QUOTIENTS OF CONE COMPLEXES

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1. Groups acting on sets

Let G be a group and X a set. We can define the quotient X/G to be the set of orbits of G in X, or (equivalently) as the initial object in the set of maps $X \to Y$ which are G-equivariant (with the trivial G action on Y).

An alternative definition is to take the quotient *groupoid* $[X/G]$; this is a category whose objects are the elements of X, and where the morphisms from x to y are the elements $g \in G$ such that $gx = y$.

The naive quotient can be recovered from the groupiod quotient by taking the set of connected components. However, especially in the case of a non-free action, I would like to suggest that the groupoid quotient is the better/more useful object. For example, suppose that X and G are both finite. Then in the case of a *free* action we have

$$\#(X/G) = \#X/\#G. \tag{1}$$

However, for a non-free action this is far from true. What about a groupoid version? Here $[X/G]$ is a category, so how to definition its cardinality? Note that, if $C \in \pi_0([X/G])$ is a connected component of a groupoid, then any two elements of C have isomorphic automorphisms groups; in particular, it makes sense to talk about the *size* of the automorphism group of any element of C. We then define

$$\#[X/G] = \sum_{C \in \pi_0([X/G])} \frac{1}{|\operatorname{Aut} C|}. \tag{2}$$

With this definition we find that the formula

$$\#[X/G] = \#X/\#G \tag{3}$$

always holds, regardless of whether the action is free.

2. Cones and cone complexes

Rather than taking quotients of sets, we want to think about quotients of cones. Here I want to think of cones as subsets of $\mathbb{R}^n$ cut out by intersecting a finite number of rational half-spaces through the origin. If I want to do this formally I probably define a cone as a sharp integral saturated finitely generated monoid, but let's not worry about this for now.

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The goal of this project is to understand how to extend the above discussion of quotients of sets be groups, to understanding quotients of cones by groups.

For example, take the cone $\sigma = \mathbb{R}^2_{\geq 0}$ and the group $G = \mathbb{Z}/2\mathbb{Z}$. Let G act on σ by reflection in the line $x = y$. Then the naive quotient is a 'folded' cone, but there is also a groupoid quotient; formally this is defined as a category fibred in groupoids over the category of cones; or even better, a cone *stack* for a suitable Grothendieck topology.

3. Why?

The original motivation for this was in studying moduli spaces of metric graphs. Roughly speaking, this should be some kind of 'space' whose points correspond to graphs given edge lengths which are positive real numbers. For example, consider a circle graph with two edges and two vertices. Then one might expect the moduli space to be $\mathbb{R}^2_{>0}$, but this does not quite work, because there is an automorphism of the graph swapping the two edges. So really the moduli space should be the quotient $[\mathbb{R}^2_{>0}/(\mathbb{Z}/2\mathbb{Z})]$; the above machinery is used to make all this precise.