Symplectic manifolds with symmetries and symplectic reduction

Bachelor project, supervisor: F.Pasquotto

Suppose we consider a manifold M (think of spheres or hypersurfaces). At first we are interested in the topology of the manifold, but at some point we may also want to introduce some additional structure: a Riemannian structure (or metric) g, for example, prescribes at every point a way of multiplying two tangent vectors to get a real number (an inner product): this operation is symmetric and gives us the notions of distance, length, and angles. A *symplectic structure* ω defines an anti-symmetric product of vectors: this necessarily vanishes on all 1-dimensional subspaces, so instead of 1-dimensional measurements we have 2-dimensional measurements.

Symplectic structures made their first appearance in the study of classical mechanical systems. Many interesting physical phenomena are described by the solutions of a system of *Hamiltonian differential equations*: these solutions live in the so called *phase space* (for example, the space of positions and velocities in the case of the motion of a planet or a particle), which is in a natural way a symplectic manifold. Illustrating how these problems historically originated in celestial mechanics, the solutions of the Hamiltonian equations are usually called *orbits*. In particular, due to a *conservation of energy* principle, each solution can be seen to lie on a given *energy surface* (think of the surface of the unitary ball in $\mathbb{R}^{2n}$).

The aim of *symplectic reduction* is to find a way to take quotients of symplectic manifolds under group actions (*symmetries*). If the action is *Hamiltonian*, one can first restrict to a level set of the *momentum map*, and then take the quotient of this new manifold: it turns out that this quotient carries a natural symplectic structure.

We will start this project by learning about symplectic manifolds, group actions on symplectic manifolds, definition and properties of the momentum map, and Marsden-Weinstein quotients. The ultimate goal is to understand and compute some explicit examples of reduction (spherical pendulum, n-dimensional rigid body...)

Prerequisites: a working knowledge of differentiable manifolds, tangent and cotangent bundles, vector fields and their flow, Lie groups and Lie algebras.

Literature:

- A. Cannas da Silva, *Lectures on Symplectic Geometry*, Springer Lecture Notes in Mathematics (2006)
- J. Marsden and A. Weinsten, *Reduction of symplectic manifolds with symmetries*, Reports on Mathematical Physics (1974)
- M. Jeffs, *Classical mechanics and symplectic geometry*, Lecture notes (2022)