Systolic inequalities on surfaces

Bachelor project, supervisor: F.Pasquotto

Suppose Σ is a surface. A Riemannian metric on Σ is a suitable choice of an inner product on the tangent space at each point of Σ: it enables us to speak about length, area, and *curvature*. Understanding $\mathcal{R}(\Sigma)$, the space of Riemannian metrics on Σ, is an interesting and challenging problem. One possible approach to this problem is to measure and compare the size of different objects with respect to elements $g \in \mathcal{R}(\Sigma)$.

At the end of the 19th century, Hadamard proved that if Σ is not simply connected ($\Sigma \neq S^2$), for any metric $g \in \mathcal{R}(\Sigma)$ there exists a non-contractible loop γ_g of minimal length. Such a curve is called a *systole* of g and its length is denoted by $\mathrm{sys}(g)$: it gives us one way to measure the "size" of a surface.

Roughly speaking, the natural question which arises is the following: can we have long systoles on a small surface? Of course, in order to make the question precise, we need to define what we mean by *small* surface! For instance, if we refer to the diameter of the surface, it is easy to see that the following inequality must hold: $\mathrm{sys}(g) \leq 2\mathrm{diam}_g(\Sigma)$, so we cannot have long systoles on surfaces with small diameter. But what is the relationship between $\mathrm{sys}(g)$ and the total area of Σ, calculated with respect to g? This leads us to the following notion: for all $g \in \mathcal{R}(\Sigma)$ we define the *systolic ratio* to be

$$\sigma(g) = \frac{\mathrm{sys}(g)}{\mathrm{area}_g(\Sigma)}$$

and ask ourselves questions such as:

- Is $\sigma(g)$ bounded for all $g \in \mathcal{R}(\Sigma)$?
- Does there exist $g \in \mathcal{R}(\Sigma)$ maximizing $\sigma(g)$?

The answer to both questions is for instance affirmative in the case of flat tori.

We will start the project by learning about geometry of surfaces (metric, curvature, geodesics). The ultimate goal of the project is to discuss existence of upper bounds for the systolic ratio on surfaces of different genus, and in the case of the torus construct the metric which maximizes this ratio.

Literature:

- J. Stillwell, *Geometry of surfaces*, Springer (1992)
- M. Berger, *What is... a systole?*, Notices of the AMS 55 (2008)
- G. Benedetti, *First steps into the world of systolic inequalities*, notes written for the DDT&G seminar (2020)