Character variety of finite upper-triangular matrices

Let Σ_g be an orientable, compact surface of genus g. The size of the set of group homomorphisms $\operatorname{Hom}(\pi_1(\Sigma_g), G)$ from the fundamental group of the surface to a FINITE group G satisfies an interesting identity.

Theorem 1. *(Counting formula). The classical counting formula gives*

$$\frac{|\operatorname{Hom}(\pi_1(\Sigma_g), G)|}{|G|} = \sum_{\lambda \in \hat{G}} \left(\frac{\dim \lambda}{|G|}\right)^{2-2g}$$

where $\hat{G}$ is the set of irreducible representations of G, $\dim \lambda$ is the dimension of the irreducible representation $\lambda \in \hat{G}$.

This theorem has interesting special cases. In the $g = 0$ case, we obtain the well-known formula in representation theory:

$$|G| = \sum_{\lambda \in \hat{G}} (\dim \lambda)^2.$$

Furthermore, in the $g = 1$ case, the formula is a special case of Burnside's lemma: we have that

$$\frac{|\{(g,h) \in G^2 | gh = hg\}|}{|G|}$$

equals the number of conjugacy classes. (Use Burnside's lemma when G acts on G by conjugation)

The goal of the project is twofold. First, you will need to prove this formula, second, you will need to compute the right-hand side of this formula for the groups of upper triangular matrices over finite fields, $U_n(\mathbb{F}_q)$.

References

[1] Motohico Mulase, Geometry of character varieties of surface groups