Topics in ANT

LECTURE 9.

Last Time: $L \subset \mathbb{C}_p$ complete

$$\{\text{Continuous } f: \mathbb{Z}_p \to L\} \supset \{\text{Analytic } f: \mathbb{Z}_p \to L\}$$

Mahler's thm

Radius of convergence R
Zeroes.

Newton polygons:

Useful tool! Get information about zeroes of polynomials / power series from unreasonably small amount of work.

1. Polynomials

Let $f = a_n x^n + \cdots + a_1 x + a_0 \in K[x]$

$K \subset \mathbb{C}_p$ compl.

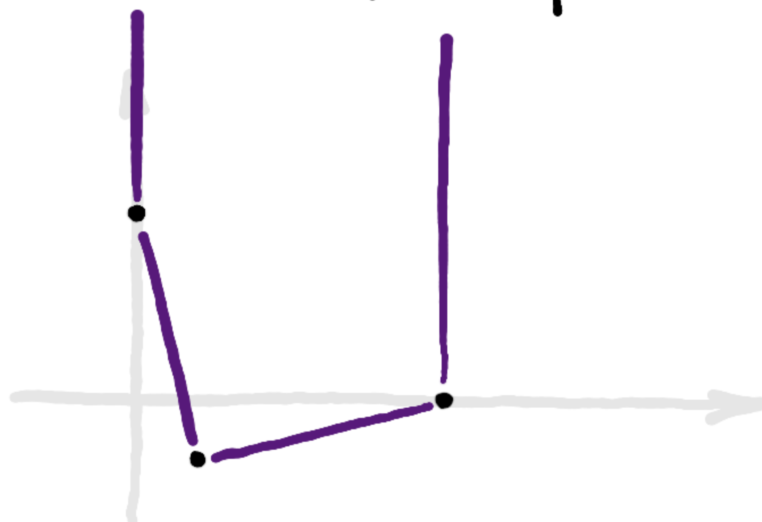
Its Newton polygon $NP(f)$ is the lower convex hull of the set

$$S = \{(i, \operatorname{ord}_p(a_i)) : i = 0, 1, \ldots, n\}$$

$\subset \mathbb{R}^2.$

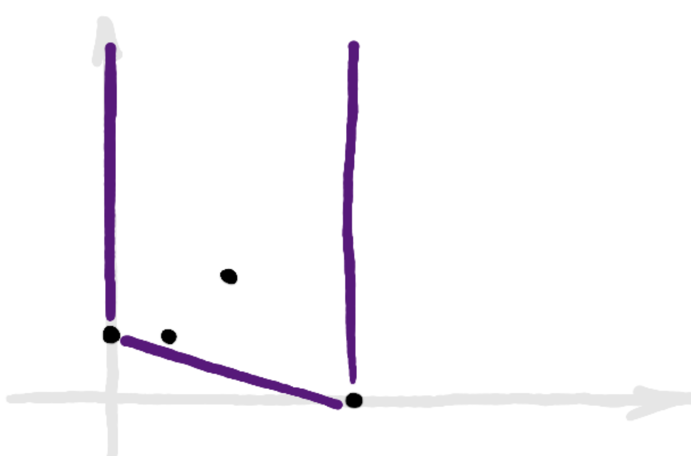
Example 1: $f = -p^3 + \frac{x}{p} + x^5$.

NP(f)



Example 2: $f = x^5 + p^2x^2 + px - p$.

NP(f)



Eisenstein polynomial.

Every segment of the Newton polygon has

$$\begin{cases} \text{slope } \lambda & : \quad \text{Lies on } y = \lambda x + c \\ \text{multiplicity } m_\lambda & : \quad \text{Length of projection to } x\text{-axis.} \end{cases}$$

Example 1: Slopes $\lambda = -4$ mult $m_\lambda = 1$
$= 1/4$ $= 4$.

Example 2: Slopes $\lambda = -1/5$ mult $m_\lambda = 5$.

Thm: Let $f \in K[x]$, write $f = a_n \prod_{\lambda \in \mathbb{Q}} f_\lambda$

where

$$f_\lambda = \prod_{\substack{f(\alpha) = 0 \\ -\mathrm{ord}_p(\alpha) = \lambda}} (x - \alpha)$$

Then ① $f_\lambda \in K[x]$

② $\deg f_\lambda = m_\lambda$, multiplicity of λ as a slope of $NP(f)$.

Pf: ① The action of $\mathrm{Gal}(\overline{K}/K)$ preserves valuations so it permutes roots of each f_λ

$\Rightarrow$ f_λ is defined over K.

② Assume WLOG

$$f = 1 + a_1 x + \cdots + a_n x^n$$

$$= \prod_{i=1}^{n} (1 + \beta_i x)$$

labelled such that

$$\mathrm{ord}_p(\beta_1) \leq \cdots \leq \mathrm{ord}_p(\beta_n)$$

and

$$\begin{aligned}
\mathrm{ord}_p(\beta_1) = \cdots &= \mathrm{ord}_p(\beta_{\kappa_1}) = \nu_1 \\
&\qquad\qquad\qquad\quad \wedge \\
\mathrm{ord}_p(\beta_{\kappa_1+1}) = \cdots &= \mathrm{ord}_p(\beta_{\kappa_1+\kappa_2}) = \nu_2 \\
&\qquad\qquad\qquad\quad \vdots \\
&\qquad\qquad\qquad\quad \wedge \\
\mathrm{ord}_p(\beta_{\kappa_1+\cdots+\kappa_{\ell-1}+1}) = \cdots &= \mathrm{ord}_p(\beta_{\kappa_1+\cdots+\kappa_\ell}) = \nu_\ell
\end{aligned}$$

We have

$$(*) \qquad a_s = \sum_{1 \leq i_1 < \cdots < i_s \leq n} \beta_{i_1} \cdots \beta_{i_s} \qquad 1 \leq s \leq n$$

By ultrametric inequality,

A. If $s = \kappa_1 + \kappa_2 + \cdots + \kappa_\rho$, some $1 \leq \rho \leq l$

then

$$\begin{aligned} \operatorname{ord}_p(a_s) &= \operatorname{ord}_p(\beta_1 \cdots \beta_s) \\ &= \nu_1 \kappa_1 + \nu_2 \kappa_2 + \cdots + \nu_\rho \kappa_\rho \end{aligned}$$

B. If $\kappa_1 + \cdots + \kappa_\rho < s < \kappa_1 + \cdots + \kappa_{\rho+1}$ some $0 \leq \rho \leq l$

then

$$\begin{aligned} \operatorname{ord}_p(a_s) &\geq \operatorname{ord}_p(\beta_1 \cdots \beta_s) \\ &= \nu_1 \kappa_1 + \nu_2 \kappa_2 + \cdots + \nu_\rho \kappa_\rho \\ &\quad + \nu_{\rho+1}(s - \kappa_1 - \kappa_2 - \cdots - \kappa_\rho) \end{aligned}$$

In case A. we have $\underbrace{(s, \nu_1 \kappa_1 + \cdots + \nu_\rho \kappa_\rho)}_{P_\rho} \in S$

$0 \leq \rho \leq n$

Set defining $NP(f)$

The inequality for $\mathrm{ord}_p(a_s)$ in case B is equivalent to saying that $(s, \mathrm{ord}_p(a_s))$ lies on or above line segment $[P_\ell, P_{\ell+1}]$. $\square$

2. Power series

Newton polygons also work for power series! Let

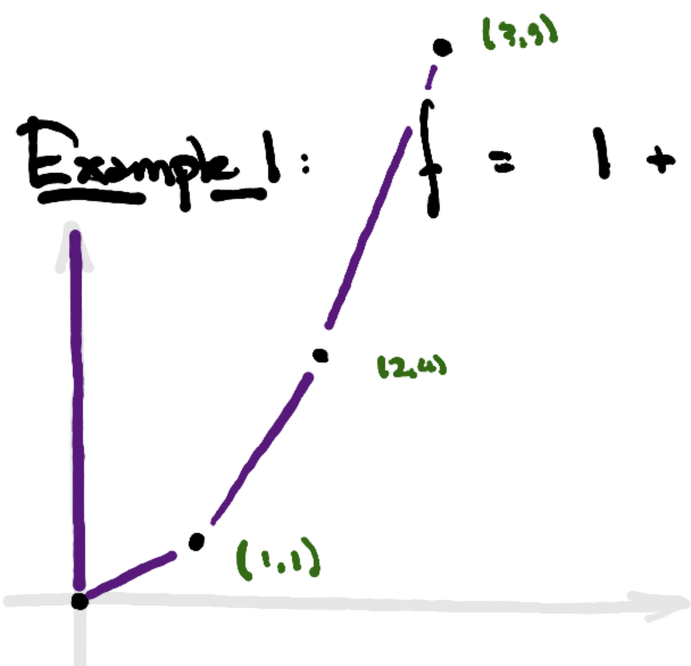
$$f = a_0 + a_1 x + \cdots \quad \in \quad K[\![x]\!]$$

then its Newton polygon $NP(f)$ is the lower convex hull of the set

$$S = \{ (i, \mathrm{ord}_p(a_i)) : \quad i = 0, 1, \ldots \} \subset \mathbb{R}^2 .$$

<u>Example 1</u>:

$$f = 1 + px + p^4 x^2 + \cdots + p^{n^2} x^n + \cdots$$

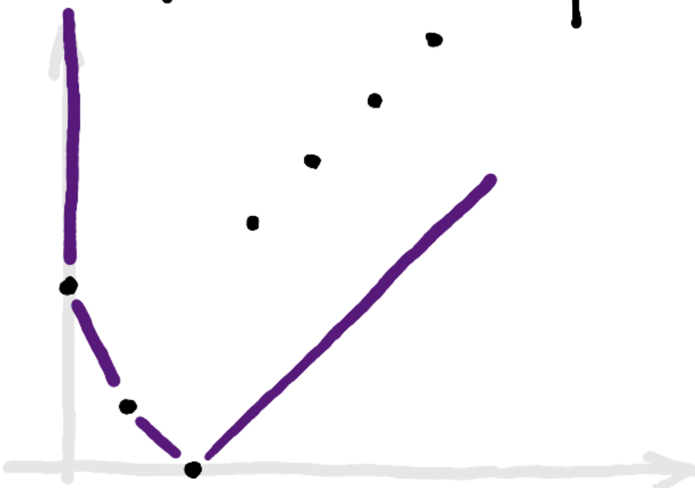


Has finite slopes

$$\lambda = 1, 3, 5, 7, \ldots$$

$$m_\lambda = 1, 1, 1, 1, \ldots$$

Example 2: $f = p^3 + px + x^2 + p^4 x^3 + \dots + p^{n+1} x^n + \dots$



Has slopes

$\lambda = -2, -1, 1$

$m_\lambda = 1, 1, \infty$

Thm: Suppose λ = slope of $NP(f)$

m_λ = multiplicity $< \infty$.

Then $f(x) = P(x) \cdot Q(x)$, with

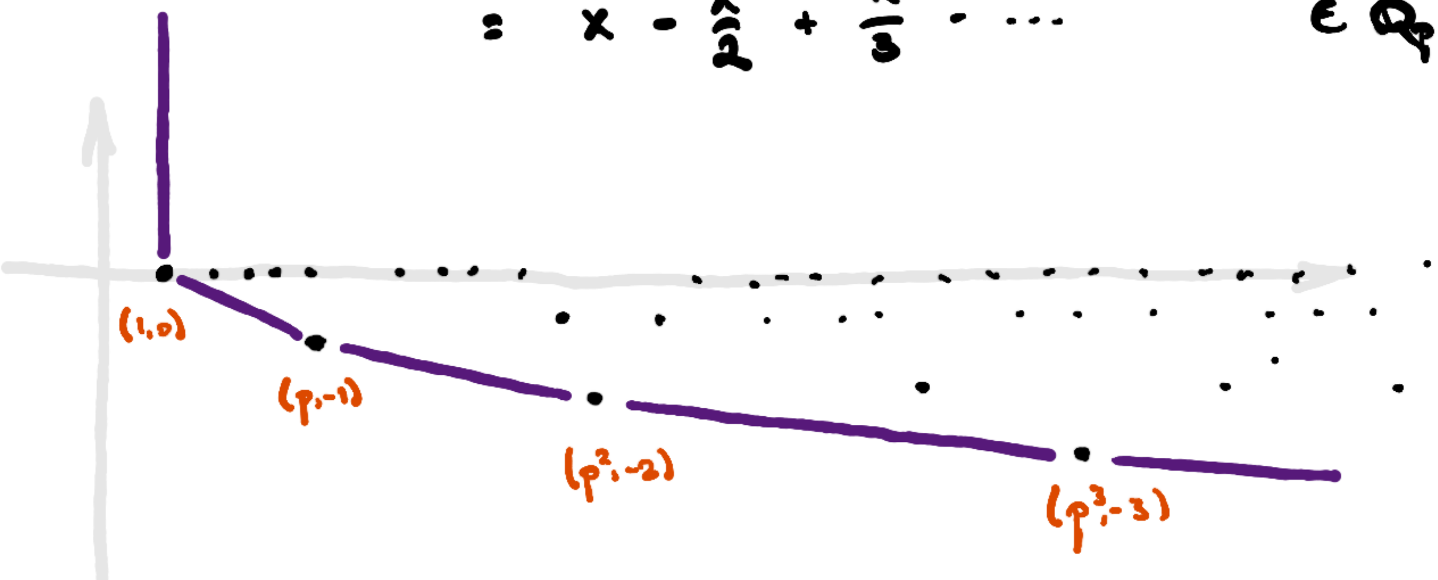
① $P(x) \in K[x]$ degree $= m_\lambda$

roots α_i of $\mathrm{ord}_p(\alpha_i) = -\lambda$.

② $Q(x)$ has no zeroes of $\mathrm{ord}_p = -\lambda$ $\square$

Example: $f = \log_p(1+x)$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \in \mathbb{Q}_p[[x]]$$



Slopes $\lambda = \infty, \ \frac{-1}{p-1}, \ \frac{-1}{p^2-p}, \ \frac{-1}{p^3-p^2}, \ \ldots$

$m_\lambda = 1, \ p-1, \ p^2-p, \ p^3-p^2$

Zeroes $x = 0, \ \zeta_p^i - 1, \ \zeta_{p^2}^i - 1, \ \ldots$

$i \in (\mathbb{Z}/p\mathbb{Z})^\times \quad i \in (\mathbb{Z}/p^2\mathbb{Z})^\times$

Corollary: (Weierstraß preparation)

Suppose K is discretely valued

ϖ = uniformiser

$\mathcal{O}_K$ = valuation ring

Let $f \in \mathcal{O}_K[\![x]\!]$ non-zero, then we can uniquely write

$$f(x) = \varpi^{\mu} \cdot P(x) \cdot Q(x)$$

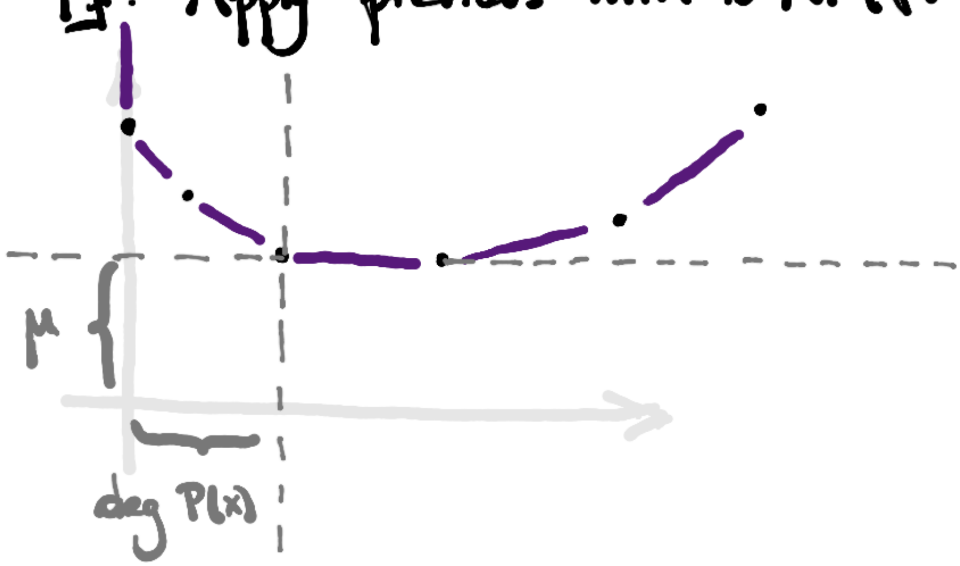
with

$\mu \geq 0$ integer

"Distinguished polynomial" $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$, $\operatorname{ord}_p(a_i) > 0$

$Q(x)$ = unit, i.e in $\mathcal{O}_K[\![x]\!]^{\times}$.

Pf: Apply previous thm to $NP(f)$



$P(x)$ = product of polynomials attached to all negative slopes.

Chapter 3: Distributions & Measures

Let $G = \mathbb{Z}_p$ or $\mathbb{Z}_p^\times$, and $L \subset \mathbb{C}_p$ complete.
Will study dual of $\mathrm{Cont}(G, L)$, the space of measures.

§1. Distributions

Define L-vector space

$$LC(G,L) = \{ f : G \to L \text{ locally constant} \}$$

$$\mathrm{Dist}(G,L) = LC(G,L)^\vee$$

$$= \{ \mu : LC(G,L) \to L \text{ linear} \}$$

Elements $\mu \in \mathrm{Dist}(G,L)$ are called distributions.
We denote

$$\int_G f(x)\cdot \mu(x) = \int_G f \cdot \mu := \mu(f).$$

$f \in LC(G,L)$

Since G is compact, any locally constant function is a <u>finite</u> linear combination of characterist fns

$$\mathbb{1}_U(x) := \begin{cases} 0 & \text{if } x \notin U \\ 1 & \text{if } x \in U \end{cases}$$

with U compact open.

Therefore any distribution is determined by the function

$$\mu(U) := \mu(\mathbb{1}_U) \qquad U \text{ compact open.}$$

which is finite additive, i.e if U = disjoint union of $U_1, \dots, U_k$, then

$$\mu(U) = \mu(U_1) + \dots + \mu(U_k).$$

Conversely, any such function on compact opens determines a distribution!

<u>Example 1</u>: Dirac distribution δ_a, $a \in G$

$$\int_G f \cdot \delta_a = f(a)$$

Example 2: Haar distribution $\mu_{Haar} \in Dist(\mathbb{Z}_p, \mathbb{Q}_p)$

$$\mu_{Haar}(a + p^n \mathbb{Z}_p) = p^{-n}.$$

Example 3: Mazur distribution $\mu_{Mazur} \in Dist(\mathbb{Z}_p, \mathbb{Q})$

$$\mu_{Mazur}(a + p^n \mathbb{Z}_p) = \frac{a}{p^n} - \frac{1}{2}$$

$a = 0, 1, \dots, p^n - 1$

§2. Measures

Define space of measures $Meas(G,L)$ to be the continuous dual of $Cont(G,L)$, i.e

$$\mu : Cont(G,L) \to L \quad \text{continuous.}$$

Since $$LC(G,L) \overset{\text{dense}}{\subset} Cont(G,L)$$

we get $$Dist(G,L) \supset Meas(G,L)$$

Example 1: The Dirac distribution is actually a measure!