BACHELORPROJECT: GRADED BRAUER GROUPS

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Non-commutative rings, albeit underrepresented in every bachelor curriculum plays an important role in mathematics and physics. They show up in unexpected contexts; in topology, mirror symmetry, algebraic geometry and string theory. The main examples of such rings are given by matrix algebras.

Example 1. For some fixed positive integer n, the matrix ring (or algebra over some field k) is given by $n \times n$ matrices with entries in k. One could multiply two such matrices in two different ways and we all know that the answers are not always the same. Let A be a matrix, such that for *any* other matrix B, we obtain $AB = BA$ then it can be shown that A is a diagonal matrix, i.e. $A = c \cdot \mathrm{Id}_{n\times n}$ for some constant $c \in k$. Such elements are called *central elements*. Notice that the set of such elements, called *the center*, is nothing but our field k. Next, with a bit of matrix multiplications, one can observe the following: if we start with any $n \times n$ non-zero matrix A then the ideal generated by multiplying A on *both* sides is the entire matrix algebra!

This is a prototype of a more general concept known as the *central simple algebra*.

Definition 2 (Central Simple Algebra)**.** A central simple algebra (CSA) R over a field k is a finite-dimensional vector space with a multiplication operation such that (i) for any $a, b, c \in R$ the multiplication is associative, i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, (ii) the center of R is exactly k and (iii) there are no proper two sided ideals of R.

Such algebras are not far from matrix algebras (Artin-Wedderburn). Despite their innocent appearance, the set of their equivalence classes, known as *Brauer group*, dictates the geometry of projective spaces. This project aims to establish a strong foundation that is essential for comprehending the geometric aspect.

Very related non-commutative rings are the *Clifford algebras*. In algebraic geometry they are omnipresent. For example, they are useful in studying modules over commutative rings, in telling apart certain 4-dimensional manifolds. They also induce spinor modules, which is motivated by Physics.

Clifford algebras are constructed from bilinear forms on vector spaces. They have two pieces, one of which often forms a CSA, making them special examples of the so-called *graded CSAs*. The set of their equivalence classes is known as the *Brauer–Wall* group.

Project: The goal of this project is to understand Brauer and Brauer–Wall groups well with lots of examples. We will start by learning about non-commutative rings focusing on CSAs. Reference for this is [Art99, Chapter 1,3]. Next, vector spaces with quadratic forms leading to Clifford algebras. This will be covered by [Lam05, Chap. 1, 3, 5] and [Che54, Chap. 1,2]. After this we move to classifications and study Brauer groups and and Brauer–Wall groups. The main literature here is Wall's original paper [Wal64]. We will also follow [Lam05, Chapter 4].

References

[Art99] Michael Artin. Noncommutative rings. *Class Notes, fall* pages 75–77, 1999. URL `https://math.mit.edu/~etingof/artinnotes.pdf`.

[Che54] Claude Chevalley. The algebraic theory of spinors. New York: Columbia University Press. viii, 131 p. (1954)., 1954.

[Lam05] T. Y. Lam. *Introduction to quadratic forms over fields*, volume 67 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2005.

[Wal64] C. T. C. Wall. Graded Brauer groups. *J. Reine Angew. Math.* 213, pp. 187–199, 1964. doi:10.1515/crll.1964.213.187.