Bachelor Project: Dedekind Sums

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Richard Dedekind (1831–1916) introduced and studied the properties of the so-called Dedekind sums $s(a,c)$ defined for integers a and $c \neq 0$ with $\gcd(a,c) = 1$ by

$$s(a,c) = \sum_{n=1}^{|c|-1} \left(\left(\frac{n}{c}\right)\right)\left(\left(\frac{na}{c}\right)\right).$$

Here $((\cdot))\colon \mathbb{R} \longrightarrow \mathbb{R}$ is the sawtooth function defined by the formula

$$((x)) = \begin{cases} x - \lfloor x \rfloor - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}, \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the floor function (e.g., $\lfloor 1.7 \rfloor = 1$). Due to their remarkable arithmetic properties, Dedekind sums are ubiquitous in number theory and deeply connected with the theories of

- modular forms (Dedekind η-function, Ramanujan Δ-function)
- special values of L-functions (ζ-functions of real quadratic fields)
- linking numbers (of modular knots with the trefoil knot).

Recall that the matrix group $\mathrm{SL}_2(\mathbb{Z})$ is defined as

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

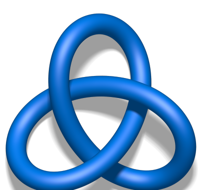
The Dedekind cocycle is a function $\Phi\colon \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathbb{Q}$ defined, for a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, by

$$\Phi(A) = \begin{cases} \frac{b}{d} & \text{if } c = 0 \\ \frac{a+d}{c} - 12\,\mathrm{sign}(c) s(a,c) & \text{if } c \neq 0. \end{cases}$$

Remarkably, this function takes its values in $\mathbb{Z}$ and is *almost* a homomorphism of groups: if $A, B \in \mathrm{SL}_2(\mathbb{Z})$, then

$$\Phi(AB) = \Phi(A) + \Phi(B) - 3\,\mathrm{sign}(c_A c_B c_{AB}),$$

where c_M denotes the lower left entry of a matrix $M \in \mathrm{SL}_2(\mathbb{Z})$.



If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is hyperbolic with $a + d > 2$, then it turns out that the quantity

$$\lim_{n\to\infty} \frac{\Phi(A^n)}{n} \tag{1}$$

is equal to the linking number of a certain knot k_A on the 3-manifold $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ with the trefoil knot. In particular, it is an integer.

Question: How fast does the quantity (1) converge? What happens when A is not hyperbolic?

The student will conduct numerical experiments using SageMath or MAGMA to study this question. Later in the project, it is possible to explore connections with modular forms, L-series, and/or linking numbers depending on the interests of the student.

Prerequisites: Group theory, real/complex analysis (optional), differential geometry (optional).

References

1. W. Duke, O. Imamoḡlu, A. Tóth, *Modular cocycles and linking numbers*, Duke Math. J. **166** (2017), 1179–1210.
2. E. Ghys, *Knots and dynamics*, International Congress of Mathematicians. Vol. I (2007), 247–277.
3. H. Rademacher, E. Grosswald, *Dedekind sums*, The Carus Mathematical Monographs, No. 16 (1972), xvi+102.