \rtimes \langle \mathrm{Frob}_p \rangle$$
$$G_2 \simeq \mathrm{Cl}_2^+ \rtimes \langle \mathrm{Frob}_p \rangle$$

where the pair of isomorphisms $\mathrm{Cl}_i^+ \simeq \mathrm{Gal}(H_i/K_i)$ for $i = 1, 2$ is provided by the global Artin map from class field theory. We make the following assumption:

Assumption: $(p-1)$ divides 12.

Equivalently, we assume that $X_0(p)$ has genus 0, or explicitly, that

$$p \in \{2, 3, 5, 7, 13\}.$$

This assumption is made to get the most straightforward statements of the main conjectures. We see why it is natural: when $X_0(p)$ has genus zero, the group of lifting obstructions is essentially trivial. Any concerned reader who feels the urge to object to such strong restrictions on p should keep in mind that:

- the work of Gross–Zagier restricts to $p \in \{\infty\}$,
- we address later in § 4.5 what happens for general primes p.

With the above notation and assumptions, we conjecture the following.

Conjecture 4.17. *For any quadratic forms* $(F, G) \in \mathcal{F}_{\Delta_1} \times \mathcal{F}_{\Delta_2}$ *we have*

$$\Theta_p^\times[F, G] \in H_1 H_2.$$

Furthermore, the set of these invariants is finite, and permuted simply transitively by the action of the Galois group $\mathrm{Gal}(H_1H_2/L) \simeq \mathrm{Cl}_1^+ \times \mathrm{Cl}_2^+$.

For a more precise version of the reciprocity law, see [20, Conjecture 3.14]. It is analogous to the well-known Shimura reciprocity law for CM singular moduli.

The factorisations of these algebraic numbers relate to intersection numbers of geodesics on *Shimura curves*. Let q be a prime, and $R \subset B_{pq}$ a maximal order in the indefinite quaternion algebra over $\mathbf{Q}$ ramified at p and q. The Shimura curve

$$X_{pq} := R_1^{\times} \backslash \mathcal{H}_{\infty}$$

is a compact algebraic curve defined over $\mathbf{Q}$, where $R_1^{\times}$ is the group of units of norm one, viewed as a subgroup of $\mathrm{SL}_2(\mathbf{R})$ via some chosen embedding.

Consider a quadratic order $\mathcal{O}$. An *optimal embedding* is an injective ring homomorphism $\mathcal{O} \hookrightarrow R$ that does not extend to a larger order in $\mathrm{Frac}(O)$. Choose a pair of optimal embeddings of the orders $\mathcal{O}_1, \mathcal{O}_2$ of discriminants Δ_1, Δ_2

$$\alpha_1 : \mathcal{O}_1 \hookrightarrow R, \qquad \alpha_2 : \mathcal{O}_2 \hookrightarrow R$$

Note that for the set of such embeddings to be non-empty, the primes p and q must both be nonsplit with respect to both Δ_1 and Δ_2.

- Choose an embedding $R \subset M_2(\mathbf{R})$, then there are associated oriented geodesics $\mathrm{geo}(\alpha_1)$ and $\mathrm{geo}(\alpha_2)$ in the upper half plane $\mathcal{H}_{\infty}$, connecting the fixed points of the images of the order. Define the *intersection*

$$\mathrm{sgn}(\alpha_1, \alpha_2) \in \{0, 1, -1\}$$

  to be the intersection of the oriented geodesics $\mathrm{geo}(\alpha_1)$ and $\mathrm{geo}(\alpha_2)$.
- The (q-adic) *multiplicity* of the embeddings

$$m_q(\alpha_1, \alpha_2) \in \mathbf{Z}_{\geqslant 1}$$

  is defined to be the largest integer t such that the images of the embeddings α_1 and α_2 coincide in the ring $R/q^{t-1}R$.

Finally, we define the q-weighted intersection number by

$$\mathrm{Int}_q(\alpha_1, \alpha_2) \quad := \sum_{b \in \Gamma_1 \backslash R_1^{\times} / \Gamma_2} \mathrm{sgn}(\alpha_1, b\alpha_2 b^{-1}) \cdot m_q(\alpha_1, b\alpha_2 b^{-1})$$

where Γ_1 and Γ_2 are the (infinite) stabilisers of the images of the two embeddings in the group $R_1^{\times}$. If we were to omit the q-adic multiplicity m_q, we would recover the usual intersection pairing of the homology classes generated by the images of $\mathrm{geo}(\alpha_1)$ and $\mathrm{geo}(\alpha_2)$ in $H_1(X_{pq}, \mathbf{Z})$ on the Shimura curve X_{pq}.

Conjecture 4.18. *With the notation and assumptions as above, for every prime* $\mathfrak{q}$ *above* q *in* $H_1H_2/\mathbf{Q}$*, there is a pair of optimal embeddings* (α_1, α_2) *such that*

$$\begin{aligned} \mathrm{ord}_{\mathfrak{q}}\left(\Theta_p^{\times}[F, G]\right) &= \mathrm{Int}_q(\alpha_1, \alpha_2) \\ \mathrm{ord}_{\mathfrak{q}}\left(\Theta_p^{\times}[w_p F, G]\right) &= \mathrm{Int}_q(w_p \alpha_1, \alpha_2) \end{aligned}$$

where $w_p \alpha_1$ *is the embedding* α_1 *conjugated by the Atkin–Lehner involution at* p*.*

The factorisation conjecture [20, Conjecture 3.27] is more precise, and stipulates how the embeddings (α_1, α_2) are put in correspondence with the different choices of primes $\mathfrak{q}$ above q. The conjecture can be phrased as a statement about G-sets, where $G = \mathrm{Gal}(H_1H_2/\mathbf{Q})$ acts on the set of optimal embeddings, as described in the work of Eichler [31], see Gross [38] and Voight [64, Chapter 30].

Remark 4.19. Besides extensive amounts of explicit examples, this conjecture was inspired by the algebraic proof of the factorisation of differences of CM singular moduli [39]. The expression for $\mathrm{ord}_{\mathfrak{q}}\,(j(\tau_1)-j(\tau_2))$ is related to ours by interchanging ∞ and p, i.e. by counting embeddings of *imaginary* quadratic orders

$$\begin{aligned} \alpha_1 \,:\, \mathcal{O}_1 &\hookrightarrow R_1 \subset B_{\infty q} \\ \alpha_2 \,:\, \mathcal{O}_2 &\hookrightarrow R_2 \subset B_{\infty q} \end{aligned}$$

in maximal orders of a *definite* quaternion algebra, of which there are finitely many, up to conjugation. The *intersection* $\mathrm{sgn}(\alpha_1, \alpha_2)$ measures whether the embeddings land in the same maximal order up to conjugacy. For example, consider

$$\begin{aligned} j\left(\frac{1+\sqrt{-67}}{2}\right) - j\left(\frac{1+\sqrt{-163}}{2}\right) &= -2^{15}\cdot 3^3\cdot 5^3\cdot 11^3 + 2^{18}\cdot 3^3\cdot 5^3\cdot 23^3\cdot 29^3 \\ &= 2^{15}\cdot 3^7\cdot 5^3\cdot 7^2\cdot 13\cdot 139\cdot 331. \end{aligned}$$

The prime $q = 3$ appears in this factorisation, and the exponent may be predicted as follows: The quaternion algebra $B_{\infty 3}$ has a unique maximal order up to conjugation, making the intersection $\mathrm{sgn}(-,-)$ is identically 1. For the 3-weighted intersection number, we find

$$\begin{aligned} \mathrm{Int}_3(\alpha_1, \alpha_2) &= \sum_{b\,\in\,\Gamma_1\backslash R_1^\times/\Gamma_2} \mathrm{sgn}(\alpha_1, b\alpha_2 b^{-1})\cdot m_3(\alpha_1, b\alpha_2 b^{-1}) \\ &= 1+1+1+1+1+2 = 7. \end{aligned}$$

Here $R_1^\times$ is of order 12 and we have $\Gamma_1 = \Gamma_2 = \{\pm 1\}$. The unique term with multiplicity 2 comes from the embeddings (with congruent images modulo 3)

$$\begin{cases} \alpha_1 & : \frac{-1+\sqrt{-67}}{2} \mapsto -\frac{1+x+8y}{2} \\ b\alpha_2 b^{-1} & : \frac{-1+\sqrt{-163}}{2} \mapsto -\frac{1+7x-4y}{2} \end{cases}$$

where we use generators $B_{3\infty} = \langle 1, x, y, z\rangle$ with $x^2 = z^2 = -3, y^2 = -1$.

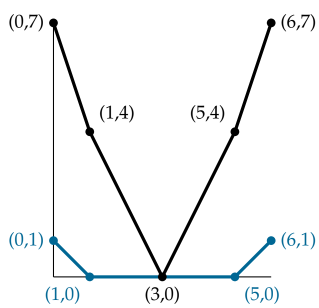
Example 4.20. Efficient methods for computing intersection numbers of geodesics on Shimura curves were developed by Rickards [58,59], complete with a PARI/GP implementation that yielded hundreds of pages of explicit data to work with. We revisit Example 4.16 for $p = 2$, where

$$(\Delta_1, \Delta_2) = (13, 621).$$

Due to practical challenges in recognising algebraic numbers $\Theta_p^\times[F,G]$ quite this gargantuan, we found it more feasible in practice to instead recognise both

$$\Theta_p^\times[F,G] \div \Theta_p^\times[w_p F,G] \qquad \text{and} \qquad \Theta_p^\times[F,G] \times \Theta_p^\times[w_p F,G].$$

The polynomial for the former was written out (for better or worse) in Example 4.16. We spare the reader the polynomial for the latter. The conjecture concerns the slopes of their Newton polygons, depicted here for the prime $q = 7$.



On the genus one Shimura curve

$$X_{14} \;:\; y^2 = -x^4 + 13x^2 - 128$$

the 7-weighted intersection numbers between geodesics of discriminants 13 and 621 are

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -2 & -1 & 1 & 2 & 1 \end{pmatrix}$$

where the rows are interchanged by w_p.

Observe that the Newton slopes of the two polynomials coincide with the difference and sum of the two rows of intersection numbers, respectively. This agrees with the prediction made by Conjecture 4.18.

4.5. General primes. Without the hypothesis that $X_0(p)$ has genus zero, we do not expect $\Theta_p^\times[F,G]$ to be algebraic. The work of Gross–Zagier on singular moduli [39] is in essence the local computation of the height pairing of Heegner divisors

$$\langle P_{\Delta_1}, P_{\Delta_2} \rangle = 0 \qquad \text{on } \; X_0(1).$$

For us, the underlying geometric object is $X_0(p)$, which yields the closest analogues to [39] in situations where this curve is also of genus $g = 0$. For general primes p, this suggests the quantity $\Theta_p^\times[F,G]$ should perhaps be thought of as the analytic contribution to a certain, yet to be defined, height pairing. This viewpoint is further developed in the works [23, 24].

In the absence of algebraic cycles, it is difficult to foresee how this will pan out, but in the present context we might view the problems as being caused by the fact that the group of *lifting obstructions* $H^2(\Gamma, \mathbf{C}_p^\times)$ is a torus of rank $2g+1$, and stands in the way of meaningfully lifting the theta cocycle $\Theta_F^\times$ to a rigid cocycle. When $g = 0$, it was shown in [20] that this may be done up to powers of a fundamental unit. If we take this more pedestrian view of our difficulties to obtain algebraic numbers for general p, there are two options to address them:

(1) **Kill the lifting obstruction.** For instance, let $p = 11$ and choose

$$F = \langle 1, 3, -3 \rangle, \qquad \text{discriminant } \; \Delta_1 = 13,$$
$$G = \langle 1, 4, -4 \rangle, \qquad \text{discriminant } \; \Delta_2 = 32.$$

If we attempt to find a minimal polynomial for $\Theta_{11}^{\times}[F,G]$ using LLL, the results are unconvincing, and no clear algebraic numbers have so far been recognised. Note that the space of weight two cusp forms on $\Gamma_0(11)$ is killed by the Hecke operator $(w_p - 1)$. We find that the invariant

$$\Theta_{11}^{\times}[w_{11}F,G] \div \Theta_{11}^{\times}[F,G]$$

computed modulo 11^{50} satisfies

$$13x^4 + 12x^3 + 14x^2 + 12x + 13,$$

with splitting field $\mathbf{Q}(\sqrt{-1},\sqrt{-3})$. The splitting field over K_1K_2 is H_1H_2.

In general, when T is a Hecke operator that kills the space of weight two cusp forms on $\Gamma_0(p)$, the quantity $\Theta_p^{\times}[TF,G]$ defined multiplicatively in the obvious way, is expected to be an algebraic number in H_1H_2.

(2) **Cherish the lifting obstruction.** A more gentle approach is preferable. The lifting obstructions of *analytic* theta cocycles, i.e. cocycles for Γ valued in $\mathcal{A}^{\times}/\mathbf{C}_p^{\times}$ where $\mathcal{A}^{\times}$ is the multiplicative group of invertible analytic functions on $\mathcal{H}_p$, form a multiplicative lattice Λ in $H^2(\Gamma,\mathbf{C}_p^{\times})$. The quotient can be identified, up to an elementary factor coming from the Eisenstein line, with two copies of the Jacobian of $X_0(p)$.

$$\begin{array}{ccc} H^1(\Gamma,\mathcal{A}^{\times}/\mathbf{C}_p^{\times}) & \longrightarrow & \Lambda \\ \cap & & \cap \\ H^1(\Gamma,\mathcal{M}^{\times}/\mathbf{C}_p^{\times}) & \longrightarrow & H^2(\Gamma,\mathbf{C}_p^{\times}) \\ & & \downarrow \\ & & (\mathbf{C}_p^{\times}/\langle p\rangle)\times J_0(p)^2 \end{array}$$

When $p = 11$, it is an elliptic curve, with minimal Weierstraß equation

$$J_0(11): y^2 + y = x^3 - x^2 - 10x - 20.$$

In [22] we consider the image of the lifting obstruction of $\Theta_F^{\times}$. For instance, when $F = \langle 1,3,-3\rangle$ it equals $(\alpha,P,0)$ mod 11^{100}, where

$$\alpha = \frac{-103+24\sqrt{-7}}{121}, \qquad P = \left(\frac{-3-\sqrt{-7}}{2}, \frac{-3-\sqrt{-7}}{2}\right) \in J_0(11).$$

We see that the lifting obstruction is of independent interest, and appears to encode global points on modular Jacobians. This alternative viewpoint on Stark–Heegner points [15] is explored further in [17,22].

4.6. Recent analytic approaches Whereas the *geometric* nature of RM singular moduli remains elusive, we mention recent works that have explored the more accessible (p-adic) *analytic* structures analogous to those in CM theory.

Borcherds products. The work of Zagier [70] on *traces* of CM singular moduli has unveiled that their generating series may be understood in terms of the theory of *Borcherds products*, which provide a morphism

$$\Psi_{\mathrm{Bor}} \;:\; M^{+,!}_{1/2}(\Gamma_0(4)) \;\longrightarrow\; H^0(\mathrm{SL}_2(\mathbf{Z}), \mathrm{Mer}^{\times}_{\infty}).$$

valued in meromorphic functions on $\mathcal{H}_\infty$ with poles at CM divisors. It is tempting to view rigid cocycles, whose poles occur at RM divisors on the p-adic upper half plane, as analogues of these Borcherds products. This analogy was explored in [22], where a similar morphism is constructed:

$$\Psi_{\mathrm{Rig}} \;:\; M^{+,!}_{1/2}(\Gamma_0(4p)) \;\longrightarrow\; H^1(\Gamma, \mathrm{Mer}^{\times}).$$

The natural generalisation suggested by this analogy with Borcherds products is to consider rigid meromorphc cocycles for orthogonal groups $O(r,s)$ of general signatures, realised in Darmon–Gehrmann–Lipnowski [16]. This suggests a natural place for rigid cocycles in an emerging p-adic Kudla programme.

Analytic families of modular forms. The recent papers [17, 18] have explored analogues of the analytic arguments of Gross–Zagier [39] in the degenerate case where $F = \langle 0, 1, 0 \rangle$ is the split form of discriminant 1. We refer to the corresponding theta cocycle as the *winding cocycle*, denoted

$$\Theta^{\times}_{\mathrm{wind}} := \Theta^{\times}_{F} \in Z^1(\Gamma, \mathrm{Mer}^{\times}_{p} \,/\, \mathbf{C}^{\times}_{p})$$

The main conjectures for its RM values reduce to the well-known properties of p*-adic Gross–Stark units*. Since the form F is split, the factorisation conjecture predicts that it is a unit at all primes $q \neq p$. At primes above p, it predicts (with the convention that $B_{pp} = M_2(\mathbf{Q})$) that the factorisations are given by classical partial L-values, after applying Meyer's theorem, showing that the algebraic numbers we predict are precisely the p-adic Gross–Stark units in this degenerate case. Of course, we already have the more powerful and more general analytic formula for p-adic Gross–Stark units due to Dasgupta–Kakde [25, 26]. Nonetheless, this proof yields some theoretical evidence for the non-degenerate case of RM singular moduli that have been the focus of these lectures.

The strategy of [17, 18] resembles the arguments on Hecke's Eisenstein family discussed in the introduction. Consider a quadratic form G of discriminant $\Delta > 0$ for which p is inert. Define the real quadratic field $K = \mathbf{Q}(\sqrt{\Delta})$ and its narrow Hilbert class field H. Choose an odd character

$$\psi : \mathrm{Cl}^{+}_{K} \longrightarrow \bar{\mathbf{Q}}^{\times}.$$

There is a holomorphic Hilbert Eisenstein series of weight $(1,1)$ and level $\Gamma_0(p)$ over K, whose q-expansion at the class of the different is

$$E(\psi) := L_p(\psi, 0) + 4 \sum_{\nu \in \mathfrak{d}_+^{-1}} \left(\sum_{p \nmid I \mid (\nu)\mathfrak{d}} \psi(I) \right) e^{2\pi i(\nu_1 z_1 + \nu_2 z_2)}.$$

Since p is inert in K, the p-adic L-function $L_p(\psi, s)$ has an expectional zero at $s = 0$, and the series $E(\psi)$ is p-adically cuspidal. The approach is to p-adically deform this series. All p-adic deformations as *Hilbert eigenforms* are described by the local geometry of the eigenvariety $\mathscr{E} \longrightarrow \mathscr{W}$ at the parallel weight one point corresponding to $E(\psi)$. The local geometry is described by Betina–Dimitrov–Shih [3]; the cuspidal part is étale over weight space $\mathscr{W}$, and two lower-dimensional Eisenstein families, which exist only in parallel weight, intersect it transversely.

The papers [17, 18] each consider a different p-adic family specialising to $E(\psi)$. The p-adic Eisenstein family in parallel weight $(1+s, 1+s)$ is used in [17]. The advantage is that its Fourier expansion is completely explicit, but it yields more crude RM invariants. In contrast, [18] considers the cuspidal p-adic family of anti-parallel weight $(1+s, 1-s)$. This family lies deeper, and its Fourier expansion is determined from the deformation theory of the Galois representation

$$\rho = 1 \oplus \psi.$$

This representation is p-irregular, causing technical complications [2, 3, 57]. Galois cohomological arguments relate the deformation theory via global class field theory to the p-adic logarithm of the Gross–Stark unit $u_\psi \in \mathcal{O}_H[1/p]^\times \otimes \mathbf{Q}$.

For both choices of p-adic family appearing in [17, 18] one considers

(1) its diagonal restriction (vanishes at $s = 0$)
(2) its analytic first order derivative with respect to s
(3) its *ordinary projection* $\lim U_p^{n!}$, contained in the space of weight two holomorphic modular forms

$$M_2(\Gamma_0(p)).$$

This reflects the structure of the analytic proof of Gross–Zagier [39], discussed in the introduction. These operations are applied to the two p-adic families, and the first Fourier coefficient of the resulting form is explicitly computed:

- for the elementary *parallel* weight Eisenstein family [17], it is
$$\log_p \left(\mathrm{Nm}_{\mathbf{Q}_p} \Theta^{\times}_{\mathrm{wind}}[G] \right),$$
- for (a small modification of) the *anti-parallel* weight family [18], it is
$$\log_p \left(\Theta^{\times}_{\mathrm{wind}}[G] \right).$$

The resulting weight two form is contained in $M_2(\Gamma_0(p))$. Projecting onto the Eisenstein line (has no effect when p is a genus zero prime), the relation between its zeroth and first Fourier coefficients gives the (degenerate) main conjectures.

Remark 4.21. The proof of the degenerate RM conjectures in [17, 18] raises the natural question of whether the original setup of CM points on modular curves may be attacked p-adically using arguments like the above. This question is answered affirmatively in the forthcoming PhD thesis of Mike Daas [11, 12].

More precisely, Daas considers CM points on Shimura curves X_D associated to indefinite quaternion algebras of discriminant D. When the curve X_D is of genus zero, one may choose a generator j_D of its function field, and compute the (well-defined) cross ratio of its values at CM divisors. Giampietro and Darmon [35] study these quantities experimentally, and formulate a conjecture for their factorisations, which resemble those of Gross–Zagier in many respects. For instance, they compute for (P_1, P_2) of discriminants $(-43, -163)$ that

$$\mathrm{Nm}_{\mathbf{Q}} \left[\frac{j_6(P_1) - j_6(P_2))(j_6(P_1') - j_6(P_2')}{j_6(P_1) - j_6(P_2'))(j_6(P_1') - j_6(P_2)} \right] = \left(\frac{2 \cdot 29 \cdot 257 \cdot 277}{73 \cdot 137 \cdot 241} \right)^2$$

on the quaternionic Shimura curve X_6 of genus zero. Daas considers p-adic deformations, where p is a divisor of D, of the same Eisenstein series considered by Hecke and Gross–Zagier, with associated Galois representation $\rho = 1 \oplus \chi$, and computes the Fourier expansion of the form

$$\mathrm{Proj}_{\mathrm{ord}} \left[\frac{\partial}{\partial s} E(1, \chi)^D_s(z, z) \right]_{s=0} \qquad \in S_2(\Gamma_0(D))$$

to equal the p-adic logarithm of the cross-ratio invariant, together with contributions for primes q that are an explicit multiple of $\log_p(q)$. Daas proves the conjectures of Giampietro and Darmon [35] about the factorisations of these quaternionic singular moduli by showing that the above series must vanish. Remarkably, this proof does not use CM theory, and does not make reference to the QM abelian surfaces for which the Shimura curve X_D is a moduli space.

Remark 4.22. The proof of the degenerate RM conjectures in [18] also yields an *algorithm* for computing p-adic Gross–Stark units, using an idea of Hecke and Klingen–Siegel. This idea was used to compute p-adic L-functions [53], and using the anti-parallel family this was described in [4] and is upgraded to an algorithm to *simultaneously* compute p-adic Gross–Stark units and Stark–Heegner points by Håvard Damm–Johnsen in his fortcoming DPhil thesis [13, 14].

For example, let $K = \mathbf{Q}(\sqrt{136})$ which has narrow class group $\mathrm{Cl}^+(K) \simeq \mathbf{Z}/4\mathbf{Z}$. Choose the prime $p = 19$, which is inert in K. Applying the three operations to the anti-parallel weight family (the ordinary projection is computed using the algorithms of Lauder [52, 65]), Damm–Johnsen computes all the Fourier coefficients, *except for the constant term*, of the generating series of the RM values of the winding cocycle constructed in [18]

$$\log_p(u_G) + \sum_{n \geqslant 1} \log_p\left(\Theta^{\times}_{\mathrm{wind}}[T_n G]\right) q^n$$

for all choices of G of discriminant 136, up to precision 19^{50}. The computation takes less than three seconds. Projecting the result onto the Eisenstein line in the space $M_2(\Gamma_0(p))$, which can be done using only the higher Fourier coefficients, one *recovers* a numerical value for the constant term $\log_p(u_G)$. The resulting numerical value of the Gross–Stark unit u_G was recognised, using LLL routines, to be a root of the polynomial

$$361x^4 + 508x^3 + 310x^2 + 508x + 361.$$

It generates the narrow Hilbert class field over K.

It is clear that the limits of analytic arguments of the above sort should be explored further. Though significant new ideas are needed, one may hope that they could be applicable beyond the degenerate setups considered here. As far as we are from fully understanding RM singular moduli, it is clear that explicit experimentation has an important role to play in informing future progress.

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5. Exercises

(1) Prove that the set of reduced forms of a fixed discriminant $D \neq 0$ is finite. Let $F = \langle a, b, c \rangle$ be of non-square discriminant D. Show that

- if F is *definite*, then F is reduced if and only if

$$r_F \in \mathcal{D},$$

  where $\mathcal{D}$ is the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ in the Poincaré upper half plane $\mathfrak{H}_\infty := \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$.

- if F is *indefinite*, then F is reduced if and only if $\left|\sqrt{D} - 2|a|\right| < b < \sqrt{D}$. Show that this is furthermore equivalent to the following condition on the roots:

$$r'_F r_F < 0 \quad \text{and} \quad |r_F| < 1 < |r'_F|.$$

(2) Compute the set Σ_F of nearly reduced forms in the $\mathrm{SL}_2(\mathbf{Z})$-orbit of the quadratic form F, which we defined by

$$\Sigma_F := \left\{ \langle a, b, c \rangle \sim F \ : \ ac < 0 \right\},$$

for the forms $F = \langle -1, 4, 4 \rangle, \langle 5, 11, -5 \rangle$, and $\langle -4825, -15989, -13246 \rangle$. Characterise all forms F of non-square discriminant $D > 0$ for which the sets Σ_F are symmetric under the involutions

$$\begin{array}{rcccl} s_1 & : & \langle a, b, c \rangle & \longmapsto & \langle -a, -b, -c \rangle \\ s_2 & : & \langle a, b, c \rangle & \longmapsto & \langle a, -b, c \rangle \end{array}$$

in terms of the associated class in the (narrow) Picard group of $\mathbf{Z}\left[\frac{D+\sqrt{D}}{2}\right]$.

(3) Consider a multiplicative cocycle

$$\Theta \in Z^1\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^{\times}\right).$$

Its *value* at a form $G \in \mathcal{F}_D$ with $D > 0$ non-square is defined by

$$\Theta[G] := \Theta(\gamma_G)(r_G)$$

where γ_G is the automorph of G. Show that for a fixed Θ, the value $\Theta[G]$ only depends on the $\mathrm{SL}_2(\mathbf{Z})$-orbit of G.

(4) (Warm-up cocycle I) For any $c = (r, s) \in \mathbf{P}^1(\mathbf{Q})$ define the function

$$L(c) := \frac{s}{sz - r}$$

and consider the map

$$\begin{array}{rccl} p_c : & \mathrm{GL}_2(\mathbf{Q}) & \longrightarrow & \mathbf{C}(z), \\ & \gamma & \longmapsto & L(c) - L(\gamma c). \end{array}$$

Show that

- this is a 1-cocycle,
- its cohomology class

$$[p_c] \in H^1(\mathrm{GL}_2(\mathbf{Q}), \mathbf{C}(z))$$

  is independent of the choice of cusp c,
- if $F \in \mathcal{F}_D$ for $D > 0$ non-square, the multiplicative lift $p^{\times}$ of its restriction to the subgroup $\mathrm{SL}_2(\mathbf{Z})$ has value at F equal to

$$p^{\times}[F] = \varepsilon_D^{12}$$

  where $\varepsilon_D > 1$ is the fundamental unit of norm 1 in the quadratic order $\mathbf{Z}\left[\frac{D+\sqrt{D}}{2}\right]$ of discriminant D.

(5) (Warm-up cocycle II) For any $c = (r, s) \in \mathbf{P}^1(\mathbf{Q})$ define the function

$$N(c) := \frac{1}{(sz - r)^2},$$

where choose $\gcd(r,s)=1$, and consider the map

$$\begin{array}{rccl} q_c: & SL_2(\mathbf{Q}) & \longrightarrow & \mathbf{C}(z), \\ & \gamma & \longmapsto & N(c)-N(\gamma c). \end{array}$$

Show that

- this is a 1-cocycle,
- its cohomology class
$$[q_c] \in H^1(SL_2(\mathbf{Z}), \mathbf{C}(z))$$
is independent of the choice of cusp c.

Remark. Note that this cocycle is not in the image of dlog.

(6) (Knopp cocycle) Choose a cusp $c \in \mathbf{P}^1(\mathbf{Q})$. Define the map
$$kn_{c,F} : SL_2(\mathbf{Z}) \longrightarrow \mathbf{C}(z)$$
by setting
$$kn_{c,F}(\gamma) = \sum_{Q\sim F} \frac{sgn_{c,Q}(\gamma)}{z-r(Q)}.$$
where the numerator is defined by
$$sgn_{c,Q}(\gamma) := \begin{cases} 1 & \text{if } Q(c) > 0 > Q(\gamma c), \\ -1 & \text{if } Q(c) < 0 < Q(\gamma c), \\ 0 & \text{else.} \end{cases}$$

- Show that $kn_{c,F}$ is a 1-cocycle.
- Show that its cohomology class
$$[kn_{c,F}] \in H^1(SL_2(\mathbf{Z}), \mathbf{C}(z))$$
is independent of the choice of cusp c.
- Show that the multiplicative lift
$$kn_F^{\times} \in Z^1(SL_2(\mathbf{Z}), \mathbf{C}(z)^{\times})$$
satisfies
$$kn_F^{\times}(T) = \varepsilon_D^{12},$$
where $\varepsilon_D > 1$ is the fundamental unit of norm 1 in the quadratic order $\mathbf{Z}\left[\frac{D+\sqrt{D}}{2}\right]$ of discriminant D.

(7) Let $f(z) \in \mathbf{C}(z)$. Show that there exists a $\varphi \in Z^1_{par}\big(SL_2(\mathbf{Z}), \mathbf{C}(z)\big)$ with the property that $\varphi(S) = f(z)$ if and only if $f(z)$ satisfies the identities
$$\begin{cases} 0 = (1+S) \star f \\ 0 = (1+ST+(ST)^2) \star f \end{cases}$$
where S and T are the generators of $SL_2(\mathbf{Z})$ defined in the main text.

(8) Define the map $t : \mathbf{R}_{\geqslant 0} \longrightarrow \mathbf{R}_{\geqslant 0}$ by

$$t(x) := \begin{cases} x - 1 & \text{if} \quad x \geqslant 1 \\ x/(1-x) & \text{if} \quad 0 \leqslant x < 1 \end{cases}$$

Show that the periodic orbits for iteration of t are the sets $\{0\}$ and

$$\mathcal{S}_F := \{r(Q) \ : \ Q \in \Sigma_F\},$$

for $F \in \mathcal{F}_D$, with $D > 0$ non-square.

(9) Use the previous exercise to show that the cocycles p, q, kn_F introduced above generate the group of parabolic (additive) rational cocycles

$$Z^1_{\text{par}}\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)\right).$$

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