MODULAR EIGENFORMS AT THE BOUNDARY OF WEIGHT SPACE

JAN VONK

Abstract. Andreatta, Iovita, and Pilloni recently introduced $\mathbf{F}_p(\!(\mathrm{t})\!)$-Banach spaces of overconvergent t-adic modular forms, whose weight may be considered a "boundary" point of weight space. In an effort to make them concrete and accessible to explicit experimentation, we construct orthonormal bases, deduce t-adic analogues of certain p-adic results in the literature, and exhibit explicit examples.

Contents

Introduction

This note investigates the $\mathbf{F}_p(\!(\mathrm{t})\!)$-Banach spaces of overconvergent modular forms defined in [AIP15] motivated by a desire to make them as concrete as possible. We explicitly construct an orthonormal basis, which implies that there is a quadratic lower bound for the Newton polygon of the compact operator U_p on these spaces. The main virtue of our results is that they make these spaces amenable to explicit computation, and we present some first examples of finite slope eigenforms at the boundary. Finally, we include a strong t-adic analogue of an observation of Calegari on infinite slope limits of sequences of finite slope overconvergent eigenforms in §3.3.

Coleman's conjecture on boundary forms. Choose a prime $p \geq 3$, and let $\Lambda = \mathbf{Z}_p[\![\mathbf{Z}_p^\times]\!]$ be the Iwasawa algebra. Setting $\Delta \subset \mathbf{Z}_p^\times$ to be the torsion subgroup, we obtain an isomorphism $\Lambda \simeq \mathbf{Z}_p[\Delta][\![\mathrm{t}]\!]$ defined by $1+p \mapsto 1+\mathrm{t}$. Let $\Lambda^\circ = \mathbf{Z}_p[\![\mathrm{t}]\!] \subset \Lambda$ be the identity component.

The p-adic theory of modular forms [Kat73, Col97b] gives us, for any weight $\kappa \in \mathrm{Spm}\,\Lambda$, a collection of $\mathbf{Q}_p$-Banach spaces of r-overconvergent modular forms $M_\kappa^\dagger(r)$ of some fixed tame level. [1] These spaces arise as sections of rigid analytic line bundles, defined by Pilloni [Pil13] and Andreatta–Iovita–Stevens [AIS13], and they come equipped with an action of the Hecke algebra. The operator U_p is compact and, in particular, possesses a characteristic power series.

[1] We follow the notation of [AIP15], so that sections converge less far into the supersingular locus as r gets larger.

The characteristic power series of U_p varies analytically in the weight, yielding

$$P(\mathrm{t}, X) \in \mathbf{Z}_p[\![\mathrm{t}]\!]\{\{X\}\},$$

an entire power series with coefficients in Λ°. This integrality was proved by Coleman [Col97a], and led him to conjecture that an integral theory of p-adic variation of overconvergent modular forms should exist, so that reduction modulo p of this power series is a meaningful operation:

Conjecture 1 (Coleman)**.** *There exists an $\mathbf{F}_p(\!(\mathrm{t})\!)$-Banach space with an action of a compact operator U_p whose characteristic series is the reduction of $P(\mathrm{t}, X)$ modulo p.*

This conjecture was recently proved by Andreatta–Iovita–Pilloni [AIP15], who constructed such Banach spaces $\mathrm{M}^\dagger(r)$ for r large enough using the Igusa tower over $\overline{\mathbf{F}}_p$. The authors then proceed by giving an integral geometric construction of modular sheaves on certain formal schemes arising from modular curves, whose sections are shown to recover $\mathrm{M}^\dagger(r)$ in characteristic p, and the spaces $M_\kappa^\dagger(r)$ of overconvergent modular forms in characteristic 0. Analogous spaces were constructed for definite quaternion algebras in [LWX17] and for overconvergent cohomology in [JN17].

t**-Adic variation of modular forms.** The integrality of the theory of Andreatta–Iovita–Pilloni allows one to systematically investigate modular forms over regions of weight space hitherto left largely unexplored. The most mysterious of these settings is the "boundary weight" which gives rise to the $\mathbf{F}_p(\!(\mathrm{t})\!)$-Banach spaces $\mathrm{M}^\dagger$ discussed above. In order to include such forms into the discussion, we will consider the t*-adic region* $\mathcal{W}_\mathrm{t}$ of weight space, which consists of all the valuations $|\cdot| \in \mathrm{Spa}(\Lambda, \Lambda)^{\mathrm{an}}$ for which $|\,\mathrm{t}\,| \geq |p|$. Due to a need to invert p at an early stage in the theory, many results in the literature are restricted to the p-adic region $\mathcal{W}_p$. In contrast, the integrality of the theory of Andreatta–Iovita–Pilloni allows us to work in the t-adic region, which we will do below.

Orthonormal basis for $\mathrm{M}^\dagger$. The main result of this note is the construction of an explicit orthonormal basis of the $\mathbf{F}_p(\!(\mathrm{t})\!)$-Banach spaces $\mathrm{M}^\dagger(r)$ for large enough r, which shares similarities with the basis of *Katz expansions* considered in [Kat73]. The particular nature of this basis immediately implies the following effective version of the compactness of U_p, by an argument due to Wan [Wan98].

Corollary. *Let $p \geq 3$. There is a quadratic lower bound for the Newton polygon of the characteristic series of U_p on $\mathrm{M}^\dagger(r)$ for any $r \geq 2$.*

Explicit experimentation. The $\mathbf{F}_p(\!(\mathrm{t})\!)$-Banach spaces $\mathrm{M}^\dagger(r)$ are of clear interest, but it is at present not possible to write down interesting examples beyond reductions of Hida families. As a by-product of the construction of our basis, we obtain an algorithm for computing $\mathrm{M}^\dagger$ in the spirit of Lauder [Lau11]. We present some first examples of boundary eigenforms. We then address the phenomenon of t-adic congruences between eigenforms, and elaborate on a t-adic analogue of an observation of Calegari in the p-adic setting related to infinite slope limits of sequences of finite slope eigenforms.

We note that explicit experimentation with the *characteristic series* $P(\mathrm{t}, X)$ has long been possible, and has led to the discovery of an abundance of symmetry in the slopes of modular forms, the p-adic valuation of U_p-eigenvalues. Such computations were performed by Bergdall–Pollack [BP16] using a formula for the coefficients due to Koike, or in the author's doctoral dissertation, using the work of Lauder [Lau11, Von15]. The approach of this note goes further, in that it computes the operator U_p

directly in an explicit basis, rather than just its characteristic series.

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1. The spectral halo

We start by recalling some of the main definitions and results from [AIP15]. This section contains only preliminary material, and no new results.

1.1. **The weight space.** Let $\Lambda = \mathbf{Z}_p[\![\mathbf{Z}_p^\times]\!]$ be the Iwasawa algebra. If we denote $\Delta \subset \mathbf{Z}_p^\times$ for the torsion subgroup, then we have an isomorphism $\Lambda \simeq \mathbf{Z}_p[\Delta][\![\mathrm{t}]\!]$ defined by $1+p \mapsto 1+\mathrm{t}$. For any adic space X, we denote X^{an} for its set of analytic points, which are those whose corresponding valuation has non-open support. The *weight space* is

$$\mathcal{W} = \mathrm{Spa}(\Lambda, \Lambda)^{\mathrm{an}},$$

whose points contain the weight characters $\mathrm{Hom}_{\mathrm{cts}}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$, as well as the *boundary points* χ corresponding to the finite set of valuations

$$\chi : \Lambda \xrightarrow{\mathrm{red}} \mathbf{F}_p[\Delta][\![\mathrm{t}]\!] \xrightarrow{\overline{\chi}} \mathbf{F}_p[\![\mathrm{t}]\!] \xrightarrow{|\cdot|_{\mathrm{t}}} \mathbf{R},$$

where $v_{\mathrm{t}} : \mathbf{F}_p(\!(\mathrm{t})\!) \to \mathbf{R}$ is the t-adic valuation, and $\overline{\chi} : \Delta \to \mathbf{F}_p^\times$ is any character. There is a bijection between the characters $\overline{\chi}$, the boundary points χ, and the connected components $\mathcal{W}^\chi$ of weight space. Any connected component $\mathcal{W}^\chi$ has $\Lambda^\chi \simeq \mathbf{Z}_p[\![\mathrm{t}]\!]$ as its ring of functions, and we simply write Λ° for the ring corresponding to the trivial character.

The regions of weight space defined by $|p| \geq |\,\mathrm{t}\,|$, which we call the *$p$-adic region* $\mathcal{W}_p$, and $|p| \leq |\,\mathrm{t}\,|$, which we call the *$\mathrm{t}$-adic region* $\mathcal{W}_{\mathrm{t}}$, correspond to the $\mathbf{Z}_p$-algebras

$$\begin{cases} \Lambda_p &= \Lambda\langle \mathrm{t}/p\rangle, \qquad \mathcal{W}_p = \mathrm{Spa}(\Lambda_p, \Lambda_p)^{\mathrm{an}} \\ \Lambda_{\mathrm{t}} &= \Lambda\langle p/\mathrm{t}\rangle, \qquad \mathcal{W}_{\mathrm{t}} = \mathrm{Spa}(\Lambda_{\mathrm{t}}, \Lambda_{\mathrm{t}})^{\mathrm{an}} \end{cases}$$

We denote $\Lambda_{\mathrm{t}}^\chi$ and $\Lambda_{\mathrm{t}}^\circ$ for the direct factors attached to $\overline{\chi}$ and the trivial character, respectively.

1.2. **Modular curves.** Let $\mathfrak{X} \to \mathrm{Spf}\,\mathbf{Z}_p$ be the formal completion along the special fibre of the modular curve of full level $N \geq 3$ at $p \nmid 2N$, and let $\omega = s^*\Omega_{\mathcal{E}/\mathfrak{X}}$ be defined via the identity section $s : \mathfrak{X} \to \mathcal{E}$ to the universal generalised elliptic curve. For any $r \geq 1$, we choose a section

$$\widetilde{\mathrm{Ha}^{p^r}} \in \mathrm{H}^0(\mathfrak{X}, \omega^{\otimes p^r(p-1)})$$

that lifts the p^r-th power of the Hasse invariant. Define for $\star \in \{p, \mathrm{t}\}$ the opens $\mathfrak{X}_{\star,r}$ of the admissible blow-ups of $\mathfrak{X} \times_{\mathrm{Spf}\,\mathbf{Z}_p} \mathrm{Spf}\,\Lambda_\star^\circ$ along $(\widetilde{\mathrm{Ha}^{p^r}}, \star)$ by the following conditions: Locally on any open affinoid $\mathrm{Spf}\, A$ such that $\omega\mid_{\mathrm{Spf}\, A} \simeq A$, we have

$$\begin{cases} \mathfrak{X}_{p,r} \times_{(\mathfrak{X}\times \mathrm{Spf}\,\Lambda_p^\circ)} \mathrm{Spf}\, A &= \mathrm{Spf}\, A\langle x\rangle/(x\widetilde{\mathrm{Ha}^{p^r}} - p) \\ \mathfrak{X}_{\mathrm{t},r} \times_{(\mathfrak{X}\times \mathrm{Spf}\,\Lambda_\mathrm{t}^\circ)} \mathrm{Spf}\, A &= \mathrm{Spf}\, A\langle x\rangle/(x\widetilde{\mathrm{Ha}^{p^r}} - \mathrm{t}) \end{cases} \tag{1}$$

These formal schemes come equipped with natural inclusion maps and lifts of Frobenius

$$\iota : \mathfrak{X}_{\star,r+1} \hookrightarrow \mathfrak{X}_{\star,r}, \qquad \varphi : \mathfrak{X}_{\star,r+1} \to \mathfrak{X}_{\star,r}, \qquad \text{where } \star \in \{p, \mathrm{t}\}.$$

1.3. **Modular sheaves.** The $\mathfrak{X}_{\star,r}$ have a moduli interpretation, and one may construct Igusa coverings on them, which are overconvergent analogues of the coverings in the Igusa tower $\mathfrak{IG}^{\infty}_{\star,\mathrm{ord}} \to \mathfrak{X}_{\star,\mathrm{ord}}$ over the ordinary locus $\mathfrak{X}_{\star,\mathrm{ord}}$ of $\mathfrak{X}_{\star,r}$. Using these Igusa towers, Andreatta–Iovita–Pilloni construct invertible sheaves $\mathfrak{w}_p$ on $\mathfrak{X}_{p,r}$ and $\mathfrak{w}_\mathrm{t}$ on $\mathfrak{X}_{\mathrm{t},r}$ for any $r \geq 3$. These sheaves glue on the overlap of $\mathfrak{X}_{p,r}$ and $\mathfrak{X}_{\mathrm{t},r}$, and interpolate the tensor powers of the modular sheaf ω on $\mathfrak{X}$ in the sense that on the fibre of

$$k : \Lambda^\circ \to \mathbf{Z}_p : \mathrm{t} \mapsto (1+p)^k - 1$$

the sheaf $\mathfrak{w}_p$ is isomorphic to $\omega^{\otimes k}$. The line bundles $\mathfrak{w}_\star$ come equipped with a Frobenius operator, see [AIP15, Theorem 6.2], which is an isomorphism

$$F : \iota^*\mathfrak{w}_\star \simeq \varphi^*\mathfrak{w}_\star, \qquad \star \in \{p, \mathrm{t}\}. \tag{2}$$

1.4. **Spaces of t-adic forms.** Define the space of t*-adic forms* by

$$\mathcal{M}^\dagger_\mathrm{t}(r) = \mathfrak{M}^\dagger_\mathrm{t}(r)[1/\mathrm{t}], \qquad \text{where} \qquad \mathfrak{M}^\dagger_\mathrm{t}(r) := \mathrm{H}^0(\mathfrak{X}_{\mathrm{t},r}, \mathfrak{w}_\mathrm{t}).$$

Then $\mathfrak{M}^\dagger_\mathrm{t}(r)$ is a (t)-adically complete and separated Λ°_t-module, and the space of t-adic forms is a Banach module over $\Lambda^\circ_\mathrm{t}[1/\mathrm{t}]$. It was shown in [AIP15, Prop. 6.9] that it is a projective Banach module. We will show in the next section that it is in fact orthonormalisable, by constructing a basis for it. Finally, we define the $\mathbf{F}_p(\!(\mathrm{t})\!)$-Banach space of boundary forms

$$\mathrm{M}^\dagger(r) = \mathfrak{M}^\dagger_\mathrm{t}(r) \otimes_{\Lambda^\circ_\mathrm{t}} \mathbf{F}_p(\!(\mathrm{t})\!).$$

Alternatively, it will be denoted by $\mathrm{M}^\dagger(N, r)$ when we want to explicate the implicit tame level.

The space of t-adic modular forms has a Hecke action via the usual correspondences on modular curves. Of particular interest for us is the operator U_p, which is defined via the correspondence

$$U_p : \quad \begin{tikzcd} & \mathfrak{X}_{\mathrm{t},r+1} \arrow[dl, "\iota"'] \arrow[dr, "\varphi"] & \\ \mathfrak{X}_{\mathrm{t},r} & & \mathfrak{X}_{\mathrm{t},r} \end{tikzcd}$$

The operator U_p is defined as the resulting map

$$U_p : \mathrm{H}^0(\mathfrak{X}_{\mathrm{t},r}, \mathfrak{w}_\mathrm{t}) \xrightarrow{\mathrm{res}} \mathrm{H}^0(\mathfrak{X}_{\mathrm{t},r+1}, \iota^*\mathfrak{w}_\mathrm{t}) \simeq \mathrm{H}^0(\mathfrak{X}_{\mathrm{t},r+1}, \varphi^*\mathfrak{w}_\mathrm{t}) \xrightarrow{p^{-1}\mathrm{Tr}_\varphi} \mathrm{H}^0(\mathfrak{X}_{\mathrm{t},r}, \mathfrak{w}_\mathrm{t}), \tag{3}$$

where the Tate trace $p^{-1}\mathrm{Tr}_\varphi$ was constructed in [AIP15, Section 6.3]. This defines a compact operator which has the usual effect on q-expansions.

1.5. **Ordinary t-adic forms.** We now recall the definition of Λ°-adic modular forms, and establish that an ordinary Λ°-adic modular form arises as the q-expansion of a unique section of the line bundle $\mathfrak{w}_{\mathrm{t}}$ on the ordinary locus. This will be applied to the Eisenstein family in the next section.

A Λ°*-adic modular form* of level $\Gamma(N)$ is an element of $\Lambda^\circ_{\mathcal{O}}[\![q]\!]$, whose specialisation at the ideal

$$\mathfrak{p}_{k,\psi} = \left(1 + \mathrm{t} - \psi(1+p)(1+p)^k\right)$$

with $k \in \mathbf{Z}$ sufficiently large is the q-expansion of a classical weight k modular form of level $\Gamma(N) \cap \Gamma_0(p)$ with character $\tau^{-k}\psi$, where τ is the Teichmüller character. Here, $\mathcal{O}$ is the ring of integers in a finite extension of $\mathbf{Q}_p$ and $\Lambda^\circ_{\mathcal{O}} \simeq \mathcal{O}[\![\mathrm{t}]\!]$ is the corresponding finite extension of Λ°. The image of the ordinary projector $e^{\mathrm{ord}} = \lim_{k\to\infty} U_p^{k!}$ is called the space of *ordinary* Λ°*-adic forms*. The space of Λ°-adic modular forms of level $\Gamma(N)$ is denoted by $M(\Lambda^\circ, N)$.

Hida gave a geometric interpretation of the space of Λ°-adic modular forms in terms of the Igusa tower over the ordinary locus of $\mathfrak{X}$. More precisely, define the Igusa tower

$$\pi : \mathfrak{IG}^\infty_{\mathrm{ord}} = \underline{\mathrm{Isom}}\left(\underline{\mathbf{Z}}_p, (T_pE)^{\text{ét}}\right) \longrightarrow \mathfrak{X}_{\mathrm{ord}}$$

to be the $\mathbf{Z}_p^\times$-torsor of trivialisations for the étale part of the Tate module of the universal ordinary elliptic curve E. If we let $\kappa^{\mathrm{univ}} : \mathbf{Z}_p^\times \to (\Lambda^\circ)^\times$ be the universal character, then:

Theorem 1.1 (Hida). *The q-expansion map from sections that transform via $(\kappa^{\mathrm{univ}})^{-1}$*

$$\mathrm{H}^0(\mathfrak{X}_{\mathrm{ord}}, \pi_*\mathcal{O}_{\mathfrak{IG}^\infty_{\mathrm{ord}}})[(\kappa^{\mathrm{univ}})^{-1}] \longrightarrow M(\Lambda^\circ, N),$$

is a Hecke equivariant isomorphism. The space of ordinary Λ°-adic forms is a finite free Λ°-module, and any specialisation of an ordinary Λ°-adic form at integer weight $k \geq 1$, is overconvergent.

The main idea of [AIP15] is to make an overconvergent construction of an "Igusa tower" analogous to $\mathfrak{IG}^\infty_{\mathrm{ord}}$, trading off the local analyticity of the universal character near the centre of weight space, against the existence of higher canonical subgroups near the boundary. As a consequence of the construction, as well as the theorem of Hida above, it follows that any Λ°-adic modular form arises as the q-expansion of a unique section of $\mathfrak{w}_{\mathrm{t}}$. More precisely, we have

Corollary 1.2. *Let $\chi : \Delta \to \mathbf{C}_p^\times$, then we have by construction*

$$\mathfrak{w}_{\mathrm{t}} \mid_{\mathfrak{X}_{\mathrm{t,ord}}} \simeq \Lambda^\circ_{\mathrm{t}} \otimes \pi_*\mathcal{O}_{\mathfrak{IG}^\infty_{\mathrm{ord}}}[(\kappa^{\mathrm{univ}})^{-1}].$$

In particular, any ordinary Λ°-modular form is the q-expansion of a unique section of $\mathfrak{w}_{\mathrm{t}}$ over $\mathfrak{X}_{\mathrm{t,ord}}$.

We mention also that the results of [AIP15, Section 6] imply that in fact, one may likewise identify non-ordinary q-expansions of Λ°-adic modular forms with t-adic modular forms. This follows from the Riemann Hebbarkeitssatz, known in the context of rigid spaces as the Remmert–Stein theorem, which guarantees that a bounded function on a dense Zariski open of a normal rigid space must necessarily extend to a global analytic function. This result is known not to hold for general adic spaces, but for the sheaves under consideration, it was proved by Andreatta–Iovita–Pilloni.

1.6. **Remark.** We briefly point out how our notation compares to that of [AIP15]. The development of the theory of t-adic forms in *loc. cit.* requires a plethora of objects which we did not recall in this note, allowing us to work with less heavy notation. We have still opted to mimic their notation fairly closely. The following table compares the notation for some of the objects considered here.

This paper	$\mathfrak{X}_{\mathrm{t},r}$	$\mathfrak{X}_{p,r}$	$\mathfrak{IG}_{\mathrm{t},r}$	$\mathfrak{w}_{\mathrm{t}}$
[AIP15]	$\mathfrak{X}_{r,[1,\infty]}$	$\mathfrak{X}_{r,[0,1]}$	$\mathfrak{IG}_{1,r,[1,\infty]}$	$\mathfrak{w}_{[1,\infty]}$

2. Orthonormal bases

In this section, we will show that $\mathcal{M}_{\mathrm{t}}^{\dagger}(r)$ is an orthonormalisable Banach module for large r, by exhibiting an explicit basis for it. We will do this by first trivialising $\mathfrak{w}_{\mathrm{t}}$ by a t-adic Eisenstein series, and then finding an explicit description of the functions on $\mathfrak{X}_{\mathrm{t},r}$.

2.1. **The t-adic Eisenstein family.** We start by showing that the q-expansion of the Eisenstein family defines an overconvergent t-adic modular form in the sense of Andreatta–Iovita–Pilloni [AIP15]. Let $\zeta_p(\mathrm{t}) \in \mathrm{t}^{-1}(\Lambda^{\circ})^{\times}$ be the p-adic zeta function with trivial tame character. The prototypical example of an ordinary Λ°-adic modular form is the level 1 Eisenstein family

$$\mathbf{E}(q) = 1 + \frac{2}{\zeta_p(\mathrm{t})} \sum_{n \geq 1} \boldsymbol{\sigma}(n) q^n \in \Lambda^{\circ}[\![q]\!],$$

where $\boldsymbol{\sigma}$ is the power series that specialises to $\sum_{d|n,(p,d)=1} \psi(d) d^{k-1}$ at $\mathfrak{p}_{k,\psi}$. We show that in fact, this is the q-expansion of an element in $\mathfrak{M}_{\mathrm{t}}^{\dagger}(r)$ for any $r \geq 3$.

Proposition 2.1. *For any $r \geq 3$, there is a unique form $\mathbf{E} \in \mathfrak{M}_{\mathrm{t}}^{\dagger}(r)$ whose q-expansion is the series $\mathbf{E}(q)$ defined above. It trivialises the line bundle $\mathfrak{w}_{\mathrm{t}}$ over $\mathfrak{X}_{\mathrm{t},r}$ for some large enough r.*

Proof. From Corollary 1.2, we know that the formal series $\mathbf{E}(q) \in \Lambda^{\circ}[\![q]\!]$ is the q-expansion of a unique section in $\mathrm{H}^0(\mathfrak{X}_{\mathrm{t,ord}}, \mathfrak{w}_{\mathrm{t}})$. From the natural inclusion $\mathfrak{X}_{\mathrm{t,ord}} \hookrightarrow \mathfrak{X}_{\mathrm{t},r}$, we obtain a morphism on U_p-ordinary sections:

$$\psi : e^{\mathrm{ord}} \mathrm{H}^0(\mathfrak{X}_{\mathrm{t},r}, \mathfrak{w}_{\mathrm{t}}) \longrightarrow e^{\mathrm{ord}} \mathrm{H}^0(\mathfrak{X}_{\mathrm{t,ord}}, \mathfrak{w}_{\mathrm{t}}).$$

For all $r \geq 2$, this is an isomorphism after we tensor with $\Lambda_{\mathrm{t}}^{\circ}/\mathfrak{p}_{k,\chi}\Lambda_{\mathrm{t}}^{\circ}$ when $k \geq 2$. It follows from the topological Nakayama lemma that ψ is an isomorphism of $\Lambda_{\mathrm{t}}^{\circ}$-modules, from which it follows that $\mathbf{E}(q)$ is the q-expansion of a section over $\mathfrak{X}_{\mathrm{t},r}$ for all $r \geq 3$. Since $\mathbf{E}(q) \equiv 1 \pmod{\mathrm{t}}$, it defines an invertible section on $\mathfrak{X}_{\mathrm{t,ord}}$, which by the maximum principle means it is an invertible section on $\mathfrak{X}_{\mathrm{t},r}$ for some large enough value of r. □

2.2. **A basis for $\mathfrak{M}_{\mathrm{t}}^{\dagger}(r)$.** We will now find an explicit basis for the space of t-adic modular forms, for r large enough. We note that the basis we construct depends on a choice of splittings of multiplication by our lift of the Hasse invariant, and is not canonical.

Proposition 2.2. *For r large enough, $\mathfrak{M}_{\mathrm{t}}^{\dagger}(r)$ admits an orthonormal $\Lambda_{\mathrm{t}}^{\circ}$-basis of the form*

$$\left\{ \left(\frac{\mathrm{t}}{\widetilde{\mathrm{Ha}^{p^r}}} \right)^n \mathbf{E}\, b_{m,n} \right\}_{m,n} \tag{4}$$

where the $b_{m,n}$ are classical modular forms of weight $np^r(p-1)$.

Proof. Let $\pi : \mathfrak{L} \to \mathfrak{X}_{\mathrm{t}}$ the line bundle attached to the invertible sheaf $\omega^{\otimes p^r(p-1)}$. The partial blow-up $\iota : \mathfrak{X}_{\mathrm{t},r} \to \mathfrak{X}_{\mathrm{t}}$ gives an exact sequence

$$0 \to \pi_* \mathcal{O}_{\mathfrak{L}} \xrightarrow{\cdot(\widetilde{\mathrm{Ha}^{p^r}} - \mathrm{t})} \pi_* \mathcal{O}_{\mathfrak{L}} \longrightarrow \iota_* \mathcal{O}_{\mathfrak{X}_{\mathrm{t},r}} \to 0$$

of sheaves on $\mathfrak{X}_{\mathrm{t}}$. The sheaf $\pi_* \mathcal{O}_{\mathfrak{L}}$ is the direct sum of the line bundles $\omega^{\otimes i p^r(p-1)}$ for $i \geq 0$. When $i \geq 1$, the Kodaira–Spencer isomorphism shows that $\omega^{\otimes i p^r(p-1)}$ has degree at least $2g - 1$, where g is the genus of $\mathfrak{X}$. By Serre duality,

$$\mathrm{H}^1(\mathfrak{X}_{\mathrm{t}}, \omega^{\otimes i p^r(p-1)}) = 0, \qquad \text{for } i > 0.$$

From the long exact sequence in cohomology, we extract the 4-term exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathfrak{X}_{\mathrm{t}}, \pi_* \mathcal{O}_{\mathfrak{L}})/(\widetilde{\mathrm{Ha}^{p^r}} - \mathrm{t}) \longrightarrow \mathrm{H}^0(\mathfrak{X}_{\mathrm{t},r}, \mathcal{O}_{\mathfrak{X}_{\mathrm{t},r}}) \longrightarrow \mathrm{H}^1(\mathfrak{X}_{\mathrm{t}}, \mathcal{O}_{\mathfrak{X}_{\mathrm{t}}}) \xrightarrow{\cdot(-\mathrm{t})} \mathrm{H}^1(\mathfrak{X}_{\mathrm{t}}, \mathcal{O}_{\mathfrak{X}_{\mathrm{t}}})$$

As $\mathrm{H}^1(\mathfrak{X}_{\mathrm{t}}, \mathcal{O}_{\mathfrak{X}_{\mathrm{t}}})$ is a finite free $\Lambda_{\mathrm{t}}^\circ$-module, multiplication by $-\mathrm{t}$ is injective. It follows that the first injection is an isomorphism, so that the ring of functions on $\mathfrak{X}_{\mathrm{t},r}$ is isomorphic to

$$\bigoplus_{n \geq 0} \mathrm{H}^0(\mathfrak{X}_{\mathrm{t}}, \omega^{\otimes n p^r(p-1)})/(\widetilde{\mathrm{Ha}^{p^r}} - \mathrm{t}).$$

From this description, we obtain an explicit basis in essentially the same way as Katz [Kat73], by noting that the maps of finite free $\mathbf{Z}_p$-modules

$$\cdot\widetilde{\mathrm{Ha}^{p^r}} : \mathrm{H}^0(\mathfrak{X}, \omega^{\otimes n p^r(p-1)}) \longrightarrow \mathrm{H}^0(\mathfrak{X}, \omega^{\otimes (n+1) p^r(p-1)})$$

given by multiplication by $\widetilde{\mathrm{Ha}^{p^r}}$ are split, since the cokernel on sheaves reduces to a skyscraper sheaf mod p, which is acyclic, see [Kat73, Lemma 2.6.1]. We may therefore choose a basis $\{a_{m,n}\}_{m,n}$ for a complementary subspace of the image, consequently giving us a basis for the space of functions on $\mathfrak{X}_{\mathrm{t},r}$. □

Remark. The above result resolves the tension observed by Coleman [Col97a, Col13] in the rate of p-adic overconvergence of the explicit orthonormal bases of Katz as one approaches the boundary of weight space, by working with the t-adic topology instead of the p-adic topology.

2.3. **Specialisation to the boundary.** By the previous result, we obtain the q-expansions

$$\{\mathrm{t}^n \, \mathbf{E} \, b_{m,n}\}_{m,n} \tag{5}$$

of an explicit orthonormal basis of $\mathrm{M}^\dagger(r)$, for large enough r, by reduction modulo p of the basis for the space of t-adic modular forms. We now make a few observations about this basis.

We start by noting that a finite slope eigenform in $\mathrm{M}^\dagger(r)$ may be analytically continued to $r = 2$ using standard arguments. Indeed, if we assume that $U_p f = \lambda f$, for some $\lambda \neq 0$. Then

$$f = \lambda^{-1} U_p f \in \mathrm{M}^\dagger(r-1)$$

when $r \geq 3$, from the contractiveness of U_p. This shows that f analytically continues to an eigenform in $\mathrm{M}^\dagger(2)$. We would be able to continue it further, but the boundary modular sheaves were only defined for $r \geq 2$ in [AIP15]. This analytic continuation compensates the lack of control on r in Proposition 4, and we will henceforth often drop r from the notation.

Following an argument of Wan [Wan98], we may 'quantify' the compactness of the operator U_p on $\mathfrak{M}^\dagger(r)$, and prove that there is a quadratic lower bound for its Newton slopes. It is quite possible that one could have deduced such a bound prior to the work of Andreatta–Iovita–Pilloni. However, the estimate (6) appearing in the proof is an ingredient for the algorithms discussed below.

Corollary 2.1. *Let $p \geq 3$. There exists a quadratic lower bound for the Newton polygon of the characteristic series of U_p on $\mathrm{M}^\dagger(N, r)$ for any level $N \geq 3$.*

Proof. Choose $r \geq 2$ large enough, and choose a basis $\{e_{m,n}\}_{m,n}$ of $\mathrm{M}^\dagger(N, r)$ as in the statement of Proposition 4. We write

$$U_p(e_{m,n}) = \sum_{a,b} c_{m,n}^{a,b} e_{a,b},$$

for some $c_{m,n}^{a,b} \in \mathbf{F}_p[\![\mathrm{t}]\!]$, which we think of as the matrix entries of U_p with respect to our chosen basis. Since U_p maps $\mathrm{M}^\dagger(r)$ to $\mathrm{M}^\dagger(r-1)$, we get the estimate

$$v_{\mathrm{t}}\left(c_{m,n}^{a,b}\right) \geq b(p-1)p^{-r}. \tag{6}$$

The dimension of the spaces $\mathrm{H}^0(\mathfrak{X}, \omega^{\otimes np^r(p-1)})$ of classical mod p modular forms has linear growth in n by Riemann–Roch, so that after taking determinants, we obtain a quadratic lower bound on the Newton polygon of U_p, see also [Wan98, Lemma 3.1]. $\square$

This result is consistent with - though much weaker than - the expectation that the slope sequence of boundary forms is a superposition of a finite number of arithmetic progressions.

2.4. **Non-trivial characters.** We make some comments about the spaces of boundary forms $\mathrm{M}_\chi^\dagger(r)$ at boundary points χ corresponding to the $p-2$ non-trivial characters $\overline{\chi} : \Delta \to \mathbf{F}_p^\times$. To define the spaces $\mathrm{M}_\chi^\dagger(r)$, denote the first layer of the aforementioned overconvergent Igusa tower by

$$h : \mathfrak{IG}_{\star,r} \longrightarrow \mathfrak{X}_{\star,r}$$

which is a Galois cover with group Δ. Furthermore, [AIP15, Section 6.8] defines for any character $\overline{\chi} : \Delta \to \mathbf{F}_p^\times$ the coherent sheaf

$$\boldsymbol{\tau}_\star^\chi = h_* \mathcal{O}_{\mathfrak{IG}_{\star,r}}[\overline{\chi}^{-1}]$$

whose analytic fibre is an invertible sheaf, and $\mathfrak{w}_\star^\chi = \mathfrak{w}_\star \otimes \boldsymbol{\tau}_\star^\chi$. There is a Frobenius operator F as in (2) on the analytic fibre of the sheaves $\mathfrak{w}_\star^\chi$, though such a structure is not available integrally.

We define the space of t-adic forms with nebentype χ by

$$\mathcal{M}_{\mathrm{t}}^\dagger(r)^\chi = \mathfrak{M}_{\mathrm{t}}^\dagger(r)^\chi[1/\mathrm{t}], \qquad \text{where} \qquad \mathfrak{M}_{\mathrm{t}}^\dagger(r)^\chi := \mathrm{H}^0(\mathfrak{X}_{\mathrm{t},r}, \mathfrak{w}_{\mathrm{t}}^\chi).$$

and likewise we define the $\mathbf{F}_p(\!(\mathrm{t})\!)$-Banach space of boundary forms

$$\mathrm{M}_\chi^\dagger(r) = \mathfrak{M}_{\mathrm{t}}^\dagger(r)^\chi \otimes_{\Lambda_{\mathrm{t}}^\circ} \mathbf{F}_p(\!(\mathrm{t})\!).$$

The spaces $\mathcal{M}_{\mathrm{t}}^\dagger(r)^\chi$ are very similar to the spaces $\mathcal{M}_{\mathrm{t}}^\dagger(r)$ considered above, and a similar analysis should lead to an explicit basis of the shape of (4), where the $b_{m,n}$ are mod p modular forms of weight $np^r(p-1)$ and character $\overline{\chi}$ on the Igusa curve. Some issues arise when making this into an algorithm, most notably that [AIP15, Theorem 6.2.4] does not guarantee that U_p preserves the integral structures $\mathfrak{M}_{\mathrm{t}}^\dagger(r)^\chi$. This might be resolved when bypassing $\mathfrak{M}_{\mathrm{t}}^\dagger(r)^\chi$ and using the direct construction of the spaces $\mathrm{M}^{\dagger}_{\chi}$ in characteristic p given in [AIP15, Section 4]. We will not undertake a careful study of these issues here, as we expect to be able to see most interesting behaviour for the trivial character already. Ignoring these issues, we still obtain an algorithm for non-trivial characters, see §3.1.

3. t-Adic Hecke eigenfunctions

Our results were primarily motivated by the desire to find a method for explicit experimentation with the space of boundary forms $\mathrm{M}^{\dagger}$. Inspired by the work of Lauder [Lau11], see also [Von15], we have implemented an algorithm in `magma` [BCP97], and will present some first examples here.

3.1. **The algorithm of Lauder.** The algorithm of Lauder [Lau11] in the setting of p-adic modular forms may be adapted to compute with the spaces of boundary forms $\mathrm{M}^{\dagger}$. We will not discuss the complexity or precision bounds. For simplicity, we focus on working in level 1.

Approximating boundary forms is computationally challenging, and whereas the work of Lauder is aided by heavily optimised linear algebra routines over $\mathbf{Z}/p^m$ in `magma`, we work over $\mathbf{F}_p[\![\mathrm{t}]\!]/\mathrm{t}^m$, affecting the speed and memory requirements. One may use an explicit presentation of the ring of modular forms to obtain an efficient algorithm in level 1, which is available on the author's webpage, in the hope that it might be of use to someone. We were able to do some computations in higher level, though making our implementation practical in general would require more work.

When $\overline{\chi}$ is not trivial, in addition to the theoretical gaps in §2.4, there are other justifications necessary, e.g. in our computation of forms on the mod p Igusa curve. Ignoring all issues, we do seem to obtain output as expected. As the computation requires working in level $\Gamma_1(p)$, obtaining eigenforms is prohibitively difficult, though we do obtain some slope sequences that behave as expected.

3.2. t-**Adic eigenfunctions.** Given an eigenform $f = \sum_{n \geq 0} a_n q^n \in \mathrm{M}^{\dagger}_{\chi}(N)$ for all Hecke operators which has non-zero U_p-eigenvalue, we may find a t-adic family $\mathbf{f}$ passing through f. The Galois representations associated to a dense set of specialisations at arithmetic weights in characteristic 0 interpolate to give a semi-simple continuous representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\,\mathbf{Q}) \longrightarrow \mathrm{GL}_2\left(\overline{\mathbf{F}}_p(\!(u)\!)\right),$$

where $\overline{\mathbf{F}}_p(\!(u)\!)$ is a finite extension of $\overline{\mathbf{F}}_p(\!(\mathrm{t})\!)$. This representation is unramified outside Np, and for all primes $l \nmid Np$ we have that $\mathrm{Tr}\ \rho_f(\mathrm{Frob}_l) = a_l$.

Example 1. Let $p = 5$ and $\overline{\chi}$ both the trivial character, and the quadratic Dirichlet character of modulus 5. We compute that the compact operator U_5 on $\mathrm{M}^{\dagger}_{\chi}(\mathrm{SL}_2(\mathbf{Z}))$ has t-adic slope sequences

$$\begin{array}{rcl} \overline{\chi} = \mathbf{1} & : & \mathbf{0}_1, \mathbf{1}_1, \mathbf{3}_1, \mathbf{4}_1, \mathbf{6}_1, \mathbf{8}_1, \mathbf{9}_1, \mathbf{11}_1, \mathbf{12}_1, \mathbf{14}_1, \mathbf{16}_1, \mathbf{17}_1, \mathbf{19}_1, \mathbf{20}_1, \mathbf{22}_1, \mathbf{24}_1, \ldots \\ \overline{\chi} = \left(\frac{\cdot}{5}\right) & : & \mathbf{0}_1, \mathbf{2}_1, \mathbf{4}_1, \mathbf{5}_1, \mathbf{7}_1, \ldots \end{array}$$

where the slope is given by the bold number, and the subscript denotes its multiplicity as a zero of the characteristic series. We remind the reader that the computation for the non-trivial character relies on a number of unchecked assertions. As was pointed out by an anonymous referee, it is worth noting that as predicted by [LWX17, Theorem 1.5] the first sequence has period $(p-1)^2/2 = 8$, and is obtained from the second by adding 4. We compute that the unique eigenform f of slope 4 has the following Fourier coefficients, where we have normalised the form so that $a_1(\mathrm{t}) = 1$:

n	Fourier coefficient $a_n(\mathrm{t})$, for $\chi = \mathbf{1}$
2	$1+3\,\mathrm{t}+4\,\mathrm{t}^2+4\,\mathrm{t}^4+3\,\mathrm{t}^5+4\,\mathrm{t}^6+3\,\mathrm{t}^7+2\,\mathrm{t}^8+3\,\mathrm{t}^9+\mathrm{O}(\mathrm{t}^{10})$
3	$2+\mathrm{t}+4\,\mathrm{t}^2+\mathrm{t}^3+3\,\mathrm{t}^4+2\,\mathrm{t}^7+2\,\mathrm{t}^8+\mathrm{t}^9+\mathrm{O}(\mathrm{t}^{10})$
4	$3+4\,\mathrm{t}^2+4\,\mathrm{t}^3+4\,\mathrm{t}^4+3\,\mathrm{t}^5+4\,\mathrm{t}^6+2\,\mathrm{t}^7+\mathrm{t}^9+\mathrm{O}(\mathrm{t}^{10})$
5	$4\,\mathrm{t}^4+2\,\mathrm{t}^5+3\,\mathrm{t}^6+\mathrm{t}^9+\mathrm{O}(\mathrm{t}^{10})$
6	$2+2\,\mathrm{t}+2\,\mathrm{t}^3+3\,\mathrm{t}^5+4\,\mathrm{t}^6+3\,\mathrm{t}^7+\mathrm{t}^8+3\,\mathrm{t}^9+\mathrm{O}(\mathrm{t}^{10})$
7	$1+\mathrm{t}^5+3\,\mathrm{t}^6+4\,\mathrm{t}^7+\mathrm{t}^8+3\,\mathrm{t}^9+\mathrm{O}(\mathrm{t}^{10})$
8	$4\,\mathrm{t}+3\,\mathrm{t}^2+3\,\mathrm{t}^3+4\,\mathrm{t}^7+4\,\mathrm{t}^8+4\,\mathrm{t}^9+\mathrm{O}(\mathrm{t}^{10})$
9	$2+\mathrm{t}+4\,\mathrm{t}^3+3\,\mathrm{t}^4+4\,\mathrm{t}^5+4\,\mathrm{t}^7+\mathrm{t}^8+4\,\mathrm{t}^9+\mathrm{O}(\mathrm{t}^{10})$
10	$4\,\mathrm{t}^4+4\,\mathrm{t}^5+2\,\mathrm{t}^7+3\,\mathrm{t}^8+\mathrm{t}^9+\mathrm{O}(\mathrm{t}^{10})$

The Hecke algebra relations are visibly satisfied for the coefficients tabulated here, we see for instance that $a_2(\mathrm{t})a_3(\mathrm{t}) \equiv a_6(\mathrm{t}) \pmod{\mathrm{t}^{10}}$. All Hecke algebra relations were in fact satisfied for the first 200 coefficients, which we computed up to precision t^{100}. The author finds this a very convincing confirmation of the correctness of our algorithms, which solely involves the operator U_5.

Example 2. We compute that the operator U_{11} on $\mathrm{M}^\dagger(\mathrm{SL}_2(\mathbf{Z}))$ has slope sequence

$$\mathbf{0}_1, \mathbf{1}_1, \mathbf{2}_1, \mathbf{3}_1, \mathbf{4}_2, \mathbf{5}_1, \mathbf{6}_1, \mathbf{7}_1, \mathbf{9}_2, \ldots$$

We computed the eigenforms f, g of slopes 3 and 5, which have the same residual representations, but different first order deformations. The first few prime coefficients of f and g are:

n	$a_n(f)$	$a_n(g)$
2	$6+4\,\mathrm{t}+7\,\mathrm{t}^2+9\,\mathrm{t}^3+7\,\mathrm{t}^4+4\,\mathrm{t}^5+2\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$	$6+3\,\mathrm{t}+8\,\mathrm{t}^2+7\,\mathrm{t}^3+7\,\mathrm{t}^4+5\,\mathrm{t}^5+\mathrm{O}(\mathrm{t}^7)$
3	$8+8\,\mathrm{t}+7\,\mathrm{t}^2+10\,\mathrm{t}^3+4\,\mathrm{t}^4+8\,\mathrm{t}^5+10\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$	$8+5\,\mathrm{t}+6\,\mathrm{t}^2+10\,\mathrm{t}^3+8\,\mathrm{t}^4+\mathrm{t}^5+8\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$
5	$9+3\,\mathrm{t}^2+4\,\mathrm{t}^3+4\,\mathrm{t}^5+4\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$	$9+9\,\mathrm{t}+5\,\mathrm{t}^2+5\,\mathrm{t}^3+8\,\mathrm{t}^4+9\,\mathrm{t}^5+7\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$
7	$6+5\,\mathrm{t}+\mathrm{t}^2+7\,\mathrm{t}^3+\mathrm{t}^4+2\,\mathrm{t}^5+\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$	$6+5\,\mathrm{t}^2+3\,\mathrm{t}^3+6\,\mathrm{t}^4+2\,\mathrm{t}^5+4\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$
11	$\mathrm{t}^3+8\,\mathrm{t}^4+9\,\mathrm{t}^5+3\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$	$\mathrm{t}^5+8\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$
13	$6+10\,\mathrm{t}+5\,\mathrm{t}^2+6\,\mathrm{t}^3+2\,\mathrm{t}^4+7\,\mathrm{t}^5+6\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$	$6+10\,\mathrm{t}+\mathrm{t}^2+\mathrm{t}^3+2\,\mathrm{t}^4+7\,\mathrm{t}^5+3\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$
17	$1+10\,\mathrm{t}^2+3\,\mathrm{t}^3+4\,\mathrm{t}^4+5\,\mathrm{t}^5+10\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$	$1+9\,\mathrm{t}+8\,\mathrm{t}^2+5\,\mathrm{t}^3+2\,\mathrm{t}^4+6\,\mathrm{t}^5+9\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$
19	$9+4\,\mathrm{t}+9\,\mathrm{t}^3+9\,\mathrm{t}^4+8\,\mathrm{t}^5+3\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$	$9+8\,\mathrm{t}+4\,\mathrm{t}^2+2\,\mathrm{t}^4+4\,\mathrm{t}^5+8\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$
23	$2+8\,\mathrm{t}+6\,\mathrm{t}^2+7\,\mathrm{t}^3+6\,\mathrm{t}^4+5\,\mathrm{t}^5+9\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$	$2+3\,\mathrm{t}+7\,\mathrm{t}^2+10\,\mathrm{t}^3+2\,\mathrm{t}^4+4\,\mathrm{t}^5+9\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$
29	$6+8\,\mathrm{t}+6\,\mathrm{t}^2+\mathrm{t}^3+8\,\mathrm{t}^4+9\,\mathrm{t}^5+\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$	$6+9\,\mathrm{t}+7\,\mathrm{t}^2+7\,\mathrm{t}^3+8\,\mathrm{t}^4+7\,\mathrm{t}^5+4\,\mathrm{t}^6+\mathrm{O}(\mathrm{t}^7)$

3.3. **Limits of infinite t-adic slope.** It was proved in [BC05] by an explicit parametrisation of the relevant region in $X_0(2)$, which has genus 0, that when $k = 0$, the n-th slope of 2-adic eigenforms in $M_0^\dagger$ is equal to

$$v_2(\lambda_n) = 1 + 2v_2\left(\frac{(3n)!}{n!}\right),$$

with multiplicity 1. Calegari [Cal13] observes that if one lets φ_n be the 2-adic overconvergent eigenform in $M_0^\dagger$ of eigenvalue λ_n, then it seems that on the level of q-expansions we have

$$\varphi_{2^n} \longrightarrow \varphi_\infty = \sum_{n=1}^{\infty}\left(\sum_{2\nmid d|n}\frac{1}{d}\right)q^n,$$

which is the 2-depletion of the weight 0 Eisenstein series, a form of infinite slope. This has the flavour of the results of Coleman–Stein [CS04] on the approximation of infinite slope twists of finite slope eigenforms by sequences of finite slope eigenforms of various weights, except that the weight in

Calegari's observation *remains constant* throughout. Below, we will describe a similar phenomenon at the boundary. It is not hard to show that in the particular case we will consider, a convergent sequence of eigenforms must exist:

Proposition 3.1. *If all finite slope eigenforms $f_i \in \mathrm{M}^\dagger_\chi(N)$ are defined over $\mathbf{F}_p[\![\mathrm{t}]\!]$, then the sequence of q-expansions with the uniform p-adic topology $\{f_i(q)\}_i$ contains a convergent subsequence.*

Proof. Consider a modular representation $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_p)$ unramified outside Np. The functor from the category of complete local Noetherian $\mathbf{Z}_p$-algebras to the category of sets, which sends an object A to the set of continuous representations $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(A)$ such that ρ is unramified outside Np, and reduces to $\overline{\rho}$ modulo the maximal ideal, is represented by

$$\rho^\square : G_{\mathbf{Q}} \to \mathrm{GL}_2(R^\square_{\overline{\rho}}),$$

for a topologically finitely generated $\mathbf{Z}_p$-algebra $R^\square_{\overline{\rho}}$. This implies that the set of $\mathbf{F}_p[\![\mathrm{t}]\!]$-points on $\mathrm{Spf}\ R^\square_{\overline{\rho}}$ is profinite. Any eigenform $f \in \mathrm{M}^\dagger_\chi(N)$ whose associated representation reduces to $\overline{\rho}$ gives rise to a point of $\mathrm{Spf}\ R^\square_{\overline{\rho}}$, and as the number of residual representation unramified outside Np is finite, the compactness of the set of $\mathbf{F}_p[\![\mathrm{t}]\!]$-points forces the sequence of q-expansions of eigenforms in $\mathrm{M}^\dagger_\chi(N)$ to have a convergent subsequence. □

Experimentally, we observe even more: a strong form of continuity in the index set, and very rapid convergence. Let $p = 3$, then we compute that the t-adic slope sequence of U_3 on $\mathrm{M}^\dagger(\mathrm{SL}_2(\mathbf{Z}))$ is

$$\mathbf{0}_1, \mathbf{2}_1, \mathbf{4}_1, \mathbf{6}_1, \mathbf{8}_1, \mathbf{10}_1, \mathbf{12}_1, \mathbf{14}_1, \ldots$$

It was shown by Roe [Roe14] that indeed $v_3(\lambda_n) = 2n$, where $U_p f_n = \lambda_n \varphi_n$ is ordered by ascending slope. In fact, a computation of the first 12 eigenforms suggests that the leading coefficient is

$$\lambda_n = (-1)^n\, \mathrm{t}^{2n} + \mathrm{O}(\mathrm{t}^{2n+1}).$$

All φ_n are then (provably) defined over $\mathbf{F}_p[\![\mathrm{t}]\!]$, so that there certainly must be a subsequence of $(\varphi_n)_n$ that converges as a q-expansion. Our computations suggest something much more precise. For instance, we compute the following table:

n	$a_n(\varphi_0)$	$a_n(\varphi_1)$
2	$t + 2t^2 + 2t^3 + t^4 + 2t^5 + 2t^9 + O(t^{10})$	$2t + 2t^2 + 2t^4 + t^5 + 2t^6 + t^7 + t^8 + 2t^9 + O(t^{10})$
3	1	$2t^2 + 2t^4 + t^5 + t^6 + 2t^7 + O(t^{12})$
5	$2t + t^3 + t^4 + 2t^6 + 2t^7 + t^9 + O(t^{10})$	$t + 2t^2 + 2t^3 + t^6 + t^8 + O(t^{10})$
7	$2 + 2t + t^2 + 2t^3 + t^4 + 2t^5 + t^6 + 2t^7 + t^8 + O(t^{10})$	$2 + 2t + t^2 + 2t^4 + 2t^7 + 2t^8 + O(t^{10})$
11	$t + 2t^2 + t^9 + O(t^{10})$	$2t + 2t^2 + 2t^3 + t^5 + 2t^8 + O(t^{10})$
n	$a_n(\varphi_3)$	$a_n(\varphi_4)$
2	$2t + t^2 + t^3 + 2t^4 + t^5 + t^6 + t^7 + 2t^9 + O(t^{10})$	$t + t^2 + t^4 + 2t^5 + O(t^{10})$
3	$2t^6 + t^9 + 2t^{10} + 2t^{11} + t^{12} + 2t^{13} + t^{14} + t^{15}\, O(t^{16})$	$t^8 + t^{10} + t^{11} + 2t^{13} + 2t^{14} + 2t^{16} + O(t^{18})$
5	$t + 2t^3 + 2t^4 + 2t^7 + 2t^8 + t^9 + O(t^{10})$	$2t + t^2 + t^3 + O(t^{10})$
7	$2 + 2t + t^2 + 2t^3 + t^4 + 2t^5 + t^6 + 2t^9 + O(t^{10})$	$2 + 2t + t^2 + 2t^4 + t^8 + 2t^9 + O(t^{10})$
11	$2t + t^2 + t^6 + t^7 + t^9 + O(t^{10})$	$t + t^2 + t^3 + 2t^5 + 2t^6 + t^7 + 2t^8 + t^9 + O(t^{10})$
n	$a_n(\varphi_9)$	$a_n(\varphi_{10})$
2	$2t + t^2 + t^3 + 2t^4 + t^5 + t^9 + O(t^{10})$	$t + t^2 + t^4 + 2t^5 + t^6 + 2t^7 + 2t^8 + t^9 + O(t^{10})$
3	$2t^{18} + O(t^{19})$	$t^{20} + O(t^{21})$
5	$t + 2t^3 + 2t^4 + t^6 + t^7 + 2t^9 + O(t^{10})$	$2t + t^2 + t^3 + 2t^6 + 2t^8 + O(t^{10})$
7	$2 + 2t + t^2 + 2t^3 + t^4 + 2t^5 + t^6 + 2t^7 + t^8 + O(t^{10})$	$2 + 2t + t^2 + 2t^4 + 2t^7 + 2t^8 + O(t^{10})$
11	$2t + t^2 + 2t^9 + O(t^{10})$	$t + t^2 + t^3 + 2t^5 + t^8 + O(t^{10})$

As the reader may observe, this suggests, amongst other things, the continuity of the map $\mathbf{N} \to \mathrm{Spf}\ R_{\overline{\rho}}^{\square}$ at the boundary. We calculated the first 300 coefficients up to precision $O(t^{200})$, and compiled some experimentally observed congruences between the first 12 eigenforms below. We will use the notation $f^{(i)}$ for the twist of a form f by the i-th power of the Teichmüller character:

i	j	$v_t\left(\varphi_i^{(i)} - \varphi_j^{(j)}\right)$	i	j	$v_t\left(\varphi_i^{(i)} - \varphi_j^{(j)}\right)$	i	j	$v_t\left(\varphi_i^{(i)} - \varphi_j^{(j)}\right)$
0	1	2	1	2	2	2	3	2
0	2	2	1	3	2	2	4	2
0	3	6	1	4	6	2	5	6
0	4	2	1	5	2	2	6	2
0	5	2	1	6	2	2	7	2
0	6	6	1	7	6	2	8	6
0	7	2	1	8	2	2	9	2
0	8	2	1	9	2	2	10	2
0	9	18	1	10	18	2	11	18

This table suggests that any Teichmüller twist of any boundary eigenform, which is of infinite slope, arises as the limit of a convergent sequence of finite slope eigenforms at the boundary. Moreover, it is suggested by our data that perhaps the rate of convergence is linear, in the sense that

$$v_3(i-j) \geq n \quad \Rightarrow \quad \varphi_i^{(i)} \equiv \varphi_j^{(j)} \pmod{t^{2\cdot 3^n}}$$

We note that the simplicity of the expression $2\cdot 3^n$ is in stark contrast with the observations of Calegari in the p-adic case. Finally, one is led to speculate that similar infinite slope limits might exist even in Coleman families, if one considers the (p, t)-adic topology on q-expansions.

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Department of Mathematics and Statistics, Burnside Hall, 805 Sherbrooke Street West, Montreal, QC, Canada, H3A 0B9

E-mail address: jan.vonk@mcgill.ca