Topics in ANT

Lecture 5

TODAY: K complete discretely valued field.

Study finite extensions L/K

Will see that we can split up into

$$K \subset L_0 \subset L_1 \subset L$$

"unramified" "tamely ram." "wildly ram."

§1. Unramified extensions

All come from extensions of residue field.

Prop: K complete, residue field k.
L/K finite, residue field ext^n l/k separable.

There is a unique unramified subextension

$$L / L_0 / K$$

with residue field l/k.

Pf: Write $\ell = k(\bar{x})$.

Choose monic polynomial $f \in A_K[z]$ lifting the characteristic polynomial of $\bar{x}$ over k.

Then

- $f \mod m_K$ has simple root $\bar{x}$
- Lifts to simple root $x \in L$ by Hensel's lemma.
- Since f is irreducible over K, get $L_0 := K(x)$ is subextension of degree
$$\deg(L_0/K) = \deg(f) = \deg(\ell/k)$$
- Its residue field is ℓ/k, and
$$\deg(L_0/K) = e \cdot f = e \cdot \deg(\ell/k) = e \cdot \deg(L_0/K)$$

so L_0/K is unramified.

Any such subextension contains x, so L_0 must be unique

$\square$

Remark: The proof implies that construction of L_0/K is functorial in L/K (see notes) and the compositum of two unramified extensions is again unramified.

Get maximal unramified extension K^{unr}/K

$$K^{unr} := \bigcup_{\substack{F \subset K^{sep} \\ F/K \text{ unramified.}}} F$$

Suppose K is non-archimedean, $\overset{\text{local field}}{}$ then k is finite.

$\Rightarrow$ For every $n \geq 1$, there is a unique ext^n k_n/k of degree n.
It is Galois with group $\mathbb{Z}/n\mathbb{Z}$.

$\Rightarrow$ For every $n \geq 1$, there is unique unramified ext^n K_n/K of degree n.

It is Galois with group $\mathbb{Z}/n\mathbb{Z}$.

In particular, $K^{unr} := \bigcup_{n \geq 1} K_n$, have

$$\mathrm{Gal}(K^{unr}/K) \simeq \mathrm{Gal}(\bar{k}/k) \simeq \hat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$$

§2. Tamely ramified extensions

L/K fin ext. non-arch

Def: Say L/K is tamely ramified if

$$p = \mathrm{char}(k) \nmid e.$$

Theorem: K complete wrt discrete val.
L/K totally and tamely ramified.

Then there exists uniformiser ϖ of K s.t.

$$L = K(\sqrt[e]{\varpi}).$$

Proof: Pick uniformisers $\pi_L \in L$
$\pi_K \in K$

then $\pi_L^e = u \cdot \pi_K$, $u \in A_L^\times$

Since $l = k$, we can find $v \in A_K^\times$

s.t $u \equiv v \mod m_L$.

and therefore

$$X^e - \left(\frac{v\pi_K}{\pi_L^e}\right) \in A_L[X]$$

has a simple root α which reduces to 1 in ℓ. (Here we used $\mathrm{char}(k) \nmid e$)

It follows that $L = K(\pi_L)$

$$= K(\alpha \pi_L)$$

$$= K\left(\sqrt[e]{v \cdot \pi_K}\right)$$

(L is tot. ram., i.e. $\deg(L/K) = e$. Also $K(\pi_L)/K$ has ram index at least e.)

so theorem is proved, with

$$\varpi = v \cdot \pi_K \quad \text{uniformiser in } K \qquad \square$$

Examples: ① In exercise 2.15 we used

$$\mathbb{Q}_p^\times \simeq p^{\mathbb{Z}} \times \mathbb{Z}/_{(p-1)\mathbb{Z}} \times (1 + p\mathbb{Z}_p)$$

$$\simeq p^{\mathbb{Z}} \times \mathbb{Z}/_{(p-1)\mathbb{Z}} \times \mathbb{Z}_p$$

and deduced that if $p > 2$ we have precisely 3 quadratic extensions of $\mathbb{Q}_p$

$$\mathbb{Q}_{p^2} := \mathbb{Q}_p(\sqrt{\alpha}), \quad \mathbb{Q}_p(\sqrt{p}), \quad \mathbb{Q}_p(\sqrt{\alpha p})$$

where $\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2 = \{1, \alpha, p, \alpha p\}$.

② The extension $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p$ is totally and tamely ramified.

Can write

$$\mathbb{Q}_p(\zeta_p) = \mathbb{Q}_p\left(\sqrt[p-1]{-p}\right) \quad \text{(Check this!)}$$

§3. Wildly ramified extensions

L/K finite non-arch.

Def: Say L/K is <u>wildly ramified</u> if $p = \mathrm{char}(k) \mid e$

Recall that a polynomial $f \in A_K[x]$ is Eisenstein if it is of form

$$f = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_i \in \mathfrak{m}_K$

$a_0 \notin \mathfrak{m}_K^2$.

We know that Eisenstein polynomials are irred.

Prop: K complete wrt discrete valuation

Then ① L/K totally ramified
then $L = K(\pi_L)$ for any unif. $\pi_L \in L$
and char poly of π_L is Eisenstein.

② Every root of an Eisenstein polynomial over K generates a totally ramified extension.

Pf: For ① note that $K(\pi_L)/K$ is totally ramified of degree $e = \deg(L/K)$, so $L = K(\pi_L)$.

Every root of f = char. polynomial π_L in normal closure has same valuation as π_L,
so by ultrametric inequality

$$f = \prod (x - \pi_L^{\sigma})$$

$$= x^e + a_{e-1}x^{e-1} + \cdots + a_1 x + a_0$$

must be contained in m_K.

The constant term

$$a_0 = (-1)^e \cdot \mathrm{Nm}_{L/K}(\pi_L)$$

has valuation $\varphi(\pi_L)^e = \varphi(\pi_K)$.

Conversely, the valuation of the root α of an Eisenstein polynomial of degree e is

$$\varphi\left(\mathrm{Nm}_{K(\alpha)/K}(\alpha)\right)^{1/e} = \varphi(a_0)^{1/e}$$

so $K(\alpha)/K$ has ramification index e. $\square$

§4. Krasner's Lemma

We saw there are precisely 3 extensions of $\mathbb{Q}_p$ of degree 2, when $p>2$.

K finite extension of $\mathbb{Q}_p$.

Lemma: (Krasner)

Let $\alpha \in \overline{K}$ with Galois conjugates

$$\alpha_1 = \alpha, \quad \alpha_2, \dots, \alpha_n.$$

If an element $\beta \in \overline{K}$ is such that

$$\varphi(\alpha - \beta) < \varphi(\alpha_i - \beta), \quad \forall i > 1.$$

then $K(\alpha) \subseteq K(\beta)$.

Pf: Exercise.

It follows from this lemma that two monic polynomials of the same degree have same splitting field over K if their coefficients are close enough.

Thm: There are only finitely many L/K of degree n inside $\overline{\mathbb{Q}}_p$.

Pf: True for unramified extensions (unique!) so may assume L/K is totally ramified.

$$L = K(\pi_L)$$

where π_L is a uniformiser, with char. polynomial

$$f = X^e + a_{e-1}X^{e-1} + \dots + a_1 X + a_0$$

Then f is Eisenstein!

The coefficient vector $(a_0, \dots, a_{e-1}) \in \overbrace{m_K \times \dots \times m_K}^{e \text{ times}}$ determines L/K. Moreover the equivalence classes of

$(a_0, \dots, a_{e-1}) \sim (b_0, \dots, b_{e-1})$ iff their assoc. extensions coincide

are open by Krasner's Lemma.

Since m_K, and hence $m_K \times \dots \times m_K$ is compact there are finitely many equivalence classes, and hence finitely many extensions L/K. $\square$

Argument is topological! Not very constructive.
Suppose we want to know how many extensions of $\mathbb{Q}_p$ of degree n there are.

Then

a) ask Alex Bartel

b) read Krasner's papers

c) read about Serre's mass formula (notes).