Coefficients of cyclotomic polynomials

For n an integer, the nth cyclotomic polynomial is defined to be:

$$\Phi_n(x) = \prod_{\substack{1 \leqslant j \leqslant n \\ (j,n)=1}} (x - \zeta_n^j),$$

for $\zeta_n = e^{2\pi i/n}$ an n^{th} root of unity. The cyclotomic polynomial can be written explicitly

$$\Phi_n(x) = \sum_{k=0}^{\varphi(n)} a_n(k) x^k,$$

where $\varphi(n) = \sum_{\substack{1 \leqslant j \leqslant n \\ (j,n)=1}} 1$ is Euler's totient function. The coefficients $a_n(k)$ are integers, as $\Phi_n(x)$ it is the minimal polynomial of the n^{th} root of unity ζ_n.

The behavior of the coefficients is quite well understood if:

- $n = p$ prime: $\Phi_p(x) = 1 + x + \cdots + x^{p-1}$, thus $a_p(k) = 1$ for $k = 0, \ldots, p-1$.
- $n = pq$ for p, q primes: $\Phi_{pq}(x) = \sum_{k=0}^{(p-1)(q-1)} a_{pq}(k) x^k$, with $a_{pq}(k) \in \{0, \pm 1\}$ (Lam and Leung, [?])

For $n = pqr$, p, q, r primes we finally obtain coefficients that are not $0, \pm 1$. In [?], Kaplan finds an explicit formula for the coefficients a_{pqr} based on the equality:

$$\Phi_{pqr}(x) = (1 + x^{pq} + x^{2pq} + \ldots)(1 + x + \cdots + x^{p-1} - x^q - \cdots - x^{q+p-1})\Phi_{pq}(x^r).$$

Using Kaplan's Lemma, there are several possible directions to proceed for a project:

(1) When p is a fixed prime, and q, r vary among the primes, it is expected that the coefficients are bounded by:

$$a_{pqr} < 2p/3.$$

Using Kaplan's Lemma for various primes q, r, one can explore both directly or/and using programming how close to the bound one can get.

(2) Using properties of the cyclotomic polynomials and Kaplan's lemma, it should be possible to compute formulas for coefficients of Φ_{pqrs}, when $n = pqrs$ is the product of 4 primes. This can be explored numerically or done for particular cases.

References

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