A COMPUTATIONAL APPROACH TO PROVING MATHEMATICAL EXPRESSIONS

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> The ultimate goal of mathematics is to eliminate any need for intelligent thought.
>
> *Alfred N. Whitehead*

1. Introduction

About 4000 years ago, the Babylonian tax office employed professional mathematicians to solve quadratic equations. Our modern notation, on the other hand, simplifies this problem to a degree where we teach unsuspecting schoolchildren the solution to the quadratic equation in one blow:

$$ax^2 + bx + c = 0,\ a \neq 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

Developments in recent decades allow for a similar degree of systematization and simplification of what was once a respectable task for mathematicians, namely, that of discovering and proving identities between certain kinds of combinatorial sequences as well as analytic functions.

Here are two straightforward examples that fall within the purview of this technique:

$$\forall n \in \mathbb{Z}, \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2, \tag{2}$$

$$\forall x \in \mathbb{R}, \quad \cos(2x) = \cos(x)^2 - \sin(x)^2. \tag{3}$$

During this project, you will learn how to *discover and prove* identities of the form above in a *purely systematic way*. For instance, the method would discover the following recursion and differential equation, respectively, for the identities above:

$$p(n) - p(n-1) = n^3, \quad p(0) = 0, \tag{4}$$

$$f''(x) + 4f(x) = 0, \quad f(0) = 1,\ f'(0) = 0. \tag{5}$$

This proves the identities above since both sides of the expression satisfy the same recursion (or differential equation) with the same initial conditions.

As you can see above, verifying an identity becomes trivial once you discover the correct recursion or differential equation. The real problem here is to systematically discover—without thinking, as it were—recursions and differential equations for a broad class of mathematical expressions.

Date: January 14, 2024.

2. Objectives of this project

The primary objective of this project is for you to develop an effective computational understanding of certain types of recurrence sequences and ordinary differential equations. This will be an excellent complement to your theoretical prowess.

As a secondary objective, I will encourage you to write your computer programs to deepen your understanding of this computational perspective. Anything from small scripts to more extended program modules is possible, depending on your interest. As a mathematical programming language, you may use any one of Mathematica, Maple, Magma, SageMath, etc.

3. Necessary background

A good grasp of linear algebra and calculus will be sufficient. Absolutely no prior coding experience is necessary. Depending on what you know, we will decide on appropriate completion criteria for the project.

4. References

(1) The enjoyable book "A=B" by Petkovsek, Wilf and Zeilberger is our primary reference and starting point. Link to the book.
(2) As an advanced option, we can complement "A=B" with "Hypergeometric Summation" by Koepf. Link to the book.

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