Classifying curves of genus two

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Smooth algebraic curves appear naturally in several branches of Mathematics and they can be considered under quite a few different points of view. A classical example is a (complex) torus or genus one curve, which one can visualize as the shape of the surface of a bagel or a doughnut. More generally, one can talk about curves of any genus $g \in \mathbb{Z}_{\geq 0}$, where the genus of a curve is defined as the maximal number of linearly independent (l.i.) one-forms on the curve, or l.i. holomorphic sections of the canonical bundle.

Since two curves of different genera cannot be isomorphic, we see that the genus is an important invariant of the curve. In particular, it makes sense to ask: Can we construct a space $\mathcal{M}_g$ whose points are in bijection with isomorphism classes of curves of genus g?

In fact we can, and the spaces $\mathcal{M}_g$ are one of the most widely studied objects in modern Algebraic Geometry. They are one of the most relevant examples of moduli spaces.

A classical construction is that of $\mathcal{M}_1$, which parameterizes (smooth and complete) genus one curves. Every such curve is completely determined by an invariant, called the $j-$invariant, which takes values in the complex numbers. Because every complex number appears as a possible invariant the moduli space is an entire complex line.

In general, moduli spaces can be thought of as geometric solutions to geometric classification problems. It was Mumford (winner of the Fields Medal in 1974) in the 1960s the first one to give precise definitions of what a moduli space is and to describe a method for constructing them, namely geometric invariant theory (GIT). A rich theory that later was shown to have interactions with Symplectic Geometry and Equivariant Topology.

The project

In this project we will learn how GIT can be used to construct the moduli space $\mathcal{M}_2$, parameterizing curves of genus two. It is known that there are one-to-one correspondences between curves of genus two and homogeneous polynomials in two variables of degree six (with distinct roots); and between curves of genus two and pencils of quadrics in $\mathbb{P}^5$. The main idea will be to explore each of these two correspondences in great detail in order to construct the desired moduli space. That is, in order to classify all curves of genus two up to isomorphism.

Understanding this hands-on and classical example will provide the student with a concise introduction to the study of classification problems in Algebraic Geometry and the construction of moduli spaces using GIT.

Prerequisites

An introductory course in algebraic geometry and/or algebraic curves is desirable.

Suggested Bibliography

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