Topics in ANT

Lecture 10.

Last Time: Defined distributions

Let $G = \mathbb{Z}_p$ or $\mathbb{Z}_p^\times$.

$$LC(G, L) := \{ f : G \to L \text{ loc. constant} \}$$ L-vector space

$$\begin{aligned} \mathrm{Dist}(G, L) &:= LC(G, L)^\vee \\ &= \{ \mu : LC(G,L) \to L \text{ linear} \} \end{aligned}$$ dual space

Alternatively, have concrete description of distribution μ as a function

$$\mu : \{ U \subset G \text{ compact open} \} \to L$$

which are finite additive.

Ex. Haar distribution on $\mathbb{Z}_p$:

$$\mu_{\text{Haar}}(a + p^n \mathbb{Z}_p) = p^{-n}.$$

TODAY: Define and study measures.

$$\mathrm{Meas}(G,L) := \mathrm{Cont}(G,L)^\vee$$ continuous dual.

$$= \{\mu : \mathrm{Cont}(G,L) \to L \text{ linear } + \text{ continuous}\}.$$

Since $LC(G,L)$ is contained in (even dense!) $\mathrm{Cont}(G,L)$, we got that

$$\mathrm{Dist}(G,L) \supset \mathrm{Meas}(G,L).$$

Do we also have a concrete description of measures as functions on compact opens?

Yes! They are precisely those

$$\mu : \{U \subset G \text{ compact open}\} \to L$$

which are finite additive + bounded

i.e $$|\mu(U)| \leq C_\mu$$

fixed constant

for all U.

Suppose $\mu : \{ U \subset G \text{ comp. open} \} \to L$

is a distribution.

μ is a measure $\Rightarrow$ μ is bounded (Exercise)

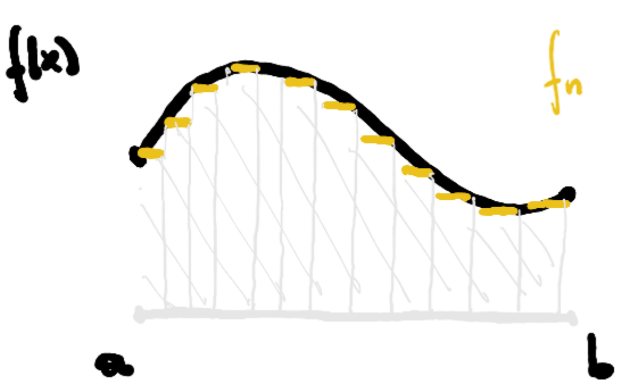
μ is bounded $\Rightarrow$ μ is a measure (Riemann sums)

Over $\mathbb{R}$: Know this idea!

Any $f : [a,b] \to \mathbb{R}$ continuous can be integrated

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

where f_n is a sequence of step functions converging to f in the sup norm.



Key in the proof:

f is uniformly continuous!

<u>Over</u> <u>p-adics</u>: Same idea!

We will define integral of $f : \mathbb{Z}_p \to L$ cont. by setting

$$\int_{\mathbb{Z}_p} f(x) \cdot \mu(x) := \lim_{n \to \infty} \int_{\mathbb{Z}_p} f_n(x) \cdot \mu(x)$$

where $f_n \in LC(\mathbb{Z}_p, L)$ converges to f in sup norm.

<u>Lemma</u>: $f \in \mathrm{Cont}(G, L)$
$\mu \in \mathrm{Dist}(G, L)$ bounded

then the Riemann sums

$$S_n := \sum_{a + p^n\mathbb{Z}_p \subset G} f(x_{a,n}) \cdot \mu(a + p^n\mathbb{Z}_p)$$

(where $x_{a,n} \in a + p^n\mathbb{Z}_p$ arbitrarily chosen)
converge to a limit as $n \to \infty$, independent of choices.

Pf: Choose $\varepsilon > 0$.

Let n be large enough, so that

$$|f(x) - f(y)| < \varepsilon$$

for all $x, y \in a + p^n\mathbb{Z}_p$, for any $a \in \mathbb{Z}_p$.

(Uniform continuity of f)

Let $m \geq n$, define S_m and S_n with respect to some choices of $x_{a,n}$ and $x_{a,m}$.

Then

$$|S_m - S_n| < \varepsilon \left| \sum_{a+p^m\mathbb{Z}_p \subset G} \mu(a+p^m\mathbb{Z}_p) \right|$$

$$\leq \varepsilon \cdot C_\mu .$$

where $|\mu(U)| \leq C_\mu$ for all U. $\square$

Example 1: Note that Haar distribution μ_{Haar} is **not** a measure!

<u>Example 2</u>: The Dirac distribution δ_a for $a \in G$, defined by

$$\int_G f(x) \cdot \delta_a(x) := f(a). \quad \in L.$$

<u>OR</u> alternatively,

$$\delta_a(U) = \begin{cases} 1 & \text{if} \quad a \in U \\ 0 & \text{else} \end{cases}$$

so δ_a is actually a measure.

<u>Remark</u>: It follows that

$$\mathrm{Meas}(G, L) = \mathrm{Meas}(G, \mathcal{O}_L) \otimes_{\mathcal{O}} L.$$

i.e suffices to study measures valued in $\mathcal{O}_L$.

§ Mahler transforms

A measure on $\mathbb{Z}_p$ specified by either

① The values $\mu(U)$ for all $U \underset{\cdot}{\subset} \mathbb{Z}_p$ comp. open

(Iwasawa algebra $\Lambda(\mathbb{Z}_p)$)

② The integrals $\int_{\mathbb{Z}_p} \binom{x}{n} \cdot \mu$, by Mahler's theorem

(Mahler transform, in $\mathcal{O}_L[[T]]$.)

A. Iwasawa algebra:

Suppose R ring

A group (finite)

then group ring $R[A]$ is R-module of formals sums

$$\sum_{g \in A} c_g \cdot [g], \qquad c_g \in R$$

with multiplization defined by $[g_1 g_2] =: [g_1] \cdot [g_2]$

The Iwasawa algebra is defined by

$$\Lambda(G) = \varprojlim_{U \text{ open subgroup}} \mathcal{O}_L[G/U]$$

e.g.: $G = \mathbb{Z}_p$:

$$\begin{array}{ccccccc} x_{n+1} \longrightarrow & x_n & \longrightarrow & \cdots & \longrightarrow & x_1 \\ & \cap & & & & \cap \\ & \mathcal{O}_L[\mathbb{Z}_p/p^n\mathbb{Z}_p] & & & & \mathcal{O}_L[\mathbb{Z}_p/p\mathbb{Z}_p] \end{array}$$

Suppose $\mu \in \mathrm{Meas}(G, \mathcal{O}_L)$, then get an associated element of $\Lambda(G)$ by

$$\mu \longmapsto \underbrace{\sum_{g \in G/U} \mu(g+U) \cdot [g]}_{\eta_U(\mu)} \quad \in \mathcal{O}_L[G/U]$$

Then ① Finite additivity of μ implies that when $U_1 \subset U_2$ we have

$$\begin{tikzcd}
\gamma_{U_1}(\mu) \arrow[r, dashed, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \gamma_{U_2}(\mu) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\mathcal{O}_L[G/U_1] \arrow[r] & \mathcal{O}_L[G/U_2]
\end{tikzcd}$$

so that we can really associate an element of the inverse limit to μ

i.e we constructed a map

$$\mathrm{Meas}(G, \mathcal{O}_L) \longrightarrow \Lambda(G).$$

② This map is an isomorphism of $\mathcal{O}_L$-modules (see notes).

<u>Note</u>: $\Lambda(G)$ is a ring!

This means we can multiply measures.

Concretely, one can check (exercises) that it is given by convolution $\mu_1 * \mu_2$

$$\int_G f \cdot (\mu_1 * \mu_2) = \int_G \left(\int_G f(x+y) \cdot \mu_2(y) \right) \cdot \mu_1(x)$$

B. <u>Mahler tranform</u>: Now $G = \mathbb{Z}_p$

Suppose $\mu \in \Lambda(\mathbb{Z}_p)$ is a measure.

Define its <u>Mahler transform</u> by

$$\mathcal{A}_\mu(T) := \int_{\mathbb{Z}_p} (1+T)^x \cdot \mu(x)$$

$$= \sum_{n \geq 0} \left(\int_{\mathbb{Z}_p} \binom{x}{n} \cdot \mu \right) T^n$$

This is a power series in $\mathcal{O}_L[[T]]$.

Thm: The Mahler transform defines an $\mathcal{O}_L$-algebra isomorphism

$$\Lambda(\mathbb{Z}_p) \longrightarrow \mathcal{O}_L[\![T]\!]$$
$$\mu \longmapsto \mathcal{A}_\mu(T)$$

Pf: $\mathcal{O}_L$-algebra morphism: Check

Injective : Mahler's theorem.

Surjective: For any $U \subset \mathbb{Z}_p$ open subgroup $a \in \mathbb{Z}_p/U$ then write

$$\mathbb{1}_{a+U} = \sum_{n \geq 0} b_{a,n} \binom{x}{n} \quad \text{(Mahler)}$$

The inverse of Mahler transform $\mathcal{A}_\mu$ is (check!)

$$a_0 + a_1 T + \dots \longrightarrow \sum_{a \in \mathbb{Z}_p/U} \underbrace{\Big(\sum_{n \geq 0} a_n b_{a,n} \Big)}_{\text{converges}} \cdot [a]$$

$$\in \mathcal{O}_L[\mathbb{Z}_p/U] \qquad \square$$

Very convenient! To define measures,
all we need is power series.

Almost ready to start doing number theory...