# Topics in ANT

**Lecture 8.**

Next few weeks: Bring Euler & Kummer into $20^{th}$ century

- Riemann zeta function $\zeta(s) = \sum_{n\geq 1} n^{-s}$ $\quad Re(s)>1$

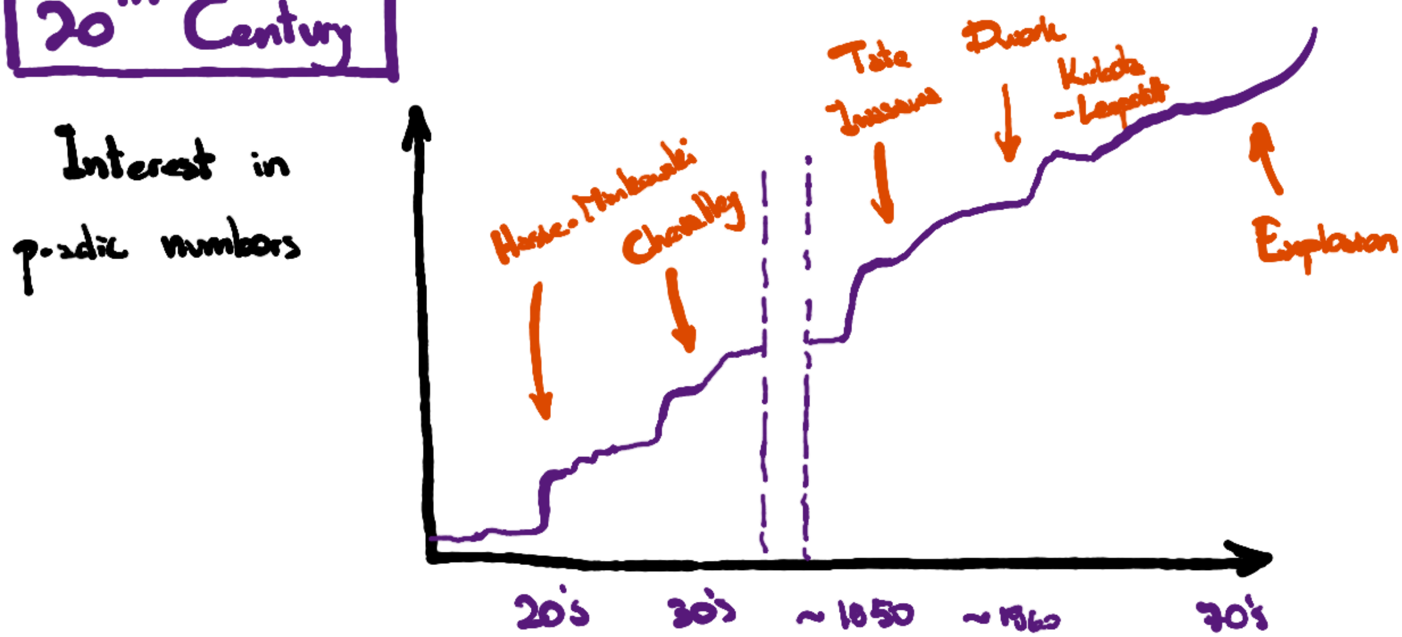
  special values $\zeta(1-2n) = -B_{2n}/2n$

- Kubota-Leopoldt zeta function $\zeta_p(s) = ??$

  special values $\zeta_p(1-2n) = \int_{\mathbb{Z}_p^\times} x^{2n} \cdot \zeta_p$

  $$= -(1-p^{2n-1})\frac{B_{2n}}{2n}$$

## $\boxed{20^{th}\text{ Century}}$



# § Foundations of p-adic analysis

$L / \mathbb{Q}_p$ finite.

Analysis over p-adics is often much more elegant than over $\mathbb{R}$!

1) An infinite series $\sum a_i$ converges in $L$
$\iff \quad a_i \to 0$

2) For any rearrangement $\{a_i^?\}$ of $\{a_i\}$ we have

$$\sum a_i^? = \sum a_i \qquad \text{if } a_i \to 0$$

3) If $b_{mn} \in L$ s.t $b_{mn} \to 0$ as $\max\{m,n\} \to \infty$

then

$$\sum_m \sum_n b_{mn} = \sum_n \sum_m b_{mn}$$

Will study continuous functions $f : \mathbb{Z}_p \to L$.
Have two simple, but very helpful, properties:

<u>A</u>. <u>$\mathbb{Z}_p$ is compact</u>: This implies that any continuous function $f : \mathbb{Z}_p \to L$ is automatically uniformly continuous i.e

$$\forall \varepsilon > 0, \ \exists \delta > 0$$

$$\text{s.t} \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Compare with theorem of Heine-Borel for continuous $f : [a, b] \to \mathbb{R}$.

<u>B</u>. <u>$\mathbb{N} \subset \mathbb{Z}_p$ is dense</u>: Allows us to construct lots of examples of continuous functions $f : \mathbb{Z}_p \to L$ "by interpolation"

<u>Lemma</u>: Suppose $f : \mathbb{N} \to L$ is uniformly cont. then $f$ uniquely extends to continuous $\mathbb{Z}_p \to L$.

For $n \in \mathbb{N}$ define polynomial function

$$\binom{x}{n} := \begin{cases} 1 & \text{if } n = 0 \\ \dfrac{x(x-1)\dots(x-n+1)}{n!} & \text{if } n \geq 1 \end{cases}$$

continuous function of $x \in \mathbb{Z}_p$.

<u>Theorem</u> (Mahler)

Let $f : \mathbb{Z}_p \to L$ be continuous, then there exist unique $a_n \in L$, $a_n \to 0$ s.t

$$f(x) = \sum_{n \geq 0} a_n \binom{x}{n} .$$

Pf: Define finite differences $\Delta^{[n]} f(x)$ by

$$\begin{cases} \Delta^{[0]} f(x) = f(x) \\ \Delta^{[n+1]} f(x) = \Delta^{[n]} f(x+1) - \Delta^{[n]} f(x) \end{cases}$$

$\underset{\sim}{n \geq 0}$

then one can show that

$$\Delta^{[n]} f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k)$$

$$= \sum_{j=0}^{m} \binom{m}{j} \Delta^{[n+j]} f(x-m)$$

$$n \geq 0$$

which implies (by setting $x = m$) that

$$(*) \quad \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k+m) = \sum_{j=0}^{m} \binom{m}{j} a_{n+j}$$

where $a_i := \Delta^{[i]} f(0)$.

Suffices to show that

I. $\sum a_n \binom{x}{n}$ converges to cont. fn on $\mathbb{Z}_p$

II. It agrees with $f(x)$ on a dense subset of $\mathbb{Z}_p$

To prove I, note that $f$ is uniformly continuous so for any $s \geq 1$, there exists $t \geq 1$ s.t

$$|x-y| \leq p^{-t} \Rightarrow |f(x)-f(y)| \leq p^{-s}$$

Setting $m = p^t$ in (*) we find

$$a_{n+p^t} = -\sum_{j=1}^{p^t-1} \underbrace{\binom{p^t}{j}}_{\equiv 0 \pmod p} a_{n+j}$$

$$+ \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \Big( \underbrace{f(k+p^t) - f(k)}_{|\cdot| \leq p^{-s}} \Big).$$

so $|a_{n+p^t}| \leq \max\left\{ \frac{1}{p}|a_n|, \frac{1}{p}|a_{n+1}|, \dots, \frac{1}{p^s} \right\}$

Can assume WLOG $f$ takes values in $\mathbb{Z}_p$,

so that
$$|a_n| \le 1/p \qquad n \ge p^t$$
$$|a_n| \le 1/p^2 \qquad n \ge 2p^t$$
$$\vdots$$
$$|a_n| \le 1/p^s \qquad n \ge sp^t$$

To prove II, note that (*) implies
$$f(m) = \sum_{j=0}^{m} a_j \binom{m}{j} \qquad \square.$$

Denote space of continuous functions $f: \mathbb{Z}_p \to L$ by

$$\mathrm{Cont}(\mathbb{Z}_p, L).$$

Has supremum norm

$$\| f \| := \sup_{x \in \mathbb{Z}_p} | f(x) |$$

which satisfies

$$\begin{cases} \| f \| \geq 0 , & \text{equality} \Leftrightarrow f = 0 \\ \| f + g \| \leq \max \{ \| f \|, \| g \| \} \\ \| \lambda f \| = | \lambda | \cdot \| f \| \end{cases}$$

+ $\mathrm{Cont}(\mathbb{Z}_p, L)$ is complete wrt supremum norm

( People say : Banach space )

The supremum norm is related to Mahler coefficients

$$\| f \| = \sup_{n \geq 0} | a_n |$$

(Exercise)

We say a function $f: \mathbb{Z}_p \longrightarrow L$ is analytic if it is of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \in L[[x]]$$

when $a_i \to 0$ this always converges to continuous function on $\mathbb{Z}_p$, but not every cont. function is analytic!

An analytic function has domain much larger than $\mathbb{Z}_p$! What exactly?

(Exercises)

$$\begin{cases} \overline{\mathbb{Q}}_p & \text{alg closure of } \mathbb{Q}_p\text{, } \underline{\text{NOT}} \text{ complete} \\ \mathbb{C}_p & \text{completion of } \overline{\mathbb{Q}}_p\text{, algebraically closed!} \end{cases}$$

__Lemma__: $\mathbb{C}_p$ is alg. closed

Pf: Suppose $F(x) \in \mathbb{C}_p[x]$ roots $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$

Choose $G(x) \in \overline{\mathbb{Q}}_p[x]$ roots $\beta_1, \beta_2, \ldots, \beta_n$.

__close to__ $F(x)$

Since $G(\alpha)$ is very small, we must have for some $i$ :

$$|\alpha - \beta_i| \text{ small } < |\alpha - \alpha_j| \quad \forall j > 1.$$

Krasner implies $\mathbb{C}_p(\alpha) \subset \mathbb{C}_p(\beta_i) = \mathbb{C}_p$ $\square$

The radius of convergence $R$ of an an analytic function

$$f(x) = a_0 + a_1 x + \cdots \quad \in \quad L[[x]]$$

is defined by

$$1/R = \lim_n \sup |a_n|^{1/n}$$

where $0 \le R \le +\infty$, then $f(x)$ converges on $x \in \mathbb{C}_p$ if and only if

$$\begin{cases} |x| \le R & \text{if } |a_n| R^n \to 0 \\ |x| < R & \text{otherwise} \end{cases}$$

**Example 1:** The p-adic logarithm

$$\log_p(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

has radius of convergence is $R=1$, and it does not converge anywhere on the boundary.

Can extend it to a function

$$\log_p : \mathbb{C}_p^\times \longrightarrow \mathbb{C}_p$$

s.t $\begin{cases} \log_p(xy) = \log_p(x) + \log_p(y) \\ \log_p(p) = 0 \end{cases}$

(Iwasawa branch of the logarithm)

**Example 2**: The p-adic exponential

$$\exp_p(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Trickier radius of convergence!
How fast does $\frac{1}{i!}$ grow p-adically?

$$\operatorname{ord}_p(i!) = \left\lfloor \frac{i}{p} \right\rfloor + \left\lfloor \frac{i}{p^2} \right\rfloor + \left\lfloor \frac{i}{p^3} \right\rfloor + \cdots$$

$$< \frac{i}{p-1}$$

Also, when $p^a \leq i < p^{a+1}$ we have

$$\left\lfloor \frac{i}{p} \right\rfloor + \left\lfloor \frac{i}{p^2} \right\rfloor + \cdots > \frac{i}{p} + \cdots + \frac{i}{p^a} - a$$

$$= \frac{i}{p-1} - a - \frac{ip^{-a}}{p-1}$$

$$> \frac{i-p}{p-1} - \frac{\log(i)}{\log(p)}.$$

so $\operatorname{ord}_p(i!) \sim \frac{i}{p-1}$, so $R = p^{-1/p-1} < 1$

**Prop:** If $|x| < p^{-1/p-1}$

then

$$\begin{cases} \log_p(\exp_p(x)) = x \\ \exp_p(\log_p(1+x)) = 1+x \end{cases}$$

**Example 3:** Power functions (p odd)

Recall Teichmüller rep's give isomorphism

$$\mathbb{Z}_p^\times \simeq \mu_{p-1} \times (1+p\mathbb{Z}_p)$$

which gives projections

$$\begin{cases} \omega : \mathbb{Z}_p^\times \longrightarrow \mu_{p-1} \\ \langle \cdot \rangle : \mathbb{Z}_p^\times \longrightarrow 1+p\mathbb{Z}_p \end{cases}$$

i.e $\omega(x) :=$ Teichmüller lift of $x \bmod p$

$\langle x \rangle := x \cdot \omega(x)^{-1}$.

Then have, for every $a \in \mathbb{Z}_p^*$

$$s \longmapsto \langle a \rangle^s := \exp_p\left(s \cdot \log_p(\langle a \rangle)\right)$$

Converges on $|s| < 1$.

<u>Conclusion:</u> Have 2 spaces

$$\text{Analytic functions} \subset \text{Cont}(\mathbb{Z}_p, L)$$

Very structured!
Study convergence
+ zeroes (next time, Newton polygons)

Very large!
Its dual space is tremendously important.