# Jacobson's Commutativity Theorem

Mike Daas, Universiteit Leiden

In the year 1945, the American mathematician Nathan Jacobson published a paper [Jac45] containing the following surprising theorem.

**Theorem 1.** *Let $R$ be a ring with the property that, for any $x \in R$, there exists some integer $n(x) \geq 2$ such that $x^{n(x)} = x$. Then $R$ must be commutative.*

Both its conception and its proof followed from a long series of developments in ring theory that aimed to describe the structure of rings in as great a generality as possible. However, as was observed later, special cases of this theorem allowed for much more elementary proofs, and multiple papers were later published in which such shorter or easier proofs were found.

The aims for this project would be two-fold and would allow for the student to choose the second half of the project to their own preference. The first part would be to understand Jacobson's original proof as outlined in T.Y. Lam's book [Lam91] to produce a fully self-contained proof of Jacobson's Theorem from first principles. This would include, but is not limited to, treatments of the following:

- The Jacobson radical $\mathrm{Rad}(R)$ of a ring $R$ and modules over rings;
- Wedderburn's Theorem for finite division rings;
- Herstein's Lemma for division rings of finite characteristic;
- The definition of and structure theorem for semi-primitive rings;
- The definition of and structure theorem for left-primitive rings.

The second part of the project would entail exploring one or more of the following research questions:

- Many "shorter" or "more elementary" proofs of Jacobson's theorem have been published since its first appearance. However, at times clarity was sacrificed for the purpose of conciseness. It would be valuable to take some of these shorter proofs and to work them out in detail; comparing them to Jacobson's original proof. Perhaps inspiration can be gathered to find a novel, short proof.
- An interesting weaker version of Jacobson's theorem is the case that the integer $n(x) = n$ is fixed. For example, for $n = 2$, this would recover the elementary exercise that all Boolean rings are commutative. For a significant density of even $n$, purely equational proofs of Jacobson's theorem are known. It would be interesting to explore the known methods for even exponents and try to improve on them, or analyse the odd exponent case in which much less is known or understood.
- A variant on Jacobson's theorem would be to analyse for which polynomials $p \in \mathbb{Z}[X]$ it is true that any ring in which $p(x) = 0$ for all $x \in R$, it would hold that $R$ must necessarily be commutative. Jacobson's theorem asserts that any polynomial of the form $p(x) = x^n - x$ for $n \geq 2$ would work. Could one find more polynomials for which this implication holds? Or maybe classify them all?
- The center $Z(R)$ of a ring is defined as $Z(R) = \{x \in R \mid xy = yx \ \forall y \in R\}$. If $p \in \mathbb{Z}[X]$, it is a much weaker requirement for a ring to ask that merely $p(x) \in Z(R)$ for all $x \in R$. For which $p \in \mathbb{Z}[X]$ can one still conclude that $R$ itself must be commutative? The degree at most 2 case is known, but for higher degree much less so, and as such this could be interesting to explore.

## References


[Jac45] Nathan Jacobson. Structure theory for algebraic algebras of bounded degree. *Annals of Mathematics*, 46(4):695–707, 1945.

[Lam91] Tsit-Yuen Lam. *A first course in noncommutative rings*, volume 131. Springer, 1991.